MSDS 596 Regression & Time Series

ARMA and ARIMA Models

Department of Statistics
Rutgers University

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Models for Stationary Time Series

Let $\{y_t\}$ be an observed time series and $\{e_t\}$ an unobserved white noise. $\{y_t\}$ is said to be a **general linear process** if it can be represented as

$$y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots$$

- In order for the above expression to make sense, we require that

$$\sum_{i=1}^{\infty} \psi_i^2 < \infty.$$

- Suppose $\psi_0 = 1$ throughout.
- Autoregressive (**AR**), moving average (**MA**) and **ARMA** processes.

Example: a general linear process

$\{y_t\}$ is a **general linear process** if it can be represented as

$$y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots$$

Suppose $\psi_i = \phi^i$ for some $|\phi| < 1$. That is,

$$y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots$$

- $E(Y_t) = 0$;
- $Var(Y_t) = \sigma^2/(1-\phi^2)$, where $Var(e_t) = \sigma^2$;
    - *Geometric series sum: $1 + a + a^2 + \cdots = 1/(1-a)$ for $|a| < 1$;
- $Cov(Y_t, Y_{t-k}) = \phi^k \sigma^2/(1-\phi^2)$, and $Corr(Y_t, Y_{t-k}) = \phi^k$;

For GLPs, can show that

$$E(Y_t) = 0, \qquad \gamma_k = Cov(Y_t, Y_{t-k}) = \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}.$$

Moving Average Processes

A **moving average (MA) process** is a general linear process with a finite number of nonzero ψ weights. In particular,

- The time series $\{Y_t\}$ is an MA process of order q, denoted MA(q), if for every t
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},$$
where $\{e_t\}$ is WN$(0, \sigma^2)$.
- MA processes are always stationary.

MA(1) Process

$$Y_t = e_t - \theta e_{t-1}.$$

- $E(Y_t) = 0$;
- $Var(Y_t) = (1 + \theta^2)\sigma^2 = \gamma_0$;
- $Cov(Y_t, Y_{t-1}) = -\theta\sigma^2 = \gamma_1$ and
  $Cov(Y_t, Y_{t-k}) = 0 = \gamma_k$ for $k \geq 2$;
- Therefore, $\rho_1 = -\theta/(1 + \theta^2)$, and $\rho_k = 0$ for $k \geq 2$.
  (Same derivation as for the two-point moving average.)
- Notice that
  - MA(1) process has no correlation beyond lag 1;
  - ρ_1 decreases as θ increases;

MA(2) Process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

- $E(Y_t) = 0$;
- $Var(Y_t) = (1 + \theta_1^2 + \theta_2^2)\sigma^2 = \gamma_0$;
- $Cov(Y_t, Y_{t-1}) = (-\theta_1 + \theta_1\theta_2)\sigma^2 = \gamma_1$,
  $Cov(Y_t, Y_{t-2}) = -\theta_2\sigma^2 = \gamma_2$, and
  $Cov(Y_t, Y_{t-k}) = 0 = \gamma_k$ for $k \geq 3$;
- $\rho_k = \begin{cases} (-\theta_1 + \theta_1\theta_2) \,/\, \left(1 + \theta_1^2 + \theta_2^2\right) & k = 1 \\ -\theta_2 / \left(1 + \theta_1^2 + \theta_2^2\right) & k = 2 \\ 0 & k \geq 3 \end{cases}$.
- The ACF cuts off after lag 2.

MA(q) Process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.$$

- $E(Y_t) = 0$;
- $\gamma_0 = (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma^2$;
- $\rho_k = \begin{cases} \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \cdots + \theta_q^2} & k = 1, 2, \ldots, q \\ 0 & k \geq q + 1 \end{cases}.$
- Again, the ACF cuts off after lag q.

Autoregressive Processes

The time series $\{Y_t\}$ is an **autoregressive (AR) process** of order p, denoted AR(p), if for every t

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t,$$

where $\{e_t\}$ is $\mathrm{WN}(0, \sigma^2)$.

- Assume $E(Y_t) = 0$ (because if not, can replace Y_t with $Y_t - \mu$);

AR(1)

$$Y_t = \phi Y_{t-1} + e_t \qquad \leftarrow \textit{the "defining recursion"}$$

- $Var(Y_t) = \phi^2 Var(Y_{t-1}) + \sigma^2$, where $Var(Y_t) = Var(Y_{t-1})$ if stationary;
- Thus, $\gamma_0 = Var(Y_t) = \sigma^2/(1-\phi^2)$, if $|\phi| < 1$;
- $Cov(Y_t, Y_{t-k}) = E(Y_t Y_{t-k}) = \phi E(Y_{t-1} Y_{t-k}) + E(e_t Y_{t-k})$;
- Thus for $k = 1, 2, \ldots$,

$$\gamma_k = \phi\gamma_{k-1} = \phi(\phi\gamma_{k-2}) = \cdots = \phi^k \gamma_0.$$

- Similarly, $\rho_k = \gamma_k/\gamma_0 = \phi^k$. That is, the ACF of AR(1) exhibits exponential decay (up to sign) according to the damping factor ϕ.
- These equations called the Yule-Walker equations for AR(1).

Recap so far

- General linear process
- Moving average
- AR (1)

Is $AR(1)$ a generalized linear process?

AR(1): GLP representation

- Write the defining recursion of AR(1) as

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t = \phi\left(\phi Y_{t-2} + e_{t-1}\right) + e_t = \cdots \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k} \\ &\vdots \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} \ldots \end{aligned}$$

  which is the GLP representation as we've seen before.
- Since we require $\sum_{i=1}^{\infty} \psi_i^2 < \infty$ for a GLP,

$$|\phi| < 1 \qquad \Longleftrightarrow \qquad \text{AR(1) is stationary.}$$

- In general, an AR process is said to be causal if it possesses a GLP representation.

AR(2)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

Stationarity?

- Introduce the **characteristic polynomial** of AR(2):

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2,$$

  and the **characteristic equation**: $\phi(z) = 0$.
- The characteristic equation has two **roots**: $\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$, which may be real or complex.
- An AR(2) process is stationary if and only if both roots are **larger than 1 in modulus**.
  - Equivalently, $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, $|\phi_2| < 1$.

AR(2)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

The ACF of AR(2):

- Consider $E(Y_t Y_{t-k}) = E(Y_{t-k}(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t))$.
- Yule-Walker equations for AR(2):

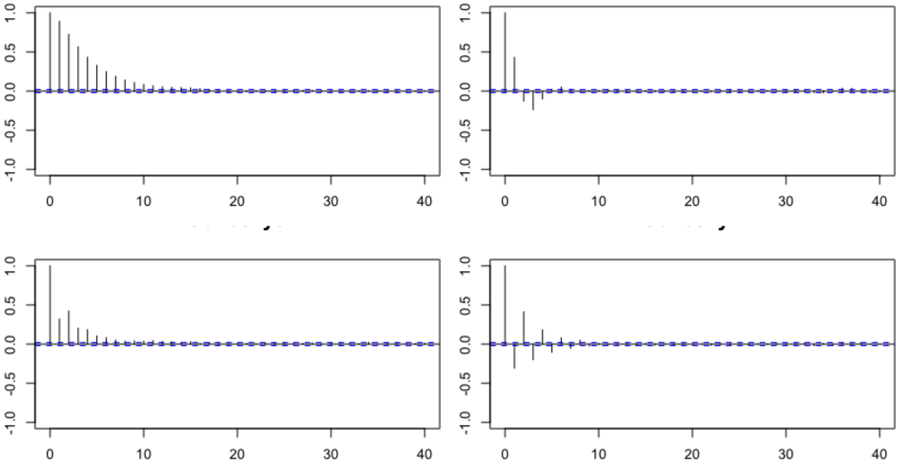
$$\gamma_k = \begin{cases} \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2 & \text{for } k = 0, \\ \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} & \text{for } k > 0, \end{cases}$$

  and the ACF satisfies $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$ for $k > 0$.
- Use $k = 1$, $\rho_0 = 1$, and $\rho_{-1} = \rho_1$ to start the recursion:

$$\begin{aligned} \rho_1 &= \phi_1 / (1 - \phi_2) \\ \rho_2 &= \phi_1\rho_1 + \phi_2\rho_0 = \left\{\phi_2 (1 - \phi_2) + \phi_1^2\right\} / (1 - \phi_2) \\ &\vdots \end{aligned}$$

Autoregressive process: AR(2)

AR(2), clockwise from top left:
$(\phi_1, \phi_2) = (1.2, -.35)\,;\ (.6, -.4)\,;\ (-.2, -.35)\,;\ (-.2, .35)\,.$



AR(p)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t,$$

Stationarity:

- The characteristic polynomial for an AR(p) process is

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$

  and the characteristic equation: $\phi(z) = 0$.
- The AR(p) process is stationary if and only if all roots to the characteristic equation are larger than 1 in modulus.
- In particular, stationarity implies that

$$\phi_1 + \phi_2 + \cdots + \phi_p < 1, \quad \& \quad |\phi_p| < 1,$$

  although these alone aren't enough.

AR(p)

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t,$$

The ACF of AR(p):

- Again consider $E(Y_t Y_{t-k})$, noticing that $E(e_t Y_t) = \sigma^2$.
- Yule-Walker equations for AR(p):

$$\begin{cases} \gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_k \gamma_{k-p} + \sigma^2 & \text{when } k = 0, \\ \gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_k \gamma_{k-p} & \text{when } k > 0. \end{cases}$$

- Solve the first $p+1$ equations corresponding to $k = 0, 1, \ldots, p$ to get $\gamma_0, \gamma_1, \ldots, \gamma_p$, and then compute the γ_k recursively for $k \geq p+1$.

Invertibility

Take the MA(1) process, $Y_t = e_t - \theta e_{t-1}$.

- The ACF is

$$\rho_1 = -\theta/(1+\theta^2), \& \quad \rho_k = 0,\ k \geq 2.$$

  Both θ and its reciprocal $1/\theta$ result in the same ρ, creating non-identifiability of the model.

- Can the MA process be expressed as an AR process?

$$\begin{aligned} e_t &= Y_t + \theta\left(Y_{t-1} + \theta e_{t-2}\right) = Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} = \cdots \\ &= Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \cdots \end{aligned}$$

  Or alternatively,

$$Y_t = \left(-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} \cdots\right) + e_t$$

- If $|\theta| < 1$, the MA(1) process has been "inverted" into an AR(∞) process, and we say that the MA(1) process is invertible.

Invertibility

For MA(q) process,

- The characteristic polynomial for an MA(q) process is

$$\theta(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q,$$

  and the characteristic equation: $\theta(z) = 0$.
- The MA(q) process $\{Y_t\}$ is said to be invertible if and only if all roots to the characteristic equation are larger than 1 in modulus.
- For a given ACF, there is a unique set of parameter values that yield an invertible MA process.
- For this reason, we require MA processes to be invertible.

(Mixed) Autoregressive Moving Average Processes

The time series $\{Y_t\}$ is said to be an **(mixed) autoregressive moving average process** of order (p, q), denoted ARMA(p, q), if for every t

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}, \end{aligned}$$

where $\{e_t\}$ is a $\mathrm{WN}(0, \sigma^2)$.

ARMA(1, 1)

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}.$$

- Notice $E(e_t Y_t) = \sigma^2$, and $E(e_{t-1} Y_t) = E(e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})) = (\phi - \theta)\sigma^2.$
- Calculate $E(Y_t Y_{t-k})$ to get the Yule-Walker equations

$$\begin{aligned}
\gamma_0 &= \phi\gamma_1 + \{1 - \theta(\phi - \theta)\}\sigma^2 \\
\gamma_1 &= \phi\gamma_0 - \theta\sigma^2 \\
\gamma_k &= \phi\gamma_{k-1} \qquad \text{for } k \geq 2
\end{aligned}$$

- We obtain

$$\begin{aligned}
\gamma_0 &= \left(1 - 2\phi\theta + \theta^2\right)\sigma^2 / \left(1 - \phi^2\right) \\
\rho_k &= (1 - \theta\phi)(\phi - \theta)\phi^k / \left(1 - \phi^2\right) \qquad \text{for } k \geq 1.
\end{aligned}$$

Notice that starting from lag 2, the ACF exhibits exponential decay w/ damping factor ϕ.

ARMA(1, 1)

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}.$$

- The GLP representation of ARMA(1, 1) is

$$Y_t = e_t + (\phi - \theta) \sum_{i=1}^{\infty} \phi^{i-1} e_{t-i},$$

  that is, $\psi_i = (\phi - \theta)\, \phi^{i-1}$ for $i \geq 1$.
- The ARMA(1, 1) process is stationary if $|\phi| < 1$, i.e. the AR(1) component is stationary.

ARMA(p, q)

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\ &\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}, \end{aligned}$$

- The characteristic polynomials for an ARMA(p,q) process are defined as

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$
$$\theta(z) = 1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q.$$

- The ARMA(p, q) process is stationary if its AR(p) component is stationary, i.e. if all roots to $\phi(z) = 0$ are larger than 1 in modulus.
- The ARMA(p, q) process is invertible if its MA(q) component is invertible, i.e. if all roots to $\theta(z) = 0$ are larger than 1 in modulus.
- We require ARMA processes to be stationary and invertible.

ARMA(p, q)

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\
&\quad -\theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},
\end{aligned}
$$

- The GLP representation of ARMA(p, q) is

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= -\theta_1 + \phi_1 \\
\psi_2 &= -\theta_2 + \phi_2 + \phi_1\psi_1 \\
&\vdots \\
\psi_k &= -\theta_k + \phi_p\psi_{k-p} + \phi_{p-1}\psi_{k-p+1} + \cdots + \phi_1\psi_{k-1}
\end{aligned}
$$

  assuming $\psi_k = 0$ for all $k < 0$, and $\theta_k = 0$ for all $k > q$.
- The ACF of ARMA(p, q) satisfies for all $k > q$,

$$
\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \cdots + \phi_p\rho_{k-p}.
$$

  For $k \leq q$, ρ_k also involves the θ terms.

AR(p) estimation: Yule-Walker equations

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{p-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \cdots \\ \gamma_p \end{pmatrix},$$

and

$$\sigma^2 = \gamma_0 - \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_p \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \cdots \\ \gamma_p \end{pmatrix}.$$

The estimates of ϕ and σ^2 can be obtained by plugging in the sample autocovariances $\hat{\gamma}_0, \hat{\gamma}_1, \ldots \hat{\gamma}_p$ and solving the equations.

Nonstationary time series

- Example: a time series $\{Y_t\}$ is called a random walk if it satisfies:

$$Y_t = Y_{t-1} + e_t,$$

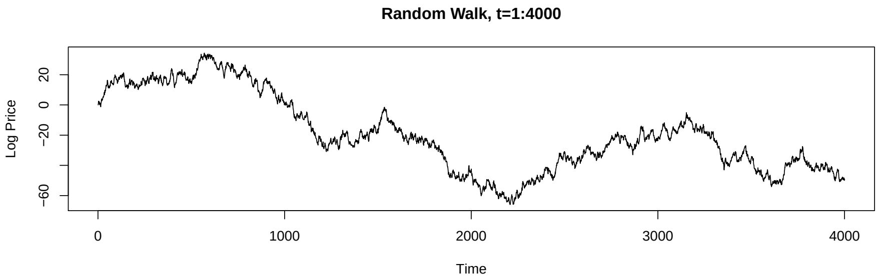
where e_t are i.i.d. with mean zero and variance σ^2.

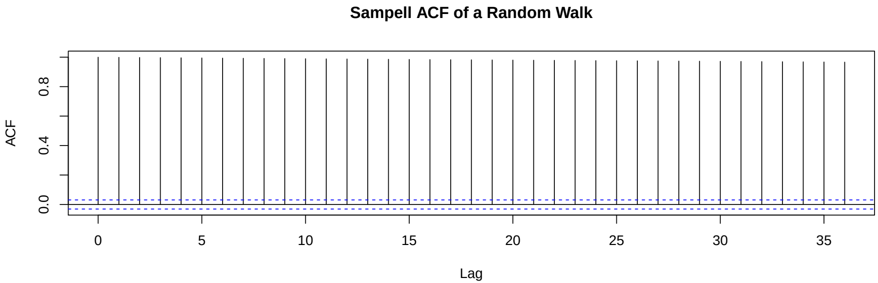
- It is an AR(1) model with coefficient $\phi_1 = 1$ and $\phi_0 = 0$.
    - Nonstationary: the variance diverges as t increases.
    - Strong memory: the sample ACF approaches 1 for any finite lag.
    - Unpredictable: l-step ahead forecast is $\hat{r}_h(l) = Y_h$, and $\mathrm{Var}(e_h(l)) = l\sigma^2$.
- A random walk with drift takes the form $Y_t = \mu + Y_{t-1} + e_t$.
    - Same properties as a random walk;
    - In addition, it has a time trend with slope μ:

$$Y_t = \mu t + Y_0 + e_1 + e_2 + \cdots + e_t.$$

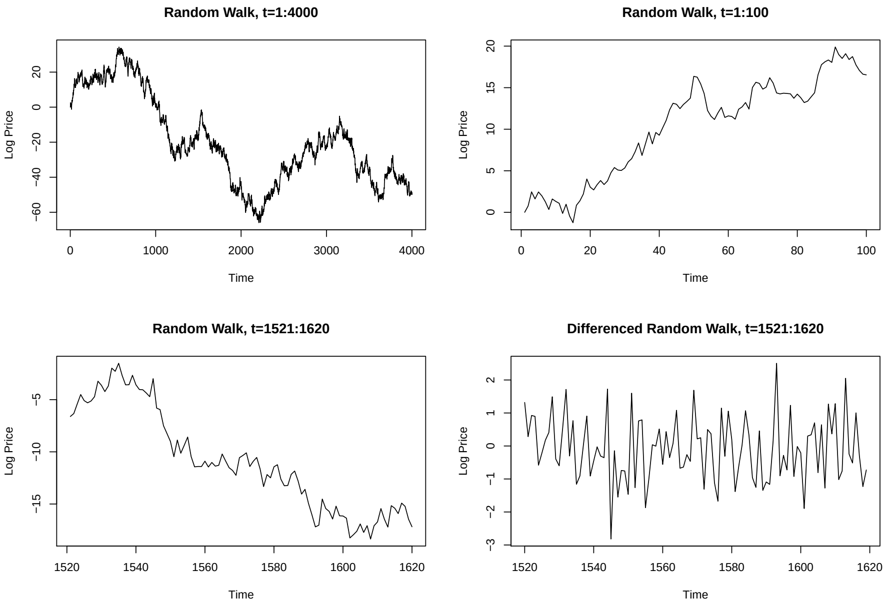
- Differencing: $\nabla Y_t := Y_t - Y_{t-1}$ leads to a white noise (with nonzero mean, if there is a drift).

Example: ACF does not decay

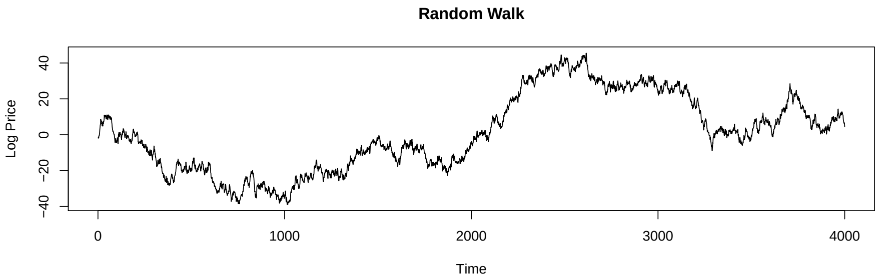


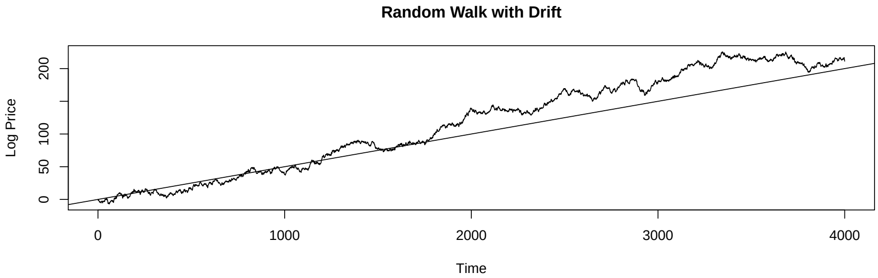


Example: Trend over a Short Time Period



Example: Random Walk with a Drift





The Backshift Operator

The **backshift operator** B operates on the time index of a series and shifts time back one time unit to form a new series. That is,

$$Y_{t-1} = BY_t.$$

- B is linear: $B(aY_t + bX_t + c) = aBY_t + bX_t + c$;
- For any positive integer m, $B^m Y_t = Y_{t-m}$.
- A general AR(p) process can be written as

  $$\phi(B)Y_t = e_t$$

  where $\phi(B)$ is the AR characteristic polynomial evaluated at B;
- A general MA(q) process can be written as

  $$Y_t = \theta(B)e_t$$

  where $\theta(B)$ is the MA characteristic polynomial evaluated at B;
- Combining the two, a general ARMA(p, q) process can be written as

  $$\phi(B)Y_t = \theta(B)e_t$$

The Differencing Operator

The **differencing operator** $\nabla = 1 - B$. Therefore,

$$\nabla Y_t = (1 - B)Y_t = Y_t - Y_{t-1}.$$

- The **second difference**

$$\nabla^2 Y_t = (1-B)^2 Y_t = (1-B)(Y_t - Y_{t-1}) = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2});$$

- The *d*-**th difference** $\nabla^d Y_t = (1 - B)^d Y_t$;
- To be distinguished from the **seasonal difference** of period s:

$$\nabla_s Y_t = (1 - B^s)Y_t = Y_t - Y_{t-s}.$$

ARIMA Models

- A time series $\{Y_t\}$ is said to be an autoregressive integrated moving average process of order $(p, 1, q)$, denoted ARIMA$(p, 1, q)$, if the differenced series $W_t = \nabla Y_t = Y_t - Y_{t-1}$ follows a ARMA(p, q) model.
- In general, a time series $\{Y_t\}$ is said to be an autoregressive integrated moving average process of order (p, d, q), denoted by ARIMA(p, d, q), if the d-th difference $\nabla^d Y_t$ follows a ARMA(p, q) model.

$$\begin{aligned}
\nabla Y_t &= Y_t - Y_{t-1} \\
\nabla^2 Y_t &= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) = Y_t - 2Y_{t-1} + Y_{t-2} \\
\nabla^3 Y_t &= (Y_t - 2Y_{t-1} + Y_{t-2}) - (Y_{t-1} - 2Y_{t-2} + Y_{t-3})
\end{aligned}$$

Examples of ARIMA Models

- Random walk (ARIMA(0, 1, 0), or the I(1) process): $e_t = \nabla Y_t$.
- IMA(1, 1) = ARIMA(0, 1, 1): $W_t = \nabla Y_t = e_t - \theta e_{t-1}$;
- ARI(1, 1) = ARIMA(1, 1, 0): $\nabla Y_t = \nabla Y_{t-1} + e_t$, for $|\phi| < 1$;
- IMA(2, 2): $W_t = \nabla^2 Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$.