3rd Assignment:
Multi-input multi-output (MIMO) systems

1. Show that the transfer function matrix of the block diagram shown in Figure 1 is given by

$$z = \left[P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \right] w:$$

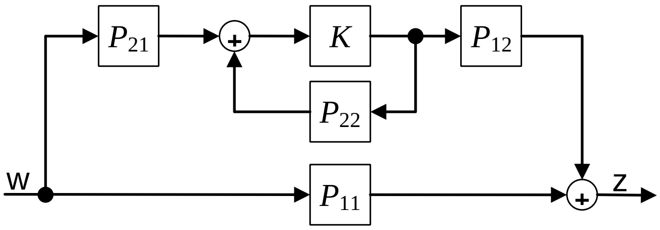

Figure 1: Block diagram of an MIMO system

2. Consider the cascaded control system shown in Figure 2. The output we want to control is y_1 while y_2 is a secondary output (extra measurement) with no associated control objective. We assume that the disturbance d is measurable. The control configuration includes a two degrees-of-freedom controller; a feed forward controller (pre-filter) to improve the performance with respect to reference signal and a feedback controller to improve the performance with respect to disturbances.

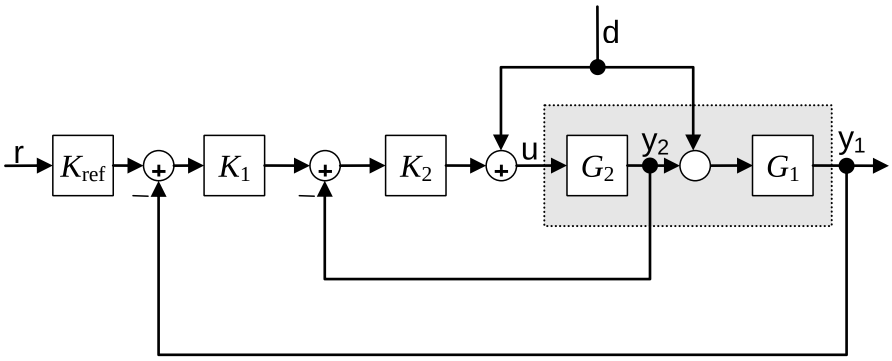

Figure 2: Block diagram of a cascaded control structure

- Derive the generalized Plant P and the generalized controller K!
- Calculate the closed-loop transfer function matrix N!

3. A process plant with two inputs and three outputs operates with certain flow-rates in the region of 100 (kmol/min). Assume we want to increase the flow rate at each input from 100 to 110 (kmol/min). These changes are effected by servo-controlled valves which rely on measurement of the flow as shown in Figure 3. Due to a flow measurement error of 1%, however, the actual change is from 100 to $109 \cdots 111$ (kmol/min). Hence, the corresponding error in the required change is about 10%. Thus our plant model, which describes changes about some operating point, is subject to errors of up to 10% on each input channel

(assuming that each input channel is a flow rate regulated as described). Furthermore, the process plant has unstructured multiplicative uncertainty Δ_{m} at its inputs with $\sigma_{\max}(\Delta_{\mathrm{m}}) \le |W_{\mathrm{m}}(s)|$.

- Reformulate the control problem in the general form with generalized plant P and uncertainty Δ satisfying $\|\Delta(s)\|_{\infty} \le 1$!

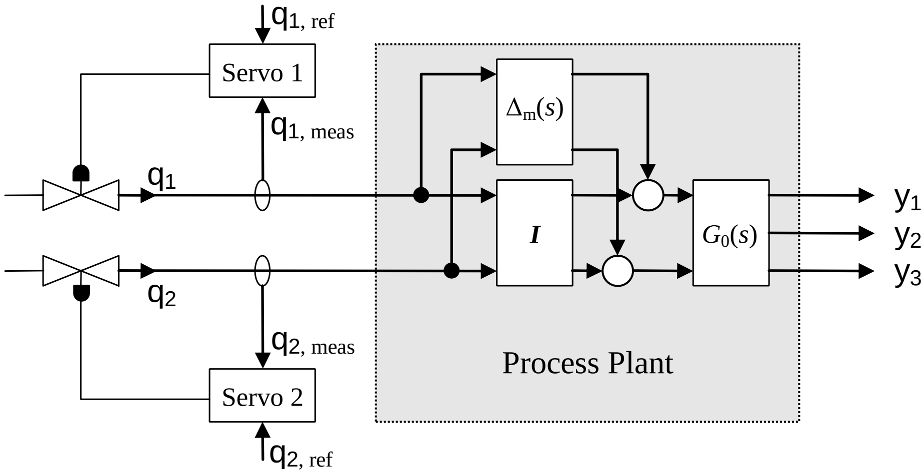

Figure 3: Perturbed process plant with two inputs and three outputs

4. State the small gain theorem and its scaled & generalized versions! Is it useful for robust control analysis and synthesis?

5. Consider a feedback system with multiplicative uncertainty as shown in Figure 4.

- Derive the generalized plant P with the inputs $[u_{\Delta}\ \mathrm{w}\ \mathrm{u}]^T$ and outputs $[\mathrm{y}_{\Delta}\ \mathrm{z}\ \mathrm{v}]^T$!
- Derive the closed-loop system matrix N!
- From Figure 4 one can easily observe that the uncertain transfer function from w to z is given by $F = W_{\mathrm{P}} S_p = W_{\mathrm{P}} \left(I + G_0 (I + \Delta W) K\right)^{-1}$. Show that this is identical to $F_u(N, \Delta)$ where N is derived in the previous step!

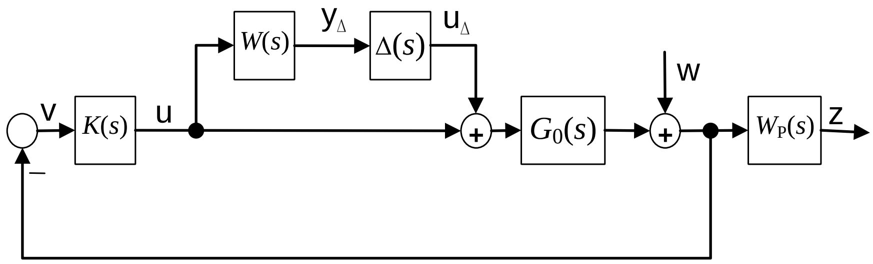

Figure 4: Multiplicative uncertainty description with a performance measure

6. What do necessary conditions, sufficient conditions, and necessary & sufficient conditions in general mean? Visualize the relationship between each of them and the others!

7. Consider the matrix $M = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ and the diagonal uncertainty matrix $\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$.

   Prove that the smallest diagonal Δ, measured by the $\mathcal{H}_\infty$ norm, which makes $\det(\mathrm{I} - M\Delta) = 0$ is given by

$$\Delta^* = \begin{pmatrix} 1/3 & 0 \\ 0 & -1/3 \end{pmatrix}!$$

// End of Assignment //