Chapter 3 (Part 2)

Discrete-time, linear time-invariant systems

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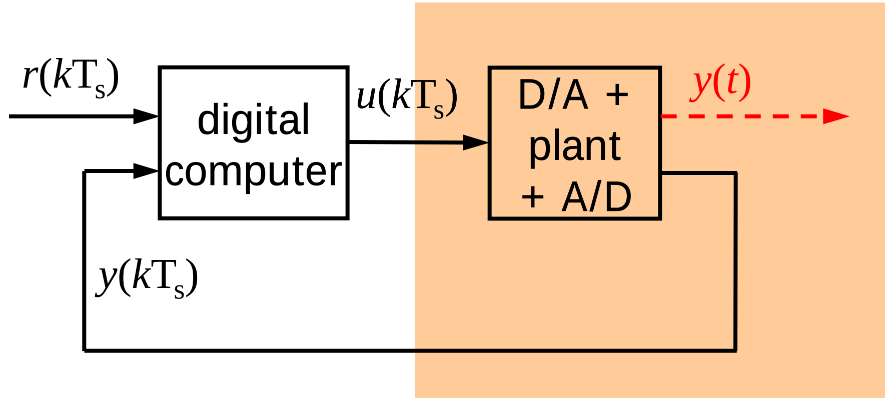
Necessity of discrete-time description of LTI systems for control design



From the viewpoint of the design of digital control, the behavior of the plant (process) preceded by A/D and D/A converters is obviously discrete in time (i. e., only present at the normally equidistant sampling instants). It is therefore important to discribe this discrete-time behavior with a mathematical model which characterizes the relation between the input and output at the sampling instants.

The LTI properties remain unchanged in the discrete version as far as the sampling period is constant.

Difference equation as LTI system description

A SISO LTI system in the discrete-time form is described by a linear *difference equation* (*recurrance equation*) with constant coefficients which relates the input u to the output y at the sampling instants $k\mathrm{T}_s$ (often denoted as k)

$$y_k = -\alpha_1 y_{k-1} - \alpha_2 y_{k-2} - \cdots - \alpha_n y_{k-n} + \beta_0 u_k + \beta_1 u_{k-1} + \cdots + \beta_p u_{k-p} \qquad (3T2.1)$$

For causal systems it is $n \geq p$. The number n of the maximum recursion is said to be the order of the system.

Eq. (3T2.1) is called the *discrete input-output relationship* of a linear, time-invariant dynamic system and is one of the standard forms for the mathematical system models. The linearity and time-invariance are defined in a similar way as in case of continuous-time systems.

Unlike in case of continuous systems the input and output of a discrete-time system are series of values, i. e.

$$\begin{aligned} \{u_k\} &= \{u_1, u_2, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots\} \\ \{y_k\} &= \{y_1, y_2, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots\} \end{aligned} \qquad (3T2.2)$$

Compared to continuous systems the system response of a discrete-time system can be numerically very easily calculated by Eq. (3T2.1).

The z-transform of a discrete-time signal

A compact description of a discrete-time signal as sampled series of an analogous signal can be obtained using the z-transform.

The (one-sided or unilateral) z-transform of a causal discrete signal $\{f_k\}$ is defined as

$$F(z) = \mathcal{Z}\{f_k\} = \sum_{k=0}^{\infty} f_k z^{-k} \tag{3T2.3}$$

$$f_k = \mathcal{Z}^{-1}\{F(z)\} \frac{1}{2\pi \mathrm{j}} \oint F(z) z^{k-1} \mathrm{d}z \tag{3T2.4}$$

or abbreviated as

$$\{f_k\} \circ\!\!-\!\!\bullet \; F(z) \; . \tag{3T2.5}$$

Obviously, the z-transform of a discrete signal does only exist, if the series Eq. (3T2.4) converges which is, fortunately, the case for most signals of engineering interest.

Calculation of z-transform

Exp. 3T2-1 The input series $\{f_k\}$ are taken as samples from the time signal e^{-at} für $t \geq 0$ with the sampling period T_s. Find the z-transform of this sampled signal.

Solution Application of (3T2.3) yields

$$F(z) = \mathcal{Z}\{f_k\} = \sum_{k=0}^{\infty} e^{-akT_s} z^{-k} = \sum_{k=0}^{\infty} \left(e^{-aT_s} z^{-1}\right)^k$$

$$= \frac{1}{1-e^{-aT_s} z^{-1}} = \frac{z}{z-e^{-aT_s}} \qquad \text{für} \quad |z| > e^{-aT_s}$$

As in the case of the Laplace transform, we don't need to worry about the calculation effort of the z-transform at this point because z-transform of almost all signals in control engineering can be found in a z-transform table. The direct and inverse z-transform can be calculated then easily, sometimes based on the partial fraction expansion.

Some properties of the z-transform

$$f_k \;\circ\!-\!\bullet\; F(z); \qquad g_k \;\circ\!-\!\bullet\; G(z)$$

Linearity:	$a_1 f_k + a_2 g_k \;\circ\!-\!\bullet\; a_1 F(z) + a_2 G(z)$	(3T2.6)
Right shift (delay): Left shift (advance):	$f_{k-m} \;\circ\!-\!\bullet\; z^{-m} F(z)$ $f_{k+m} \;\circ\!-\!\bullet\; z^{m} F(z) - z^{m} \sum_{\mu=0}^{m-1} f_\mu z^{-\mu}$	(3T2.7) (3T2.8)
Time convolution:	$f_k * g_k = \sum_{l=-\infty}^{\infty} f_l \cdot g_{k-l} \;\circ\!-\!\bullet\; F(z) \cdot G(z)$	(3T2.9)
Scaling in the z-plane:	$r^{-k} f_k \;\circ\!-\!\bullet\; F(rz)$	(3T2.10)
Initial-value theorem: Final-value theorem:	$f_0 = \lim_{z \to \infty} F(z)$ $\lim_{k \to \infty} f_k = \lim_{z \to 1} (z-1) F(z)$ if the limits exist	(3T2.11) (3T2.12)

z-transform table (1)

No.	$f(t)$	$F(s)$	$\{f_k\}$	$F(z)$
1	$\delta(t)$	1	$1, k=0; 0, k \neq 0$	1
2	$\sigma(t)$	$\frac{1}{s}$	1	$\frac{z}{z-1}$
3	$t\sigma(t)$	$\frac{1}{s^2}$	$k\mathrm{T_s}$	$\frac{\mathrm{T_s} z}{(z-1)^2}$
4	$\frac{t^2}{2}\sigma(t)$	$\frac{1}{s^3}$	$\frac{1}{2}(k\mathrm{T_s})^2$	$\frac{\mathrm{T_s}^2}{2}\frac{z(z+1)}{(z-1)^3}$
5	$e^{-at}\sigma(t)$	$\frac{1}{s+a}$	$e^{-ak\mathrm{T_s}}$	$\frac{z}{z-e^{-a\mathrm{T_s}}}$
6	$t \cdot e^{-at}\sigma(t)$	$\frac{1}{(s+a)^2}$	$k\mathrm{T_s} e^{-ak\mathrm{T_s}}$	$\frac{\mathrm{T_s} z e^{-a\mathrm{T_s}}}{(z-e^{-a\mathrm{T_s}})^2}$

z-transform table (2)

No.	$f(t)$	$F(s)$	$\{f_k\}$	$F(z)$
7	$1 - e^{-at}$	$\frac{a}{s\cdot(s+a)}$	$1-e^{-akT_s}$	$\frac{z(1-e^{-aT_s})}{(z-1)(z-e^{-aT_s})}$
8	$e^{-at} - e^{-bt}$	$\frac{b-a}{(s+a)\cdot(s+b)}$	$e^{-akT_s} - e^{-bkT_s}$	$\frac{(e^{-aT_s} - e^{-bT_s})z}{(z-e^{-aT_s})(z-e^{-bT_s})}$
9	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\sin(\omega k T_s)$	$\frac{z\sin\omega T_s}{z^2-(2\cos\omega T_s)z+1}$
10	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\cos(\omega k T_s)$	$\frac{z(z-\cos\omega T_s)}{z^2-(2\cos\omega T_s)z+1}$
11	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$	$e^{-akT_s}\sin(\omega k T_s)$	$\frac{ze^{-aT_s}\sin\omega T_s}{z^2-(2e^{-aT_s}\cos\omega T_s)z+e^{-2aT_s}}$
12	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$	$e^{-akT_s}\cos(\omega k T_s)$	$\frac{z(z-e^{-aT_s}\cos\omega T_s)}{z^2-(2e^{-aT_s}\cos\omega T_s)z+e^{-2aT_s}}$

Solution of difference equation using z-transform

Exp. 3T2-2: Find the solution of the difference equation $y_{k+1} = 0.5y_k + u_{k+1}$ with the initial condition $y_0 = 8$ and the input $u_k = k^2$ $(k \geq 0)$.

Solution: z-transform of the difference equation with the initial condition by applying the shift property yields:

$$zY(z) - zy_0 - 0.5Y(z) = zU(z) - zu_0$$

or

$$Y(z) = \frac{z}{z-0.5}U(z) + \frac{z}{z-0.5}(y_0 - u_0) = \underbrace{\frac{z}{z-0.5} \cdot \frac{z(z+1)}{(z-1)^3}}_{\text{zero-state response}} + \underbrace{\frac{8z}{z-0.5}}_{\text{zero-input response}}$$

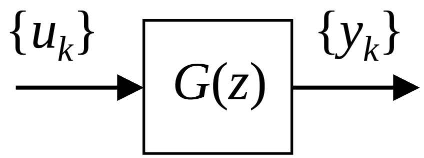
Applying partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{2}{z-0.5} + \frac{6}{z-1} - \frac{2}{(z-1)^2} + \frac{4}{(z-1)^3}$$

After inverse z-transform we obtain finally

$$y_k = 2 \cdot (0.5)^k + 6 - 2k + 2k(k-1)$$

Zero-state response and the discrete transfer function



Analogously to the definition of continuous transfer function, the discrete transfer function is defined as the ratio of output $Y(z)$ to input $U(z)$ (assuming all initial conditions zero):

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\mathcal{Z}\left[zero - state\ response\right]}{\mathcal{Z}\left[input\right]} \tag{3T2.13}$$

For example, the discrete transfer function of the system in Exp. 3T2-2 is

$$G(z) = \frac{z}{z-0.5}$$

General z-transfer function of discrete LTI system

For the more general relation given by Eq. (3T2.1), it is readily verified by the same techniques that

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 + \beta_1 z^{-1} + \cdots + \beta_p z^{-p}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_n z^{-n}} \tag{3T2.14}$$

And if $n \geq p$ (causal system), we can write this as a ratio of polynomials in z as

$$G(z) = \frac{\beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_p z^{n-p}}{z^n + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \cdots + \alpha_n} = \frac{Z(z)}{N(z)} \tag{3T2.15}$$

Completely factorized, the transfer function can also be given in the form of zeros and poles

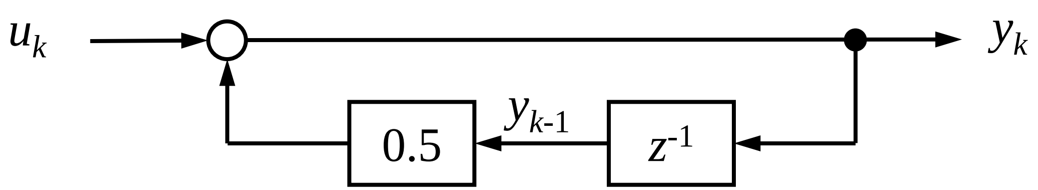
$$G(z) = K \frac{\prod_{i=1}^{p} \overbrace{(z - z_i)}^{\text{Zeros}}}{\prod_{j=1}^{n} \underbrace{(z - p_j)}_{\text{Poles}}} \tag{3T2.16}$$

z^{-1} as delay operator

Does the operator z have a physical meaning at all? We assume that all coefficients in (3T2.14) are zero except b_1 and we take b_1 to be 1. Then the transfer function $G(z) = z^{-1}$ and the difference equation in the time-domain becomes

$$y_k = u_{k-1} \quad .$$

The present value of the output, , equals the input delayed by one sample period. Thus a transfer function of z^{-1} is a delay operator of one time unit (*unit delay*). Since a delay is always causal, it is usually used in discrete system block diagrams in the same way as an integrator s^{-1} in continuous system block diagrams. For example, the system treated in Exp. 3T2-2 can be described by the following block diagram:



More generally, z is a *shift operator*.

Transfer function and unit pulse response

In (3T2.13) , suppose we deliberately select u_k to be the *unit discrete pulse* defined by

$$u_k = \delta_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} . \tag{3T2.17}$$

Then it follows from (3T2.3) that $U(z) = 1$ and therefore that $Y(z) = G(z)$. Clearly, the transfer function can be seen to be the z-*transform* of the system response (called unit-pulse response g_k) to a unit-pulse input δ_k.

From the definition of the z-transform (Eq. (3T2.3)) we know that

$$U(z) = \sum_{k=0}^{\infty} u_k z^{-k} = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \cdots \tag{3T2.18}$$

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} = g_0 + g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3} + \cdots \tag{3T2.19}$$

Thus the general system response is obtained by multiplying the infinite polynomials as shown in the following table.

System response as discrete convolution

$U(z)$ $G(z)$	$u_0 + u_1 z^{-1}$ $g_0 + g_1 z^{-1}$	$+u_2 z^{-2}$ $+g_2 z^{-2}$	$+u_3 z^{-3}$ $+g_3 z^{-3}$	$+\cdots$ $+\cdots$
	$u_0 g_0 + u_1 g_0 z^{-1}$ $+u_0 g_1 z^{-1}$	$+u_2 g_0 z^{-2}$ $+u_1 g_1 z^{-2}$ $+u_0 g_2 z^{-2}$	$+u_3 g_0 z^{-3}$ $+u_2 g_1 z^{-3}$ $+u_1 g_2 z^{-3}$ $+u_0 g_3 z^{-3}$	
$Y(z)$	$u_0 g_0 + (u_0 g_1 + u_1 g_0) z^{-1} + (u_0 g_2 + u_1 g_1 + u_2 g_0) z^{-2} + (u_0 g_3 + u_1 g_2 + u_2 g_1 + u_3 g_0) z^{-3} + \cdots$			

It follows from $Y(z) = \sum_{k=0}^{\infty} y_k z^{-k}$ that

$$y_0 = u_0 g_0$$
$$y_1 = u_0 g_1 + u_1 g_0$$
$$y_2 = u_0 g_2 + u_1 g_1 + u_2 g_0$$
$$y_3 = u_0 g_3 + u_1 g_2 + u_2 g_1 + u_3 g_0$$

In general, the system response to an arbitrary input is given by the *discrete convolution*

$$y_k = \sum_{j=0}^{k} u_j g_{k-j} = \sum_{i=0}^{k} g_i u_{k-i} \qquad (3T2.20)$$

The unit step response

Just as in the case of continuous systems, the system response to the *unit step*, defined by

$$u_k = \sigma_k = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} , \tag{3T2.21}$$

is characteristic for the dynamic behavior. The z-transform of the unit step input

$$\sigma(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} . \tag{3T2.22}$$

Obviously, the z-domain unit step response is given by $H(z) = G(z)\dfrac{z}{z-1}$ which the time-domain step response is the inverse transform from.

For the stationary final-value of the unit step response (*dc gain* of the system) we have

$$\lim_{k\to\infty} h_k = \lim_{z\to 1}(z-1)H(z) = \lim_{z\to 1} G(z) = \frac{\sum_{j=0}^{p} \beta_j}{1+\sum_{i=1}^{n} \alpha_i} = V \tag{3T2.23}$$

Relation between Laplace and z-transform

A discrete signal can be thought as the result of sampling a continuous-time signal $f(t)$. By utilizing the sifting property of the unit impulse a continuous-time representation of a discrete signal is obtained

$$f_s(t) = \sum_{k=0}^{\infty} f(k\mathrm{T}_s)\delta(t - k\mathrm{T}_s) \quad . \tag{3T2.24}$$

Taking Laplace-transform both sides of Eq. (3T2.24) with time-shift property yields

$$F_s(s) = \sum_{k=0}^{\infty} f(k\mathrm{T}_s)e^{-k\mathrm{T}_s s} = \sum_{k=0}^{\infty} f(k\mathrm{T}_s)\left(e^{\mathrm{T}_s s}\right)^{-k} \quad . \tag{3T2.25}$$

Compared with Eq. (3T2.3) it is clear that the z-transform can be considered as a Laplace transform with a change of variable

$$z = e^{\mathrm{T}_s s} \tag{3T2.26}$$

or

$$s = \frac{1}{\mathrm{T}_s}\ln z \quad . \tag{3T2.27}$$

Of course, we may consider the z-transform independently in its own right, too. In practice, the conversion between Laplace and z-transform is done using correspondence table.

Mapping the s-plane onto the z-plane

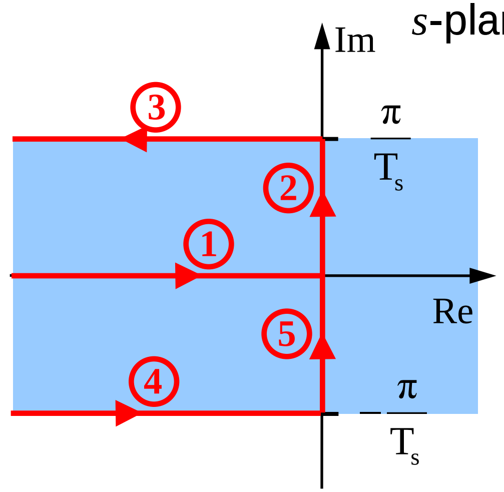
The z-plane is related to the s-plane by the simple transformation of (3T2.26). For every point $s = \sigma + j\omega$ on the s-plane we therefore can find the correspondence using

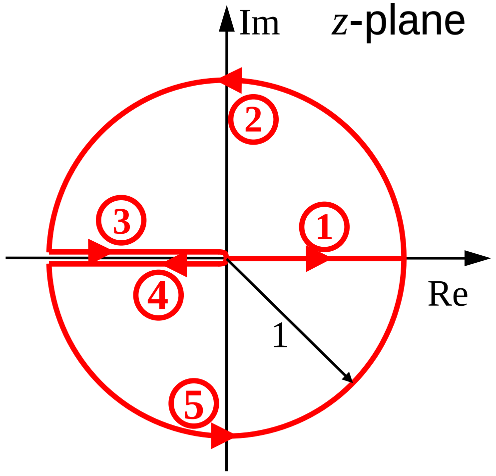
$$z = e^{T_s(\sigma + j\omega)} = e^{\sigma T_s}\left(\cos\omega T_s + j\sin\omega T_s\right) \tag{3T2.28}$$

Thus, the j-axis of the s-plane is transformed into the unit circle in the z-plane ($z = e^{j\omega T_s}$, or $|z| = 1$), the LHP (left half plane) and RHP (right half plane) into the inside and outside, respectively, of the unit circle.

Therefore, we can state that

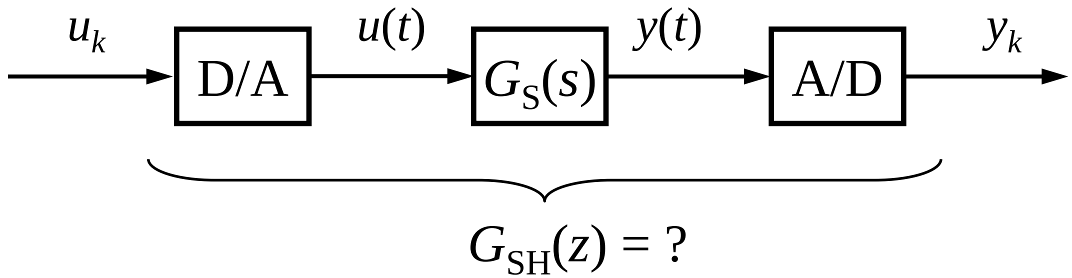
a discrete system is bibo-stable if and only if all the poles of its transfer function $G(z)$ lie within the unit circle of the z-plane.

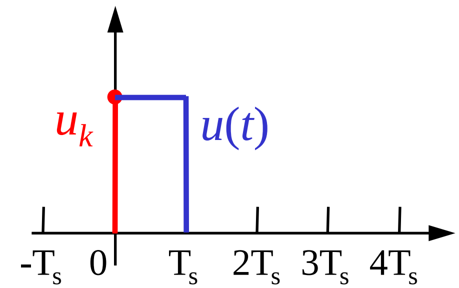




Only the main stripe of the s-plane can be mapped one-to-one onto the z-plane (aliasing)!

Discrete plant model for sampled-data control





The transfer function $G_{SH}(z)$ is the z-transform of the unit impulse response which can be, as shown before, obtained by applying Laplace transform and variable substitution (3T2.26). For $u_k = \delta_k$ the output of the D/A converter (sampler + ZOH) is a rectangular pulse. Thus

$$Y(s) = G_S(s)U(s) = (1 - e^{-sT_s})\frac{G_S(s)}{s}$$

$$G_{SH}(z) = \frac{Y(z)}{U(z)} = \mathcal{Z}\left\{(1 - e^{-sT_s})\frac{G_S(s)}{s}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{G_S(s)}{s}\right\} \tag{3T2.29}$$

Exp. 3T2-3: Find the discrete transfer function of $G(s) = a/(s+a)$ preceded by D/A and A/D.

$$G_{SH}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{a}{s(s+a)}\right\} = \frac{z-1}{z} \cdot \frac{z(1 - e^{-aT_s})}{(z-1)(z - e^{-aT_s})} = \frac{1 - e^{-aT_s}}{z - e^{-aT_s}}$$