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Chapter 2

SISO systems

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Preliminary remarks

In this chapter we discuss the issues of nominal and robust stability and performance problems in single-input single-output (SISO) systems, represented either by a transfer function or by state-space equations. The analysis is performed proceeding from stability of the nominal model to the final objective of robust control: the robust performance.

In the introductory chapter we presented classical uncertainty description, such as gain and phase margins and sensitivity functions, along with an analysis of their limitations. In this chapter we will look into more details of the robustness analysis based on more generalized formalistic descriptions, focusing rejection of sets of disturbances, noises or tracking of sets of reference signals. The methodology is introduced as a preparation for more generalized MIMO systems which contain additional issues that do not arise in the simpler SISO case.

Throughout this and the following chapters we will use the so-called packed notation to represent state-space realizations, that is,

$$G(s) = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D} \equiv \left[\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right]$$

Nominal internal stability

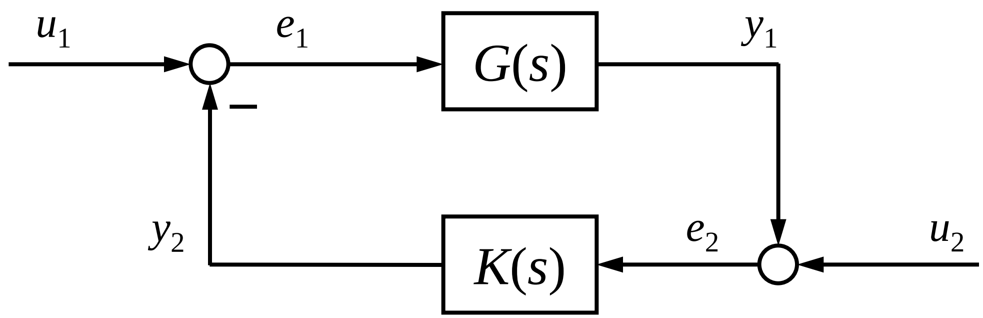


Figure 2.1: Feedback loop model to evaluate internal stability

Consider the feedback loop shown in the figure, where the input signals $[U_1(s), U_2(s)]$, the outputs $[Y_1(s), Y_2(s)]$, and the errors $[E_1(s), E_2(s)]$ satisfy the following equations:

$$Y_1(s) = G(s)E_1(s); \quad E_1(s) = U_1(s) - Y_2(s) \tag{2.1}$$

$$Y_2(s) = K(s)E_2(s); \quad E_2(s) = U_2(s) + Y_1(s) \tag{2.2}$$

It is not difficult to see that the stability of a certain input-output pair (all the poles of the corresponding transfer function in the LHP of the complex plane $\mathbb{C}$) does not guarantee that all input-output pairs will be stable (in the same sense). This is illustrated by the following simple example:

Exp. 2.1: internal stability

Exp. 2.1: Consider the following process and controller functions:

$$G(s)=\frac{s+1}{(s-1)(s+3)};\quad K(s)=\frac{s-1}{s+1}$$

connected as shown in Figure 2.1. The transfer function from the input u_2 to the output y_1 is given by

$$T_{y_1u_2}(s)=-\frac{1}{s+4}$$

which is stable in the usual sense. However, the transfer function between the input u_1 and the output y_1 is

$$T_{y_1u_1}(s)=\frac{s+1}{(s-1)(s+4)}$$

which is obviously unstable. The reason is easy to be found out: the unstable plant pole at $s = 1$ is cancelled by a zero of the controller.

This example shows that there is a difference between the stability of a certain system, considered as a mapping between its input and output, which we define as input-output stability, and stability of a feedback loop, which means that all input-output pairs are stable.

Internal stability: definition und conditions

<u>Def. 2.1:</u> The feedback loop of Figure 2.1 is *internally stable* if and only if all transfer functions obtained from all input-output pairs have their poles in the LHP (input-output stable).

<u>Lemma. 2.1:</u> Let

$$G_L(s) \equiv \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \tag{2.3}$$

be a minimal state-space realization of the closed-loop system of Figure 2.1, between the inputs $[U_1(s), U_2(\mathrm{s})]$ and outputs $[Y_1(s), Y_2(\mathrm{s})]$ (or $[E_1(s), E_2(\mathrm{s})]$). Then the following conditions are equivalent:

1. The feedback loop in Figure 2.1 is internally stable.
2. The eigenvalues of A_L are all in the open LHP (left half plane).
3. The four transfer functions obtained between inputs $[U_1(s), U_2(\mathrm{s})]$ and outputs $[Y_1(s), Y_2(\mathrm{s})]$ (or $[E_1(s), E_2(\mathrm{s})]$) have their poles in the LHP; that is, $G_L(s)$ is input-output stable.

<u>Lemma. 2.2:</u> The feedback loop in Figure 2.1 is internally stable if and only if $[1+G(s)K(s)]^{-1}$ is stable and there are no RHP pole-zero cancellations between the plant and the controller.

Exp. 2.2: more improper cancellation

Exp. 2.2: Consider the following process and controller functions:

$$G(s)=\frac{(1-s)(s^2+0.2s+1)}{(s+1)(s+3)(s+5)};\quad K(s)=\frac{(s+3)(s+5)}{s(s^2+0.2s+1)}$$

connected as shown in Figure 2.1. The design objectives are to internally stabilize the loop and to track with y_1 a step input injected at u_2 with less than 10% overshoot. With the controller given above the performance specifications can be fulfilled. The closed-loop is internally stable and the transfer function

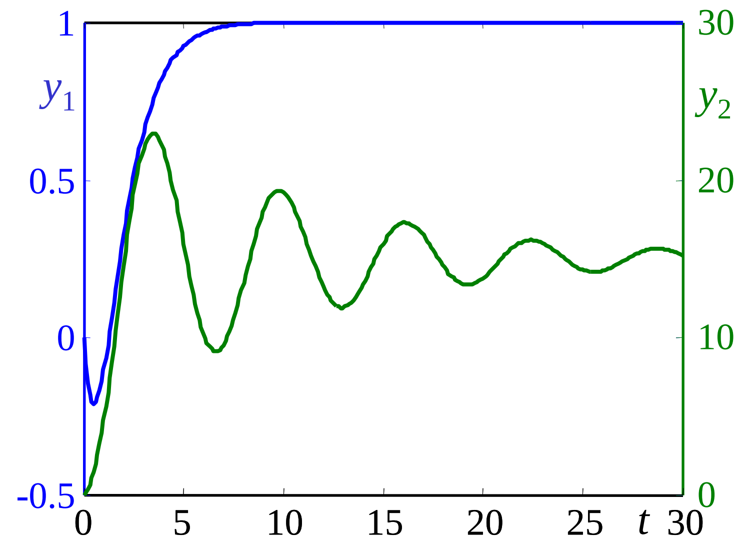
$$T_{y_1u_2}(s)=\frac{1-s}{(s+1)^2}$$

has an adequate time response. However, the transfer function

$$T_{y_2u_2}(s)=\frac{(s+3)(s+5)}{(s+1)(s^2+0.2s+1)}$$

presents undesirable oscillations due to the lightly damped poles at $s\approx$ -0.1±j.

This situation can be avoided by simply extending the concept of internal stability to cover more general regions, rather than just the (open) LHP.



Representing uncertainty

The various types of model uncertainty may be grouped into two main classes:

1. *Dynamic (frequency-dependent) uncertainty*. Here the model is in error because of missing dynamics, usually at high frequencies, either through deliberate neglect or because of a lack of understanding of the physical process.
2. *Parametric (real) uncertainty*. Here the structure of the model (including the order) is known, but some of the parameters are uncertain.

Dynamic uncertainty is somewhat less precise and thus more difficult to quantify. It appears that the frequency domain is particularly well suited for this class. This leads to complex perturbations Δ which we normalize to $|\Delta| \le 1$.

Parametric uncertainty is quantified by assuming that each uncertain parameter is bounded within some region $[\alpha_{\min}, \alpha_{\max}]$. That is, we have parameter sets of the form

$$\alpha_p = \bar{\alpha}(1 + r_\alpha \Delta)$$

where $\bar{\alpha}$ is the mean parameter value, $r_\alpha = (\alpha_{\max}-\alpha_{\min})/(\alpha_{\max}+\alpha_{\min})$ is the relative uncertainty in the parameter, and Δ is any real scalar satisfying $|\Delta| \le 1$.

Global dynamic uncertainty

Global dynamic uncertainty is a description used to provide a more realistic description of model uncertainty. The name arises since it is related to the uncertainty in the system dynamics and covers globally the complete model of the plant (not only a part of it). This type of uncertainty description can be used when the exact order of the differential equations describing the plant is unknown or the effect of linearization errors must be taken into account. Also, many physical systems may be described by partial differential equations (e.g. flexible structures) or time-delayed equations, both called infinite-dimensional systems. In these cases, an approximation of the general equation can be made so that a finite-dimensional model is obtained. The approximation error can be then interpreted as dynamic uncertainty. In particular, uncertainty in the time delay can be modeled as dynamic uncertainty in a very simple way.

The approach adopted by robust control theory is to describe a physical system by a set of models in terms of a nominal plant together with bounded uncertainty. In classical control theory the controller is designed using the nominal model. Instead, robust control theory attempts to design a single controller guaranteeing that certain properties such as stability and a given level of performance are achieved for all members of the model set. In this sense, these properties are said to be robust.

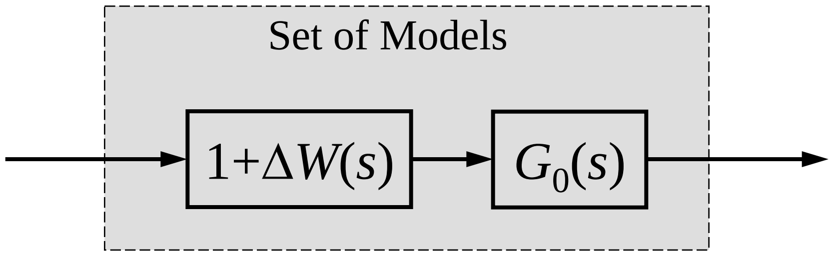
Multiplicative dynamic uncertainty (1)

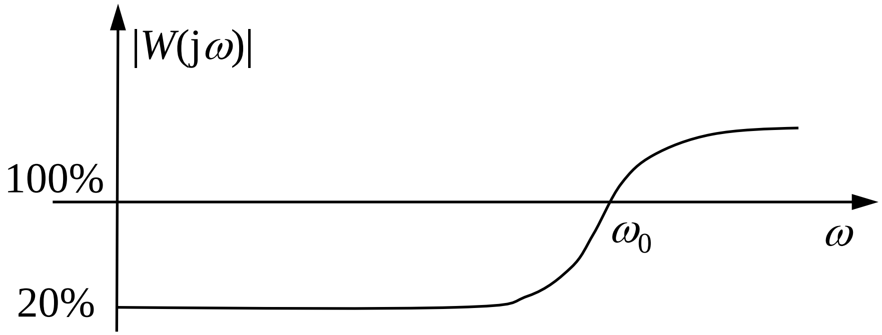
An uncertain model with *multiplicative dynamic uncertainty* is shown in Figure 2.2. It describes a physical system as a set $\mathcal{G}$ of mathematical models:

$$\mathcal{G} = \{G(s) : G(s) = G_0(s)[1 + \Delta W(s)],\ \Delta \in \mathbb{C},\ |\Delta| \leq 1\} \tag{2.4}$$

The set $\mathcal{G}$ is the family of models and is characterized by a *nominal plant* $G_0(s)$, a fixed weighting function $W(s)$, and a class of *bounded uncertainty* Δ. The nominal model $G_0(s)$ corresponds to the case where there is no uncertainty, that is, $\Delta = 0$. Without loss of generality, the bound on the uncertainty Δ can be assumed to be one, because any other bound can be absorbed into the weight $W(s)$.

The weighting function $W(s)$ represents the "dynamics" of the uncertainty, or in other words its "frequency distribution". A graphical interpretation is given in below, where a uncertainty of 20% at low frequencies and 100% or more at higher ones.





Multiplicative dynamic uncertainty (2)

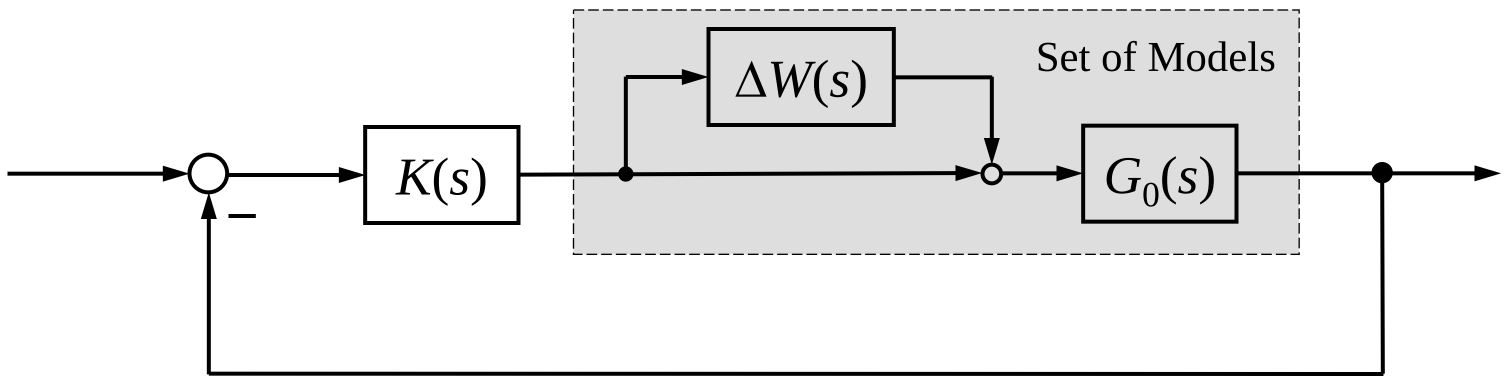
<u>Theorem. 2.1:</u> (robust stability) Assume the nominal model $G_0(s)$ is (internally) stable by a controller $K(s)$. Then all members of the set $\mathcal{G}$ will be (internally) stabilized by the same controller if and only if the following condition is satisfied:

$$\left\| T(s)W(s) \right\|_\infty \triangleq \sup_\omega \left| T(\mathrm{j}\omega)W(\mathrm{j}\omega) \right| < 1 \quad \Leftrightarrow \quad \left| T(\mathrm{j}\omega)W(\mathrm{j}\omega) \right| < 1, \quad \forall\, \omega \tag{2.5}$$

with

$$T(s) \triangleq G_0(s)K(s)\left[1 + G_0(s)K(s)\right]^{-1} \tag{2.6}$$

the complementary sensitivity function of the closed-loop system.

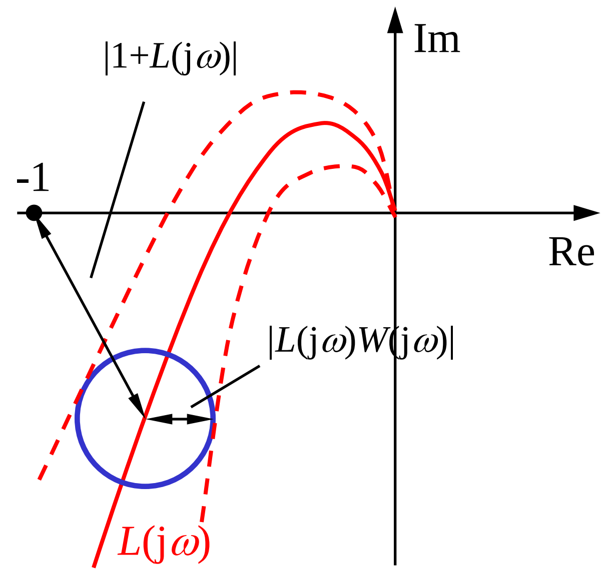


Multiplicative dynamic uncertainty (3)

The condition (2.5) can also be interpreted graphically in terms of the family of Nyquist plots corresponding to the set of loop transfer functions. We know that (2.5) is equivalent to

$$\left|1+G_0(s)K(s)\right| > \left|G_0(s)K(s)W(s)\right|, \quad \forall\ s=\mathrm{j}\omega \tag{2.7}$$

For a given frequency ω, the locus of all points $z(\mathrm{j}\omega) = G_0(\mathrm{j}\omega)K(\mathrm{j}\omega) + G_0(\mathrm{j}\omega)K(\mathrm{j}\omega)W(\mathrm{j}\omega)\Delta$, $\Delta \in \mathbb{C}$, $|\Delta| \le 1$ is a disk $\mathcal{D}(\omega)$, centered at $L(\mathrm{j}\omega) = G_0(\mathrm{j}\omega)K(\mathrm{j}\omega)$ with radius $r = |G_0(\mathrm{j}\omega)K(\mathrm{j}\omega)W(\mathrm{j}\omega)|$. Since $|1+G_0(\mathrm{j}\omega)K(\mathrm{j}\omega)|$ is the distance between the critical point and the point of the nominal Nyquist plot corresponding to the frequency ω, it follows that condition (2.5) is equivalent to requiring that, for each frequency ω, the uncertainty disk $\mathcal{D}(\omega)$ excludes the critical point $z = -1$. Therefore robust stability for SISO systems can be checked by drawing the envelope of all Nyquist plots formed by the set of circles centered at the nominal plot, with radii $|G_0(\mathrm{j}\omega)K(\mathrm{j}\omega)W(\mathrm{j}\omega)|$, and checking whether or not this envelope encloses the critical point $z = -1$.



Exp. 2.3: modeling multiplicative dynamic uncertainty

Exp. 2.3: Consider the following nominal model $G_0(s)$ and a second possible plant $G_1(s)$:

$$G_0(s) = \frac{3}{(s+1)(s+3)}; \quad G_1(s) = \frac{300}{(s+1)(s+3)(s+100)}$$

These two plants can be described using the following family, characterized by multiplicative uncertainty represented by a weight $W(s)$ and a bound on Δ.

$$\mathcal{G} = \{G(s) : G(s) = G_0(s)[1 + \Delta W(s)],\ \Delta \in \mathbb{C},\ |\Delta| \leq 1\}$$

$$W(s) = \frac{s}{s+100}$$

It is easy to verify that G_1 corresponds to Δ = -1. Note that the set $\mathcal{G}$ also includes many other plant models. For instance, for $\Delta = 1$ we obtain

$$G_2(s) = \frac{6(s+50)}{(s+1)(s+3)(s+100)}$$

Therefore, if we only need to consider the models $G_0(s)$ and $G_1(s)$, the description used above could be unnecessarily conservative. Any design that applies to all members of $\mathcal{G}$ could be conservative as well.

Exp. 2.4: modeling multiplicative dynamic uncertainty

Exp. 2.4: Consider a set of models with uncertainty in the high-frequency dynamics ($\omega > 120$ rad/s) of the numerator polynomial:

$$\mathcal{G} = \{\frac{3[1+\Delta/5+s(\Delta/100)]}{(s+1)(s+3)},\ |\Delta|\leq 1\};\quad G_0(s) = \frac{3}{(s+1)(s+3)} \quad \text{for } \Delta = 0$$

This uncertainty can be represented as multiplicative dynamic uncertainty using the weight $W(s)$

$$W(s) = \frac{s+20}{100}$$

The system is known with ca. 20% relative error up to about 10 rad/s. Above 100 rad/s the model has no information on the system that may be useful for control design ($|W(\mathrm{j}\omega)| > 1$). According to condition (2.7), this frequency is also the upper limit for the bandwidth of the complementary sensitivity function $T(s)$, to achieve robust stability.

Exp. 2.5: multiplicative uncertainty and robust stability

<u>Exp. 2.5:</u> Consider the following nominal plant and PI controller (cp. Exp. 1.2):

$$G_0(s) = \frac{3(-2s+1)}{(5s+1)(10s+1)}; \; K(s) = K_C \frac{12.7s+1}{12.7s}$$

Initially, we select $K_C = K_{C1} = 1.14$ as suggested by the Ziegler-Nichols tuning rule. It results in a nominally stable closed-loop system. Suppose that one "extreme" uncertain plant is

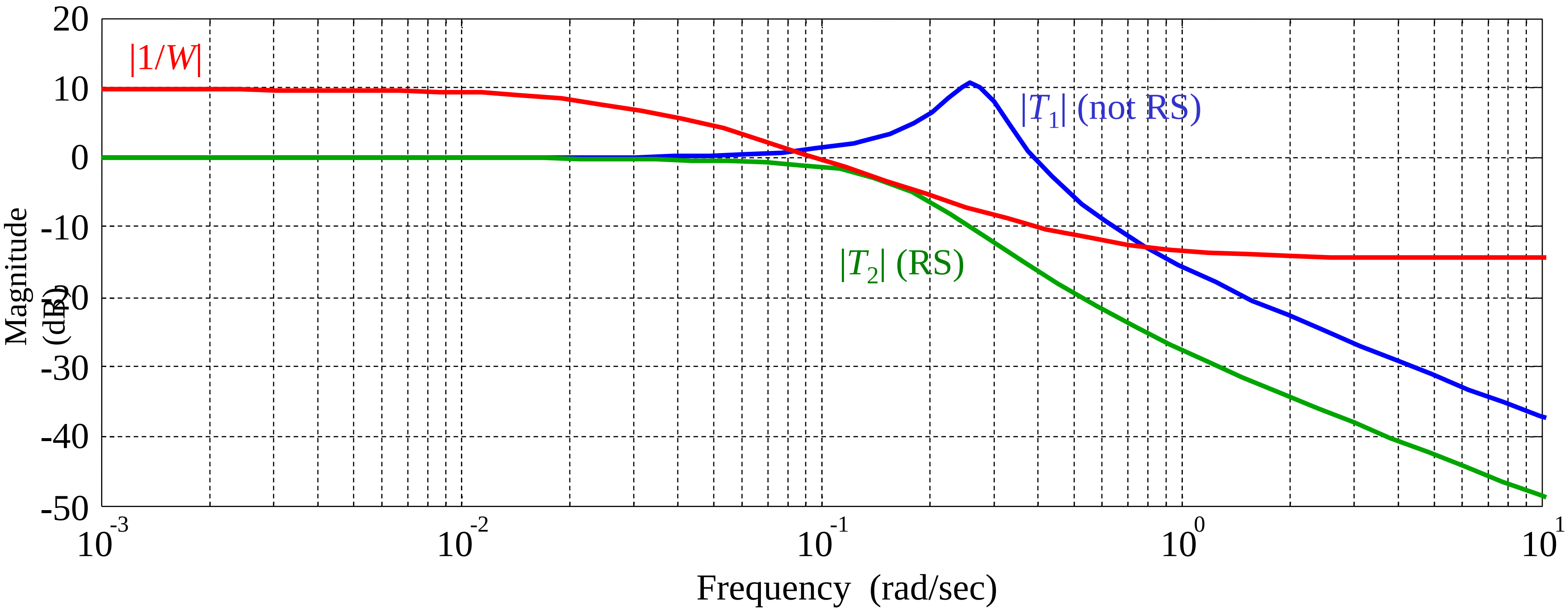
$$G'(s) = \frac{4(-3s+1)}{(4s+1)^2}$$

For this plant the relative error $|(G'-G)/G|$ is 0.33 at low frequencies; it is 1 at about 0.1 rad/s and 5.25 at high frequencies. We choose now the following uncertainty weighting function

$$W(s) = \frac{10s+0.33}{(10/5.25)s+1}$$

which closely matches this relative error. We now evaluate the robust stability for all possible plants as given by $G = G_0(1+\Delta W)$, where Δ is any perturbation satisfying $|\Delta| \leq 1$. According to (2.5) we check the stability condition by computing $T_1 = G_0K_1/(1+G_0K_1)$ as a function of frequency (s. next page). Since $|T_1|$ exceeds $|W|$ over a wide frequency range, we can conclude that the system is not robustly stable!

Exp. 2.5: multiplicative uncertainty and robust stability



The worst-case frequency is about $\omega = 0.26$, where $|T_1|$ is a factor of $1/0.14 = 7.14$ larger than $|W|$. This means, reducing the uncertainty weights W by a factor 7.14 would give stability.

By trial and error we can reduce the controller gain to $K_{C2} = 0.31$ to just achieve robust stability, as can be seen from the Bode plot for $T_2 = G_0K_2/(1+G_0K_2)$ in the figure.

Additive dynamic uncertainty

Besides multiplicative dynamic uncertainty, other uncertainty descriptions are also available similar to (2.4) to represent specific types of uncertainty. For example, in case where a high-order (even infinite-dimensional) model $G(s)$ must be approximated by a lower-order one $G_0(s)$, the approximation error can be considered as *additive dynamic uncertainty*. The set of models "centered" at the nominal function $G_0(s)$, which includes the high-order one $G(s)$, can be defined as follows:

$$\mathcal{G}_a = \{G_0(s) + \Delta W_a(s),\ \Delta \in \mathbb{C},\ |\Delta| \leq 1\} \tag{2.8}$$

The weight $W_\mathrm{a}(s)$ can be obtained from the frequency response of the approximation error (or its upper bound) and $G(s)$.

For additive dynamic uncertainty a corresponding condition for robust stability can be formulated as

$$\left\| W_\mathrm{a}(s)K(s)S(s) \right\|_\infty \triangleq \sup_\omega \left| W_\mathrm{a}(\mathrm{j}\omega)K(\mathrm{j}\omega)S(\mathrm{j}\omega) \right| < 1 \tag{2.9}$$

where

$$S(s) \triangleq \left[1 + G_0(s)K(s)\right]^{-1} \tag{2.10}$$

is the sensitivity function of the nominal closed-loop system.

Inverse multiplicative uncertainty

Another uncertainty form, which is better suited for representing pole uncertainty, is the *inverse multiplicative uncertainty* (sometimes also called *quotient dynamic uncertainty*) described by

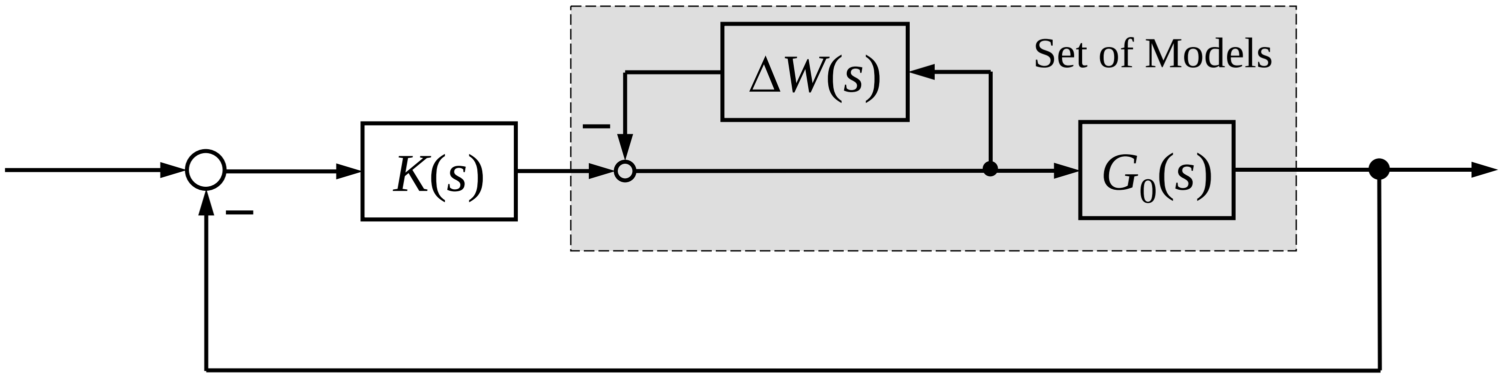
$$\mathcal{G} = \{G(s) : G(s) = G_0(s)[1 + \Delta W_i(s)]^{-1},\ \Delta \in \mathbb{C},\ |\Delta| \leq 1\} \tag{2.11}$$

Even with a stable $W_a(s)$ this form allows for uncertainty in the location of an unstable pole, and it also allows for poles crossing between the left- and right-half planes.

For inverse dynamic uncertainty a corresponding condition for robust stability can be formulated as

$$\left\| W_i(s)S(s) \right\|_\infty \triangleq \sup_\omega \left| W_i(j\omega)S(j\omega) \right| < 1 \tag{2.12}$$

where $S(s)$ again is the sensitivity function of the nominal closed-loop system.



Exp. 2.6: modeling inverse multiplicative uncertainty

Exp. 2.6: Consider the following set of mathematical models, described using inverse multiplicative uncertainty:

$$G_i(s) = \left\{ \frac{1}{[s+4(0.5+\delta)]}; \quad |\delta| \le 1 \right\} \triangleq \left\{ G_0(s)[1+\Delta_\mathrm{i} W_\mathrm{i}(s)]^{-1}, \ |\Delta_\mathrm{i}| \le 1 \right\}$$

with the following nominal model and uncertainty weight:

$$G_0(s) = \frac{1}{s+2}; \quad W_\mathrm{i}(s) = \frac{4}{s+2}$$

Taking a PI-type controller

$$K(s) = 0.5(1+\frac{1}{s})$$

the closed-loop with the nominal plant is obviously stable and the phase margin φ_m is about $97°$. The system is, however, *not robustly stable*. The magnitude of the sensitivity function $|S|$ exceeds $|1/W_\mathrm{i}|$ over the frequency range about $[0.15, 3.26]$ and violets thus the stability condition (2.12).

Weighted sensitivity

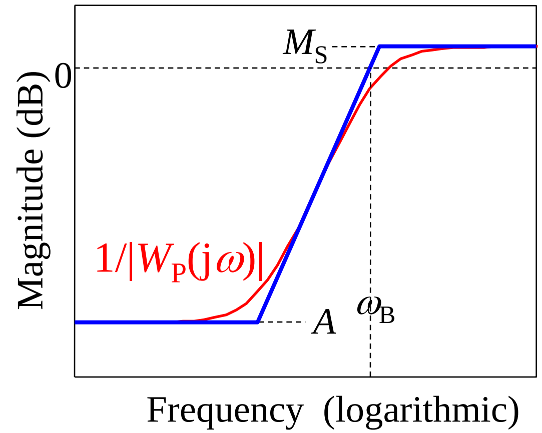
As already discussed, the sensitivity function S is a very good indicator of closed-loop performance. The main advantage of considering S is that because we ideally want S small, it is sufficient to evaluate just its magnitude $|S|$; that is, we need not worry about its phase. Typical specifications for S include:

- Minimum bandwidth frequency (defined as frequency where $|S(\mathrm{j}\omega)|$ crosses 0.707 from below).
- Maximum steady-state tracking error, A
- Shape of S over selected frequency ranges.
- Maximum peak value $M_\mathrm{S} \leq 2$.

Mathematically, these specifications may be captured by an upper bound, $1/|W_\mathrm{P}(s)|$, on the magnitude of S, where $W_\mathrm{P}(s)$ is the performance weight function. A typical selection of $W_\mathrm{P}(s)$ is

$$W_\mathrm{P}(s) = \frac{s/M_\mathrm{S} + \omega_B}{s + \omega_B A} \qquad (2.13)$$

$1/|W_\mathrm{P}(s)|$ is equal to A (typically $A \approx 0$) at low frequencies and equal to M_S at high frequencies. Its asymptote crosses 1 at the frequency ω_B, which is approximately the bandwidth requirement.



Nominal performance

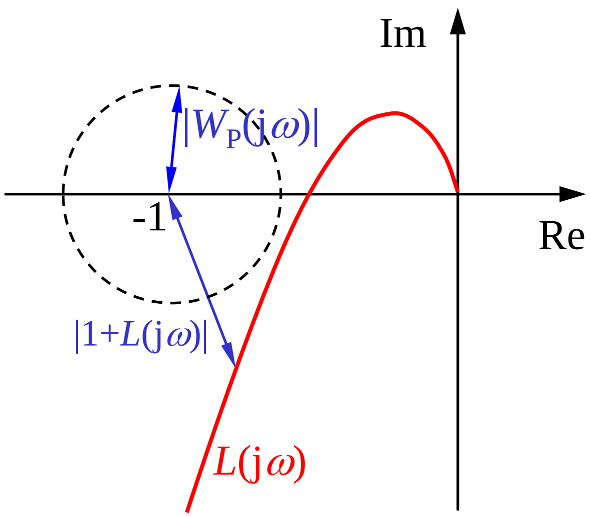
Using weighted sensitivity function as indicator, the *nominal performance* (NP) is then defined as

$$\text{NP} \quad \Leftrightarrow \quad |W_\text{P} S| < 1 \quad \forall \omega \quad \Leftrightarrow \quad |W_\text{P}| < |1 + L| \quad \forall \omega \quad \Leftrightarrow \quad \|W_\text{P} S\|_\infty < 1 \tag{2.14}$$

Since $|1+L(\mathrm{j}\omega)|$ represents at each frequency the distance of $L(\mathrm{j}\omega)$ from the point -1 in the Nyquist plot, so $L(\mathrm{j}\omega)$ must be at least a distance of $|W_\text{P}(\mathrm{j}\omega)|$ from -1, as illustrated in the figure below. It is obviously that for nominal performance $L(\mathrm{j}\omega)$ must stay outside a disc of radius $|W_\text{P}(\mathrm{j}\omega)|$ centered on -1.

Nominal performance, as defined in (2.14), is based on the sensitivity function. Depending on input-output pairs considered there are also other possible definitions of NP.

Input-output pair	Nominal performance
$n \to y, r \to e, n \to e$	$\lvert W_\text{P}(s)S(s)\rvert < 1$
$r \to u, n \to u$	$\lvert W_\text{P}(s)K(s)S(s)\rvert < 1$
$r \to y$	$\lvert W_\text{P}(s)T(s)\rvert < 1$
$d \to y$	$\lvert W_\text{P}(s)G_0(s)S(s)\rvert < 1$



Robust performance (1)

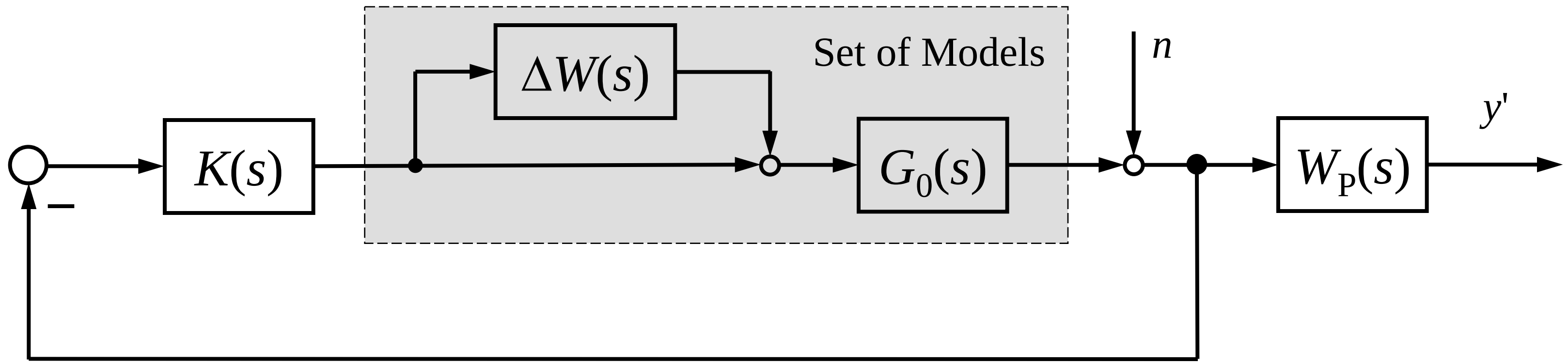
The final goal of robust control is to achieve the performance requirement for all members of the family of models. This is then called *robust performance* (RP). The condition for robust performance, based on the structure shown in the figure (multiplicative uncertainty) is

$$\text{RP} \quad \Leftrightarrow \quad |W_P S_P| < 1 \quad \forall S_P, \forall \omega \quad \Leftrightarrow \quad |W_P| < |1 + L_P| \quad \forall L_P, \forall \omega \quad \Leftrightarrow \quad \|W_P S_P\|_\infty < 1 \qquad (2.15)$$

where the set of possible loop transfer function is

$$L_P(s) = K(s)G_0(s)[1 + \Delta W(s)] = L(s)[1 + \Delta W(s)]; \quad |\Delta| \leq 1 \qquad (2.16)$$

The RP condition corresponds to requiring $|y'/n| < 1\ \forall \Delta$.



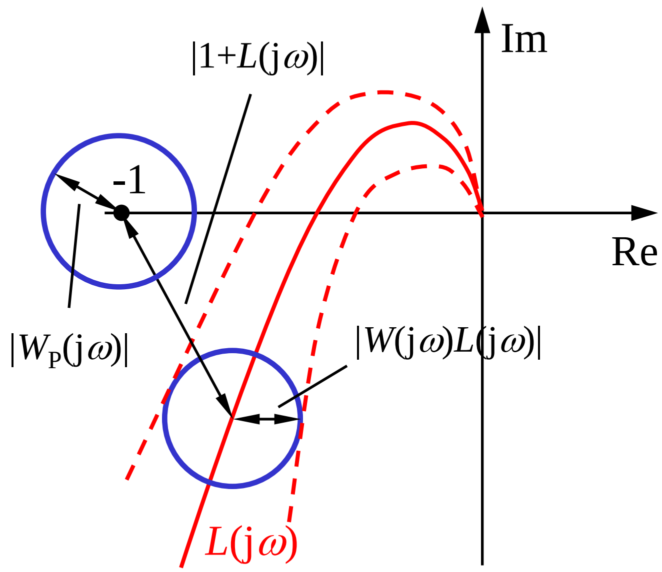
Robust performance (2)

Condition (2.15) can be interpreted graphically by the Nyquist plot. For RP we must require that all possible $|1+L_P(j\omega)|$ stay outside a disc of radius $|W_P(j\omega)|$ centered on -1. Since L_P at each frequency stays within a disc of radius $|W(j\omega)L(j\omega)|$ centered on L, we see that the condition for RP is that the two discs, with radii $|W_P|$ and $|WL|$ respectively, do not overlap. Since their centers are located a distance $|1+L|$ apart, the RP condition becomes

$$\text{RP} \quad \Leftrightarrow \quad |W_P| + |WL| < |1+L| \qquad \Leftrightarrow \quad \left|W_P(1+L)^{-1}\right| + \left|WL(1+L)^{-1}\right| < 1, \quad \forall\omega$$

or in other words

$$\text{RP} \quad \Leftrightarrow \quad \max_{\omega}\left(|W_P S| + |WT|\right) < 1 \qquad (2.17)$$



The RP condition (2.17) can be used to derive bounds on the loop shape $|L|$. At a given frequency the condition is satisfied if

$$|L| > \frac{1+|W_P|}{1-|W|} \quad \text{(at freuencies where } |W|<1) \qquad (2.18)$$

or if

$$|L| < \frac{1-|W_P|}{1+|W|} \quad \text{(at freuencies where } |W_P|<1) \qquad (2.19)$$

The relationship between NP, RS and RP

Consider a SISO system with multiplicative uncertainty, and assume that the closed-loop is nominally stable (NS). The conditions for nominal performance (NP), robust stability (RS) and robust performance (RP) can then be summarized as follows

$$\text{NP} \quad \Leftrightarrow \quad |W_P S| < 1, \quad \forall \omega$$

$$\text{RS} \quad \Leftrightarrow \quad |WT| < 1, \quad \forall \omega$$

$$\text{RP} \quad \Leftrightarrow \quad |W_P S| + |WT| < 1, \quad \forall \omega$$

A prerequisite for RP is obviously that we satisfy NP and RS. This applies in general, both for SISO and MIMO systems and for any type of uncertainty. In addition, for SISO systems, if we satisfy both RS and NP, then we have at each frequency

$$|W_P S| + |WT| \leq 2 \max\{|W_P S|, |WT|\} < 2 \tag{2.20}$$

which means that, within a factor of at most 2, we will automatically get RP when the NP and RS are satisfied. Thus, RP is not that a "big issue" for SISO systems (and therefore rarely discussed in classical control theory). On the other hand, as we will see in the next chapter, for MIMO systems we may get very poor RP even though the conditions for NP and RS are individually satisfied.