



Model Predictive Control

7. Reference Tracking and Disturbance Rejection

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Reference Tracking based on Target Calculation

- Discrete-Time Linear Time-Invariant (LTI) System

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad \text{state equation} \quad (7.1)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \quad \text{measured output equation} \quad (7.2)$$

$$\mathbf{y}_r(k) = \mathbf{C}_r\mathbf{x}(k) \quad \text{controlled output equation} \quad (7.3)$$

- Symbols

$\mathbf{x}(k) \in \mathbb{R}^n$ state vector

$\mathbf{u}(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ input vector

$\mathbf{y}(k) \in \mathbb{Y} \subseteq \mathbb{R}^p$ measured output vector

$\mathbf{y}_r(k) \in \mathbb{R}^{p_r}$ controlled output vector

$\mathbf{A} \in \mathbb{R}^{n \times n}$ system matrix

$\mathbf{B} \in \mathbb{R}^{n \times m}$ input matrix

$\mathbf{C} \in \mathbb{R}^{p \times n}$ measured output matrix

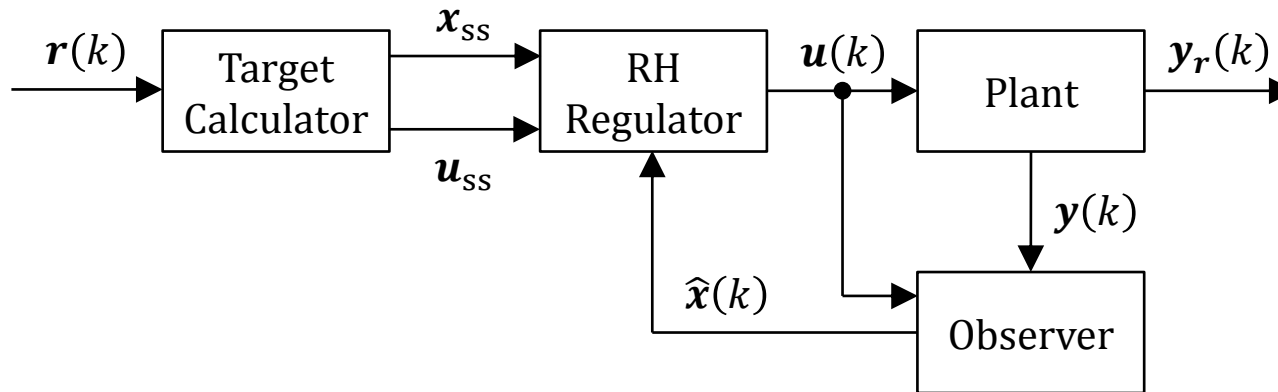
$\mathbf{C}_r \in \mathbb{R}^{p_r \times n}$ controlled output matrix

- Remarks

- The measured output $\mathbf{y}(k)$ is used for the observer
- The controlled output $\mathbf{y}_r(k)$ is considered for reference tracking

Reference Tracking based on Target Calculation

- Structure



- Objective

- Control the discrete-time LTI system such that $y_r(k) \rightarrow r(k)$ if $r(k) \rightarrow \text{const.}$ as $k \rightarrow \infty$

- Approach

- The **target calculator** computes the **target state x_{ss}** and the **target input u_{ss}**
- The **RH regulator** controls the discrete-time LTI system to the **target pair (x_{ss}, u_{ss})**

Reference Tracking based on Target Calculation

- Approach

- Consider that the discrete-time LTI system (7.1)/(7.3) is in the steady **target state** \mathbf{x}_{ss} , i.e.

$$\mathbf{x}_{ss} = \mathbf{A}\mathbf{x}_{ss} + \mathbf{B}\mathbf{u}_{ss} \Leftrightarrow (\mathbf{I}_{n \times n} - \mathbf{A})\mathbf{x}_{ss} - \mathbf{B}\mathbf{u}_{ss} = \mathbf{0}_{n \times 1}$$

$$\mathbf{y}_{r_{ss}} = \mathbf{C}_r \mathbf{x}_{ss} = \mathbf{r} \Leftrightarrow \mathbf{C}_r \mathbf{x}_{ss} = \mathbf{r}$$

- Rewrite the equations in **matrix form**, i.e.

$$\begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C}_r & \mathbf{0}_{p_r \times m} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{ss} \\ \mathbf{u}_{ss} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n \times 1} \\ \mathbf{r} \end{pmatrix} \quad (7.4)$$

- The steady **target pair** $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$ can be then calculated from (7.4) provided that a solution exists
- Introduce the **state deviation** and **input deviation**

$$\tilde{\mathbf{x}}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}_{ss}, \quad \tilde{\mathbf{u}}(k) = \mathbf{u}(k) - \mathbf{u}_{ss}$$

- This leads to the **discrete-time LTI state equation**

$$\tilde{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k+1) - \mathbf{x}_{ss} = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) - (\mathbf{A}\mathbf{x}_{ss} + \mathbf{B}\mathbf{u}_{ss}) = \mathbf{A}\tilde{\mathbf{x}}(k) + \mathbf{B}\tilde{\mathbf{u}}(k) \quad (7.5)$$

Reference Tracking based on Target Calculation

- Approach

the **discrete-time quadratic cost function**

$$\tilde{V}_N(\tilde{\mathbf{x}}(k), \tilde{\mathbf{U}}(k)) = \tilde{\mathbf{x}}^T(k+N) \mathbf{P} \tilde{\mathbf{x}}(k+N) + \sum_{i=0}^{N-1} \tilde{\mathbf{x}}^T(k+i) \mathbf{Q} \tilde{\mathbf{x}}(k+i) + \tilde{\mathbf{u}}^T(k+i) \mathbf{R} \tilde{\mathbf{u}}(k+i) \quad (7.6)$$

the **state** and **input constraints**

$$\begin{aligned} \mathbf{C}(\tilde{\mathbf{x}}(k+i) + \mathbf{x}_{ss}) &\in \mathbb{Y}, i = 1, 2, \dots, N \\ \tilde{\mathbf{u}}(k+i) + \mathbf{u}_{ss} &\in \mathbb{U}, \quad i = 0, 1, \dots, N-1 \end{aligned} \quad (7.7)$$

- Rewrite **Problem 4.1/5.1** w.r.t. the state equation (7.5), cost function (7.6) and constraints (7.7), i.e.

$$\begin{aligned} &\min_{\tilde{\mathbf{U}}(k)} \tilde{V}_N(\tilde{\mathbf{x}}(k), \tilde{\mathbf{U}}(k)) \\ &\text{subject to } \begin{cases} \tilde{\mathbf{x}}(k+i+1) = \mathbf{A}\tilde{\mathbf{x}}(k+i) + \mathbf{B}\tilde{\mathbf{u}}(k+i), i = 0, 1, \dots, N-1 \\ \mathbf{C}(\tilde{\mathbf{x}}(k+i) + \mathbf{x}_{ss}) \in \mathbb{Y}, & i = 1, 2, \dots, N \\ \tilde{\mathbf{u}}(k+i) + \mathbf{u}_{ss} \in \mathbb{U}, & i = 0, 1, \dots, N-1 \end{cases} \end{aligned} \quad (7.8)$$

Reference Tracking based on Target Calculation

- Remark on the Optimization Problem

- Problem (7.8) can be formulated as a **quadratic program** using the methods from Chapter 4 and 5
- Problem (7.8) relates to a **regulation problem**, i.e. $\tilde{\mathbf{x}}(k) \rightarrow \mathbf{0}, \tilde{\mathbf{u}}(k) \rightarrow \mathbf{0}$ as $k \rightarrow \infty$
- The **reference tracking problem** is, however, simultaneously addressed since
$$\tilde{\mathbf{x}}(k) \rightarrow \mathbf{0}, \tilde{\mathbf{u}}(k) \rightarrow \mathbf{0} \Rightarrow \hat{\mathbf{x}}(k) \rightarrow \mathbf{x}_{ss}, \mathbf{u}(k) \rightarrow \mathbf{u}_{ss} \Rightarrow \mathbf{y}_r(k) \rightarrow \mathbf{r}$$

- Remark on the Target Calculator

- Generally it is not possible to control the state $\mathbf{x}(k)$ to an arbitrary target state \mathbf{x}_{ss}
- E.g. it is not possible to maintain a constant position and a constant velocity of a car simultaneously
- A **sufficient condition** for the **existence of a solution of (7.4)** for any reference input \mathbf{r} is that

$$\begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C}_r & \mathbf{0}_{p_r \times m} \end{pmatrix} \in \mathbb{R}^{(n+p_r) \times (n+m)} \text{ has full rank } n + p_r$$

- This implies that \mathbf{C}_r must have full rank and number of controlled outputs $p_r \leq$ number of inputs m
- The solution may not be unique

Reference Tracking based on Target Calculation

- Remark on the Constrained Case

- For the constrained case target pair $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$ must fulfill output constraint \mathbb{Y} and input constraint \mathbb{U}
- For this purpose the target calculator based on (7.4) must be modified to

$$\begin{aligned} \min_{\mathbf{x}_{ss}, \mathbf{u}_{ss}} \quad & \frac{1}{2} \left((\mathbf{C}_r \mathbf{x}_{ss} - \mathbf{r})^T \mathbf{Q}_{ss} (\mathbf{C}_r \mathbf{x}_{ss} - \mathbf{r}) + (\mathbf{u}_{ss} - \mathbf{u}_{ss}^{\text{unc}})^T \mathbf{R}_{ss} (\mathbf{u}_{ss} - \mathbf{u}_{ss}^{\text{unc}}) \right) \\ \text{subject to} \quad & \begin{cases} \begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C}_r & \mathbf{0}_{p_r \times m} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{ss} \\ \mathbf{u}_{ss} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{n \times 1} \\ \mathbf{r} \end{pmatrix} \\ \mathbf{C} \mathbf{x}_{ss} \in \mathbb{Y} \\ \mathbf{u}_{ss} \in \mathbb{U} \end{cases} \end{aligned} \quad (7.9)$$

where $\mathbf{u}_{ss}^{\text{unc}}$ is the target input resulting from (7.4) for the unconstrained case and $\mathbf{Q}_{ss} = \mathbf{Q}_{ss}^T \succcurlyeq \mathbf{0}$ and $\mathbf{R}_{ss} = \mathbf{R}_{ss}^T \succ \mathbf{0}$ are weighting matrices

- Problem (7.9) can be formulated as a quadratic program
- Feasibility of Problem (7.9) is discussed in [RM09, Section 1.5.1]

Reference Tracking based on Target Calculation

- Remark on Receding Horizon Control

1. Estimate the current state $\hat{\mathbf{x}}(k)$
2. Solve Problem (7.9) for the given reference input $\mathbf{r}(k)$ to determine the target pair $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$
3. Solve Problem (7.8) for $\tilde{\mathbf{x}}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}_{ss}$ to determine the optimal input sequence $\tilde{\mathbf{U}}^*(k)$
4. Compute first element of the optimal input sequence $\tilde{\mathbf{u}}^*(k) = (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \tilde{\mathbf{U}}^*(k)$
5. Implement the optimal input $\mathbf{u}^*(k) = \tilde{\mathbf{u}}^*(k) + \mathbf{u}_{ss}$
6. Increment the time instant $k := k + 1$ and go to 1.

- Further Remarks

- The extension for state constraints and controlled output constraints is straightforward
- The structure on Slide 7-3 is essentially equivalent to the state-command structure on Slide 2-49ff
- Disturbances and uncertainties in \mathbf{A} , \mathbf{B} and \mathbf{C}_r lead to a steady-state error or offset
- More details and references are given in [RM09, Section 1.5.1], [BBM15, Section 13.6], and [MR93]

Reference Tracking based on the Delta Input Formulation

- Discrete-Time Linear Time-Invariant (LTI) System

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad \text{state equation} \quad (7.10)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \quad \text{output equation} \quad (7.11)$$

- Symbols

$\mathbf{x}(k) \in \mathbb{R}^n$ state vector

$\mathbf{u}(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ input vector

$\mathbf{y}(k) \in \mathbb{Y} \subseteq \mathbb{R}^p$ output vector

$\mathbf{A} \in \mathbb{R}^{n \times n}$ system matrix

$\mathbf{B} \in \mathbb{R}^{n \times m}$ input matrix

$\mathbf{C} \in \mathbb{R}^{p \times n}$ output matrix

- Objective

- Control the discrete-time LTI system such that $\mathbf{y}(k) \rightarrow \mathbf{r}(k)$

Reference Tracking based on the Delta Input Formulation

- Approach

- Introduce the **input increment**

$$\Delta \mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}(k-1)$$

- Introduce the **discrete-time quadratic cost function**

$$\begin{aligned} V_N(\mathbf{y}(k), \Delta \mathbf{U}(k), \mathbf{r}(k)) &= (\mathbf{y}(k+N) - \mathbf{r}(k))^T \mathbf{P} (\mathbf{y}(k+N) - \mathbf{r}(k)) \\ &+ \sum_{i=0}^{N-1} (\mathbf{y}(k+i) - \mathbf{r}(k))^T \mathbf{Q} (\mathbf{y}(k+i) - \mathbf{r}(k)) + \Delta \mathbf{u}^T(k+i) \mathbf{R} \Delta \mathbf{u}(k+i) \end{aligned} \quad (7.12)$$

with the reference input $\mathbf{r}(k)$ and the weighting matrices $\mathbf{P} = \mathbf{P}^T > \mathbf{0}$, $\mathbf{Q} = \mathbf{Q}^T > \mathbf{0}$, $\mathbf{R} = \mathbf{R}^T > \mathbf{0}$

- Note that $\mathbf{r}(k)$ is **held** over the whole prediction horizon
- Note that $\mathbf{y}(k+i) - \mathbf{r}(k)$ is the **control error**

Reference Tracking based on the Delta Input Formulation

- Approach

- Reformulate **Problem 4.1/5.1** w.r.t. the system (7.10)/(7.11), cost function (7.12) and $\Delta \mathbf{u}(k)$, i.e.

$$\begin{aligned} & \min_{\Delta \mathbf{U}(k)} V_N(\mathbf{y}(k), \Delta \mathbf{U}(k), \mathbf{r}(k)) \\ & \text{subject to } \begin{cases} \mathbf{x}(k+i+1) = \mathbf{A}\mathbf{x}(k+i) + \mathbf{B}\mathbf{u}(k+i), & i = 0, 1, \dots, N-1 \\ \mathbf{y}(k+i) = \mathbf{C}\mathbf{x}(k+i), & i = 0, 1, \dots, N \\ \mathbf{y}(k+i) \in \mathbb{Y}, & i = 1, 2, \dots, N \\ \mathbf{u}(k+i) \in \mathbb{U}, & i = 0, 1, \dots, N-1 \\ \mathbf{u}(k+i) = \mathbf{u}(k+i-1) + \Delta \mathbf{u}(k+i), & i = 0, 1, \dots, N-1 \end{cases} \end{aligned} \quad (7.13)$$

- Problem (7.13) can be formulated as a **quadratic program** using the methods from Chapter 4 and 5
- Problem (7.13) relates to a **regulation problem**, i.e. $\mathbf{y}(k) - \mathbf{r}(k) \rightarrow \mathbf{0}, \Delta \mathbf{u}(k) \rightarrow \mathbf{0}$ in steady state
- Note that the **reference tracking problem** has been transformed into a regulation problem using $\mathbf{y}(k+i) - \mathbf{r}(k)$ instead of $\mathbf{x}(k+i)$ and $\Delta \mathbf{u}(k)$ instead of $\mathbf{u}(k)$ (generally $\mathbf{u}(k) \rightarrow \mathbf{0}$ in steady state)

Reference Tracking based on the Delta Input Formulation

- Approach

- The **prediction model (4.4)**, the **cost function (4.5)**, and the **constraint model (5.1)** must be reformulated to obtain a quadratic program
- For this purpose the discrete-time LTI system (7.10)/(7.11) is augmented w.r.t. $\Delta \mathbf{u}(k)$, i.e.

$$\underbrace{\begin{pmatrix} \mathbf{x}(k+1) \\ \mathbf{u}(k) \end{pmatrix}}_{\tilde{\mathbf{x}}(k+1)} = \underbrace{\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}}_{\tilde{\mathbf{A}}} \underbrace{\begin{pmatrix} \mathbf{x}(k) \\ \mathbf{u}(k-1) \end{pmatrix}}_{\tilde{\mathbf{x}}(k)} + \underbrace{\begin{pmatrix} \mathbf{B} \\ \mathbf{I} \end{pmatrix}}_{\tilde{\mathbf{B}}} \Delta \mathbf{u}(k) \quad (7.14)$$

$$\underbrace{\mathbf{y}(k)}_{\mathbf{y}(k)} = \underbrace{(\mathbf{C} \quad \mathbf{0})}_{\tilde{\mathbf{C}}} \underbrace{\begin{pmatrix} \mathbf{x}(k) \\ \mathbf{u}(k-1) \end{pmatrix}}_{\tilde{\mathbf{x}}(k)} \quad (7.15)$$

- The **prediction model (4.4)** can then be reformulated w.r.t. the augmented system (7.14)/(7.15) as

$$\tilde{\mathbf{X}}(k) = \tilde{\Phi} \tilde{\mathbf{x}}(k) + \tilde{\Gamma} \Delta \mathbf{U}(k) \quad (7.16)$$

where $\tilde{\mathbf{X}}(k)$, $\tilde{\Phi}$, $\tilde{\Gamma}$, $\Delta \mathbf{U}(k) = (\Delta \mathbf{u}^T(k) \quad \Delta \mathbf{u}^T(k+1) \quad \dots \quad \Delta \mathbf{u}^T(k+N-1))^T$ are defined as in (4.4)

Reference Tracking based on the Delta Input Formulation

- Approach

- The **cost function** (4.5) can then be reformulated w.r.t. the augmented system (7.14)/(7.15), the prediction model (7.16), and the cost function (7.12) as

$$\begin{aligned}
 V_N(\tilde{\mathbf{x}}(k), \Delta \mathbf{U}(k), \mathbf{R}(k)) &= (\mathbf{y}(k) - \mathbf{r}(k))^T \mathbf{Q}(\mathbf{y}(k) - \mathbf{r}(k)) + (\mathbf{Y}(k) - \mathbf{R}(k))^T \mathbf{\Omega}(\mathbf{Y}(k) - \mathbf{R}(k)) + \Delta \mathbf{U}^T(k) \mathbf{\Psi} \Delta \mathbf{U}(k) \\
 &= (\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) - \mathbf{r}(k))^T \mathbf{Q}(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) - \mathbf{r}(k)) + (\tilde{\mathbf{C}}\tilde{\mathbf{X}}(k) - \mathbf{R}(k))^T \mathbf{\Omega}(\tilde{\mathbf{C}}\tilde{\mathbf{X}}(k) - \mathbf{R}(k)) + \Delta \mathbf{U}^T(k) \mathbf{\Psi} \Delta \mathbf{U}(k) \\
 &= (\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) - \mathbf{r}(k))^T \mathbf{Q}(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) - \mathbf{r}(k)) + \left(\tilde{\mathbf{C}}(\tilde{\mathbf{F}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{F}}\Delta \mathbf{U}(k)) - \mathbf{R}(k) \right)^T \mathbf{\Omega} \left(\tilde{\mathbf{C}}(\tilde{\mathbf{F}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{F}}\Delta \mathbf{U}(k)) - \mathbf{R}(k) \right) + \Delta \mathbf{U}^T(k) \mathbf{\Psi} \Delta \mathbf{U}(k) \\
 &= \dots \\
 &= \Delta \mathbf{U}^T(k) (\mathbf{\Psi} + \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \tilde{\mathbf{C}} \tilde{\mathbf{F}}) \Delta \mathbf{U}(k) + 2 \Delta \mathbf{U}^T(k) \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \tilde{\mathbf{C}} \tilde{\mathbf{F}} \tilde{\mathbf{x}}(k) - 2 \Delta \mathbf{U}^T(k) \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \mathbf{R}(k) + f(\tilde{\mathbf{x}}(k), \mathbf{R}(k)) \\
 &= \frac{1}{2} \Delta \mathbf{U}^T(k) 2 (\mathbf{\Psi} + \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \tilde{\mathbf{C}} \tilde{\mathbf{F}}) \Delta \mathbf{U}(k) + \Delta \mathbf{U}^T(k) \left(\underbrace{2 \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \tilde{\mathbf{C}} \tilde{\mathbf{F}} \tilde{\mathbf{x}}(k)}_{\tilde{\mathbf{H}}} - \underbrace{2 \tilde{\mathbf{F}}^T \tilde{\mathbf{C}}^T \mathbf{\Omega} \mathbf{R}(k)}_{\tilde{\mathbf{F}}_R} \right) + f(\tilde{\mathbf{x}}(k), \mathbf{R}(k)) \\
 &= \frac{1}{2} \Delta \mathbf{U}^T(k) \tilde{\mathbf{H}} \Delta \mathbf{U}(k) + \Delta \mathbf{U}^T(k) \left(\tilde{\mathbf{F}} \tilde{\mathbf{x}}(k) + \tilde{\mathbf{F}}_R \mathbf{R}(k) \right) + f(\tilde{\mathbf{x}}(k), \mathbf{R}(k))
 \end{aligned}$$

where $\tilde{\mathbf{C}} = \text{diag}(\tilde{\mathbf{C}}, \dots, \tilde{\mathbf{C}})$, $\mathbf{\Omega}$, $\mathbf{\Psi}$, $\mathbf{Y}(k) = (\mathbf{y}^T(k+1) \ \mathbf{y}^T(k+2) \ \dots \ \mathbf{y}^T(k+N))^T$ and $\mathbf{R}(k) = (\mathbf{r}^T(k) \ \mathbf{r}^T(k) \ \dots \ \mathbf{r}^T(k))^T$ are defined as in (4.5)

Reference Tracking based on the Delta Input Formulation

- Approach

- The constraints in standard form (cf. Slide 5-10) can be reformulated w.r.t. Problem (7.13) as

$$\mathbf{M}\mathbf{y}(k+i) + \mathbf{E}\Delta\mathbf{u}(k+i) \leq \mathbf{b}, i = 0, 1, \dots, N-1$$

$$\mathbf{M}\mathbf{y}(k+N) \leq \mathbf{b}$$

- The **constraint model** (5.1) can then be reformulated w.r.t. the augmented system (7.14)/(7.15) and the prediction model (7.16) as

$$\mathcal{D}\tilde{\mathcal{C}}\tilde{\mathbf{x}}(k) + \mathcal{M}\tilde{\mathcal{C}}\left(\tilde{\Phi}\tilde{\mathbf{x}}(k) + \tilde{\Gamma}\Delta\mathbf{U}(k)\right) + \mathcal{E}\Delta\mathbf{U}(k) \leq \mathcal{b} \quad \Leftrightarrow$$

$$(\mathcal{D}\tilde{\mathcal{C}} + \mathcal{M}\tilde{\mathcal{C}}\tilde{\Phi})\tilde{\mathbf{x}}(k) + (\mathcal{M}\tilde{\mathcal{C}}\tilde{\Gamma} + \mathcal{E})\Delta\mathbf{U}(k) \leq \mathcal{b} \quad \Leftrightarrow$$

$$\underbrace{(\mathcal{M}\tilde{\mathcal{C}}\tilde{\Gamma} + \mathcal{E})\Delta\mathbf{U}(k)}_{\tilde{\mathcal{A}} \Delta\mathbf{U}(k)} \leq \mathcal{b} + \underbrace{(-\mathcal{D}\tilde{\mathcal{C}} - \mathcal{M}\tilde{\mathcal{C}}\tilde{\Phi})\tilde{\mathbf{x}}(k)}_{\tilde{\mathcal{W}} \tilde{\mathbf{x}}(k)} \quad \Leftrightarrow$$

$$\leq \mathcal{b} + \tilde{\mathcal{W}} \tilde{\mathbf{x}}(k)$$

where \mathcal{D} , \mathcal{M} , \mathcal{E} and \mathcal{b} are defined as in (5.1)

Reference Tracking based on the Delta Input Formulation

- Approach

- Problem 4.1 is then solved by the optimal state feedback control law

$$\partial/\partial \Delta \mathbf{U}(k) V_N(\tilde{\mathbf{x}}(k), \Delta \mathbf{U}(k), \mathbf{R}(k)) = \tilde{\mathbf{H}} \Delta \mathbf{U}(k) + \tilde{\mathbf{F}} \tilde{\mathbf{x}}(k) + \tilde{\mathbf{F}}_R \mathbf{R}(k) = \mathbf{0} \Leftrightarrow \Delta \mathbf{U}^*(k) = -\tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}} \tilde{\mathbf{x}}(k) - \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}_R \mathbf{R}(k)$$

- Problem 5.1 can then be formulated as the quadratic program

$$\min_{\Delta \mathbf{U}(k)} \frac{1}{2} \Delta \mathbf{U}^T(k) \tilde{\mathbf{H}} \Delta \mathbf{U}(k) + \Delta \mathbf{U}^T(k) (\tilde{\mathbf{F}} \tilde{\mathbf{x}}(k) + \tilde{\mathbf{F}}_R \mathbf{R}(k)) + f(\tilde{\mathbf{x}}(k), \mathbf{R}(k))$$

Term is independent of $\Delta \mathbf{U}(k)$
Term is therefore not relevant!

$$\text{subject to } \tilde{\mathcal{A}} \Delta \mathbf{U}(k) \leq \mathbf{b} + \tilde{\mathcal{W}} \tilde{\mathbf{x}}(k)$$

The reference input sequence $\mathbf{R}(k)$ occurs here!

- The receding horizon controller in the unconstrained case is given by

$$\begin{aligned} \mathbf{u}^*(k) &= (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \Delta \mathbf{U}^*(k) + \mathbf{u}^*(k-1) \\ &= \underbrace{-(\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}} \tilde{\mathbf{x}}(k)}_{\tilde{\mathbf{K}}_{\text{RHC}}} - \underbrace{(\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \tilde{\mathbf{H}}^{-1} \tilde{\mathbf{F}}_R \mathbf{R}(k)}_{\tilde{\mathbf{K}}_{\text{RRHC}}} + \mathbf{u}^*(k-1) \\ &= \tilde{\mathbf{K}}_{\text{RHC}} \tilde{\mathbf{x}}(k) + \tilde{\mathbf{K}}_{\text{RRHC}} \mathbf{R}(k) + \mathbf{u}^*(k-1) \end{aligned}$$

- The receding horizon controller is an affine TI state feedback controller in the unconstrained case

Reference Tracking based on the Delta Input Formulation

- Approach

- The feedback matrices $\tilde{\mathbf{K}}_{\text{RHC}}$ and $\tilde{\mathbf{K}}_{\text{RRHC}}$ can be calculated **offline** in the unconstrained case
- The closed-loop system is globally asymptotically stable iff $\rho(\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\tilde{\mathbf{K}}_{\text{RHC}}) < 1$ (cf. Theorem 2.3)
- The **receding horizon controller** in the **constrained case** results as

$$\mathbf{u}^*(k) = (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \Delta \mathbf{U}^*(k) + \mathbf{u}^*(k-1)$$

- The receding horizon controller is a **nonlinear state feedback controller** in the constrained case

- Remarks

- The extension for state constraints is straightforward
- Constraints on $\mathbf{u}(k+i-1)$ and thus on $\mathbf{u}(k+i)$ can be written as $(\mathbf{0}_{m \times n} \quad \mathbf{I}_{m \times m}) \tilde{\mathbf{x}}(k+i) \in \mathbb{U}$
- Disturbances and uncertainties in \mathbf{A} , \mathbf{B} and \mathbf{C} do not lead to a steady-state error or offset
- More details and references are given in [Mac02] and [BBM15, Section 13.6]

Preview Control for Reference Tracking

- **Motivation**

- Sometimes the **reference input $\mathbf{r}(k + i)$** is **known** over the prediction horizon
- E.g. in motion control problems the reference sequence is usually precomputed
- The **reference input $\mathbf{r}(k + i)$** can then be included in **Problem (7.13)** and **previewed** in this way
- The **receding horizon controller** can then work **proactively**

- **Approach**

- Problem (7.13) with reference $\mathbf{r}(k + i)$ can be solved in “**batch**” way using **quadratic programming**
- Assume that the reference input $\mathbf{r}(k + i)$ is known for $i = 0, 1, \dots, N$
- The reference input $\mathbf{r}(k)$ can then be replaced by the reference input $\mathbf{r}(k + i)$ in cost function (7.12)
- The reference input sequence on Slide 7-13ff is then $\mathbf{R}(k) = (\mathbf{r}^T(k + 1) \ \mathbf{r}^T(k + 2) \ \dots \ \mathbf{r}^T(k + N))^T$
- The formulation as a quadratic program then follows analogously to Slide 7-12ff

Reference Tracking based on Target Calculation with Disturbance Estimation

- Discrete-Time Linear Time-Invariant (LTI) System

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k) \quad \text{state equation} \quad (7.17)$$

$$y(k) = Cx(k) + C_w w(k) \quad \text{measured output equation} \quad (7.18)$$

$$y_r(k) = C_r x(k) + C_{rw} w(k) \quad \text{controlled output equation} \quad (7.19)$$

- Symbols

$x(k) \in \mathbb{R}^n$ state vector

$u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ input vector

$w(k) \in \mathbb{R}^{m_w}$ disturbance vector

$y(k) \in \mathbb{Y} \subseteq \mathbb{R}^p$ measured output vector

$y_r(k) \in \mathbb{R}^{p_r}$ controlled output vector

$A \in \mathbb{R}^{n \times n}$ system matrix

$B \in \mathbb{R}^{n \times m}$ input matrix

$B_w \in \mathbb{R}^{n \times m_w}$ disturbance input matrix

$C \in \mathbb{R}^{p \times n}$ measured output matrix

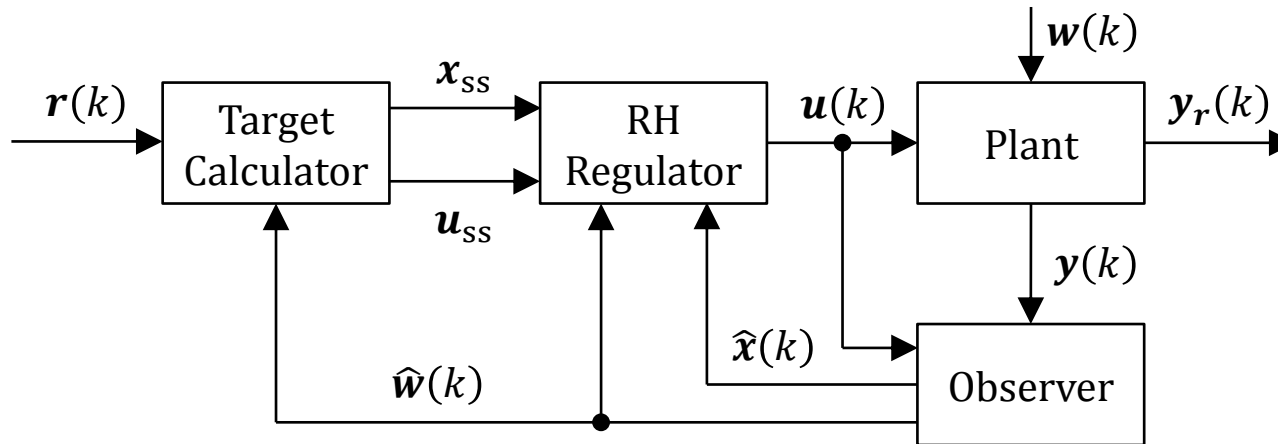
$C_r \in \mathbb{R}^{p_r \times n}$ controlled output matrix

$C_w \in \mathbb{R}^{p \times m_w}$

$C_{rw} \in \mathbb{R}^{p_r \times m_w}$

Reference Tracking based on Target Calculation with Disturbance Estimation

- Structure



- Objective

- Control the discrete-time LTI system such that $y_r(k) \rightarrow r(k)$ if $r(k), w(k) \rightarrow \text{const.}$ as $k \rightarrow \infty$

- Approach

- The **observer** estimates the **current state** $\hat{x}(k)$ and the **disturbance** $\hat{w}(k)$
- The **target calculator** and **RH regulator** are essentially utilized as outlined on Slide 7-3

Reference Tracking based on Target Calculation with Disturbance Estimation

- Approach

- Introduce the **disturbance model**

$$\mathbf{w}(k+1) = \mathbf{w}(k)$$

- Augment the discrete-time LTI system (7.17)/(7.18) by the disturbance model, i.e.

$$\begin{pmatrix} \mathbf{x}(k+1) \\ \mathbf{w}(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B}_w \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{w}(k) \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix} \mathbf{u}(k) \quad (7.20)$$

$$\mathbf{y}(k) = (\mathbf{C} \quad \mathbf{C}_w) \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{w}(k) \end{pmatrix} \quad (7.21)$$

- Design an **observer** for the augmented discrete-time LTI system (7.20)/(7.21) based on the methods developed in Chapter 2

Reference Tracking based on Target Calculation with Disturbance Estimation

- Approach

- Consider that the discrete-time LTI system (7.17)/(7.19) is in the steady **target state** \mathbf{x}_{ss} , i.e.

$$\mathbf{x}_{ss} = \mathbf{A}\mathbf{x}_{ss} + \mathbf{B}\mathbf{u}_{ss} + \mathbf{B}_w\hat{\mathbf{w}} \Leftrightarrow (\mathbf{I}_{n \times n} - \mathbf{A})\mathbf{x}_{ss} - \mathbf{B}\mathbf{u}_{ss} = \mathbf{B}_w\hat{\mathbf{w}}$$

$$\mathbf{y}_{r_{ss}} = \mathbf{C}_r\mathbf{x}_{ss} + \mathbf{C}_{rw}\hat{\mathbf{w}} = \mathbf{r} \Leftrightarrow \mathbf{C}_r\mathbf{x}_{ss} = \mathbf{r} - \mathbf{C}_{rw}\hat{\mathbf{w}}$$

- Note that the **estimated disturbance** $\hat{\mathbf{w}}$ is used since the **steady-state disturbance** \mathbf{w}_{ss} is unknown
- Rewrite the equations in **matrix form**, i.e.

$$\begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C}_r & \mathbf{0}_{p_r \times m} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{ss} \\ \mathbf{u}_{ss} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_w\hat{\mathbf{w}} \\ \mathbf{r} - \mathbf{C}_{rw}\hat{\mathbf{w}} \end{pmatrix} \quad (7.22)$$

- The steady **target pair** $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$ can be then calculated from (7.22) provided that a solution exists
- Note that the **estimated disturbance** $\hat{\mathbf{w}}$ is regarded in the **target pair** $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$
- Introduce the **state deviation** and **input deviation**

$$\tilde{\mathbf{x}}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}_{ss}, \quad \tilde{\mathbf{u}}(k) = \mathbf{u}(k) - \mathbf{u}_{ss}$$

Reference Tracking based on Target Calculation with Disturbance Estimation

- Approach

- This leads to the **discrete-time LTI state equation**

$$\tilde{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k+1) - \mathbf{x}_{ss} = \mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{B}_w\hat{\mathbf{w}}(k) - (\mathbf{A}\mathbf{x}_{ss} + \mathbf{B}\mathbf{u}_{ss} + \mathbf{B}_w\mathbf{w}_{ss}) = \mathbf{A}\tilde{\mathbf{x}}(k) + \mathbf{B}\tilde{\mathbf{u}}(k) \quad (7.23)$$

the **discrete-time quadratic cost function**

$$\tilde{V}_N(\tilde{\mathbf{x}}(k), \tilde{\mathbf{U}}(k)) = \tilde{\mathbf{x}}^T(k+N)\mathbf{P}\tilde{\mathbf{x}}(k+N) + \sum_{i=0}^{N-1} \tilde{\mathbf{x}}^T(k+i)\mathbf{Q}\tilde{\mathbf{x}}(k+i) + \tilde{\mathbf{u}}^T(k+i)\mathbf{R}\tilde{\mathbf{u}}(k+i) \quad (7.24)$$

the **state** and **input constraints**

$$\begin{aligned} \mathbf{C}(\tilde{\mathbf{x}}(k+i) + \mathbf{x}_{ss}) + \mathbf{C}_w\hat{\mathbf{w}}(k) &\in \mathbb{Y}, i = 1, 2, \dots, N \\ \tilde{\mathbf{u}}(k+i) + \mathbf{u}_{ss} &\in \mathbb{U}, \quad i = 0, 1, \dots, N-1 \end{aligned} \quad (7.25)$$

- Note that $\hat{\mathbf{w}}(k) = \mathbf{w}_{ss}$ is assumed in (7.23) as detailed in [PR03] and [RM09, Section 1.5.1]
- Note that the deviations are resubstituted in (7.25) since the constraint sets \mathbb{Y} and \mathbb{U} are considered

Reference Tracking based on Target Calculation with Disturbance Estimation

- Approach

- Rewrite **Problem 4.1/5.1** w.r.t. the state equation (7.23), cost function (7.24) and constr. (7.25), i.e.

$$\begin{aligned} & \min_{\tilde{\mathbf{u}}(k)} \tilde{V}_N(\tilde{\mathbf{x}}(k), \tilde{\mathbf{u}}(k)) \\ & \text{subject to } \begin{cases} \tilde{\mathbf{x}}(k+i+1) = \mathbf{A}\tilde{\mathbf{x}}(k+i) + \mathbf{B}\tilde{\mathbf{u}}(k+i), i = 0, 1, \dots, N-1 \\ \mathbf{C}(\tilde{\mathbf{x}}(k+i) + \mathbf{x}_{ss}) + \mathbf{C}_w \hat{\mathbf{w}}(k) \in \mathbb{Y}, & i = 1, 2, \dots, N \\ \tilde{\mathbf{u}}(k+i) + \mathbf{u}_{ss} \in \mathbb{U}, & i = 0, 1, \dots, N-1 \end{cases} \end{aligned} \quad (7.26)$$

- Remark on the Detectability

- The **augmented discrete-time LTI system** (7.20)/(7.21) is **detectable** iff

(1) (\mathbf{C}, \mathbf{A}) is detectable

(2) $\begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B}_w \\ \mathbf{C} & \mathbf{C}_w \end{pmatrix} \in \mathbb{R}^{(n+p) \times (n+m_w)}$ has full rank $n + m_w$

- This implies that number of disturbances $m_w \leq$ number of measured outputs p is required for detectability of the disturbance. Note that \mathbf{B}_w and \mathbf{C}_w can always be chosen such that (2) is fulfilled.

Reference Tracking based on Target Calculation with Disturbance Estimation

- Remark on the Constrained Case

- For the constrained case target pair $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$ must fulfill output constraint \mathbb{Y} and input constraint \mathbb{U}
- For this purpose the target calculator based on (7.22) must be modified to

$$\begin{aligned} \min_{\mathbf{x}_{ss}, \mathbf{u}_{ss}} & \frac{1}{2} \left((\mathbf{C}_r \mathbf{x}_{ss} + \mathbf{C}_{rw} \hat{\mathbf{w}} - \mathbf{r})^T \mathbf{Q}_{ss} (\mathbf{C}_r \mathbf{x}_{ss} + \mathbf{C}_{rw} \hat{\mathbf{w}} - \mathbf{r}) + (\mathbf{u}_{ss} - \mathbf{u}_{ss}^{\text{unc}})^T \mathbf{R}_{ss} (\mathbf{u}_{ss} - \mathbf{u}_{ss}^{\text{unc}}) \right) \\ \text{subject to } & \begin{cases} \begin{pmatrix} \mathbf{I}_{n \times n} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C}_r & \mathbf{0}_{p_r \times m} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{ss} \\ \mathbf{u}_{ss} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_w \hat{\mathbf{w}} \\ \mathbf{r} - \mathbf{C}_{rw} \hat{\mathbf{w}} \end{pmatrix} \\ \mathbf{C} \mathbf{x}_{ss} + \mathbf{C}_w \hat{\mathbf{w}} \in \mathbb{Y} \\ \mathbf{u}_{ss} \in \mathbb{U} \end{cases} \end{aligned} \quad (7.27)$$

where $\mathbf{u}_{ss}^{\text{unc}}$ is the target input resulting from (7.22) for the unconstrained case

and $\mathbf{Q}_{ss} = \mathbf{Q}_{ss}^T \succcurlyeq \mathbf{0}$ and $\mathbf{R}_{ss} = \mathbf{R}_{ss}^T \succ \mathbf{0}$ are weighting matrices

- Problem (7.27) can be formulated as a quadratic program

Reference Tracking based on Target Calculation with Disturbance Estimation

- Remark on Receding Horizon Control

1. Estimate the current state $\hat{\mathbf{x}}(k)$ and the disturbance $\hat{\mathbf{w}}(k)$
2. Solve Problem (7.27) for the given $\mathbf{r}(k)$ and estimated $\hat{\mathbf{w}}(k)$ to determine the target pair $(\mathbf{x}_{ss}, \mathbf{u}_{ss})$
3. Solve Problem (7.26) for $\tilde{\mathbf{x}}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}_{ss}$ to determine the optimal input sequence $\tilde{\mathbf{U}}^*(k)$
4. Compute first element of the optimal input sequence $\tilde{\mathbf{u}}^*(k) = (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \tilde{\mathbf{U}}^*(k)$
5. Implement the optimal input $\mathbf{u}^*(k) = \tilde{\mathbf{u}}^*(k) + \mathbf{u}_{ss}$
6. Increment the time instant $k := k + 1$ and go to 1.

- Further Remarks

- The remarks on the optimization problem and on the target calculator given on Slide 7-6 analogously apply to reference tracking with disturbance estimation
- Note that reference tracking with disturbance estimation is also denoted as **offset-free control**
- More details and references are given in [RM09, Section 1.5.1], [BBM15, Section 13.6], and [PR03]

Preview Control for Disturbance Rejection

- **Motivation**

- Consider the discrete-time LTI state equation (7.17) including the disturbance $\mathbf{w}(k)$
- Sometimes the disturbance $\mathbf{w}(k + i)$ can be predicted or measured over the prediction horizon
- E.g. the renewable generation in a power system can be predicted with good accuracy
- E.g. the road displacement before a car can be measured with a camera sensor
- The disturbance $\mathbf{w}(k + i)$ can then be included in Problem 4.1/5.1 and previewed in this way
- The receding horizon controller can then work proactively

- **Approach**

- Problem 4.1/5.1 with disturbance $\mathbf{w}(k + i)$ can be solved in “batch” way using quadratic programming
- The prediction model (4.4), the cost function (4.5), and the constraint model (5.1) must be reformulated w.r.t. the disturbance $\mathbf{w}(k + i)$ to this end
- Assume that the disturbance $\mathbf{w}(k + i)$ can be predicted or measured for $i = 0, 1, \dots, N - 1$

Preview Control for Disturbance Rejection

- Approach

- The **solution** of the discrete-time LTI state equation (7.17) is then given by

$$x(k+1) = Ax(k) + Bu(k) + B_w w(k)$$

$$\begin{aligned} x(k+2) &= Ax(k+1) + Bu(k+1) + B_w w(k+1) \\ &= A^2 x(k) + ABu(k) + AB_w w(k) + Bu(k+1) + B_w w(k+1) \end{aligned}$$

⋮

$$\begin{aligned} x(k+N) &= A^N x(k) + A^{N-1} Bu(k) + A^{N-1} B_w w(k) + \dots + ABu(k+N-2) + AB_w w(k+N-2) \\ &\quad + Bu(k+N-1) + B_w w(k+N-1) \end{aligned}$$

- The **prediction model** is then given by

$$\underbrace{\begin{pmatrix} x(k+1) \\ x(k+2) \\ \vdots \\ x(k+N) \end{pmatrix}}_{X(k)} = \underbrace{\begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix}}_{\Phi} x(k) + \underbrace{\begin{pmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{pmatrix}}_{\Gamma} \underbrace{\begin{pmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1) \end{pmatrix}}_{U(k)} + \underbrace{\begin{pmatrix} B_w & 0 & \dots & 0 \\ AB_w & B_w & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_w & A^{N-2}B_w & \dots & B_w \end{pmatrix}}_{\Gamma_w} \underbrace{\begin{pmatrix} w(k) \\ w(k+1) \\ \vdots \\ w(k+N-1) \end{pmatrix}}_{W(k)} \quad (7.28)$$

Preview Control for Disturbance Rejection

- Approach

- Substituting the prediction model (7.28) into the **cost function** (4.5) leads to

$$\begin{aligned}
 V_N(x(k), U(k), W(k)) &= x^T(k) Q x(k) + X^T(k) \Omega X(k) + U^T(k) \Psi U(k) \\
 &= x^T(k) Q x(k) + (\Phi x(k) + \Gamma U(k) + \Gamma_w W(k))^T \Omega (\Phi x(k) + \Gamma U(k) + \Gamma_w W(k)) + U^T(k) \Psi U(k) \\
 &= x^T(k) Q x(k) + x^T(k) \Phi^T \Omega \Phi x(k) + x^T(k) \Phi^T \Omega \Gamma U(k) + x^T(k) \Phi^T \Omega \Gamma_w W(k) \\
 &\quad + U^T(k) \Gamma^T \Omega \Phi x(k) + U^T(k) \Gamma^T \Omega \Gamma U(k) + U^T(k) \Gamma^T \Omega \Gamma_w W(k) \\
 &\quad + W^T(k) \Gamma_w^T \Omega \Phi x(k) + W^T(k) \Gamma_w^T \Omega \Gamma U(k) + W^T(k) \Gamma_w^T \Omega \Gamma_w W(k) + U^T(k) \Psi U(k) \\
 &= U^T(k) (\Psi + \Gamma^T \Omega \Gamma) U(k) + 2 U^T(k) \Gamma^T \Omega \Phi x(k) + 2 U^T(k) \Gamma^T \Omega \Gamma_w W(k) \\
 &\quad + x^T(k) (Q + \Phi^T \Omega \Phi) x(k) + W^T(k) \Gamma_w^T \Omega \Gamma_w W(k) + 2 W^T(k) \Gamma_w^T \Omega \Phi x(k) \\
 &= \frac{1}{2} U^T(k) 2 (\Psi + \Gamma^T \Omega \Gamma) U(k) + U^T(k) (2 \Gamma^T \Omega \Phi x(k) + 2 \Gamma^T \Omega \Gamma_w W(k)) \\
 &\quad + x^T(k) (\underbrace{Q + \Phi^T \Omega \Phi}_{H}) x(k) + W^T(k) \underbrace{\Gamma_w^T \Omega \Gamma_w}_{F} W(k) + 2 W^T(k) \underbrace{\Gamma_w^T \Omega \Phi}_{F_w} x(k) \\
 &= \frac{1}{2} U^T(k) H U(k) + U^T(k) (F x(k) + F_w W(k)) + f(x(k), W(k))
 \end{aligned}$$

Preview Control for Disturbance Rejection

- Approach

- Substituting the prediction model (7.28) into the **constraint model** (5.1) leads to

$$\mathcal{D}(k)x(k) + \mathcal{M}(k)(\Phi x(k) + \Gamma U(k) + \mathbf{\Gamma}_w \mathbf{W}(k)) + \mathcal{E}(k)U(k) \leq \mathbf{\ell}(k) \quad \Leftrightarrow$$

$$(\mathcal{D}(k) + \mathcal{M}(k)\Phi)x(k) + (\mathcal{M}(k)\Gamma + \mathcal{E}(k))U(k) + \mathbf{\mathcal{M}}(k)\mathbf{\Gamma}_w \mathbf{W}(k) \leq \mathbf{\ell}(k) \quad \Leftrightarrow$$

$$\underbrace{(\mathcal{M}(k)\Gamma + \mathcal{E}(k))U(k)}_{\mathcal{A}(k)} \leq \mathbf{\ell}(k) + \underbrace{(-\mathcal{D}(k) - \mathcal{M}(k)\Phi)x(k)}_{\mathbf{W}(k)} + \underbrace{(-\mathbf{\mathcal{M}}(k)\mathbf{\Gamma}_w \mathbf{W}(k))}_{\mathbf{W}_w(k)} \quad \Leftrightarrow$$

$$\mathcal{A}(k)U(k) \leq \mathbf{\ell}(k) + \mathbf{W}(k)x(k) + \mathbf{W}_w(k)\mathbf{W}(k)$$

- **Problem 4.1** is then solved by the **optimal state feedback control law**

$$\partial/\partial U(k) V_N(x(k), U(k), W(k)) = HU(k) + Fx(k) + \mathbf{F}_w \mathbf{W}(k) = \mathbf{0} \Leftrightarrow U^*(k) = -H^{-1}Fx(k) - H^{-1}\mathbf{F}_w \mathbf{W}(k)$$

- **Problem 5.1** can then be formulated as the **quadratic program**

$$\min_{U(k)} \frac{1}{2} U^T(k) H U(k) + U^T(k) (F x(k) + \mathbf{F}_w \mathbf{W}(k)) + f(x(k), \mathbf{W}(k))$$

Term is independent of $U(k)$
Term is therefore not relevant!

subject to $\mathcal{A}(k)U(k) \leq \mathbf{\ell}(k) + \mathbf{W}(k)x(k) + \mathbf{W}_w(k)\mathbf{W}(k)$ ← The disturbance sequence $\mathbf{W}(k)$ occurs here!

Preview Control for Disturbance Rejection

- Approach

- The **receding horizon controller** in the **unconstrained case** is given by

$$\begin{aligned} \mathbf{u}^*(k) &= (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \mathbf{U}^*(k) \\ &= -(\underbrace{\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}}_{\mathbf{K}_{\text{RHC}}} \mathbf{H}^{-1} \mathbf{F} \mathbf{x}(k) + \underbrace{-(\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \mathbf{H}^{-1} \mathbf{F}_w \mathbf{W}(k)}_{\mathbf{K}_{w\text{RHC}}} \mathbf{W}(k) \\ &= \mathbf{K}_{\text{RHC}} \mathbf{x}(k) + \mathbf{K}_{w\text{RHC}} \mathbf{W}(k) \end{aligned}$$

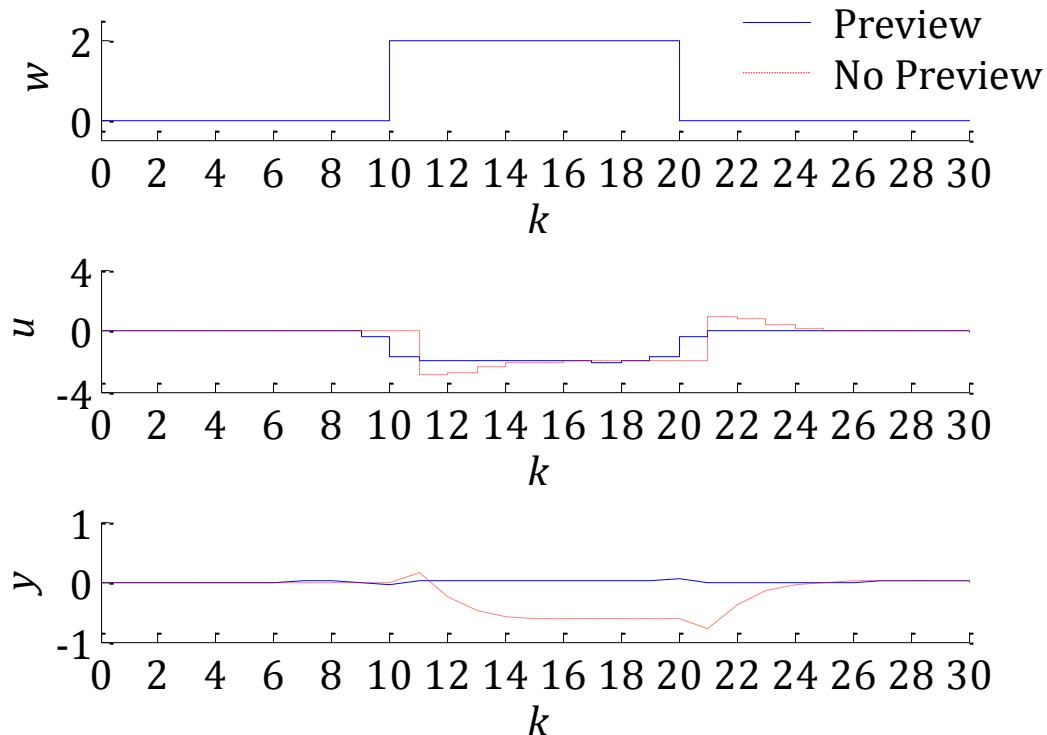
- The receding horizon controller is an **affine TI state feedback controller** in the unconstrained case
- The feedback matrices \mathbf{K}_{RHC} and $\mathbf{K}_{w\text{RHC}}$ can be calculated **offline** in the unconstrained case
- The closed-loop system is globally asymptotically stable iff $\rho(\mathbf{A} + \mathbf{B}\mathbf{K}_{\text{RHC}}) < 1$ (cf. Theorem 2.3)
- The **receding horizon controller** in the **constrained case** results as

$$\mathbf{u}^*(k) = (\mathbf{I}_{m \times m} \quad \mathbf{0}_{m \times m} \quad \cdots \quad \mathbf{0}_{m \times m}) \mathbf{U}^*(k)$$

- The receding horizon controller is an **nonlinear state feedback controller** in the constrained case

Preview Control for Disturbance Rejection

- Illustrative Example



Example from Chapter 4

$$\mathbf{x}(0) = (0 \ 0)^T$$

$$y(k) = (-1 \ 1)\mathbf{x}(k)$$

No state and input constraints

Disturbance input matrix $\mathbf{B}_w = \mathbf{B}$

Input weight $R = 0.01$

Terminal weight $\mathbf{P} = \mathbf{P}_{\text{LQR}}$

RHC (prediction horizon $N = 5$)

Preview RHC works proactively

Very good performance

Miscellaneous

- [MR93] Kenneth R. Muske and James B. Rawlings. Model predictive control with linear models. *AIChE Journal*, 39(2):262–287, 1993.
- [PR03] Gabriele Pannocchia and James B. Rawlings. Disturbance models for offset-free model predictive control. *AIChE Journal*, 49(2):426–437, 2003.