Model Predictive Control
6. Stability and Feasibility

Jun.-Prof. Dr.-Ing. Daniel Görges
Juniorprofessur für Elektromobilität
Technische Universität Kaiserslautern

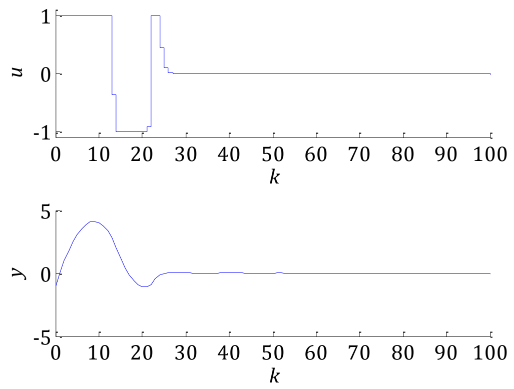
Introduction

Stability of Model Predictive Control

- **MPC without Constraints**
  - Receding horizon controller is an LTI state feedback controller in the unconstrained case
  - Stability can thus be addressed based on the eigenvalues of the closed-loop system
  - Stability is affected by the parameters N, $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ (cf. Illustrative Example on Slide 4-23ff, 4-35)
  - Closed-loop and predicted input and state sequences are identical for $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$ and arbitrary N (cf. dual mode control on Slide 4-34f)
  - Stability is guaranteed for $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$ but no formal proof has been given so far
- **MPC with Constraints**
  - Receding horizon controller is a nonlinear state feedback controller in the constrained case
  - Stability must thus be addressed based on Lyapunov's direct method
  - Closed-loop and predicted input and state sequences are not identical for $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$ and arbitrary N
  - Stability is not guaranteed for $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$ but can be guaranteed with an additional terminal constraint

Introduction

Illustrative Example



Example from Chapter 4

$\boldsymbol{x}(0) = (0.5 \quad -0.5)^T$

$y(k) = (-1 \quad 1)\boldsymbol{x}(k)$

Constraint $-1 \leq u(k) \leq 1$

Prediction horizon $N = 2$

Terminal weight $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$

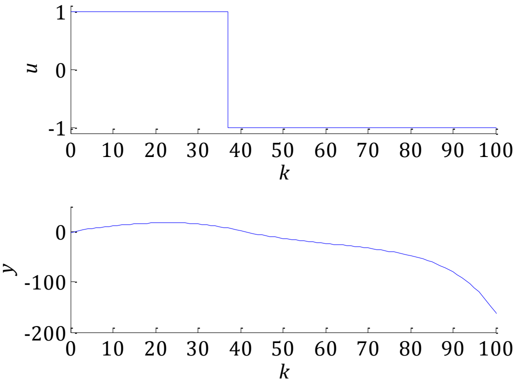
Input weight $R = 0.01$

Closed-loop system seems stable

Good performance

Introduction

Illustrative Example



Example from Chapter 4

$\boldsymbol{x}(0) = (0.8 \quad -0.8)^T$

$y(k) = (-1 \quad 1)\boldsymbol{x}(k)$

Constraint $-1 \leq u(k) \leq 1$

Prediction horizon $N = 2$

Terminal weight $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$

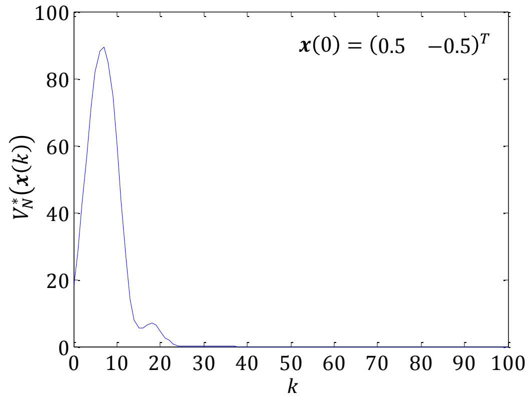
Input weight $R = 0.01$

Closed-loop system unstable

Problem 5.1 is feasible for all k, i.e. no indication for instability

Introduction

Illustrative Example



Observation

$V_N^*(\boldsymbol{x}(k))$ initially increases

Implies that energy stored in the system initially increases

Implies that closed-loop and predicted sequences differ

Conjecture

Stability guaranteed if $V_N^*(\boldsymbol{x}(k))$ is strictly decreasing over time k

$V_N^*(\boldsymbol{x}(k))$ is then a Lyapunov fcn.

Stability of MPC without Constraints

Stability Condition

Theorem 6.1 The discrete-time linear time-invariant system (4.1) with $\boldsymbol{x}(k) \in \mathbb{R}^n$ and $\boldsymbol{u}(k) \in \mathbb{R}^m$ under the receding horizon control law $\boldsymbol{u}(k) = \boldsymbol{K}_{\mathrm{RHC}}\boldsymbol{x}(k)$ is globally asymptotically stable if

- $\boldsymbol{Q}$ is positive definite
- $\boldsymbol{P}$ is positive definite and chosen such that terminal cost

$$(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}) - \boldsymbol{P} \preccurlyeq -\boldsymbol{Q} - \widetilde{\boldsymbol{K}}^T\boldsymbol{R}\widetilde{\boldsymbol{K}} \qquad (6.1)$$

where $\widetilde{\boldsymbol{K}}$ is an arbitrary matrix fulfilling $\rho(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}) < 1$.

- **Proof**
  - Let's consider the optimal cost function $V_N^*(\boldsymbol{x}(k))$ as a Lyapunov function candidate
  - The optimal cost function

$$V_N^*(\boldsymbol{x}(k)) = \boldsymbol{x}^{*T}(k+N)\boldsymbol{P}\boldsymbol{x}^*(k+N) + \sum_{i=0}^{N-1}\boldsymbol{x}^{*T}(k+i)\boldsymbol{Q}\boldsymbol{x}^*(k+i) + \boldsymbol{u}^{*T}(k+i)\boldsymbol{R}\boldsymbol{u}^*(k+i)$$

is positive definite and radially unbounded since

Stability of MPC without Constraints

Stability Condition

- **Proof**

  $V_N^*(\mathbf{0}) = 0$ since $\boldsymbol{x}(k) = \mathbf{0}$ implies $\boldsymbol{x}^*(k+i) = \mathbf{0}\ \forall i \in \{1, \dots, N\}, \boldsymbol{u}^*(k+i) = \mathbf{0}\ \forall i \in \{0, \dots, N-1\}$

  $V_N^*(\boldsymbol{x}(k)) \geq \boldsymbol{x}^T(k)\boldsymbol{Q}\boldsymbol{x}(k) > 0\ \forall \boldsymbol{x}(k) \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ since $\boldsymbol{Q} \succ \mathbf{0}$

  $V_N^*(\boldsymbol{x}(k)) \to \infty$ as $\|\boldsymbol{x}(k)\| \to \infty$

  - We must still prove that $\Delta V_N^*(\boldsymbol{x}(k)) = V_N^*(\boldsymbol{x}(k+1)) - V_N^*(\boldsymbol{x}(k))$ is negative definite
  - Consider that at time k we utilize the optimal input sequence

  $$\boldsymbol{U}^*(k) = \left(\boldsymbol{u}^{*T}(k) \quad \boldsymbol{u}^{*T}(k+1) \quad \boldsymbol{u}^{*T}(k+2) \quad \cdots \quad \boldsymbol{u}^{*T}(k+N-2) \quad \boldsymbol{u}^{*T}(k+N-1)\right)^T$$

  - Consider further that at time $k+1$ we utilize a "shifted" suboptimal input sequence

  $$\boldsymbol{U}^*(k) = \left(\underbrace{\boldsymbol{u}^{*T}(k)}_{\text{implemented}} \quad \underbrace{\boldsymbol{u}^{*T}(k+1)} \quad \underbrace{\boldsymbol{u}^{*T}(k+2)} \quad \cdots \quad \underbrace{\boldsymbol{u}^{*T}(k+N-2)} \quad \underbrace{\boldsymbol{u}^{*T}(k+N-1)}\right)^T$$

  $$\widetilde{\boldsymbol{U}}(k+1) = \left(\boldsymbol{u}^{*T}(k+1) \quad \boldsymbol{u}^{*T}(k+2) \quad \cdots \quad \boldsymbol{u}^{*T}(k+N-2) \quad \boldsymbol{u}^{*T}(k+N-1) \quad \underbrace{\left(\widetilde{\boldsymbol{K}}\boldsymbol{x}^*(k+N)\right)^T}_{\text{new "tail"}}\right)^T$$

Stability of MPC without Constraints

Stability Condition

- **Proof**
  - Note that the new tail results from the suboptimal state feedback controller $\boldsymbol{u}(k+N) = \widetilde{\boldsymbol{K}}\boldsymbol{x}^*(k+N)$
  - The suboptimal cost for the suboptimal input sequence $\widetilde{\boldsymbol{U}}(k+1)$ is given by

  $V_N(\boldsymbol{x}(k+1), \widetilde{\boldsymbol{U}}(k+1)) =$

$+V_N^*(\boldsymbol{x}(k), \boldsymbol{U}^*(k))$	old optimal cost	
$-\boldsymbol{x}^{*T}(k)\boldsymbol{Q}\boldsymbol{x}^*(k) - \boldsymbol{u}^{*T}(k)\boldsymbol{R}\boldsymbol{u}^*(k)$	old first stage cost	(6.2)
$-\boldsymbol{x}^{*T}(k+N)\boldsymbol{P}\boldsymbol{x}^*(k+N)$	old terminal cost	(6.3)
$+\boldsymbol{x}^{*T}(k+N)\left(\boldsymbol{Q} + \widetilde{\boldsymbol{K}}^T\boldsymbol{R}\widetilde{\boldsymbol{K}}\right)\boldsymbol{x}^*(k+N)$	new Nth stage cost	(6.4)
$+\boldsymbol{x}^T(k+N+1)\boldsymbol{P}\boldsymbol{x}(k+N+1)$	new terminal cost	(6.5)

  - Note that the optimal cost and the suboptimal cost at time $k+1$ are related by

  $$V_N^*(\boldsymbol{x}(k+1), \boldsymbol{U}^*(k+1)) \leq V_N(\boldsymbol{x}(k+1), \widetilde{\boldsymbol{U}}(k+1))$$

Stability of MPC without Constraints

Stability Condition

- **Proof**
  - Thus it is sufficient to prove that $V_N(\boldsymbol{x}(k+1), \widetilde{\boldsymbol{U}}(k+1)) - V_N^*(\boldsymbol{x}(k), \boldsymbol{U}^*(k))$ is negative definite
  - To this end the terms (6.2) to (6.5) must be investigated
  - The term (6.2) is negative definite
  - Thus it is sufficient to prove that the sum of the terms (6.3), (6.4), (6.5) is negative semidefinite, i.e.

$$-\boldsymbol{x}^{*T}(k+N)\boldsymbol{P}\boldsymbol{x}^*(k+N) + \boldsymbol{x}^{*T}(k+N)\big(\boldsymbol{Q} + \widetilde{\boldsymbol{K}}^T\boldsymbol{R}\widetilde{\boldsymbol{K}}\big)\boldsymbol{x}^*(k+N) + \boldsymbol{x}^T(k+N+1)\boldsymbol{P}\boldsymbol{x}(k+N+1) \leq 0 \;\forall \boldsymbol{x}(k+N)$$

  - Using that $\boldsymbol{x}(k+\mathrm{N}+1) = \big(\boldsymbol{A} + \boldsymbol{B}\widetilde{\boldsymbol{K}}\big)\boldsymbol{x}^*(k+N)$ leads to

$$\boldsymbol{x}^{*T}(k+N)\Big(\big(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\big)^T\boldsymbol{P}\big(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\big) - \boldsymbol{P}\Big)\boldsymbol{x}^*(k+N) \leq \boldsymbol{x}^{*T}(k+N)\big(-\boldsymbol{Q} - \widetilde{\boldsymbol{K}}^T\boldsymbol{R}\widetilde{\boldsymbol{K}}\big)\boldsymbol{x}^*(k+N) \;\forall \boldsymbol{x}(k+N)$$

  - This inequality is fulfilled if (6.1) is fulfilled
  - This completes the proof

Stability of MPC without Constraints

Stability Condition

- **Interpretation**
  - The suboptimal state feedback controller $\boldsymbol{u}(k+N) = \widetilde{\boldsymbol{K}}\boldsymbol{x}^*(k+N)$ evidently corresponds to the stabilizing control law utilized in mode 2 in dual mode control (cf. Slide 4-30)
  - The terminal weighting matrix $\boldsymbol{P}$ fulfilling (6.1) is used when solving Problem 4.1
  - The suboptimal feedback matrix $\widetilde{\boldsymbol{K}}$ is only introduced for the proof and not used anymore

- **Remarks**
  - For an arbitrary $\widetilde{\boldsymbol{K}}$ fulfilling $\rho\big(\boldsymbol{A} + \boldsymbol{B}\widetilde{\boldsymbol{K}}\big) < 1$ we can choose $\boldsymbol{P}$ as the solution $\widetilde{\boldsymbol{P}}$ of the DLE (4.8)
  - For $\widetilde{\boldsymbol{K}} = \boldsymbol{K}_{\mathrm{LQR}}$ we can choose $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$
  - For a globally asymptotically stable discrete-time linear time-invariant system (4.1) we have $\rho(\boldsymbol{A}) < 1$ and can thus choose $\widetilde{\boldsymbol{K}} = \boldsymbol{0}$ and $\boldsymbol{P}$ as the solution $\widetilde{\boldsymbol{P}}$ of the DLE (4.8)
  - $\boldsymbol{Q}$ positive definite can be replaced by $\big(\boldsymbol{Q}^{1/2}, \boldsymbol{A}\big)$ observable in Theorem 6.1
- **Can we formulate a similar stability condition for model predictive control with constraints?**

Stability of MPC with Constraints

Feasibility Condition

- **Observations**
  - The stability condition in Theorem 6.1 in principle also applies to MPC with constraints
  - The feasibility must, however, additionally be guaranteed
  - Assume that the optimal input sequence $\boldsymbol{U}^*(k)$ and state sequence $\boldsymbol{X}^*(k)$ at time k are feasible
  - The suboptimal input sequence and state sequence at time $k+1$ then obey

$$\tilde{\boldsymbol{U}}(k+1) = \begin{pmatrix} \boldsymbol{u}^{*T}(k+1) & \boldsymbol{u}^{*T}(k+2) & \cdots & \boldsymbol{u}^{*T}(k+N-1) & \left(\tilde{\boldsymbol{K}}\boldsymbol{x}^*(k+N)\right)^T \end{pmatrix}^T$$

$$\tilde{\boldsymbol{X}}(k+1) = \Big(\underbrace{\boldsymbol{x}^{*T}(k+2) \quad \boldsymbol{x}^{*T}(k+3) \quad \cdots \quad \boldsymbol{x}^{*T}(k+N)}_{\text{feasible (by assumption)}} \quad \underbrace{\left((\boldsymbol{A}+\boldsymbol{B}\tilde{\boldsymbol{K}})\boldsymbol{x}^*(k+N)\right)^T}_{\text{possibly infeasible}}\Big)^T$$

  - Impose terminal constraint $\boldsymbol{x}^*(k+N) \in \mathbb{X}_N$ to guarantee feasibility
  - Note that the terminal constraint is related to mode 2 in dual mode control
  - How must we choose the terminal constraint set $\mathbb{X}_N$ to guarantee feasibility?

Technische Universität Kaiserslautern

Stability of MPC with Constraints

Feasibility Condition

- **Assumption**
  - The constraints are time-invariant, i.e. $\mathbb{X}(k+i) = \mathbb{X},\ \mathbb{U}(k+i) = \mathbb{U}\ \ \forall i \in \{0,\dots,N-1\}\ \forall k \in \mathbb{N}_0$
  - E.g. for standard form $\boldsymbol{M}(k+i) = \boldsymbol{M},\ \boldsymbol{E}(k+i) = \boldsymbol{E},\ \boldsymbol{b}(k+i) = \boldsymbol{b}\ \ \forall i \in \{0,\dots,N-1\}\ \forall k \in \mathbb{N}_0$

Definition 6.1 A set $\mathbb{S} \subseteq \mathbb{R}^n$ is an invariant set for the discrete-time nonlinear time-invariant system

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k)) \tag{6.6}$$

if

$$\boldsymbol{x}(0) \in \mathbb{S} \Rightarrow \boldsymbol{f}(\boldsymbol{x}(k)) \in \mathbb{S}\ \ \forall k \in \mathbb{N}_0.$$

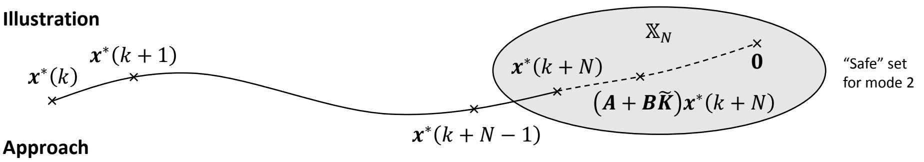
Definition 6.2 A set $\mathbb{S} \subseteq \mathbb{R}^n$ is an admissible set for the discrete-time nonlinear time-invariant system (6.6) under the state feedback control law $\boldsymbol{u}(k) = \boldsymbol{f}_C(\boldsymbol{x}(k))$, the state constraint $\mathbb{X}$ and the input constraint $\mathbb{U}$ if

$$\boldsymbol{x}(k) \in \mathbb{S} \Rightarrow \left(\boldsymbol{x}(k), \boldsymbol{f}_C(\boldsymbol{x}(k))\right) \in \mathbb{X} \times \mathbb{U}$$

Stability of MPC with Constraints

Feasibility Condition

- **Illustration**



- **Approach**
  - The terminal constraint set $\mathbb{X}_N$ must be constructed such that

    $\boldsymbol{x}^*(k+N) \in \mathbb{X}_N \Rightarrow \left(\boldsymbol{x}^*(k+N), \widetilde{\boldsymbol{K}}\boldsymbol{x}^*(k+N)\right) \in \mathbb{X} \times \mathbb{U}$ admissible set

    $\boldsymbol{x}^*(k+N) \in \mathbb{X}_N \Rightarrow (\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})\boldsymbol{x}^*(k+N) \in \mathbb{X}_N$ invariant set

  - For the standard form the terminal constraint set $\mathbb{X}_N$ is represented by $\boldsymbol{M}_N\boldsymbol{x}(k+N) \leq \boldsymbol{b}_N$ and must thus be constructed such that

    $\boldsymbol{M}_N\boldsymbol{x}^*(k+N) \leq \boldsymbol{b}_N \Rightarrow \left(\boldsymbol{M}+\boldsymbol{E}\widetilde{\boldsymbol{K}}\right)\boldsymbol{x}^*(k+N) \leq \boldsymbol{b}$ admissible set

    $\boldsymbol{M}_N\boldsymbol{x}^*(k+N) \leq \boldsymbol{b}_N \Rightarrow \boldsymbol{M}_N\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)\boldsymbol{x}^*(k+N) \leq \boldsymbol{b}_N$ invariant set

Stability of MPC with Constraints

Feasibility Condition

Theorem 6.2 Consider Problem 5.1 used for the receding horizon control law $\boldsymbol{u}^*(k)$ according to (5.2). If the terminal constraint set $\mathbb{X}_N$ is invariant and admissible for the closed-loop system

$$\boldsymbol{x}(k+1) = \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)\boldsymbol{x}(k)$$

where $\widetilde{\boldsymbol{K}}$ is an arbitrary feedback matrix fulfilling $\rho\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right) < 1$ and Problem 5.1 is feasible for $k=0$, then Problem 5.1 is feasible for all $k > 0$ if the receding horizon control law $\boldsymbol{u}^*(k)$ is used.

- **Proof**
  - The proof follows immediately from the discussion on the previous slides
- **Remark**
  - The invariant and admissible terminal constraint set $\mathbb{X}_N$ can be constructed with efficient algorithms, see [BBM15, Chapter 11 and Section 13.2.1] for a detailed discussion
  - The invariant and admissible terminal constraint set $\mathbb{X}_N$ can be constructed under MATLAB with the Multi-Parametric Toolbox [KGB+04]

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Box Constraints**

  $\underline{\boldsymbol{u}} \leq \boldsymbol{u}(k+i) \leq \overline{\boldsymbol{u}}$

  $\underline{\boldsymbol{x}} \leq \boldsymbol{x}(k+i) \leq \overline{\boldsymbol{x}}$

- **Approach**
  - Recall that the constraints must be fulfilled over the entire prediction horizon for mode 2, i.e.

    $\underline{\boldsymbol{u}} \leq \boldsymbol{u}(k+i) \leq \overline{\boldsymbol{u}} \quad \forall i \in \{N, N+1, \dots\}$

    $\underline{\boldsymbol{x}} \leq \boldsymbol{x}(k+i) \leq \overline{\boldsymbol{x}} \quad \forall i \in \{N, N+1, \dots\}$
  - Using that $\boldsymbol{u}(k+i) = \widetilde{\boldsymbol{K}}\boldsymbol{x}(k+i)$ and $\boldsymbol{x}(k+i) = \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{i-N}\boldsymbol{x}(k+N)$ leads to

    $$\underline{\boldsymbol{u}} \leq \widetilde{\boldsymbol{K}}\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{i-N}\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{u}} \quad \forall i \in \{N, N+1, \dots\} \tag{6.7}$$

    $$\underline{\boldsymbol{x}} \leq \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{i-N}\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{x}} \quad \forall i \in \{N, N+1, \dots\} \tag{6.8}$$
  - We must essentially check (6.7), (6.8) over an infinite horizon which is clearly intractable

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Approach**
  - We can show that (6.7), (6.8) must only be checked over a constraint checking horizon $N \leq N_{\mathrm{cc}} < \infty$
  - This means that (6.7), (6.8) are ensured for all $i \geq N_{\mathrm{cc}}$
  - The proof relies on $\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{i-N} \to \boldsymbol{0}$ for $i \to \infty$ since $\rho\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right) < 1$
  - The terminal constraint set $\mathbb{X}_N$ can be constructed iteratively, i.e.

    $$\mathbb{X}_N^{(0)} = \left\{\boldsymbol{x}(k+N) \middle| \underline{\boldsymbol{u}} \leq \widetilde{\boldsymbol{K}}\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^0\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{u}}, \underline{\boldsymbol{x}} \leq \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^0\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{x}}\right\}$$

    $$\mathbb{X}_N^{(1)} = \mathbb{X}_N^{(0)} \cap \left\{\boldsymbol{x}(k+N) \middle| \underline{\boldsymbol{u}} \leq \widetilde{\boldsymbol{K}}\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^1\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{u}}, \underline{\boldsymbol{x}} \leq \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^1\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{x}}\right\}$$

    $$\vdots$$

    $$\mathbb{X}_N^{(N_{\mathrm{cc}})} = \mathbb{X}_N^{(N_{\mathrm{cc}}-1)} \cap \left\{\boldsymbol{x}(k+N) \middle| \underline{\boldsymbol{u}} \leq \widetilde{\boldsymbol{K}}\left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{N_{\mathrm{cc}}-N}\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{u}}, \underline{\boldsymbol{x}} \leq \left(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}\right)^{N_{\mathrm{cc}}-N}\boldsymbol{x}(k+N) \leq \overline{\boldsymbol{x}}\right\}$$

    The iteration can be stopped if $\mathbb{X}_N^{(N_{\mathrm{cc}})} = \mathbb{X}_N^{(N_{\mathrm{cc}}+1)}$

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Approach**
  - Problem 5.1 then becomes

$$\min_{\boldsymbol{U}(k)} V_N(\boldsymbol{x}(k), \boldsymbol{U}(k))$$

$$\text{subject to} \begin{cases} \boldsymbol{x}(k+i+1) = \boldsymbol{A}\boldsymbol{x}(k+i) + \boldsymbol{B}\boldsymbol{u}(k+i), & i = 0,1,\dots,N-1 \\ \underline{\boldsymbol{x}} \le \boldsymbol{x}(k+i) \le \overline{\boldsymbol{x}}, & i = 1,2,\dots,N \\ \underline{\boldsymbol{u}} \le \boldsymbol{u}(k+i) \le \overline{\boldsymbol{u}}, & i = 0,1,\dots,N-1 \\ \underline{\boldsymbol{x}} \le (\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{i-N}\boldsymbol{x}(k+N) \le \overline{\boldsymbol{x}}, & i = N, N+1,\dots,N_{\mathrm{cc}} \\ \underline{\boldsymbol{u}} \le \widetilde{\boldsymbol{K}}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{i-N}\boldsymbol{x}(k+N) \le \overline{\boldsymbol{u}}, & i = N, N+1,\dots,N_{\mathrm{cc}} \end{cases}$$

- **Remarks**
  - Problem 5.1 can still be written as a quadratic program with additional constraints
  - The terminal constraint set depends only on $\boldsymbol{A}, \boldsymbol{B}, \widetilde{\boldsymbol{K}}, \underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}, \underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}$ and N_{cc} but not on $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ and N
  - The constraint checking horizon N_{cc} can be computed by checking $\mathbb{X}_N^{(N_{\mathrm{cc}})} = \mathbb{X}_N^{(N_{\mathrm{cc}}+1)}$ during iteration

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Algorithm for the Computation of N_{cc} (for $\mathbb{X} = \mathbb{R}^n$ and $m = 1$)**
  1. Set $N_{\mathrm{cc}} := 0$
  2. Determine

$$u_{\max} := \max_{\boldsymbol{x}(k+N)} \underbrace{\widetilde{\boldsymbol{K}}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{N_{\mathrm{cc}}+1}\boldsymbol{x}(k+N)}_{\boldsymbol{u}(k+N_{\mathrm{cc}}+1)}$$

$$\text{subject to } \underline{u} \le \underbrace{\widetilde{\boldsymbol{K}}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{i-N}\boldsymbol{x}(k+N)}_{\boldsymbol{u}(k+i)} \le \overline{u}, i = N, N+1,\dots,N_{\mathrm{cc}}$$

$$u_{\min} := \min_{\boldsymbol{x}(k+N)} \widetilde{\boldsymbol{K}}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{N_{\mathrm{cc}}+1}\boldsymbol{x}(k+N)$$

$$\text{subject to } \underline{u} \le \widetilde{\boldsymbol{K}}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^{i-N}\boldsymbol{x}(k+N) \le \overline{u}, i = N, N+1,\dots,N_{\mathrm{cc}}$$

linear programs

  3. If $u_{\max} \le \overline{u}$ and $u_{\min} \ge \underline{u}$ then
     stop
     else
     set $N_{\mathrm{cc}} := N_{\mathrm{cc}} + 1$ and goto 2.

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Illustrative Example**
  - Reconsider the Illustrative Example from Chapter 4 (cf. Slide 4-11) with the input constraint $-1 \leq u(k) \leq 1$, the input weight $R = 1$ and $\widetilde{\boldsymbol{K}} = \boldsymbol{K}_{\mathrm{LQR}}$
  - The terminal constraint set $\mathbb{X}_N$ follows from

$\mathbb{X}_N^{(0)} = \{\boldsymbol{x}(k+N) | -1 \leq (-1.19 \quad -7.88)\boldsymbol{x}(k+N) \leq 1\}$ intersection of 2 half-spaces

$\mathbb{X}_N^{(1)} = \mathbb{X}_N^{(0)} \cap \{\boldsymbol{x}(k+N) | -1 \leq (-0.57 \quad -4.98)\boldsymbol{x}(k+N) \leq 1\}$ intersection of 4 half-spaces

$\mathbb{X}_N^{(2)} = \mathbb{X}_N^{(1)} \cap \{\boldsymbol{x}(k+N) | -1 \leq (-0.16 \quad -2.78)\boldsymbol{x}(k+N) \leq 1\}$ intersection of 6 half-spaces

$\mathbb{X}_N^{(3)} = \mathbb{X}_N^{(2)} \cap \{\boldsymbol{x}(k+N) | -1 \leq (0.08 \quad -1.24)\boldsymbol{x}(k+N) \leq 1\}$ intersection of 8 half-spaces

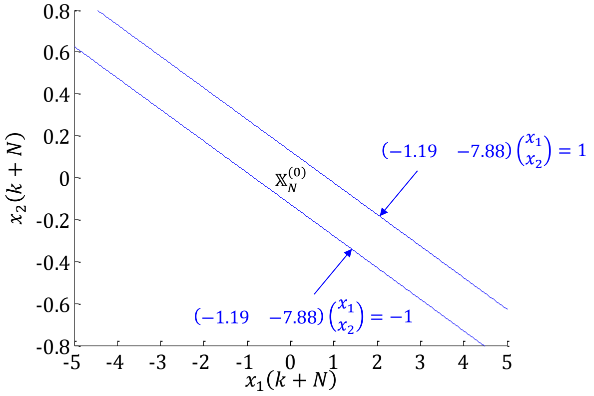
$\mathbb{X}_N^{(4)} = \mathbb{X}_N^{(3)} \cap \{\boldsymbol{x}(k+N) | -1 \leq (0.21 \quad -0.25)\boldsymbol{x}(k+N) \leq 1\}$ intersection of 10 half-spaces

  - We can show that $\mathbb{X}_N^{(i)} = \mathbb{X}_N^{(4)}$ for all $i > 4$ and thus $N_{\mathrm{cc}} = 4$

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Illustrative Example**

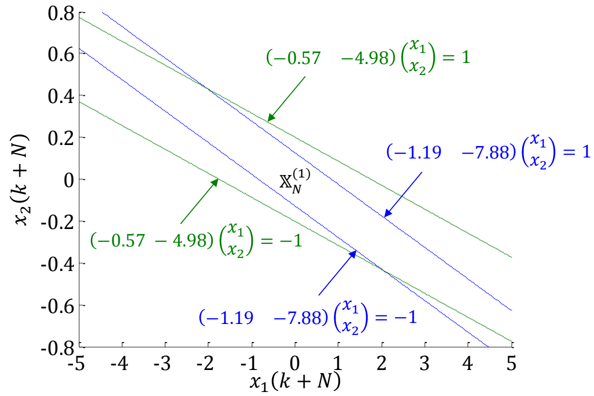


TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Illustrative Example**

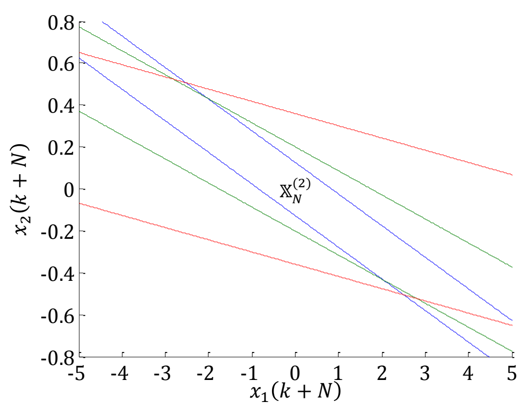


TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

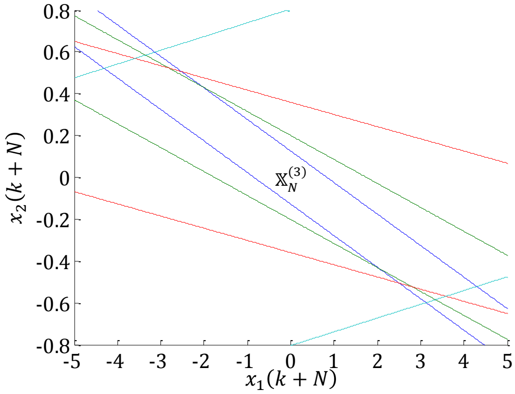
- **Illustrative Example**



Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

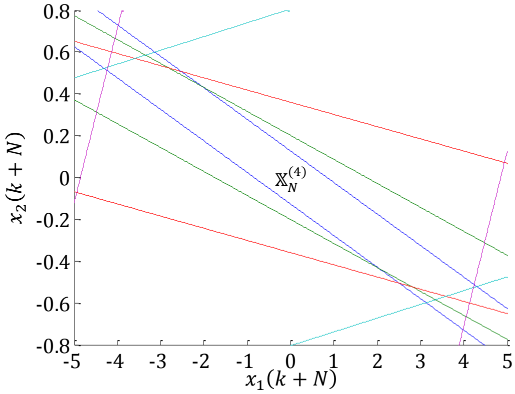
- **Illustrative Example**



Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Illustrative Example**

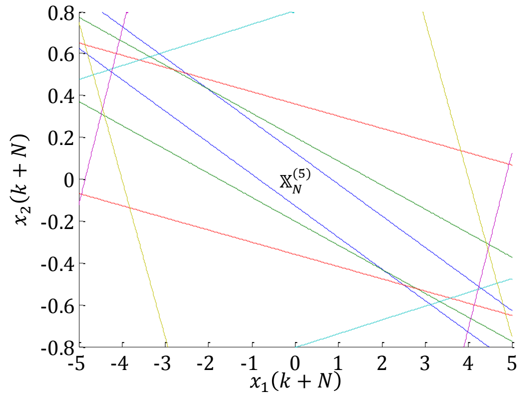


TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Stability of MPC with Constraints

Terminal Constraint Set for Box Constraints

- **Illustrative Example**



TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Stability of MPC with Constraints

Stability Condition

Theorem 6.3 The discrete-time linear time-invariant system (4.1) with $\boldsymbol{x}(k) \in \mathbb{X}$ and $\boldsymbol{u}(k) \in \mathbb{U}$ under the receding horizon control law $\boldsymbol{u}^*(k)$ according to (5.2) is asymptotically stable if

- $\boldsymbol{Q}$ is positive definite
- $\boldsymbol{P}$ is positive definite and chosen such that — terminal cost

  $$(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})^T\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}) - \boldsymbol{P} \preccurlyeq -\boldsymbol{Q} - \widetilde{\boldsymbol{K}}^T\boldsymbol{R}\widetilde{\boldsymbol{K}} \quad (6.1)$$

  where $\widetilde{\boldsymbol{K}}$ is an arbitrary matrix fulfilling $\rho(\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}}) < 1$
- $\boldsymbol{x}(k+N) \in \mathbb{X}_N$ — terminal constraint

  where $\mathbb{X}_N$ is invariant and admissible for $\boldsymbol{x}(k+1) = (\boldsymbol{A}+\boldsymbol{B}\widetilde{\boldsymbol{K}})\boldsymbol{x}(k)$.

The domain of attraction is $\mathbb{D} = \{\boldsymbol{x}(0) \in \mathbb{X} | \exists \boldsymbol{U}(0): \boldsymbol{x}(i) \in \mathbb{X}, \boldsymbol{u}(i) \in \mathbb{U}\ \forall i \in \{0, \dots, N-1\}, \boldsymbol{x}(N) \in \mathbb{X}_N\}$.

- **Proof**
  - The proof follows immediately from the discussion on the previous slides

Stability of MPC with Constraints

Stability Condition

- **Remark on the Domain of Attraction**
  - The domain of attraction $\mathbb{D}$ increases with the prediction horizon N and terminal constraint set $\mathbb{X}_N$
  - For a given prediction horizon N the domain of attraction $\mathbb{D}$ should ideally be as large as possible
  - This is achieved for the maximal invariant and admissible terminal constraint set $\mathbb{X}_N$
- **Remark on the Selection of the Terminal Constraint**
  - The terminal constraint $\boldsymbol{x}(k+N) = \mathbf{0}$ satisfies the conditions in Theorem 6.3 trivially since then the “tail” is always feasible (cf. Slide 6-11)
  - This terminal constraint has been proposed in [KG88] and is commonly considered as the first stability condition presented for MPC with constraints
  - This terminal constraint is unfortunately very restrictive and usually impairs performance
  - This terminal constraint is still useful if the construction of a terminal constraint set is difficult, e.g. for nonlinear systems

Stability of MPC with Constraints

Stability Condition

- **Remark on the Need for a Terminal Constraint**
  - The terminal constraint is not needed if $N \geq N_{\text{stab}}$ for a given $\boldsymbol{x}(0)$ since then $\mathbb{X}_N$ is inactive
  - Computing the stabilizing prediction horizon N_{stab} is, however, involved and subject to research
  - Note that the stabilizing prediction horizon N_{stab} depends on the initial state $\boldsymbol{x}(0)$
  - Note furthermore that for $N \geq N_{\text{stab}}$ also the closed-loop cost does not change anymore
- **Remark on the Influence of the Terminal Constraint**
  - The terminal constraint influences the performance
  - We generally have that

    large computation time $\Leftrightarrow$ large N $\Leftrightarrow$ large $\mathbb{X}_N$ $\Leftrightarrow$ good performance

    small computation time $\Leftrightarrow$ small N $\Leftrightarrow$ small $\mathbb{X}_N$ $\Leftrightarrow$ poor performance
  - Constructing the maximal invariant and admissible terminal constraint set is thus crucial
- **More details on stability of MPC can be found in the seminal paper [MRR+00]**

Stability of MPC with Constraints

Stability Condition

- **Illustrative Example**
  - Reconsider the Illustrative Example from Chapter 4 (cf. Slide 4-11) with $\boldsymbol{x}(0) = (-7 \quad 0.5)^T$, $-1 \leq u(k) \leq 1$, $R = 1$, $\widetilde{\boldsymbol{K}} = \boldsymbol{K}_{\mathrm{LQR}}$ and $\boldsymbol{P} = \boldsymbol{P}_{\mathrm{LQR}}$
  - Compute the closed-loop cost $V_\infty(\boldsymbol{x}(0))$ and the optimal predicted cost $V_N^*(\boldsymbol{x}(0))$ for different N

N	5	6	7	10	> 10
$V_\infty(\boldsymbol{x}(0))$	295.2	287.7	286.8	286.6	286.6
$V_N(\boldsymbol{x}(0))$	286.7	286.7	286.6	286.6	286.6

  - Evidently the closed-loop cost $V_\infty(\boldsymbol{x}(0))$ and optimal predicted cost $V_N^*(\boldsymbol{x}(0))$ are equal for $N \geq 10$
  - This implies that the terminal constraint $\boldsymbol{x}(k+N) \in \mathbb{X}_N$ is inactive for $N \geq 10$
  - This implies in turn that $N_{\mathrm{stab}} = 10$
  - The receding horizon controller for $N \geq N_{\mathrm{stab}}$ is called constrained linear quadratic regulator (CLQR)

Literature

Miscellaneous

[KG88] S. S. Keerthi and E. G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, 57(2):265–293, 1988.

[KGB+04] Michal Kvasnica, Pascal Grieder, Mato Baotić, and Manfred Morari. Multi-Parametric Toolbox (MPT). In *Proceedings of the 7th International Workshop on Hybrid Systems: Computation and Control*, pages 448–462, Philadelphia, PA, USA, 2004. – control.ee.ethz.ch/~mpt/3/

[MRR+00] David Q. Mayne, James B. Rawlings, Christopher V. Rao, and Pierre O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.