CS430 Homework 1

Due: 6pm, Sep. 7, 2023.

1. Compute n^{15} using 6 multiplications.

2. Sort 5 distinct integers using at most 7 comparisons.

3. Suppose $T(0) = 0$ and $T(n) \leq \sqrt{n}T(\sqrt{n}) + n$ for each $n \geq 1$. Show that $T(n) = O(n \log \log n)$.

4. The following recurrence relation arises from the average-case analysis of Quicksort: $T(0) = T(1) = 0$; and for each $n \geq 2$,

$$T(n) = n - 1 + \frac{1}{n} \sum_{i=1}^{n} [T(i-1) + T(n-i)]$$

   Show that for each $n \geq 1$, $T(n) = 2(n+1)H(n) - 4n$ where $H(n) = \sum_{i=1}^{n} \frac{1}{k}$ is the n-th harmonic number.

5. The Fibonacci numbers are defined by the recurrence relation: $F_0 = 0$, $F_1 = 1$; and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Suppose the Euclidean algorithm for the positive integer pair $a > b$ takes n iterations.

   - Prove by induction on n that $a \geq F_{n+2}$ and $b \geq F_{n+1}$.
   - Show that $n \leq 5 \log_{10} b$.

6. Given a sequence of n integers, deciding whether all integers in the sequence are unique is known as **Element Uniqueness**. Suppose any algorithm for **Element Uniqueness** requires $\Omega(n \log n)$ time. Show that computing the closest pair of n points in an Euclidean plane requires $\Omega(n \log n)$ time.