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Process Modelling, Identification, and Control
I

Models and dynamic characteristics of continuous processes

Slovak University of Technology in Bratislava

This publication deals with mathematical modelling, dynamical process characteristics and properties. The intended audience of this book includes graduate students but can be of interest of practising engineers or applied scientists that are interested in modelling, identification, and process control.

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Preface

This publication is the first part of a book that deals with mathematical modelling of processes, their dynamical properties and dynamical characteristics. The need of investigation of dynamical characteristics of processes comes from their use in process control. The second part of the book will deal with process identification, optimal, and adaptive control.

The aim of this part is to demonstrate the development of mathematical models for process control. Detailed explanation is given to state-space and input-output process models.

In the chapter Dynamical properties of processes, process responses to the unit step, unit impulse, harmonic signal, and to a random signal are explored.

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About the Authors

J. Mikleš obtained the degree Ing. at the Mechanical Engineering Faculty of the Slovak University of Technology (STU) in Bratislava in 1961. He was awarded the title PhD. and DrSc. by the same university. Since 1988 he has been a professor at the Faculty of Chemical Technology STU. In 1968 he was awarded the Alexander von Humboldt fellowship. He worked also at Technische Hochschule Darmstadt, Ruhr Universität Bochum, University of Birmingham, and others.

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The publication activity of Dr. Fikar includes more than 60 works and he is co-author of one book. In his scientific work he deals with predictive control, constraint handling, system identification, optimisation, and process control.

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Chapter 1

Introduction

This chapter serves as an introduction to process control. The aim is to show the necessity of process control and to emphasize its importance in industries and in design of modern technologies. Basic terms and problems of process control and modelling are explained on a simple example of heat exchanger control. Finally, a short history of development in process control is given.

1.1 Topics in Process Control

Continuous technologies consist of unit processes, that are rationally arranged and connected in such a way that the desired product is obtained effectively with certain inputs.

The most important technological requirement is safety. The technology must satisfy the desired quantity and quality of the final product, environmental claims, various technical and operational constraints, market requirements, etc. The operational conditions follow from minimum price and maximum profit.

Control system is the part of technology and in the framework of the whole technology which is a guarantee for satisfaction of the above given requirements. Control systems in the whole consist of technical devices and human factor. Control systems must satisfy

- disturbance attenuation,
- stability guarantee,
- optimal process operation.

Control is the purposeful influence on a controlled object (process) that ensures the fulfillment of the required objectives. In order to satisfy the safety and optimal operation of the technology and to meet product specifications, technical, and other constraints, tasks and problems of control must be divided into a hierarchy of subtasks and subproblems with control of unit processes at the lowest level.

The lowest control level may realise *continuous-time control* of some measured signals, for example to hold temperature at *constant value*. The second control level may perform static *optimisation* of the process so that optimal values of some signals (flows, temperatures) are calculated in certain time instants. These will be set and remain constant till the next optimisation instant. The optimisation may also be performed continuously. As the unit processes are connected, their operation is *coordinated* at the third level. The highest level is influenced by *market, resources*, etc.

The fundamental way of control on the lowest level is *feedback control*. Information about process output is used to calculate control (manipulated) signal, i.e. process output is fed back to process input.

There are several other methods of control, for example feed-forward. *Feed-forward control* is a kind of control where the effect of control is not compared with the desired result. In this case we speak about *open-loop control*. If the feedback exists, *closed-loop system* results.

Process design of "modern" technologies is crucial for successful control. The design must be developed in such a way, that a "sufficiently large number of degrees of freedom" exists for the purpose of control. The control system must have the ability to operate the whole technology or the unit process in the required technology regime. The processes should be "well" controllable and the control system should have "good" information about the process, i.e. the design phase of the process should include a selection of suitable *measurements*. The use of computers in the process control enables to choose optimal structure of the technology based on claims formulated in advance. Projectants of "modern" technologies should be able to include all aspects of control in the design phase.

Experience from control praxis of "modern" technologies confirms the importance of assumptions about dynamical behaviour of processes and more complex control systems. The control centre of every "modern" technology is a place, where all information about operation is collected and where the operators have contact with technology (through keyboards and monitors of control computers) and are able to correct and interfere with technology. A good knowledge of technology and process control is a necessary assumption of qualified human influence of technology through control computers in order to achieve optimal performance.

All of our further considerations will be based upon *mathematical models of processes*. These models can be constructed from a physical and chemical nature of processes or can be abstract. The investigation of dynamical properties of processes as well as whole control systems gives rise to a need to look for effective means of differential and difference equation solutions. We will carefully examine dynamical properties of open and closed-loop systems. A fundamental part of each procedure for effective control design is the process *identification* as the real systems and their physical and chemical parameters are usually not known perfectly. We will give procedures for design of *control algorithms* that ensure effective and safe operation.

One of the ways to secure a high quality process control is to apply *adaptive control laws*. Adaptive control is characterised by gaining information about unknown process and by using the information about on-line changes to process control laws.

1.2 An Example of Process Control

We will now demonstrate problems of process dynamics and control on a simple example. The aim is to show some basic principles and problems connected with process control.

1.2.1 Process

Let us assume a heat exchanger shown in Fig. 1.2.1. Inflow to the exchanger is a liquid with a flow rate q and temperature ϑ_v. The task is to heat this liquid to a higher temperature ϑ_w. We assume that the heat flow from the heat source is independent from the liquid temperature and only dependent from the heat input ω. We further assume ideal mixing of the heated liquid and no heat loss. The accumulation ability of the exchanger walls is zero, the exchanger holdup, input and output flow rates, liquid density, and specific heat capacity of the liquid are constant. The temperature on the outlet of the exchanger ϑ is equal to the temperature inside the exchanger. The exchanger that is correctly designed has the temperature ϑ equal to ϑ_w. *The process* of heat transfer realised in the heat exchanger is defined as our *controlled system*.

1.2.2 Steady-State

The inlet temperature ϑ_v and the heat input ω are *input variables* of the process. The outlet temperature ϑ is process *output variable*. It is quite clear that every change of input variables ϑ_v, ω results in a change of output variable ϑ. From this fact follows *direction of information*

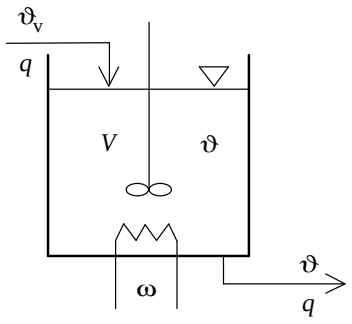

Figure 1.2.1: A simple heat exchanger.

transfer of the process. The process is in the *steady state* if the input and output variables remain constant in time t.

The heat balance in the steady state is of the form

$$q\rho c_p(\vartheta^s - \vartheta_v^s) = \omega^s \tag{1.2.1}$$

where

ϑ^s is the output liquid temperature in the steady state,

ϑ_v^s is the input liquid temperature in the steady state,

ω^s is the heat input in the steady state,

q is volume flow rate of the liquid,

ρ is liquid density,

c_p is specific heat capacity of the liquid.

ϑ_v^s is the desired input temperature. For the suitable exchanger design, the output temperature in the steady state ϑ^s should be equal to the desired temperature ϑ_w. So the following equation follows

$$q\rho c_p(\vartheta_w - \vartheta_v^s) = \omega^s. \tag{1.2.2}$$

It is clear, that if the input process variable ω^s is constant and if the process conditions change, the temperature ϑ would deviate from ϑ_w. The change of operational conditions means in our case the change in ϑ_v. The input temperature ϑ_v is then called *disturbance variable* and ϑ_w *setpoint variable*.

The heat exchanger should be designed in such a way that it can be possible to change the heat input so that the temperature ϑ would be equal to ϑ_w or be in its neighbourhood for all operational conditions of the process.

1.2.3 Process Control

Control of the heat transfer process in our case means to influence the process so that the output temperature ϑ will be kept close to ϑ_w. This influence is realised with changes in ω which is called *manipulated variable.* If there is a deviation ϑ from ϑ_w, it is necessary to adjust ω to achieve smaller deviation. This activity may be realised by a human operator and is based on the observation of the temperature ϑ. Therefore, a thermometer must be placed on the outlet of the exchanger. However, a human is not capable of high quality control. The task of the change of ω based on error between ϑ and ϑ_w can be realised automatically by some device. Such control method is called *automatic control*.

1.2.4 Dynamical Properties of the Process

In the case that the control is realised automatically then it is necessary to determine values of ω for each possible situation in advance. To make control decision in advance, the changes of ϑ as the result of changes in ω and ϑ_v must be known. The requirement of the knowledge about *process response* to changes of input variables is equivalent to knowledge about *dynamical properties of the process*, i.e. description of the process in *unsteady state*. The heat balance for the heat transfer process for a very short time Δt converging to zero is given by the equation

$$(q\rho c_p \vartheta_v dt + \omega dt) - (q\rho c_p \vartheta dt) = (V\rho c_p d\vartheta), \tag{1.2.3}$$

where V is the volume of the liquid in the exchanger. The equation (1.2.3) can be expressed in an abstract way as

$$\text{(inlet heat)} - \text{(outlet heat)} = \text{(heat accumulation)}$$

The dynamical properties of the heat exchanger given in Fig. 1.2.1 are given by the differential equation

$$V\rho c_p \frac{d\vartheta}{dt} + q\rho c_p \vartheta = q\rho c_p \vartheta_v + \omega, \tag{1.2.4}$$

The heat balance in the steady state (1.2.1) may be derived from (1.2.4) in the case that $\frac{d\vartheta}{dt} = 0$. The use of (1.2.4) will be given later.

1.2.5 Feedback Process Control

As it was given above, process control may by realised either by human or automatically via *control device*. The control device performs the control actions practically in the same way as a human operator, but it is described exactly according to *control law*. The control device specified for the heat exchanger utilises information about the temperature ϑ and the desired temperature ϑ_w for the calculation of the heat input ω from formula formulated in advance. The difference between ϑ_w and ϑ is defined as *control error*. It is clear that we are trying to minimise the control error. The task is to determine the feedback control law to remove the control error optimally according to some criterion. The control law specifies the structure of the feedback controller as well as its properties if the structure is given.

The considerations above lead us to controller design that will change the heat input proportionally to the control error. This control law can be written as

$$\omega(t) = q\rho c_p(\vartheta_w - \vartheta_v^s) + Z_R(\vartheta_w - \vartheta(t)) \tag{1.2.5}$$

We speak about *proportional control* and *proportional controller*. Z_R is called the *proportional gain*. The proportional controller holds the heat input corresponding to the steady state as long as the temperature ϑ is equal to desired ϑ_w. The deviation between ϑ and ϑ_w results in nonzero control error and the controller changes the heat input proportionally to this error. If the control error has a plus sign, i.e. ϑ is greater as ϑ_w, the controller decreases heat input ω. In the opposite case, the heat input increases. This phenomenon is called *negative feedback*. The output signal of the process ϑ brings to the controller information about the process and is further transmitted via controller to the process input. Such kind of control is called *feedback control*. The quality of feedback control of the proportional controller may be influenced by the choice of controller gain Z_R. The equation (1.2.5) can be with the help of (1.2.2) written as

$$\omega(t) = \omega^s + Z_R(\vartheta_w - \vartheta(t)). \tag{1.2.6}$$

1.2.6 Transient Performance of Feedback Control

Putting the equation (1.2.6) into (1.2.4) we get

$$V\rho c_p \frac{d\vartheta}{dt} + (q\rho c_p + Z_R)\vartheta = q\rho c_p \vartheta_v + Z_R \vartheta_w + \omega^s. \tag{1.2.7}$$

This equation can be arranged as

$$\frac{V}{q}\frac{d\vartheta}{dt} + \frac{q\rho c_p + Z_R}{q\rho c_p}\vartheta = \vartheta_v + \frac{Z_R}{q\rho c_p}\vartheta_w + \frac{1}{q\rho c_p}\omega^s. \tag{1.2.8}$$

The variable $V/q = T_1$ has dimension of time and is called *time constant* of the heat exchanger. It is equal to time in which the exchanger is filled with liquid with flow rate q. We have assumed that the inlet temperature ϑ_v is a function of time t. For steady state ϑ_v^s is the input heat given as ω^s. We can determine the behaviour of $\vartheta(t)$ if ϑ_v, ϑ_w change. Let us assume that the process is controlled with feedback controller and is in the steady state given by values of $\vartheta_v^s, \omega^s, \vartheta^s$. In some time denoted by zero, we change the inlet temperature with the increment $\Delta\vartheta_v$. Idealised change of this temperature may by expressed mathematically as

$$\vartheta_v(t) = \begin{cases} \vartheta_v^s + \Delta\vartheta_v & t \geq 0 \\ \vartheta_v^s & t < 0 \end{cases} \tag{1.2.9}$$

To know the response of the process with the feedback proportional controller for the step change of the inlet temperature means to know the solution of the differential equation (1.2.8). The process is at $t = 0$ in the steady state and the initial condition is

$$\vartheta(0) = \vartheta_w. \tag{1.2.10}$$

The solution of (1.2.8) if (1.2.9), (1.2.10) are valid is given as

$$\vartheta(t) = \vartheta_w + \Delta\vartheta_v \frac{q\rho c_p}{q\rho c_p + Z_R}(1 - e^{-\frac{q\rho c_p + Z_R}{q\rho c_p}\frac{q}{V}t}) \tag{1.2.11}$$

The response of the heat transfer process controlled with the proportional controller for the step change of inlet temperature ϑ_v given by Eq. (1.2.9) is shown in Fig. 1.2.2 for several values of the controller gain Z_R. The investigation of the figure shows some important facts. The outlet temperature ϑ converges to some new steady state for $t \to \infty$. If the proportional controller is used, *steady state error* results. This means that there exists a difference between ϑ_w and ϑ at the time $t = \infty$. The steady state error is the largest if $Z_R = 0$. If the controller gain Z_R increases, steady state error decreases. If $Z_R = \infty$, then the steady state error is zero. Therefore our first intention would be to choose the largest possible Z_R. However, this would break some other closed-loop properties as will be shown later.

If the disturbance variable ϑ_v changes with time in the neighbourhood of its steady state value, the choice of large Z_R may cause large control deviations. However, it is in our interest that the control deviations are to be kept under some limits. Therefore, this kind of disturbance requires rather smaller values of controller gain Z_R and its choice is given as a compromise between these two requirements.

The situation may be improved if the controller consists of a proportional and integral part. Such a controller may remove the steady state error even with smaller gain.

It can be seen from (1.2.11) that $\vartheta(t)$ cannot grow beyond limits. We note however that the controlled system was described by the first order differential equation and was controlled with a proportional controller.

We can make the process model more realistic, for example, assuming the accumulation ability of its walls or dynamical properties of temperature measurement device. The model and the feedback control loop as well will then be described by a higher order differential equation. The solution of such a differential equation for similar conditions as in (1.2.11) can result in ϑ growing into infinity. This case represents *unstable response* of the closed loop system. The problem of stability is usually included into the general problem of *control quality*.

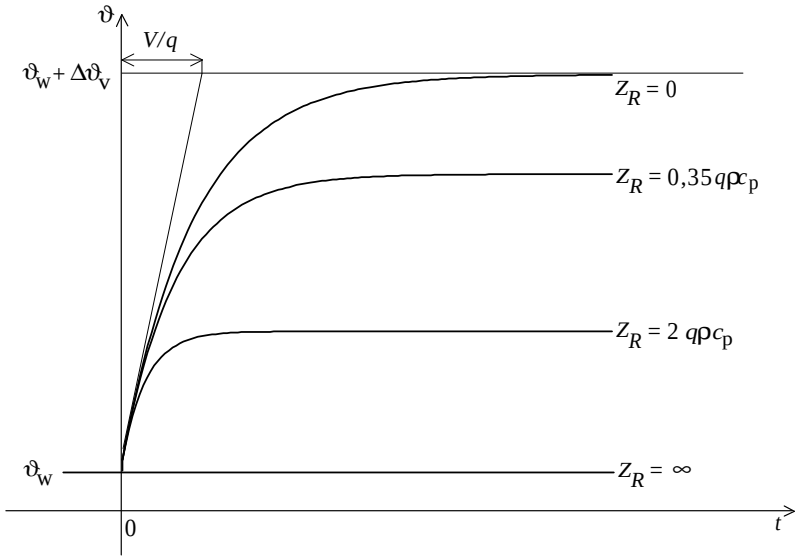

Figure 1.2.2: Response of the process controlled with proportional feedback controller for a step change of disturbance variable ϑ_v.

1.2.7 Block Diagram

In the previous sections the principal problems of feedback control were discussed. We have not dealt with technical issues of the feedback control implementation.

Consider again feedback control of the heat exchanger in Fig. 1.2.1. The necessary assumptions are i) to measure the outlet temperature ϑ and ii) the possibility of change of the heat input ω. We will assume that the heat input is realised by an electrical heater.

If the feedback control law is given then the feedback control of the heat exchanger may be realised as shown in Fig. 1.2.3. This scheme may be simplified for needs of analysis. Parts of the scheme will be depicted as blocks. The block scheme in Fig. 1.2.3 is shown in Fig. 1.2.4. The scheme gives physical interconnections and the information flow between the parts of the closed loop system. The signals represent physical variables as for example ϑ or instrumentation signals as for example m. Each block has its own input and output signal.

The outlet temperature is measured with a thermocouple. The thermocouple with its transmitter generates a voltage signal corresponding to the measured temperature. The dashed block represents the entire temperature controller and $m(t)$ is the input to the controller. The controller realises three activities:

1. the desired temperature ϑ_w is transformed into voltage signal m_w,

2. the control error is calculated as the difference between m_w and $m(t)$,

3. the control signal m_u is calculated from the control law.

All three activities are realised within the controller. The controller output $m_u(t)$ in volts is the input to the electric heater producing the corresponding heat input $\omega(t)$. The properties of each block in Fig. 1.2.4 are described by algebraic or differential equations.

Block schemes are usually simplified for the purpose of the investigation of control loops. The *simplified block scheme* consists of 2 blocks: control block and controlled object. Each block of the detailed block scheme must be included into one of these two blocks. Usually the simplified control block realizes the control law.

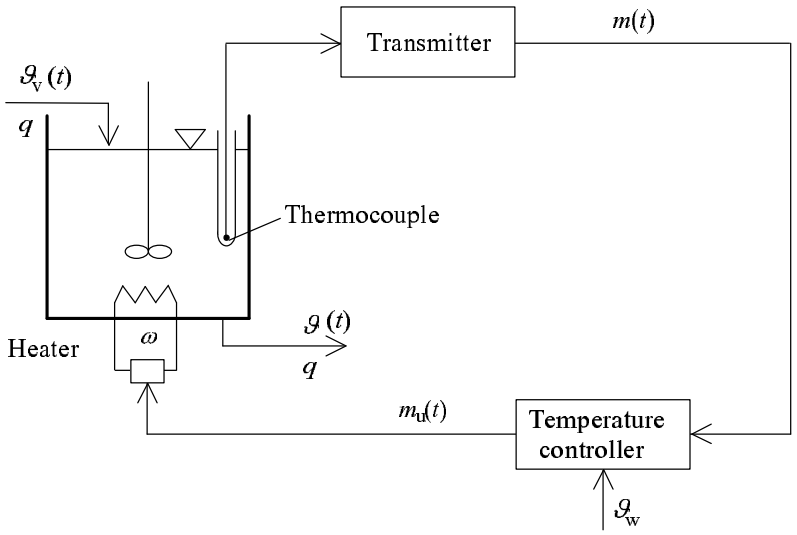

Figure 1.2.3: The scheme of the feedback control for the heat exchanger.

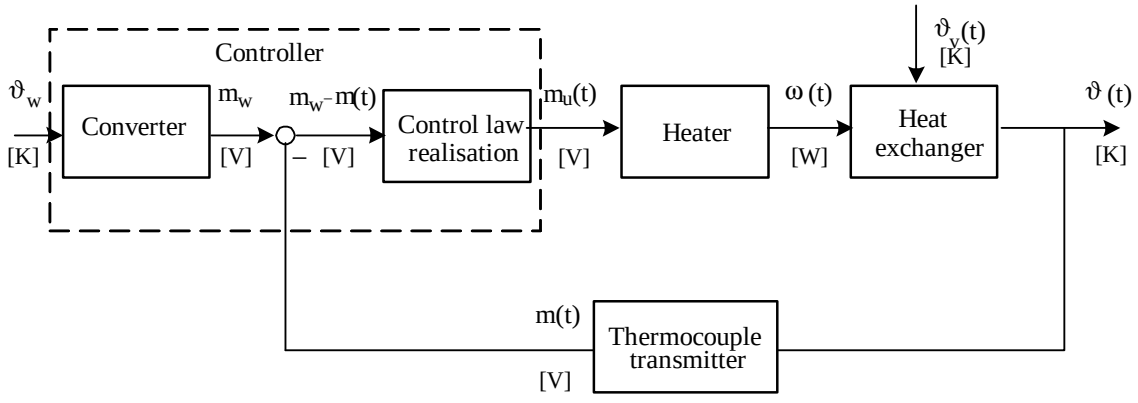

Figure 1.2.4: The block scheme of the feedback control of the heat exchanger.

1.2.8 Feedforward Control

We can also consider another kind of the heat exchanger control when the disturbance variable ϑ_v is measured and used for the calculation of the heat input ω. This is called *feedforward control*. The effect of control is not compared with the expected result. In some cases of process control it is necessary and/or suitable to use a combination of feedforward and feedback control.

1.3 Development of Process Control

The history of automatic control began about 1788. At that time J. Watt developed a revolution controller for the steam engine. An analytic expression of the influence between controller and controlled object was presented by Maxwell in 1868. Correct mathematical interpretation of automatic control is given in the works of Stodola in 1893 and 1894. E. Routh in 1877 and Hurwitz in 1895 published works in which stability of automatic control and stability criteria were dealt with. An important contribution to the stability theory was presented by Nyquist (1932). The works of Oppelt (1939) and other authors showed that automatic control was established as an independent scientific branch.

Rapid development of discrete time control began in the time after the second world war. In continuous time control, the theory of transformation was used. The transformation of sequences defined as $\mathcal{Z}$-transform was introduced independently by Cypkin (1950), Ragazzini and Zadeh (1952).

A very important step in the development of automatic control was the state-space theory, first mentioned in the works of mathematicians as Bellman (1957) and Pontryagin (1962). An essential contribution to state-space methods belongs to Kalman (1960). He showed that the linear-quadratic control problem may be reduced to a solution of the Riccati equation. Paralel to the optimal control, the stochastic theory was being developed.

It was shown that automatic control problems have an algebraic character and the solutions were found by the use of polynomial methods (Rosenbrock, 1970).

In the fifties, the idea of *adaptive control* appeared in journals. The development of adaptive control was influenced by the theory of dual control (Feldbaum, 1965), parameter estimation (Eykhoff, 1974), and recursive algorithms for adaptive control (Cypkin, 1971).

The above given survey of development in automatic control also influenced development in process control. Before 1940, processes in the chemical industry and in industries with similar processes, were controlled practically only manually. If some controller were used, these were only very simple. The technologies were built with large tanks between processes in order to attenuate the influence of disturbances.

In the fifties, it was often uneconomical and sometimes also impossible to build technologies without automatic control as the capacities were larger and the demand of quality increased. The controllers used did not consider the complexity and dynamics of controlled processes.

In 1960-s the process control design began to take into considerations dynamical properties and bindings between processes. The process control used knowledge applied from astronautics and electrotechnics.

The seventies brought the demands on higher quality of control systems and integrated process and control design.

In the whole process control development, knowledge of processes and their *modelling* played an important role.

The development of process control was also influenced by the development of computers. The first ideas about the use of *digital computers* as a part of control system emerged in about 1950. However, computers were rather expensive and unreliable to use in process control. The first use was in *supervisory control*. The problem was to find the optimal operation conditions in the sense of static optimisation and the mathematical models of processes were developed to solve this task. In the sixties, the continuous control devices began to be replaced with digital equipment, the so called *direct digital process control*. The next step was an introduction of mini and microcomputers

in the seventies as these were very cheap and also small applications could be equipped with them. Nowadays, the computer control is decisive for quality and effectivity of all modern technology.

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Chapter 2

Mathematical Modelling of Processes

This chapter explains general techniques that are used in the development of mathematical models of processes. It contains mathematical models of liquid storage systems, heat and mass transfer systems, chemical, and biochemical reactors. The remainder of the chapter explains the meaning of systems and their classification.

2.1 General Principles of Modelling

Schemes and block schemes of processes help to understand their qualitative behaviour. To express quantitative properties, mathematical descriptions are used. These descriptions are called *mathematical models*. Mathematical models are abstractions of real processes. They give a possibility to characterise behaviour of processes if their inputs are known. The validity range of models determines situations when models may be used. Models are used for control of continuous processes, investigation of process dynamical properties, optimal process design, or for the calculation of optimal process working conditions.

A process is always tied to an apparatus (heat exchangers, reactors, distillation columns, etc.) in which it takes place. Every process is determined with its physical and chemical nature that expresses its mass and energy bounds. Investigation of any typical process leads to the development of its mathematical model. This includes basic equations, variables and description of its static and dynamic behaviour. *Dynamical model* is important for control purposes. By the construction of *mathematical models of processes* it is necessary to know the problem of investigation and it is important to understand the investigated phenomenon thoroughly. If computer control is to be designed, a developed mathematical model should lead to the simplest control algorithm. If the basic use of a process model is to analyse the different process conditions including safe operation, a more complex and detailed model is needed. If a model is used in a computer simulation, it should at least include that part of the process that influences the process dynamics considerably.

Mathematical models can be divided into three groups, depending on how they are obtained:

Theoretical models developed using chemical and physical principles.

Empirical models obtained from mathematical analysis of process data.

Empirical-theoretical models obtained as a combination of theoretical and empirical approach to model design.

From the process operation point of view, processes can be divided into *continuous* and *batch*. It is clear that this fact must be considered in the design of mathematical models.

Theoretical models are derived from mass and energy balances. The balances in an unsteady-state are used to obtain dynamical models. Mass balances can be specified either in total mass of

the system or in component balances. Variables expressing quantitative behaviour of processes are *natural state variables*. Changes of state variables are given by *state balance equations*. Dynamical mathematical models of processes are described by differential equations. Some processes are *processes with distributed parameters* and are described by partial differential equations (p.d.e). These usually contain first partial derivatives with respect to time and space variables and second partial derivatives with respect to space variables. However, the most important are dependencies of variables on one space variable. The first partial derivatives with respect to space variables show an existence of transport while the second derivatives follow from heat transfer, mass transfer resulting from molecular diffusion, etc. If *ideal mixing* is assumed, the modelled process does not contain changes of variables in space and its mathematical model is described by ordinary differential equations (o.d.e). Such models are referred to as *lumped parameter* type.

Mass balances for lumped parameter processes in an unsteady-state are given by the law of mass conservation and can be expressed as

$$\frac{d(\rho V)}{dt} = \sum_{i=1}^{m} \rho_i q_i - \sum_{j=1}^{r} \rho q_j \tag{2.1.1}$$

where

ρ, ρ_i - density,

V - volume,

q_i, q_j - volume flow rates,

m - number of inlet flows,

r - number of outlet flows.

Component mass balance of the k-th component can be expressed as

$$\frac{d(c_k V)}{dt} = \sum_{i=1}^{m} c_{ki} q_i - \sum_{j=1}^{r} c_k q_j + r_k V \tag{2.1.2}$$

where

c_k, c_{ki} - molar concentration,

V - volume,

q_i, q_j - volume flow rates,

m - number of inlet flows,

r - number of outlet flows,

r_k - rate of reaction per unit volume for k-th component.

Energy balances follow the general law of energy conservation and can be written as

$$\frac{d(\rho V c_p \vartheta)}{dt} = \sum_{i=1}^{m} \rho_i q_i c_{pi} \vartheta_i - \sum_{j=1}^{r} \rho q_j c_p \vartheta + \sum_{l=1}^{s} Q_l \tag{2.1.3}$$

where

ρ, ρ_i - density,

V - volume,

q_i, q_j - volume flow rates,

c_p, c_{pi} - specific heat capacities,

ϑ, ϑ_i - temperatures,

Q_l - heat per unit time,

m - number of inlet flows,

r - number of outlet flows,

s - number of heat sources and consumptions as well as heat brought in and taken away not in inlet and outlet streams.

To use a mathematical model for process simulation we must ensure that differential and algebraic equations describing the model give a unique relation among all inputs and outputs. This is equivalent to the requirement of unique solution of a set of algebraic equations. This means that the number of unknown variables must be equal to the number of independent model equations. In this connection, the term *degree of freedom* is introduced. Degree of freedom N_v is defined as the difference between the total number of unspecified inputs and outputs and the number of independent differential and algebraic equations. The model must be defined such that

$$N_v = 0 \tag{2.1.4}$$

Then the set of equations has a unique solution.

An approach to model design involves the finding of known constants and fixed parameters following from equipment dimensions, constant physical and chemical properties and so on. Next, it is necessary to specify the variables that will be obtained through a solution of the model differential and algebraic equations. Finally, it is necessary to specify the variables whose time behaviour is given by the process environment.

2.2 Examples of Dynamic Mathematical Models

In this section we present examples of mathematical models for liquid storage systems, heat and mass transfer systems, chemical, and biochemical reactors. Each example illustrates some typical properties of processes.

2.2.1 Liquid Storage Systems

Single-tank Process

Let us examine a liquid storage system shown in Fig. 2.2.1. Input variable is the inlet volumetric flow rate q_0 and state variable the liquid height h. Mass balance for this process yields

$$\frac{d(Fh\rho)}{dt} = q_0\rho - q_1\rho \tag{2.2.1}$$

where

t - time variable,

h - height of liquid in the tank,

q_0, q_1 - inlet and outlet volumetric flow rates,

F - cross-sectional area of the tank,

ρ - liquid density.

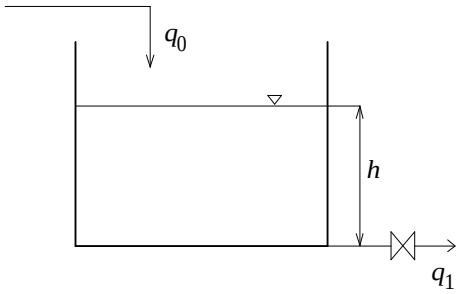

Figure 2.2.1: A liquid storage system.

Assume that liquid density and cross-sectional area are constant, then

$$F\frac{dh}{dt} = q_0 - q_1 \tag{2.2.2}$$

q_0 is independent of the tank state and q_1 depends on the liquid height in the tank according to the relation

$$q_1 = k_1 f_1 \sqrt{2g}\sqrt{h} \tag{2.2.3}$$

where

k_1 - constant,

f_1 - cross-sectional area of outflow opening,

g - acceleration gravity.

or

$$q_1 = k_{11}\sqrt{h} \tag{2.2.4}$$

Substituting q_1 from the equation (2.2.4) into (2.2.2) yields

$$\frac{dh}{dt} = \frac{q_0}{F} - \frac{k_{11}}{F}\sqrt{h} \tag{2.2.5}$$

Initial conditions can be arbitrary

$$h(0) = h_0 \tag{2.2.6}$$

The tank will be in a steady-state if

$$\frac{dh}{dt} = 0 \tag{2.2.7}$$

Let a steady-state be given by a constant flow rate q_0^s. The liquid height h^s then follows from Eq. (2.2.5) and (2.2.7) and is given as

$$h^s = \frac{(q_0^s)^2}{(k_{11})^2} \tag{2.2.8}$$

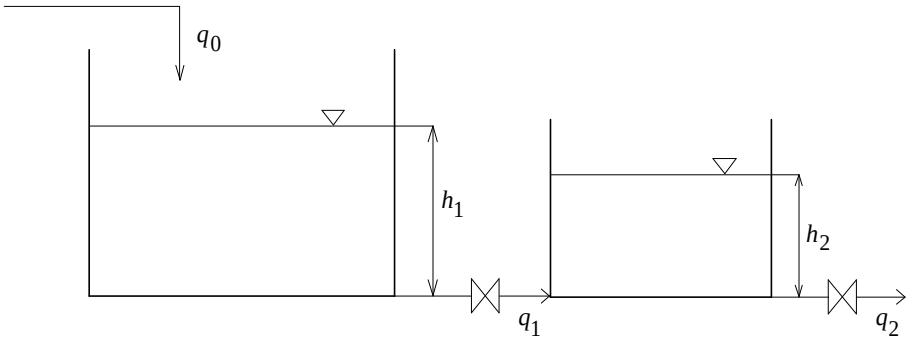

Figure 2.2.2: An interacting tank-in-series process.

Interacting Tank-in-series Process

Consider the interacting tank-in-series process shown in Fig. 2.2.2. The process input variable is the flow rate q_0.

The process state variables are heights of liquid in tanks h_1, h_2. Mass balance for the process yields

$$\frac{d(F_1 h_1 \rho)}{dt} = q_0\rho - q_1\rho \tag{2.2.9}$$

$$\frac{d(F_2 h_2 \rho)}{dt} = q_1\rho - q_2\rho \tag{2.2.10}$$

where

t - time variable,

h_1, h_2 - heights of liquid in the first and second tanks,

q_0 - inlet volumetric flow rate to the first tank,

q_1 - inlet volumetric flow rate to the second tank,

q_2 - outlet volumetric flow rate from the second tank,

F_1, F_2 - cross-sectional area of the tanks,

ρ - liquid density.

Assuming that ρ, F_1, F_2 are constant we can write

$$F_1 \frac{h_1}{dt} = q_0 - q_1 \tag{2.2.11}$$

$$F_2 \frac{h_2}{dt} = q_1 - q_2 \tag{2.2.12}$$

Inlet flow rate q_0 is independent of tank states whereas q_1 depends on the difference between liquid heights

$$q_1 = k_1 f_1 \sqrt{2g}\sqrt{h_1 - h_2} \tag{2.2.13}$$

where

k_1 - constant,

f_1 - cross-sectional area of the first tank outflow opening.

Outlet flow rate q_2 depends on liquid height in the second tank

$$q_2 = k_2 f_2 \sqrt{2g}\sqrt{h_2} \tag{2.2.14}$$

where

k_2 - constant,

f_2 - cross-sectional area of the second tank outflow opening.

Equations (2.2.13) and (2.2.14) can then be written as

$$q_1 = k_{11}\sqrt{h_1 - h_2} \tag{2.2.15}$$
$$q_2 = k_{22}\sqrt{h_2} \tag{2.2.16}$$

Substituting q_1 from Eq. (2.2.15) and q_2 from (2.2.16) into (2.2.11), (2.2.12) we get

$$\frac{dh_1}{dt} = \frac{q_0}{F_1} - \frac{k_{11}}{F_1}\sqrt{h_1 - h_2} \tag{2.2.17}$$
$$\frac{dh_2}{dt} = \frac{k_{11}}{F_1}\sqrt{h_1 - h_2} - \frac{k_{22}}{F_2}\sqrt{h_2} \tag{2.2.18}$$

with arbitrary initial conditions

$$h_1(0) = h_{10} \tag{2.2.19}$$
$$h_2(0) = h_{20} \tag{2.2.20}$$

The tanks will be in a steady-state if

$$\frac{dh_1}{dt} = 0 \tag{2.2.21}$$
$$\frac{dh_2}{dt} = 0 \tag{2.2.22}$$

Assume a steady-state flow rate q_0^s. The steady-state liquid levels in both tanks can be calculated from Eqs (2.2.17), (2.2.18), (2.2.21), (2.2.22) as

$$h_1^s = (q_0^s)^2 \left(\frac{1}{(k_{11})^2} + \frac{1}{(k_{22})^2} \right) \tag{2.2.23}$$
$$h_2^s = (q_0^s)^2 \frac{1}{(k_{22})^2} \tag{2.2.24}$$

2.2.2 Heat Transfer Processes

Heat Exchanger

Consider a heat exchanger for the heating of liquids shown in Fig. 2.2.3. The input variables are the temperatures ϑ_v, ϑ_p. The state variable is temperature ϑ.

Assume that the wall accumulation ability is small compared to the liquid accumulation ability and can be neglected. Further assume spatially constant temperature inside of the tank as the heater is well mixed, constant liquid flow rate, density, and heat capacity. Then the heat balance equation becomes

$$V\rho c_p \frac{d\vartheta}{dt} = q\rho c_p \vartheta_v - q\rho c_p \vartheta + \alpha F(\vartheta_p - \vartheta) \tag{2.2.25}$$

where

t - time variable,

ϑ - temperature inside of the exchanger and in the outlet stream,

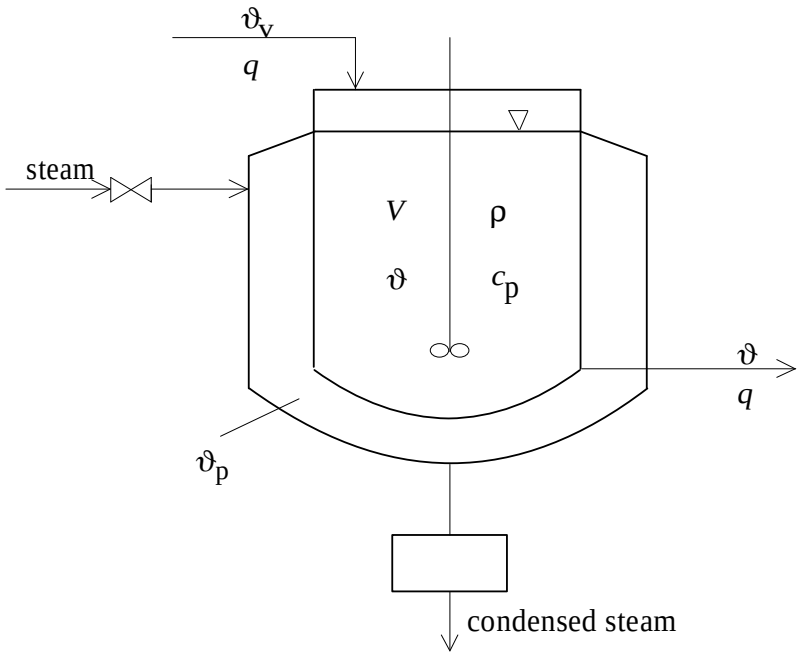

Figure 2.2.3: Continuous stirred tank heated by steam in jacket.

ϑ_v - temperature in the inlet stream,

ϑ_p - jacket temperature,

q - liquid volumetric flow rate,

ρ - liquid density,

V - volume of liquid in the tank,

c_p - liquid specific heat capacity,

F - heat transfer area of walls,

α - heat transfer coefficient.

Equation (2.2.25) can be rearranged as

$$\frac{V\rho c_p}{q\rho c_p + \alpha F}\frac{d\vartheta}{dt} = -\vartheta + \frac{\alpha F}{q\rho c_p + \alpha F}\vartheta_p + \frac{q\rho c_p}{q\rho c_p + \alpha F}\vartheta_v \tag{2.2.26}$$

or as

$$T_1\frac{d\vartheta}{dt} = -\vartheta + Z_1\vartheta_p + Z_2\vartheta_v \tag{2.2.27}$$

where $T_1 = \frac{V\rho c_p}{q\rho c_p + \alpha F}$, $Z_1 = \frac{\alpha F}{q\rho c_p + \alpha F}$, $Z_2 = \frac{q\rho c_p}{q\rho c_p + \alpha F}$. The initial condition of Eq. (2.2.26) can be arbitrary

$$\vartheta(0) = \vartheta_0 \tag{2.2.28}$$

The heat exchanger will be in a steady-state if

$$\frac{d\vartheta}{dt} = 0 \tag{2.2.29}$$

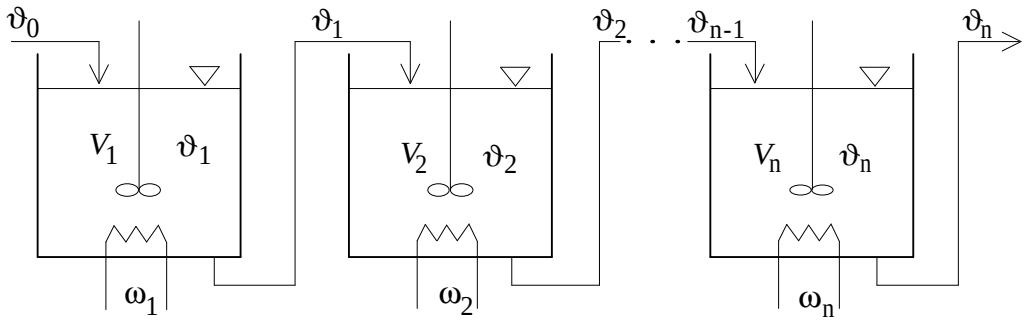

Figure 2.2.4: Series of heat exchangers.

Assume steady-state values of the input temperatures $\vartheta_p^s, \vartheta_v^s$. The steady-state outlet temperature ϑ^s can be calculated from Eqs. (2.2.26), (2.2.29) as

$$\vartheta^s = \frac{\alpha F}{q\rho c_p + \alpha F}\vartheta_p^s + \frac{q\rho c_p}{q\rho c_p + \alpha F}\vartheta_v^s \tag{2.2.30}$$

Series of Heat Exchangers

Consider a series of heat exchangers where a liquid is heated (Fig. 2.2.4). Assume that heat flows from heat sources into liquid are independent from liquid temperature. Further assume ideal liquid mixing and zero heat losses. We neglect accumulation ability of exchangers walls. Hold-ups of exchangers as well as flow rates, liquid specific heat capacity are constant.

Under these circumstances following heat balances result

$$\begin{aligned} V_1\rho c_p\frac{d\vartheta_1}{dt} &= q\rho c_p\vartheta_0 - q\rho c_p\vartheta_1 + \omega_1 \\ V_2\rho c_p\frac{d\vartheta_2}{dt} &= q\rho c_p\vartheta_1 - q\rho c_p\vartheta_2 + \omega_2 \\ &\vdots \\ V_n\rho c_p\frac{d\vartheta_n}{dt} &= q\rho c_p\vartheta_{n-1} - q\rho c_p\vartheta_n + \omega_n \end{aligned} \tag{2.2.31}$$

where

t - time variable,

$\vartheta_1, \ldots, \vartheta_n$ - temperature inside of the heat exchangers,

ϑ_0 - liquid temperature in the first tank inlet stream,

$\omega_1, \ldots, \omega_n$ - heat inputs,

q - liquid volumetric flow rate,

ρ - liquid density,

$V_1, \ldots, V_n$ - volumes of liquid in the tanks,

c_p - liquid specific heat capacity.

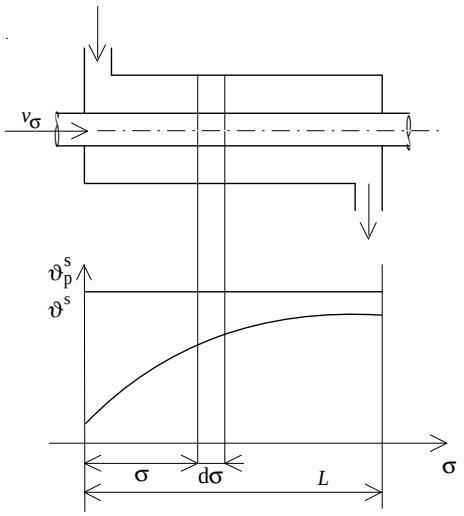

Figure 2.2.5: Double-pipe steam-heated exchanger and temperature profile along the exchanger length in steady-state.

The process input variables are heat inputs ω_i and inlet temperature ϑ_0. The process state variables are temperatures $\vartheta_1, \ldots, \vartheta_n$ and initial conditions are arbitrary

$$\vartheta_1(0) = \vartheta_{10}, \ldots, \vartheta_n(0) = \vartheta_{n0} \tag{2.2.32}$$

The process will be in a steady-state if

$$\frac{d\vartheta_1}{dt} = \frac{d\vartheta_2}{dt} = \cdots = \frac{d\vartheta_n}{dt} = 0 \tag{2.2.33}$$

Let the steady-state values of the process inputs ω_i, ϑ_0 be given. The steady-state temperatures inside the exchangers are

$$\begin{aligned} \vartheta_1^s &= \vartheta_0^s + \frac{\omega_1^s}{q\rho c_p} \\ \vartheta_2^s &= \vartheta_1^s + \frac{\omega_2^s}{q\rho c_p} \\ &\vdots \\ \vartheta_n^s &= \vartheta_{n-1}^s + \frac{\omega_n^s}{q\rho c_p} \end{aligned} \tag{2.2.34}$$

Double-pipe Heat Exchanger

Figure 2.2.5 represents a single-pass, double-pipe steam-heated exchanger in which a liquid in the inner tube is heated by condensing steam. The process input variables are $\vartheta_p(t), \vartheta(0,t)$. The process state variable is the temperature $\vartheta(\sigma,t)$. We assume the steam temperature to be a function only of time, heat transfer only between inner and outer tube, plug flow of the liquid and zero heat capacity of the exchanger walls. We neglect heat conduction effects in the direction of liquid flow. It is further assumed that liquid flow, density, and specific heat capacity are constant.

Heat balance equation on the element of exchanger length $d\sigma$ can be derived according to Fig. 2.2.6

$$F_\sigma d\sigma \rho c_p \frac{\partial \vartheta}{\partial t} = q\rho c_p \vartheta - q\rho c_p \left(\vartheta + \frac{\partial \vartheta}{\partial \sigma} d\sigma\right) + \alpha F_d d\sigma (\vartheta_p - \vartheta) \tag{2.2.35}$$

where

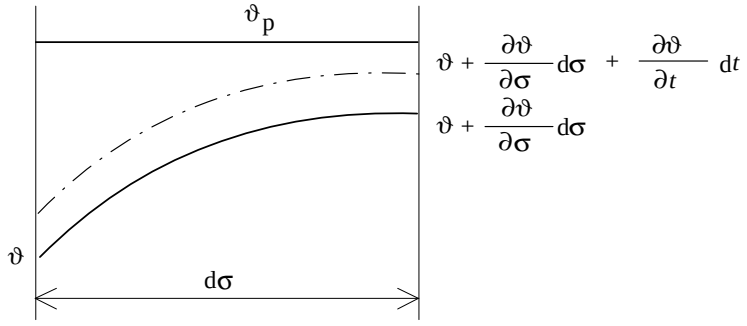

Figure 2.2.6: Temperature profile of ϑ in an exchanger element of length $d\sigma$ for time dt.

t - time variable,

σ - space variable,

$\vartheta = \vartheta(\sigma, t)$ - liquid temperature in the inner tube,

$\vartheta_p = \vartheta_p(t)$ - liquid temperature in the outer tube,

q - liquid volumetric flow rate in the inner tube,

ρ - liquid density in the inner tube,

α - heat transfer coefficient,

c_p - liquid specific heat capacity,

F_d - area of heat transfer per unit length,

F_σ - cross-sectional area of the inner tube.

The equation (2.2.35) can be rearranged to give

$$\frac{F_\sigma \rho c_p}{\alpha F_d}\frac{\partial \vartheta}{\partial t} = -\frac{q\rho c_p}{\alpha F_d}\frac{\partial \vartheta}{\partial \sigma} - \vartheta + \vartheta_p \tag{2.2.36}$$

or

$$T_1\frac{\partial \vartheta}{\partial t} = -v_\sigma T_1\frac{\partial \vartheta}{\partial \sigma} - \vartheta + \vartheta_p \tag{2.2.37}$$

where $T_1 = \dfrac{F_\sigma \rho c_p}{\alpha F_d}$, $v_\sigma = \dfrac{q}{F_\sigma}$.

Boundary condition of Eq. (2.2.37) is

$$\vartheta(0, t) = \vartheta^0(t) \tag{2.2.38}$$

and initial condition is

$$\vartheta(\sigma, 0) = \vartheta_0(\sigma) \tag{2.2.39}$$

Assume a steady-state inlet liquid temperature ϑ^{0s} and steam temperature ϑ_p^s. The temperature profile in the inner tube in the steady-state can be derived if

$$\frac{\partial \vartheta}{\partial t} = 0 \tag{2.2.40}$$

as

$$\vartheta^s(\sigma) = \vartheta_p^s - (\vartheta_p^s - \vartheta^{0s})e^{-\dfrac{\sigma}{v_\sigma T_1}} \tag{2.2.41}$$

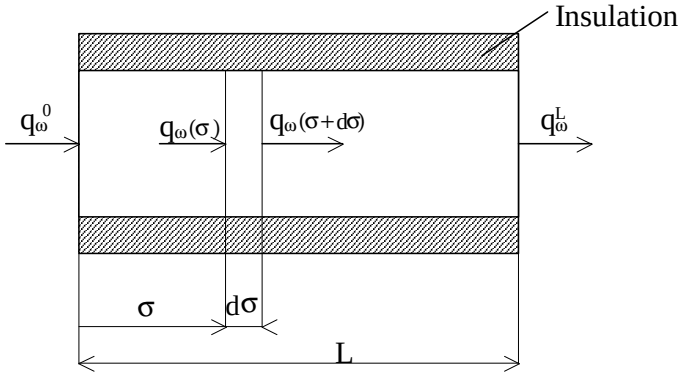

Figure 2.2.7: A metal rod.

If $\alpha = 0$, Eq. (2.2.35) reads

$$\frac{\partial \vartheta}{\partial t} = -v_\sigma \frac{\partial \vartheta}{\partial \sigma} \tag{2.2.42}$$

while boundary and initial conditions remain the same. If the input variable is $\vartheta^0(t)$ and the output variable is $\vartheta(L,t)$, then Eq. (2.2.42) describes *pure time delay* with value

$$T_d = \frac{L}{v_\sigma} \tag{2.2.43}$$

Heat Conduction in a Solid Body

Consider a metal rod of length L in Fig. 2.2.7. Assume ideal insulation of the rod. Heat is brought in on the left side and withdrawn on the right side. Changes of densities of heat flows q_ω^0, q_L^0 influence the rod temperature $\vartheta(\sigma,t)$. Assume that heat conduction coefficient, density, and specific heat capacity of the rod are constant. We will derive unsteady heat flow through the rod. Heat balance on the rod element of length $d\sigma$ for time dt can be derived from Fig. 2.2.7 as

$$F_\sigma d\sigma \rho c_p \frac{\partial \vartheta}{\partial t} = F_\sigma [q_\omega(\sigma) - q_\omega(\sigma + d\sigma)] \tag{2.2.44}$$

or

$$F_\sigma d\sigma \rho c_p \frac{\partial \vartheta}{\partial t} = -F_\sigma \frac{\partial q_\omega}{\partial \sigma} d\sigma \tag{2.2.45}$$

where

t - time variable,

σ - space variable,

$\vartheta = \vartheta(\sigma,t)$ - rod temperature,

ρ - rod density,

c_p - rod specific heat capacity,

F_σ - cross-sectional area of the rod,

$q_\omega(\sigma)$ - heat flow density (heat transfer velocity through unit area) at length σ,

$q_\omega(\sigma + d\sigma)$ - heat flow density at length $\sigma + d\sigma$.

From the Fourier law follows

$$q_\omega = -\lambda \frac{\partial \vartheta}{\partial \sigma} \tag{2.2.46}$$

where λ is the coefficient of thermal conductivity.

Substituting Eq. (2.2.46) into (2.2.45) yields

$$\frac{\partial \vartheta}{\partial t} = a \frac{\partial^2 \vartheta}{\partial \sigma^2} \tag{2.2.47}$$

where

$$a = \frac{\lambda}{\rho c_p} \tag{2.2.48}$$

is the *factor of heat conductivity*. The equation (2.2.47) requires boundary and initial conditions. The boundary conditions can be given with temperatures or temperature derivatives with respect to σ at the ends of the rod. For example

$$\vartheta(0,t) = \vartheta^0(t) \tag{2.2.49}$$

$$\vartheta(L,t) = \vartheta^L(t) \tag{2.2.50}$$

The initial condition for Eq. (2.2.47) is

$$\vartheta(\sigma,0) = \vartheta_0(\sigma) \tag{2.2.51}$$

Consider the boundary conditions (2.2.49), (2.2.50). The process input variables are $\vartheta^0(t), \vartheta^L(t)$ and the state variable is $\vartheta(\sigma,t)$.

Assume steady-state temperatures $\vartheta^{0s}, \vartheta^{Ls}$. The temperature profile of the rod in the steady-state can be derived if

$$\frac{\partial \vartheta}{\partial t} = 0 \tag{2.2.52}$$

as

$$\vartheta^s(\sigma) = \vartheta^{0s} + \frac{\vartheta^{Ls} - \vartheta^{0s}}{L}\sigma \tag{2.2.53}$$

2.2.3 Mass Transfer Processes

Packed Absorption Column

A scheme of packed countercurrent absorption column is shown in Fig. 2.2.8 where

t - time variable,

σ - space variable,

L - height of column,

G - molar flow of gas phase,

$c_y = c_y(\sigma,t)$ - molar fraction concentration of a transferable component in gas phase,

Q - molar flow of liquid phase,

$c_x = c_x(\sigma,t)$ - molar fraction concentration of a transferable component in liquid phase.

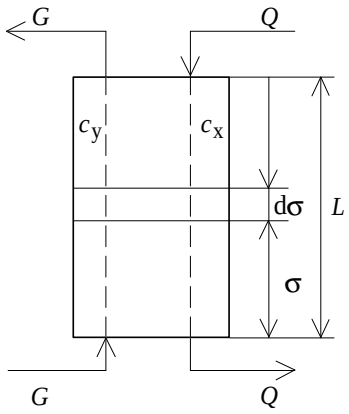

Figure 2.2.8: A scheme of a packed countercurrent absorption column.

Absorption represents a process of absorbing components of gaseous systems in liquids.

We assume ideal filling, plug flow of gas and liquid phases, negligible mixing and mass transfer in phase flow direction, uniform concentration profiles in both phases at cross surfaces, linear equilibrium curve, isothermal conditions, constant mass transfer coefficients, and constant flow rates G, Q.

Considering only the process state variables c_x, c_y and the above given simplifications and if only the physical process of absorption is considered then mass transfer is realised only in one direction. Then, the following equations result from general law of mass conservation.

For gas phase

$$-N = H_y \frac{\partial c_y}{\partial t} + G \frac{\partial c_y}{\partial \sigma} \tag{2.2.54}$$

H_y is gas molar hold-up in the column per unit length.

For liquid phase

$$N = H_x \frac{\partial c_x}{\partial t} - G \frac{\partial c_x}{\partial \sigma} \tag{2.2.55}$$

H_x is liquid molar hold-up in the column per unit length.

Under the above given conditions the following relation holds for mass transfer

$$N = K_G(c_y - c_y^*) \tag{2.2.56}$$

where

K_G - mass transfer coefficient [mol m^{-1} s^{-1}],

c_y^* - equilibrium concentration of liquid phase.

In the assumptions we stated that the equilibrium curve is linear, that is

$$c_y^* = K c_x \tag{2.2.57}$$

and K is some constant. Equations (2.2.54), (2.2.55) in conjunction with (2.2.56), (2.2.57) yield

$$H_y \frac{\partial c_y}{\partial t} + G \frac{\partial c_y}{\partial \sigma} = K_G(K c_x - c_y) \tag{2.2.58}$$

$$H_x \frac{\partial c_x}{\partial t} - G \frac{\partial c_x}{\partial \sigma} = K_G(c_y - K c_x) \tag{2.2.59}$$

In the case of the concurrent absorption column, the second term on the left side of Eq. (2.2.59) would have a positive sign, i.e. $+G(\partial c_x/\partial \sigma)$.

Boundary conditions of Eqs. (2.2.58), (2.2.59) are

$$c_y(0,t) = c_y^0(t) \tag{2.2.60}$$
$$c_x(L,t) = c_x^L(t) \tag{2.2.61}$$

and c_y^0, c_x^L are the process input variables.

Initial conditions of Eqs. (2.2.58), (2.2.59) are

$$c_y(\sigma,0) = c_{y0}(\sigma) \tag{2.2.62}$$
$$c_x(\sigma,0) = c_{x0}(\sigma) \tag{2.2.63}$$

Consider steady-state input concentration c_y^{0s}, c_x^{0s}. Profiles $c_y^s(\sigma), c_x^s(\sigma)$ can be calculated if

$$\frac{\partial c_y}{\partial t} = 0 \tag{2.2.64}$$
$$\frac{\partial c_x}{\partial t} = 0 \tag{2.2.65}$$

as solution of equations

$$G\frac{dc_y^s}{d\sigma} = K_G(Kc_x^s - c_y^s) \tag{2.2.66}$$
$$-Q\frac{dc_x^s}{d\sigma} = K_G(c_y^s - Kc_x^s) \tag{2.2.67}$$

with boundary conditions

$$c_y^s(0) = c_y^{0s} \tag{2.2.68}$$
$$c_x^s(L) = c_x^{Ls} \tag{2.2.69}$$

Binary Distillation Column

Distillation column represents a process of separation of liquids. A liquid stream is fed into the column, distillate is withdrawn from the condenser and the bottom product from the reboiler. Liquid flow falls down, it is collected in the reboiler where it is vaporised and as vapour flow gets back into the column. Vapour from the top tray condenses and is collected in the condenser. A part of the condensate is returned back to the column. The scheme of the distillation column is shown in Fig. 2.2.9.

We assume a binary system with constant relative volatility along the column with theoretical trays (100 % efficiency - equilibrium between gas and liquid phases on trays). Vapour exiting the trays is in equilibrium with the tray liquid. Feed arriving on the feed tray boils. Vapour leaving the top tray is totally condensed in the condenser, the condenser is ideally mixed and the liquid within boils. We neglect the dynamics of the pipework. Liquid in the column reboiler is ideally mixed and boils. Liquid on every tray is well mixed and liquid hold-ups are constant in time. Vapour hold-up is negligible. We assume that the column is well insulated, heat losses are zero, and temperature changes along the column are small. We will not assume heat balances. We also consider constant liquid flow along the column and constant pressure.

Mathematical model of the column consists of mass balances of a more volatile component. Feed composition is usually considered as a disturbance and vapour flow as a manipulated variable.

Situation on i-th tray is represented in Fig. 2.2.10 where

G - vapour molar flow rate,

c_{yi}, c_{yi-1} - vapour molar fraction of a more volatile component,

R - reflux molar flow rate,

F - feed molar flow rate,

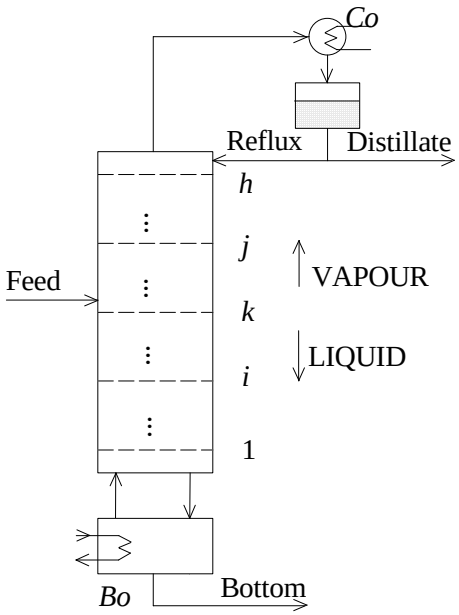

Figure 2.2.9: Scheme of a distillation column, Co - condenser; Bo - reboiler; $1, \ldots, i, \ldots, k, \ldots, j, \ldots, h$ - tray number.

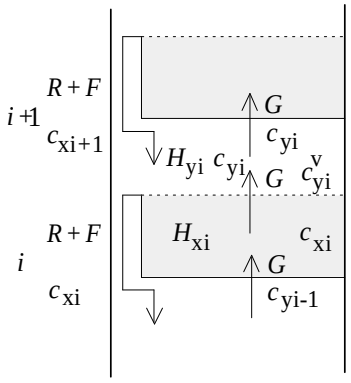

Figure 2.2.10: Model representation of i-th tray.

c_{xi}, c_{xi-1} - liquid molar fraction of a more volatile component,

H_{yi}, H_{xi} - vapour and liquid molar hold-ups on i-th tray.

Mass balance of a more volatile component in the liquid phase on the i-th tray (stripping section) is given as

$$H_{xi}\frac{dc_{xi}}{dt} = (R+F)(c_{xi+1} - c_{xi}) + G(c_{yi-1} - c^v_{yi}) \tag{2.2.70}$$

where t is time. Under the assumption of equilibrium on the tray follows

$$c^v_{yi} = c^*_{yi} = f(c_{xi}) \tag{2.2.71}$$

Mass balance of a more volatile component in the vapour phase is

$$H_{yi}\frac{dc_{yi}}{dt} = G(c^v_{yi} - c_{yi}) \tag{2.2.72}$$

We assume that vapour molar hold-up is small and the following simplification holds

$$c_{yi} \doteq c^v_{yi} \tag{2.2.73}$$

and the i-th tray is described by

$$H_{xi}\frac{dc_{xi}}{dt} = (R+F)(c_{xi+1} - c_{xi}) + G[f(c_{xi-1}) - f(c_{xi})] \tag{2.2.74}$$

Mass balance for k-th tray (feed tray) can be written as

$$H_{xk}\frac{dc_{xk}}{dt} = Rc_{xk+1} + Fc_{xF} - (R+F)c_{xk} + G[f(c_{xk-1}) - f(c_{xk})] \tag{2.2.75}$$

where c_{xF} is a molar fraction of a more volatile component in the feed stream.

Mass balances for other sections of the column are analogous:

- j-th tray (enriching section)

$$H_{xj}\frac{dc_{xj}}{dt} = R(c_{xj+1} - c_{xj}) + G[f(c_{xj-1}) - f(c_{xj})] \tag{2.2.76}$$

- h-th tray (top tray)

$$H_{xh}\frac{dc_{xh}}{dt} = R(c_{xD} - c_{xh}) + G[f(c_{xh-1}) - f(c_{xh})] \tag{2.2.77}$$

- Condenser

$$H_{xC}\frac{dc_{xD}}{dt} = -(R+D)c_{xD} + G[f(c_{xh})] \tag{2.2.78}$$

where

D - distillate molar flow,

c_{xD} - molar fraction of a more volatile component in condenser,

H_{xC} - liquid molar hold-up in condenser.

- first tray

$$H_{x1}\frac{dc_{x1}}{dt} = (R+F)(c_{x2} - c_{x1}) + G[f(c_{xW}) - f(c_{x1})] \tag{2.2.79}$$

where c_{xW} is molar fraction of a more volatile component in the bottom product.

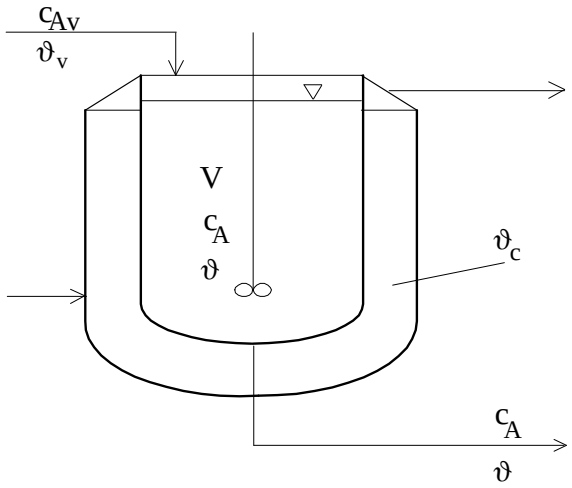

Figure 2.2.11: A nonisothermal CSTR.

- Reboiler

$$H_{xW}\frac{dc_{xW}}{dt} = (R+F)c_{x1} - Wc_{xW} - G[f(c_{xW})] \tag{2.2.80}$$

where W is a molar flow of the bottom product and H_{xW} is the reboiler molar hold-up.

The process state variables correspond to a liquid molar fraction of a more volatile component on trays, in the reboiler, and the condenser. Initial conditions of Eqs. (2.2.70)-(2.2.80) are

$$c_{xz}(0) = c_{xz0}, \quad z \in \{i,k,j,h,D,1,W\} \tag{2.2.81}$$

The column is in a steady-state if all derivatives with respect to time in balance equations are zero. Steady-state is given by the choices of G^s and c^s_{xF} and is described by the following set of equations

$$\begin{aligned}
0 &= (R+F)(c^s_{xi+1} - c^s_{xi}) + G^s[f(c^s_{xi-1}) - f(c^s_{xi})] && (2.2.82)\\
0 &= Rc^s_{xk+1} + Fc^s_{xF} - (R+F)c^s_{xk} + G^s[f(c^s_{xk-1}) - f(c^s_{xk})] && (2.2.83)\\
0 &= R(c^s_{xj+1} - c^s_{xj}) + G^s[f(c^s_{xj-1}) - f(c^s_{xj})] && (2.2.84)\\
0 &= R(c^s_{xD} - c^s_{xh}) + G^s[f(c^s_{xh-1}) - f(c^s_{xh})] && (2.2.85)\\
0 &= -(R+D)c^s_{xD} + G^s[f(c^s_{xh})] && (2.2.86)\\
0 &= (R+F)(c^s_{x2} - c^s_{x1}) + G^s[f(c^s_{xW}) - f(c^s_{x1})] && (2.2.87)\\
0 &= (R+F)c^s_{x1} - (R+F-G^s)c^s_{xW} + G^s[f(c^s_{xW})] && (2.2.88)
\end{aligned}$$

2.2.4 Chemical and Biochemical Reactors

Continuous Stirred-Tank Reactor (CSTR)

Chemical reactors together with mass transfer processes constitute an important part of chemical technologies. From a control point of view, reactors belong to the most difficult processes. This is especially true for fast exothermal processes.

We consider CSTR with a simple exothermal reaction $A \rightarrow B$ (Fig. 2.2.11). For the development of a mathematical model of the CSTR, the following assumptions are made: neglected heat capacity of inner walls of the reactor, constant density and specific heat capacity of liquid, constant reactor volume, constant overall heat transfer coefficient, and constant and equal input and output volumetric flow rates. As the reactor is well-mixed, the outlet stream concentration and temperature are identical with those in the tank.

Mass balance of component A can be expressed as

$$V\frac{dc_A}{dt} = qc_{Av} - qc_A - Vr(c_A, \vartheta) \tag{2.2.89}$$

where

t - time variable,

c_A - molar concentration of A (mole/volume) in the outlet stream,

c_{Av} - molar concentration of A (mole/volume) in the inlet stream,

V - reactor volume,

q - volumetric flow rate,

$r(c_A, \vartheta)$ - rate of reaction per unit volume,

ϑ - temperature of reaction mixture.

The rate of reaction is a strong function of concentration and temperature (Arrhenius law)

$$r(c_A, \vartheta) = kc_A = k_0 e^{-\frac{E}{R\vartheta}} c_A \tag{2.2.90}$$

where k_0 is the frequency factor, E is the activation energy, and R is the gas constant.

Heat balance gives

$$V\rho c_p \frac{d\vartheta}{dt} = q\rho c_p \vartheta_v - q\rho c_p \vartheta - \alpha F(\vartheta - \vartheta_c) + V(-\Delta H)r(c_A, \vartheta) \tag{2.2.91}$$

where

ϑ_v - temperature in the inlet stream,

ϑ_c - cooling temperature,

ρ - liquid density,

c_p - liquid specific heat capacity,

α - overall heat transfer coefficient,

F - heat transfer area,

$(-\Delta H)$ - heat of reaction.

Initial conditions are

$$c_A(0) = c_{A0} \tag{2.2.92}$$
$$\vartheta(0) = \vartheta_0 \tag{2.2.93}$$

The process state variables are concentration c_A and temperature ϑ. The input variables are $\vartheta_c, c_{Av}, \vartheta_v$ and among them, the cooling temperature can be used as a manipulated variable.

The reactor is in the steady-state if derivatives with respect to time in equations (2.2.89), (2.2.91) are zero. Consider the steady-state input variables $\vartheta_c^s, c_{Av}^s, \vartheta_v^s$. The steady-state concentration and temperature can be calculated from the equations

$$0 = qc_{Av}^s - qc_A^s - Vr(c_A^s, \vartheta^s) \tag{2.2.94}$$
$$0 = q\rho c_p \vartheta_v^s - q\rho c_p \vartheta^s - \alpha F(\vartheta^s - \vartheta_c^s) + V(-\Delta H)r(c_A^s, \vartheta^s) \tag{2.2.95}$$

Bioreactor

Consider a typical *bioprocess* realised in a fed-batch stirred bioreactor. As an example of bioprocess, alcohol *fermentation* is assumed. Mathematical models of bioreactors usually include mass balances of *biomass, substrate* and *product*. Their concentrations in the reactor are process state variables. Assuming ideal mixing and other assumptions that are beyond the framework of this section, a mathematical model of alcohol fermentation is of the form

$$\frac{dx}{dt} = \mu x - Dx \tag{2.2.96}$$

$$\frac{ds}{dt} = -v_s x + D(s_f - s) \tag{2.2.97}$$

$$\frac{dp}{dt} = v_p x - Dp \tag{2.2.98}$$

where

x - biomass concentration,

s - substrate concentration,

p - product (alcohol) concentration,

s_f - inlet substrate concentration,

D - dilution rate,

μ - specific rate of biomass growth,

v_s - specific rate of substrate consumption,

v_p - specific rate of product creation.

The symbols x, s, p representing the process state variables are used in biochemical literature. The dilution rate can be used as a manipulated variable. The process kinetic properties are given by the relations

$$\mu = \text{function1}(x, s, p) \tag{2.2.99}$$

$$v_p = \text{function2}(x, s, p) \tag{2.2.100}$$

$$v_s = \text{function3}(x, s, p) \tag{2.2.101}$$

2.3 General Process Models

A general process model can be described by a set of ordinary differential and algebraic equations or in matrix-vector form. For control purposes, linearised mathematical models are used. In this section, deviation and dimensionless variables are explained. We show how to convert partial differential equations describing processes with distributed parameters into models with ordinary differential equations. Finally, we illustrate the use of these techniques on examples.

State Equations

As stated above, a suitable model for a large class of continuous technological processes is a set of ordinary differential equations of the form

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= f_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_1(t), \ldots, r_s(t)) \\
\frac{dx_2(t)}{dt} &= f_2(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_1(t), \ldots, r_s(t)) \\
&\vdots \\
\frac{dx_n(t)}{dt} &= f_n(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_1(t), \ldots, r_s(t))
\end{aligned} \tag{2.3.1}$$

where

t - time variable,

$x_1, \ldots, x_n$ - state variables,

$u_1, \ldots, u_m$ - manipulated variables,

$r_1, \ldots, r_s$ - disturbance variables,

$f_1, \ldots, f_n$ - functions.

Typical technological processes can be described as complex systems. As processes are usually connected to other processes, the complexity of resulting systems increases. It is therefore necessary to investigate the problem of influence of processes and their contact to the environment which influences process with disturbances and manipulated variables. Process state variables are usually not completely measurable. A model of process measurement can be written as a set of algebraic equations

$$\begin{array}{rcl} y_1(t) & = & g_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_{m1}(t), \ldots, r_{mt}(t)) \\ y_2(t) & = & g_2(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_{m1}(t), \ldots, r_{mt}(t)) \\ & \vdots & \\ y_r(t) & = & g_r(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t), r_{m1}(t), \ldots, r_{mt}(t)) \end{array} \tag{2.3.2}$$

where

$y_1, \ldots, y_r$ - measurable process output variables,

$r_{m1}, \ldots, r_{mt}$ - disturbance variables,

$g_1, \ldots, g_r$ - functions.

If the vectors of *state variables* $\boldsymbol{x}$, *manipulated variables* $\boldsymbol{u}$, *disturbance variables* $\boldsymbol{r}$, and vectors of functions $\boldsymbol{f}$ are defined as

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ \boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \ \boldsymbol{r} = \begin{pmatrix} r_1 \\ \vdots \\ r_s \end{pmatrix}, \ \boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{2.3.3}$$

then the set of the equations (2.3.1) can be written more compactly

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{r}(t)) \tag{2.3.4}$$

If the vectors of *output variables* $\boldsymbol{y}$, *disturbance variables* $\boldsymbol{r}_m$, and vectors of functions $\boldsymbol{g}$ are defined as

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}, \ \boldsymbol{r}_m = \begin{pmatrix} r_{m1} \\ \vdots \\ r_{mt} \end{pmatrix}, \ \boldsymbol{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} \tag{2.3.5}$$

then the set of the algebraic equations is rewritten as

$$\boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{r}_m(t)) \tag{2.3.6}$$

There are two approaches of control design for processes with distributed parameters. The first approach called *late pass* to lumped parameter models uses model with partial differential equations (p.d.e.) for control design and the only exception is the final step - numerical solution. This approach preserves the nature of distributed systems which is advantageous, however it is also more demanding on the use of the advanced control theory of distributed systems.

The second approach called *early pass* to lumped parameter models is based on space discretisation of p.d.e's at the beginning of the control design problem.

Space discretisation means the division of a distributed process to a finite number of segments and it is assumed that each segment represents a lumped parameter system. The result of space discretisation is a process model described by a set of interconnected ordinary differential equations. The lumped parameter model can also be derived when partial derivatives with respect to space variables are replaced with corresponding differences. For the state variable $x(\sigma, t)$, $0 \leq \sigma \leq L$ holds

$$\left.\frac{\partial x(\sigma,t)}{\partial \sigma}\right|_k \doteq \frac{x(\sigma_k,t) - x(\sigma_{k-1},t)}{\Delta\sigma} \tag{2.3.7}$$

$$\left.\frac{\partial^2 x(\sigma,t)}{\partial \sigma^2}\right|_k \doteq \frac{x(\sigma_{k+1},t) - 2x(\sigma_k,t) + x(\sigma_{k-1},t)}{(\Delta\sigma)^2} \tag{2.3.8}$$

where $\Delta\sigma = L/n$. L is the length of the process. The process is divided into n parts over the interval $[0, L]$, $k = 1, \ldots, n$. It can easily be shown that process models obtained directly from process segmentation and models derived by substitution of derivatives by differences are the same. There are many combinations of finite difference methods. When applied correctly, all are equivalent for $n \to \infty$.

An advantage of the early pass to lumped parameter models exists in fact that it is possible to use well developed control methods for lumped processes. However, a drawback can be found later as the controller derived does not necessarily satisfy all requirements laid on control quality. But in the majority of cases this approach produces satisfactory results. Space *discretisation* of processes with distributed parameters leads to models of type (2.3.4), (2.3.6).

The general state-variable equations (the general form of the state-space model) consist of the *state equations* (2.3.4) and the *output equations* (2.3.6).

For comparison of process properties in various situations it is advantageous to introduce *dimensionless variables*. These can be state, input, and output variables. Sometimes also dimensionless time and space variables are used.

Example 2.3.1: *Heat exchanger - state equation*

The heat exchanger shown in Fig. 2.2.3 is described by the differential equation

$$\frac{d\vartheta}{dt} = -\frac{1}{T_1}\vartheta + \frac{Z_1}{T_1}\vartheta_p + \frac{Z_2}{T_1}\vartheta_v$$

If

$$\begin{aligned} x_1 &= \vartheta \\ u_1 &= \vartheta_p \\ r_1 &= \vartheta_v \end{aligned}$$

then the state equation is

$$\frac{dx_1}{dt} = f_1(x_1, u_1, r_1)$$

where $f_1(x_1, u_1, r_1) = -\frac{1}{T_1}\vartheta + \frac{Z_1}{T_1}\vartheta_p + \frac{Z_2}{T_1}\vartheta_v$.

The output equation if temperature ϑ is measured is

$$y = x_1$$

Example 2.3.2: *CSTR - state equations*

Equations describing the dynamics of the CSTR shown in Fig. 2.2.11 are

$$\begin{aligned} \frac{dc_A}{dt} &= \frac{q}{V}c_{Av} - \frac{q}{V}c_A - r(c_A, \vartheta) \\ \frac{d\vartheta}{dt} &= \frac{q}{V}\vartheta_v - \frac{q}{V}\vartheta + \frac{\alpha F}{V\rho c_p}(\vartheta - \vartheta_c) + \frac{(-\Delta H)}{\rho c_p} r(c_A, \vartheta) \end{aligned}$$

Introducing

$$\begin{aligned} x_1 &= c_A \\ x_2 &= \vartheta \\ u_1 &= \vartheta_c \\ r_1 &= c_{Av} \\ r_2 &= \vartheta_v \end{aligned}$$

and assuming that temperature measurement of ϑ is available, state and output equations are given as

$$\begin{aligned} \frac{d\boldsymbol{x}}{dt} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}) \\ y_1 &= (0\ 1)\boldsymbol{x} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{x} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \boldsymbol{u} = u_1,\ \boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},\ \boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ f_1 &= \frac{q}{V}c_{Av} - \frac{q}{V}c_A - r(c_A, \vartheta) \\ f_2 &= \frac{q}{V}\vartheta_v - \frac{q}{V}\vartheta + \frac{\alpha F}{V\rho c_p}(\vartheta - \vartheta_c) + \frac{(-\Delta H)}{\rho c_p}r(c_A, \vartheta) \end{aligned}$$

Example 2.3.3: *Double-pipe steam-heated exchanger - state equations*

Processes with distributed parameters are usually approximated by a series of well-mixed lumped parameter processes. This is also the case for the heat exchanger shown in Fig. 2.2.5 which is divided into n well-mixed heat exchangers. The space variable is divided into n equal lengths within the interval $[0, L]$. However, this division can also be realised differently.

Mathematical model of the exchanger is of the form

$$\frac{\partial\vartheta}{\partial t} = -v_\sigma\frac{\partial\vartheta}{\partial\sigma} - \frac{1}{T_1}\vartheta + \frac{1}{T_1}\vartheta_p$$

Introduce

$$\begin{aligned} x_1(t) &= \vartheta\left(\frac{L}{n}, t\right) \\ x_2(t) &= \vartheta\left(\frac{2L}{n}, t\right) \\ &\vdots \\ x_n(t) &= \vartheta(L, t) \\ u_1(t) &= \vartheta_p(t) \\ r_1(t) &= \vartheta(0, t) \end{aligned}$$

and replace $\partial\vartheta/\partial\sigma$ with the corresponding difference. The resulting model consits of a set of ordinary differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{v_\sigma n}{L}(x_1 - r_1) - \frac{1}{T_1}x_1 + \frac{1}{T_1}u_1 \\ \frac{dx_2}{dt} &= -\frac{v_\sigma n}{L}(x_2 - x_1) - \frac{1}{T_1}x_2 + \frac{1}{T_1}u_1 \\ &\vdots \\ \frac{dx_n}{dt} &= -\frac{v_\sigma n}{L}(x_n - x_{n-1}) - \frac{1}{T_1}x_n + \frac{1}{T_1}u_1 \end{aligned}$$

The state equation is given as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u_1 + \boldsymbol{H}r_1$$

where

$$\begin{aligned}
\boldsymbol{x} &= (x_1 \ \dots \ x_n)^T \\
\boldsymbol{A} &= \begin{pmatrix} -(\frac{v_\sigma n}{L} + \frac{1}{T_1}) & 0 & 0 & \dots & 0 & 0 \\ \frac{v_\sigma n}{L} & -(\frac{v_\sigma n}{L} + \frac{1}{T_1}) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{v_\sigma n}{L} & -(\frac{v_\sigma n}{L} + \frac{1}{T_1}) \end{pmatrix} \\
\boldsymbol{B} &= \frac{1}{T_1}(1 \ \dots \ 1)^T \\
\boldsymbol{H} &= \left(\frac{v_\sigma n}{L} \ 0 \ \dots \ 0\right)^T
\end{aligned}$$

Assume that temperature is measured at $\sigma_1 = 2L/n$ and/or $\sigma_2 = L$. Then the output equation is of the form

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \boldsymbol{C} = \begin{pmatrix} 0 \ 1 \ 0 \ \dots \ 0 \\ 0 \ 0 \ 0 \ \dots \ 1 \end{pmatrix}$$

or

$$y_1 = \boldsymbol{C}\boldsymbol{x}, \ \boldsymbol{C} = (0 \ 0 \ 0 \ \dots \ 1)$$

Example 2.3.4: *Heat exchanger - dimensionless variables*

The state equation for the heat exchanger shown in Fig. 2.2.3 is

$$\frac{d\vartheta}{dt'} = -\frac{1}{T_1}\vartheta + \frac{Z_1}{T_1}\vartheta_p + \frac{Z_2}{T_1}\vartheta_v$$

where t' is time variable. The exchanger is in a steady-state if $d\vartheta/dt' = 0$. Denote steady-state temperatures $\vartheta_p^s, \vartheta_v^s, \vartheta^s$. For the steady-state yields

$$\vartheta^s = Z_1\vartheta_p^s + Z_2\vartheta_v^s$$

Define dimensionless variables

$$\begin{aligned}
x_1 &= \frac{\vartheta}{\vartheta^s} \\
u_1 &= \frac{\vartheta_p}{\vartheta_p^s} \\
r_1 &= \frac{\vartheta_v}{\vartheta_v^s} \\
t &= \frac{t'}{T_1}
\end{aligned}$$

then the state equation is given as

$$\frac{dx_1}{dt} = -x_1 + \frac{Z_1\vartheta_p^s}{\vartheta^s}u_1 + \frac{Z_2\vartheta_v^s}{\vartheta^s}r_1$$

with initial condition

$$x_1(0) = x_{10} = \frac{\vartheta(0)}{\vartheta^s}$$

2.4 Linearisation

Linearisation of nonlinear models plays an important role in practical control design. The principle of linearisation of nonlinear equations consists in supposition that process variables change very little and their *deviations from steady-state* remain small. Linear approximation can be obtained by using the Taylor series expansion and considering only linear terms. This approximation is then called *linearised model*. An advantage of linear models is their simplicity and their use can yield to analytical results.

Let us recall the *Taylor theorem*: Let a, x be different numbers, $k \geq 0$ and J is a closed interval with endpoints a, x. Let f be a function with continuous k-th derivative on J and $k+1$-th derivative within this interval. Then there exists a point ζ within J such that

$$f(x) = f(a) + \frac{\dot{f}(a)}{1!}(x-a) + \frac{\ddot{f}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x) \tag{2.4.1}$$

where $R_k(x) = \frac{f^{(k+1)}(\zeta)}{(k+1)!}(x-a)^{k+1}$ is the rest of the function f after the k-th term of the Taylor's polynomial.

Consider a process described by a set of equations

$$\frac{dx'_i}{dt} = f_i(\boldsymbol{x}', \boldsymbol{u}') = f_i(x'_1, \ldots, x'_n, u'_1, \ldots, u'_m), \quad i = 1, \ldots, n \tag{2.4.2}$$

where

$\boldsymbol{x}'$ - vector of state variables $x'_1, \ldots, x'_n$,

$\boldsymbol{u}'$ - vector of manipulated variables $u'_1, \ldots, u'_m$.

Let the process state variables x'_i change in the neighbourhood of the steady-state x'^s_i under the influence of the manipulated variables u'_i. Then it is possible to approximate the process nonlinearities. The steady-state is given by the equation

$$0 = f_i(\boldsymbol{x}'^s, \boldsymbol{u}'^s) = f_i^s \tag{2.4.3}$$

We suppose that the solution of these equations is known for some $u'^s_j, j = 1, \ldots, m$. The function $f_i(\bullet)$ is approximated by the Taylor series expansion truncated to only first order terms as

$$\begin{aligned} f_i(\boldsymbol{x}', \boldsymbol{u}') \;\doteq\; & f_i(\boldsymbol{x}'^s, \boldsymbol{u}'^s) + \\ & + \left(\frac{\partial f_i}{\partial x'_1}\right)^s (x'_1 - x'^s_1) + \cdots + \left(\frac{\partial f_i}{\partial x'_n}\right)^s (x'_n - x'^s_n) + \\ & + \left(\frac{\partial f_i}{\partial u'_1}\right)^s (u'_1 - u'^s_1) + \cdots + \left(\frac{\partial f_i}{\partial u'_m}\right)^s (u'_m - u'^s_m) \end{aligned} \tag{2.4.4}$$

$(\partial f_i/\partial x'_l)^s$, $l = 1, \ldots, n$ and $\left(\partial f_i/\partial u'_j\right)^s$, $j = 1, \ldots, m$ denote partial derivatives for $x'_l = x'^s_l$ and $u'_j = u'^s_j$, respectively. Therefore, these partial derivatives are constants

$$a_{il} = \left(\frac{\partial f_i}{\partial x'_l}\right)^s \quad l = 1, \ldots, n \tag{2.4.5}$$

$$b_{ij} = \left(\frac{\partial f_i}{\partial u'_j}\right)^s \quad j = 1, \ldots, m \tag{2.4.6}$$

From Eq. (2.4.3) follows that the first term on the right side of (2.4.4) is zero. Introducing state and manipulated deviation variables

$$x_i = x'_i - x'^s_i \tag{2.4.7}$$

$$u_j = u'_j - u'^s_j \tag{2.4.8}$$

gives

$$\begin{aligned}
\frac{dx_1'}{dt} &= \frac{dx_1}{dt} = a_{11}x_1 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\
\frac{dx_2'}{dt} &= \frac{dx_2}{dt} = a_{21}x_1 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\
&\vdots \\
\frac{dx_n'}{dt} &= \frac{dx_n}{dt} = a_{n1}x_1 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m
\end{aligned} \tag{2.4.9}$$

We denote $\boldsymbol{x}$ the vector of *deviation state variables* and $\boldsymbol{u}$ the vector of *deviation manipulated variables*. Then (2.4.9) can be written as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{2.4.10}$$

where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{pmatrix}$$

Equation (2.4.10) is a linearised differential equation. If initial state of (2.4.2) also represent steady-state of the modelled process then

$$\boldsymbol{x}(0) = \boldsymbol{0} \tag{2.4.11}$$

Equations (2.4.2), (2.4.10) describe dynamics of a process. The differences are as follows:

1. equation (2.4.10) is only an approximation,
2. equation (2.4.10) uses deviation variables,
3. equation (2.4.10) is linear with constant coefficients.

Linearisation of the process dynamics must be completed with linearisation of the output equation if this is nonlinear.

Consider the output equation of the form

$$y_k' = g_k(\boldsymbol{x}', \boldsymbol{u}'), \quad k = 1, \dots, r \tag{2.4.12}$$

where y_k' are the output variables. In the steady-state holds

$$y_k'^s = g_k(\boldsymbol{x}'^s, \boldsymbol{u}'^s) \tag{2.4.13}$$

Introducing *output deviation variables*

$$y_k = y_k' - y_k'^s \tag{2.4.14}$$

follows

$$y_k = g_k(\boldsymbol{x}'^s + \boldsymbol{x}, \boldsymbol{u}'^s + \boldsymbol{u}) - g_k(\boldsymbol{x}'^s, \boldsymbol{u}'^s) \tag{2.4.15}$$

Using the Taylor series expansion with only linear terms the following approximation holds

$$\begin{aligned} g_k(\boldsymbol{x}', \boldsymbol{u}') &\doteq g_k(\boldsymbol{x}'^s, \boldsymbol{u}'^s) + \sum_{l=1}^{n} \left(\frac{\partial g_k}{\partial x_l'} \right)^s (x_l' - x_l'^s) \\ &\quad + \sum_{j=1}^{m} \left(\frac{\partial g_k}{\partial u_j'} \right)^s (u_j' - u_j'^s) \end{aligned} \tag{2.4.16}$$

and again the partial derivatives in (2.4.16) are constants

$$c_{kl} = \left(\frac{\partial g_k}{\partial x_l'} \right)^s \quad l = 1, \ldots, n \tag{2.4.17}$$

$$d_{kj} = \left(\frac{\partial g_k}{\partial u_j'} \right)^s \quad j = 1, \ldots, m \tag{2.4.18}$$

Output deviation variables are then of the form

$$\begin{aligned} y_1 &= c_{11}x_1 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1m}u_m \\ y_2 &= c_{21}x_1 + \cdots + c_{2n}x_n + d_{21}u_1 + \cdots + d_{2m}u_m \\ &\vdots \\ y_r &= c_{r1}x_1 + \cdots + c_{rn}x_n + d_{r1}u_1 + \cdots + d_{rm}u_m \end{aligned} \tag{2.4.19}$$

If $\boldsymbol{y}$ denotes the vector of output deviation variables then the previous equation can more compactly be written as

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \tag{2.4.20}$$

where

$$\boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ c_{r1} & c_{r2} & \ldots & c_{rn} \end{pmatrix}$$

$$\boldsymbol{D} = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ d_{r1} & d_{r2} & \ldots & d_{rm} \end{pmatrix}$$

Equations (2.4.10) and (2.4.20) constitute together the *general linear state process model*. When it is obtained from the linearisation procedure, then it can only be used in the neighbourhood of the steady-state where linearisation was derived.

Example 2.4.1: *Liquid storage tank - linearisation*

Consider the liquid storage tank shown in Fig. 2.2.1. The state equation of this process is

$$\frac{dh}{dt} = f_1(h, q_0)$$

where

$$f_1(h, q_0) = -\frac{k_{11}}{F}\sqrt{h} + \frac{1}{F}q_0$$

The steady-state equation is

$$f_1(h^s, q_0^s) = -\frac{k_{11}}{F}\sqrt{h^s} + \frac{1}{F}q_0^s = 0$$

Linearised state equation for a neighbourhood of the steady-state given by q_0^s, h^s can be written as

$$\frac{dh}{dt} = \frac{d(h - h^s)}{dt} = -\frac{k_{11}}{2F\sqrt{h^s}}(h - h^s) + \frac{1}{F}(q_0 - q_0^s)$$

Introducing deviation variables

$$\begin{aligned} x_1 &= h - h^s \\ u_1 &= q_0 - q_0^s \end{aligned}$$

and assuming that the level h is measured then the linearised model of the tank is of the form

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + b_{11}u_1 \\ y_1 &= x_1 \end{aligned}$$

where

$$a_{11} = -\frac{k_{11}}{2F\sqrt{h^s}}, \quad b_{11} = \frac{1}{F}$$

Example 2.4.2: *CSTR - linearisation*

Consider the CSTR shown in Fig. 2.2.11. The state equations for this reactor are

$$\begin{aligned} \frac{dc_A}{dt} &= f_1(c_A, c_{Av}, \vartheta) \\ \frac{d\vartheta}{dt} &= f_2(c_A, \vartheta, \vartheta_v, \vartheta_c) \end{aligned}$$

where

$$\begin{aligned} f_1(c_A, c_{Av}, \vartheta) &= \frac{q}{V}c_{Av} - \frac{q}{V}c_A - r(c_A, \vartheta) \\ f_2(c_A, \vartheta, \vartheta_v, \vartheta_c) &= \frac{q}{V}\vartheta_v - \frac{q}{V}\vartheta - \frac{\alpha F}{V\rho c_p}(\vartheta - \vartheta_c) + \frac{(-\Delta H)}{\rho c_p}r(c_A, \vartheta) \end{aligned}$$

Linearised dynamics equations for the neighbourhood of the steady-state given by steady-state input variables $\vartheta_c^s, c_{Av}^s, \vartheta_v^s$ and steady-state process state variables c_A^s, ϑ^s are of the form

$$\begin{aligned} \frac{dc_A}{dt} &= \frac{d(c_A - c_A^s)}{dt} = \left(-\frac{q}{V} - \dot{r}_{c_A}(c_A^s, \vartheta^s)\right)(c_A - c_A^s) \\ &\quad + (-\dot{r}_\vartheta(c_A^s, \vartheta^s))(\vartheta - \vartheta^s) + \frac{q}{V}(c_{Av} - c_{Av}^s) \\ \frac{d\vartheta}{dt} &= \frac{d(\vartheta - \vartheta^s)}{dt} = \left(\frac{(-\Delta H)}{\rho c_p}\dot{r}_{c_A}(c_A^s, \vartheta^s)\right)(c_A - c_A^s) \\ &\quad + \left(-\frac{q}{V} - \frac{\alpha F}{V\rho c_p} + \frac{(-\Delta H)}{\rho c_p}\dot{r}_\vartheta(c_A^s, \vartheta^s)\right)(\vartheta - \vartheta^s) \\ &\quad + \frac{\alpha F}{V\rho c_p}(\vartheta_c - \vartheta_c^s) + \frac{q}{V}(\vartheta_v - \vartheta_v^s) \end{aligned}$$

where

$$\dot{r}_{c_A}(c_A^s, \vartheta^s) = \left.\frac{\partial r(c_A, \vartheta)}{\partial c_A}\right|_{\substack{c_A = c_A^s \\ \vartheta = \vartheta^s}}$$

$$\dot{r}_\vartheta(c_A^s, \vartheta^s) = \left.\frac{\partial r(c_A, \vartheta)}{\partial \vartheta}\right|_{\substack{c_A = c_A^s \\ \vartheta = \vartheta^s}}$$

Introducing deviation variables

$$\begin{aligned} x_1 &= c_A - c_A^s \\ x_2 &= \vartheta - \vartheta^s \\ u_1 &= \vartheta_c - \vartheta_c^s \\ r_1 &= c_{Av} - c_{Av}^s \\ r_2 &= \vartheta_v - \vartheta_v^s \end{aligned}$$

and considering temperature measurements of ϑ then for the linearised process model follows

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + h_{11}r_1\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + h_{22}r_2\\ y_1 &= x_2\end{aligned}$$

where

$$\begin{aligned}a_{11} &= -\frac{q}{V} - \dot{r}_{c_A}(c_A^s, \vartheta^s),\ a_{12} = -\dot{r}_{\vartheta}(c_A^s, \vartheta^s)\\ a_{21} &= \frac{(-\Delta H)}{\rho c_p}\dot{r}_{c_A}(c_A^s, \vartheta^s),\ a_{22} = -\frac{q}{V} - \frac{\alpha F}{V\rho c_p} + \frac{(-\Delta H)}{\rho c_p}\dot{r}_{\vartheta}(c_A^s, \vartheta^s))\\ b_{21} &= \frac{\alpha F}{V\rho c_p}\\ h_{11} &= h_{22} = \frac{q}{V}\end{aligned}$$

If the rate of reaction is given as (the first order reaction)

$$r(c_A, \vartheta) = c_A k_0 e^{-\frac{E}{R\vartheta}}$$

then

$$\begin{aligned}\dot{r}_{c_A}(c_A^s, \vartheta^s) &= k_0 e^{-\frac{E}{R\vartheta^s}}\\ \dot{r}_{\vartheta}(c_A^s, \vartheta^s) &= c_A^s k_0 \frac{E}{R(\vartheta^s)^2} e^{-\frac{E}{R\vartheta^s}}\end{aligned}$$

The deviation variables have the same meaning as before: x_1, x_2 are state deviation variables, u_1 is a manipulated deviation variable, and r_1, r_2 are disturbance deviation variables.

2.5 Systems, Classification of Systems

A *deterministic single-input single-output (SISO) system* is a physical device which has only one input $u(t)$ and the result of this influence is an observable output variable $y(t)$. The same initial conditions and the same function $u(t)$ lead to the same output function $y(t)$. This definition is easily extended to *deterministic multi-input multi-output (MIMO) systems* whose input variables are $u_1(t), \ldots, u_m(t)$ and output variables are $y_1(t), \ldots, y_r(t)$. The *concept of a system* is based on the relation between cause and consequence of input and output variables.

Continuous-time (CT) systems are systems with all variables defined for all time values.

Lumped parameter systems have influence between an input and output variables given by ordinary differential equations with derivatives with respect to time. Systems with distributed parameters are described by partial differential equations with derivatives with respect to time and space variables.

If the relation between an input and output variable for deterministic CT SISO system is given by ordinary differential equations with order greater than one, then it is necessary for determination of $y(t), t > t_0$ to know $u(t), t > t_0$ and output variable $y(t_0)$ with its derivatives at t_0 or some equivalent information. The necessity of knowledge about derivatives avoids introduction of *concept of state*.

Linear systems obey the law of superposition.

The systems described in Section 2.2 are examples of *physical systems*. The systems determined only by variables that define a relation between the system elements or between the system and its environment are called *abstract*. Every physical system has a corresponding abstract model but not vice versa. A notation of oriented systems can be introduced. This is every controlled system with declared input and output variables.

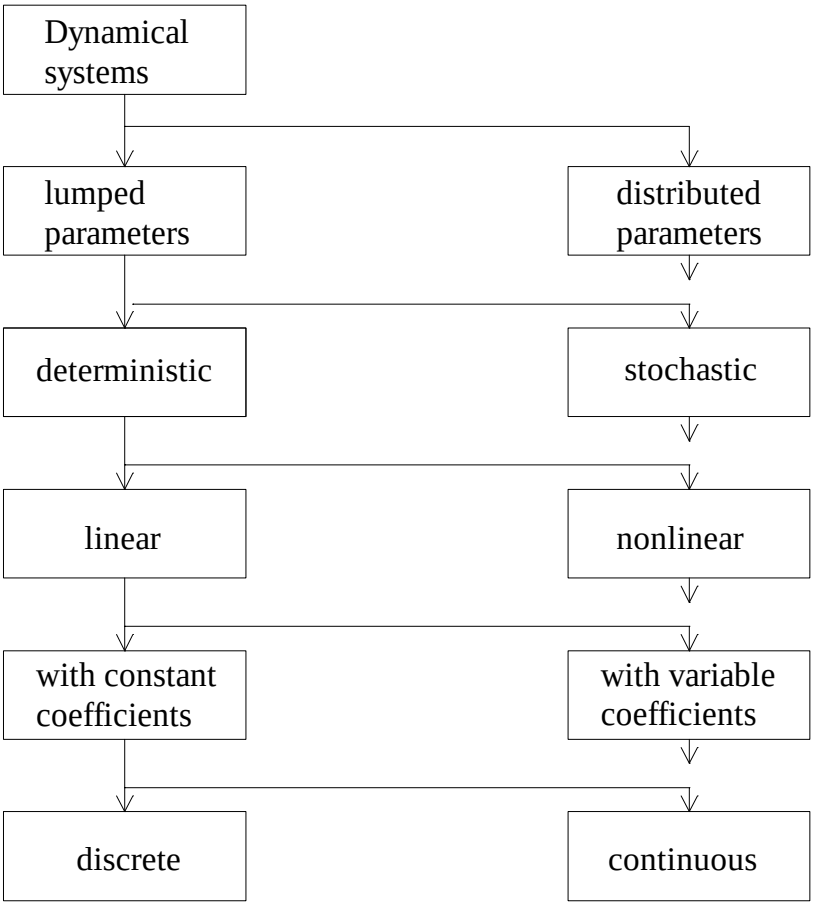

Figure 2.5.1: Classification of dynamical systems.

The relation between objects (processes) and systems can be explained as follows. If a process has defined some set of *typical important properties* significant for our investigations then we have defined a system on the process.

We note that we will not further pursue special details and differences between systems and mathematical relations describing their behaviour as it is not important for our purposes.

Analogously as continuous-time systems were defined, *discrete-time (DT) systems* have their variables defined only in certain time instants.

The process model examples were chosen to explain the procedure for simplification of models. Usually, two basic steps were performed. Models given by partial differential equations were transformed into ordinary differential equations and nonlinear models were linearised. Step-wise simplifications of process models led to models with linear differential equations. As computer control design is based on DT signals, the last transformation is toward DT systems.

A *Stochastic system* is characterised by variables known only with some probability.

Therefore, classification of *dynamical systems* can be clearly given as in Fig. 2.5.1.

2.6 References

Mathematical models of processes are discussed in more detail in works:

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B. Wittenmark, J. K. Åström, and S. B. Jørgensen. *Process Control*. Lund Institute of Technology - Technical University of Denmark, Lyngby, 1992.

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V. Strejc. *State-space Theory of Linear Control*. Academia, Praha, 1978. (in czech).

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2.7 Exercises

Exercise 2.7.1:
Consider the liquid storage tank shown in Fig. 2.7.1. Assume constant liquid density and constant flow rate q_1. Flow rate q_2 can be expressed as

$$q_2 = k_{10}h + k_{11}\sqrt{h}$$

Find:

1. state equation,
2. linearised process model.

Exercise 2.7.2:
A double vessel is used as a heat exchanger between two liquids separated by a wall (Fig. 2.7.2).

Assume heating of a liquid with a constant volume V_2 with a liquid with a constant volume V_1. Heat transfer is considered only in direction vertical to the wall with temperature $\vartheta_w(t)$, volume

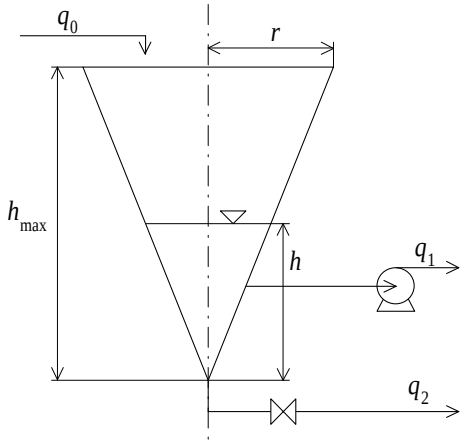

Figure 2.7.1: A cone liquid storage process.

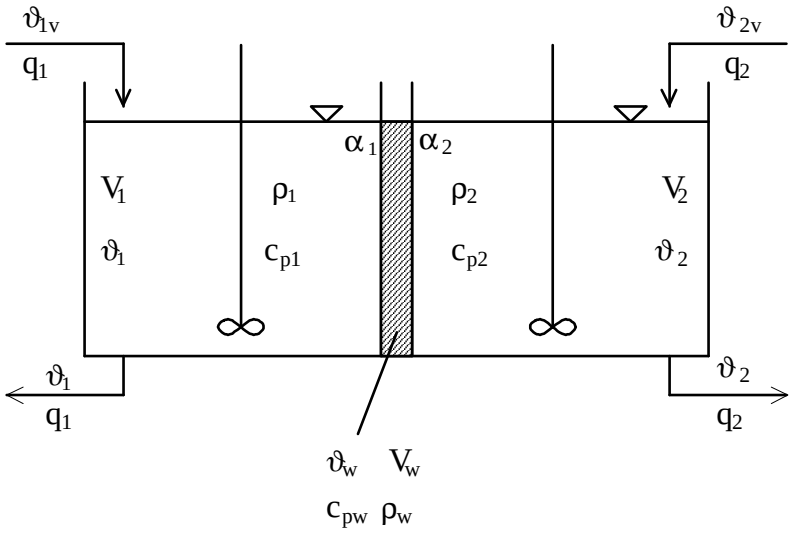

Figure 2.7.2: Well mixed heat exchanger.

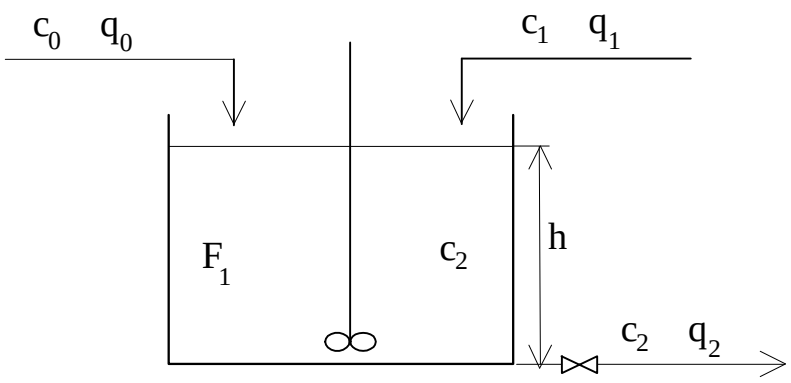

Figure 2.7.3: A well mixed tank.

V_w, density ρ_w, and specific heat capacity c_{pw}. Heat transfer from the process toits environment is neglected. Further, assume spatially constant temperatures ϑ_1 and ϑ_2, constant densities ρ_1, ρ_2, flow rates q_1, q_2, specific heat capacities c_{p1}, c_{p2}. α_1 is the heat transfer coefficient from liquid to wall and α_2 is the heat transfer coefficient from wall to liquid. The process state variables are $\vartheta_1, \vartheta_2, \vartheta_w$. The process input variables are $\vartheta_{1v}, \vartheta_{2v}$.

1. Find state equations,
2. introduce dimensionless variables and rewrite the state equations.

Exercise 2.7.3:
A tank is used for blending of liquids (Fig. 2.7.3). The tank is filled up with two pipelines with flow rates q_1, q_2. Both streams contain a component with constant concentrations c_0, c_1. The outlet stream has a flow rate q_2 and concentration c_2. Assume that the concentration within tank is c_2.

Find:

1. state equations,
2. linearised process model.

Exercise 2.7.4:
An irreversible reaction $A \rightarrow B$ occurs in a series of CSTRs shown in Fig. 2.7.4. The assumptions are the same as for the reactor shown in Fig. 2.2.11.

1. Find state equations,
2. construct linearised process model.

Exercise 2.7.5:
Consider the gas tank shown in Fig. 2.7.5. A gas with pressure $p_0(t)$ flows through pneumatic resistance (capillary) R_1 to the tank with volume V. The pressure in the tank is $p_1(t)$. Molar flow rate G of the gas through resistance R_1 is

$$G = \frac{p_0 - p_1}{R_1}$$

Assume that the ideal gas law holds. Find state equation of the tank.

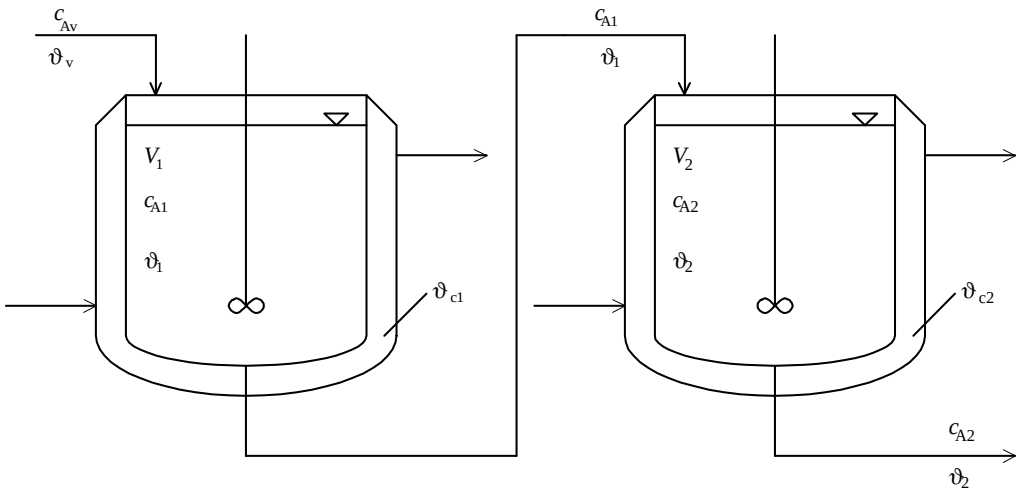

Figure 2.7.4: Series of two CSTRs.

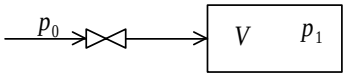

Figure 2.7.5: A gas storage tank.

Chapter 3

Analysis of Process Models

Mathematical models describing behaviour of a large group of technological processes can under some simplifications be given by linear differential equations with constant coefficients. Similarly, other blocks of control loops can also be described by linear differential equations with constant coefficients. For investigation of the dynamical properties of processes it is necessary to solve differential equations with time as independent variable. Linear differential equations with constant coefficients can be very suitably solved with the help of the Laplace transform.

Analysis of dynamical systems is based on their state-space representation. The spate-space representation is closely tied to input-output representation of the systems that are described by input-output process models. In this chapter we will define the Laplace transform and show how to solve by means of it linear differential equations with constant coefficients. We introduce the definition of transfer function and transfer function matrix. Next, the concept of states and connection between state-space and input-output models will be given. We examine the problem of stability, controllability, and observability of continuous-time processes.

3.1 The Laplace Transform

The Laplace transform offers a very simple and elegant vehicle for the solution of differential equations with constant coefficients. It further enables to derive input-output models which are suitable for process identification and control. Moreover, it simplifies the qualitative analysis of process responses subject to various input signals.

3.1.1 Definition of The Laplace Transform

Consider a function $f(t)$. *The Laplace transform* is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt \tag{3.1.1}$$

where $\mathcal{L}$ is an operator defined by the integral, $f(t)$ is some function of time. The Laplace transform is often written as

$$F(s) = \mathcal{L}\{f(t)\} \tag{3.1.2}$$

The function $f(t)$ given over an interval $0 \leq t < \infty$ is called the time *original* and the function $F(s)$ its *Laplace transform*. The function $f(t)$ must satisfy some conditions. It must be piecewise continuous for all times from $t = 0$ to $t = \infty$. This requirement practically always holds for functions used in modelling and control. It follows from the definition integral that we transform the function from the time domain into s domain where s is a complex variable. Further it is clear that the Laplace transform of a function exists if the definition integral is bounded. This condition is fulfilled for all functions we will deal with.

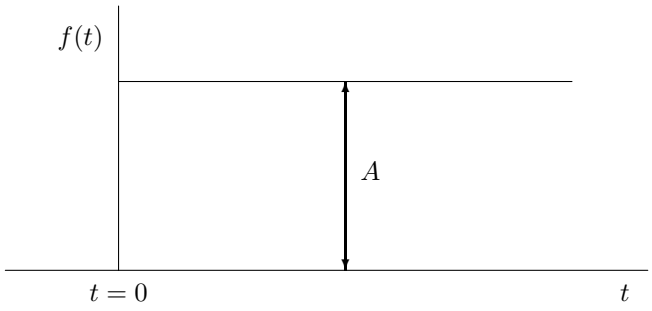

Figure 3.1.1: A step function.

The function $F(s)$ contains no information about $f(t)$ for $t < 0$. This is no real obstacle as t is the time variable usually defined as positive. Variables and systems are then usually defined such that

$$f(t) \equiv 0 \quad \text{for} \quad t < 0 \tag{3.1.3}$$

If the equation (3.1.3) is valid for the function $f(t)$, then this is uniquely given except at the points of incontinuities with the $\mathcal{L}$ transform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{3.1.4}$$

This equation defines the *inverse Laplace transform.*

The Laplace transform is a *linear operator* and satisfies the *principle of superposition*

$$\mathcal{L}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 \mathcal{L}\{f_1(t)\} + k_2 \mathcal{L}\{f_2(t)\} \tag{3.1.5}$$

where k_1, k_2 are some constants. The proof follows from the definition integral

$$\begin{aligned} \mathcal{L}\{k_1 f_1(t) + k_2 f_2(t)\} &= \int_0^\infty [k_1 f_1(t) + k_2 f_2(t)] e^{-st} dt \\ &= k_1 \int_0^\infty f_1(t) e^{-st} dt + k_2 \int_0^\infty f_2(t) e^{-st} dt \\ &= k_1 \mathcal{L}\{f_1(t)\} + k_2 \mathcal{L}\{f_2(t)\} \end{aligned}$$

An important advantage of the Laplace transform stems from the fact that operations of derivation and integration are transformed into algebraic operations.

3.1.2 Laplace Transforms of Common Functions

Step Function

The Laplace transform of step function is very important as step functions and unit step functions are often used to investigate the process dynamical properties and in control applications.

The step function shown in Fig. 3.1.1 can be written as

$$f(t) = A1(t) \tag{3.1.6}$$

and $1(t)$ is *unit step function.* This is defined as

$$1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{3.1.7}$$

The Laplace transform of step function is

$$\mathcal{L}\{A1(t)\} = \frac{A}{s} \tag{3.1.8}$$

Proof:

$$\begin{aligned}\mathcal{L}\{A1(t)\} &= \int_0^\infty A1(t)e^{-st}dt = A\int_0^\infty e^{-st}dt \\ &= A\left[-\frac{1}{s}e^{-st}\right]_0^\infty = A\frac{1}{(-s)}(e^{-s\infty}-e^{-s0}) \\ &= \frac{A}{s}\end{aligned}$$

The Laplace transform of the unit step functions is

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \tag{3.1.9}$$

Exponential Function

The Laplace transform of an exponential function is of frequent use as exponential functions appear in the solution of linear differential equations. Consider an exponential function of the form

$$f(t) = e^{-at}1(t) \tag{3.1.10}$$

hence $f(t) = e^{-at}$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. The Laplace transform of this function is

$$\mathcal{L}\left\{e^{-at}1(t)\right\} = \frac{1}{s+a} \tag{3.1.11}$$

Proof :

$$\begin{aligned}\mathcal{L}\left\{e^{-at}1(t)\right\} &= \int_0^\infty e^{-at}1(t)e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt \\ &= -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty \\ &= \frac{1}{s+a}\end{aligned}$$

From (3.1.10) follows that

$$\mathcal{L}\left\{e^{at}1(t)\right\} = \frac{1}{s-a} \tag{3.1.12}$$

Ramp Function

Consider a ramp function of the form

$$f(t) = at1(t) \tag{3.1.13}$$

The Laplace transform of this function is

$$\mathcal{L}\{at1(t)\} = \frac{a}{s^2} \tag{3.1.14}$$

Proof :

$$\mathcal{L}\{at1(t)\} = \int_0^\infty at1(t)e^{-st}dt$$

Let us denote $u = at$ and $\dot{v} = e^{-st}$ and use the rule of integrating by parts

$$\begin{aligned}(u\dot{v}) &= u\dot{v} + \dot{u}v \\ \int u\dot{v}dt &= uv - \int \dot{u}vdt\end{aligned}$$

As $\dot{u} = a$ and $v = -\frac{1}{s}e^{-st}$, the Laplace transform of the ramp function is

$$\begin{aligned}\mathcal{L}\{at1(t)\} &= \left[at\frac{1}{(-s)}e^{-st}\right]_0^\infty - a\int_0^\infty \frac{1}{(-s)}e^{-st}dt \\ &= (0-0) + \frac{a}{s}\left[at\frac{1}{(-s)}e^{-st}\right]_0^\infty \\ &= \frac{a}{s^2}\end{aligned}$$

Trigonometric Functions

Functions $\sin\omega t$ and $\cos\omega t$ are used in investigation of dynamical properties of processes and control systems. The process response to input variables of the form $\sin\omega t$ or $\cos\omega t$ is observed, where ω is the frequency in radians per time. The Laplace transform of these functions can be calculated using integration by parts or using the Euler identities

$$\begin{aligned}e^{j\omega t} &= \cos\omega t + j\sin\omega t \\ e^{-j\omega t} &= \cos\omega t - j\sin\omega t \\ e^{j\omega t} + e^{-j\omega t} &= 2\cos\omega t \\ e^{j\omega t} - e^{-j\omega t} &= 2j\sin\omega t\end{aligned} \tag{3.1.15}$$

Consider a trigonometric function of the form

$$f(t) = (\sin\omega t)1(t) \tag{3.1.16}$$

The Laplace transform of this function is

$$\mathcal{L}\{(\sin\omega t)1(t)\} = \frac{\omega}{s^2+\omega^2} \tag{3.1.17}$$

Proof :

$$\begin{aligned}\mathcal{L}\{(\sin\omega t)1(t)\} &= \int_0^\infty (\sin\omega t)1(t)e^{-st}dt = \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j}e^{-st}dt \\ &= \int_0^\infty \frac{1}{2j}e^{-(s-j\omega)t}dt - \int_0^\infty \frac{1}{2j}e^{-(s+j\omega)t}dt \\ &= \frac{1}{2j}\left[\frac{e^{-(s-j\omega)t}}{-(s-j\omega)}\right]_0^\infty + \frac{1}{2j}\left[\frac{e^{-(s+j\omega)t}}{-(s+j\omega)}\right]_0^\infty \\ &= \frac{1}{2j}\left(\frac{1}{s-j\omega}\right) - \frac{1}{2j}\left(\frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}\end{aligned}$$

The Laplace transform of other functions can be calculated in a similar manner. The list of the most commonly used functions together with their Laplace transforms is given in Table 3.1.1.

3.1.3 Properties of the Laplace Transform

Derivatives

The Laplace transform of derivatives are important as derivatives appear in linear differential equations. The transform of the first derivative of $f(t)$ is

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0) \tag{3.1.18}$$

Proof :

$$\begin{aligned}\mathcal{L}\left\{\frac{df(t)}{dt}\right\} &= \int_0^\infty \dot{f}(t)e^{-st}dt \\ &= \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)e^{-st}(-s)dt \\ &= sF(s) - f(0)\end{aligned}$$

Table 3.1.1: The Laplace transforms for common functions

$f(t)$	$F(s)$
$\delta(t)$ - unit impulse function	1
$1(t)$ - unit step function	$\frac{1}{s}$
$1(t)-1(t-T_v),\ T_v$ is a time constant	$\frac{1-e^{-sT_v}}{s}$
$at1(t)$, a is a constant (ramp)	$\frac{a}{s^2}$
$at^{n-1}1(t),\ n>1$	$a\frac{(n-1)!}{s^n}$
$e^{-at}1(t)$	$\frac{1}{s+a}$
$\frac{1}{T_1}e^{-\frac{t}{T_1}}1(t)$	$\frac{1}{T_1 s+1}$
$(1-e^{-at})1(t)$	$\frac{a}{s(s+a)}$
$(1-e^{-\frac{t}{T_1}})1(t)$	$\frac{1}{s(T_1 s+1)}$
$\left(\frac{1}{a-b}(e^{-bt}-e^{-at})\right)1(t)$, a,b are constants	$\frac{1}{(s+a)(s+b)}$
$\left(\frac{c-a}{b-a}e^{-at}+\frac{c-b}{a-b}e^{-bt}\right)1(t)$, c is a constant	$\frac{s+c}{(s+a)(s+b)}$
$\frac{t^{n-1}e^{-at}}{(n-1)!}1(t), n\geq 1$	$\frac{1}{(s+a)^n}$
$\left(\frac{1}{a}t-\frac{1-e^{-at}}{a^2}\right)1(t)$	$\frac{1}{s^2(s+a)}$
$\left(\frac{1}{ab}+\frac{1}{a(a-b)}e^{-at}+\frac{1}{b(b-a)}e^{-bt}\right)1(t)$	$\frac{1}{s(s+a)(s+b)}$
$\left(\frac{c}{ab}+\frac{c-a}{a(a-b)}e^{-at}+\frac{c-b}{b(b-a)}e^{-bt}\right)1(t)$	$\frac{s+c}{s(s+a)(s+b)}$
$\sin\omega t\ 1(t)$, ω is a constant	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t\ 1(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t\ 1(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t\ 1(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\left\{1-e^{-\frac{\zeta t}{T_k}}\left[\cos\left(\sqrt{1-\zeta^2}\frac{t}{T_k}\right)+\frac{\zeta}{1-\zeta^2}\sin\left(\sqrt{1-\zeta^2}\frac{t}{T_k}\right)\right]\right\}1(t)$	$\frac{1}{s(T_k^2 s^2+2\zeta T_k s+1)}$
$0\leq\lvert\zeta\rvert<1$	
$\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\zeta t}{T_k}}\sin\left(\sqrt{1-\zeta^2}\frac{t}{T_k}+\varphi\right)\right]1(t)$	$\frac{1}{s(T_k^2 s^2+2\zeta T_k s+1)}$
$\varphi=\arctan\frac{\sqrt{1-\zeta^2}}{\zeta}, 0\leq\lvert\zeta\rvert<1$	

The Laplace transform of the second derivative of $f(t)$ is

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0) \tag{3.1.19}$$

Proof : Let us define a new function $\bar{f}(t) = df(t)/dt$. Applying the equation (3.1.18) yields (3.1.19). Similarly for higher-order derivatives follows

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{(n-1)}(0) \tag{3.1.20}$$

Integral

The Laplace transform of the integral of $f(t)$ is

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \tag{3.1.21}$$

Proof :

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)d\tau\right] e^{-st}dt$$

Let us denote $u = \int_0^t f(\tau)d\tau, \dot{v} = e^{-st}$ and use integration by parts. Because $\dot{u} = f(t), v = \frac{1}{(-s)}e^{-st}$, the transform gives

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} &= \left[\int_0^t f(\tau)d\tau\frac{1}{(-s)}e^{-st}\right]_0^\infty - \int_0^\infty f(t)\frac{1}{(-s)}e^{-st}dt \\
&= (0-0) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\
&= \frac{F(s)}{s}
\end{aligned}$$

Convolution

The Laplace transform of convolution is important in situations when input variables of processes are general functions of time. Let functions $f_1(t)$ and $f_2(t)$ be transformed as $F_1(s)$ and $F_2(s)$ respectively. The convolution of the functions is defined as

$$f_1(t) \star f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)d\tau \tag{3.1.22}$$

The Laplace transform of convolution is

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right\} = \mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right\} = F_1(s)F_2(s) \tag{3.1.23}$$

Proof :

$$\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right\} = \int_0^\infty \int_0^t f_1(\tau)f_2(t-\tau)d\tau e^{-st}dt$$

Introduce a substitution $\eta = t - \tau, d\eta = dt$. Then

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right\} &= \int_{\eta=-\tau}^\infty \int_{\tau=0}^\infty f_1(\tau)f_2(\eta)e^{-s(\eta+\tau)}d\tau d\eta \\
&= \int_0^\infty f_1(\tau)e^{-s\tau}d\tau \int_0^\infty f_2(\eta)e^{-s\eta}d\eta \\
&= F_1(s)F_2(s)
\end{aligned}$$

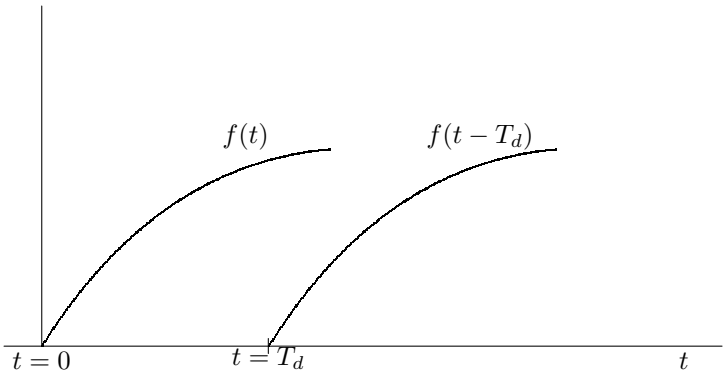

Figure 3.1.2: An original and delayed function.

Final Value Theorem

The asymptotic value of $f(t), t \to \infty$ can be found (if $\lim_{t\to\infty} f(t)$ exists) as

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s=0}[sF(s)] \tag{3.1.24}$$

Proof : To prove the above equation we use the relation for the transform of a derivative (3.1.18)

$$\int_0^\infty \frac{f(t)}{dt} e^{-st} dt = sF(s) - f(0)$$

and taking the limit as $s \to 0$

$$\begin{aligned} \int_0^\infty \frac{f(t)}{dt} \lim_{s\to 0} e^{-st} dt &= \lim_{s\to 0}[sF(s) - f(0)] \\ \lim_{t\to\infty} f(t) - f(0) &= \lim_{s\to 0}[sF(s)] - f(0) \\ \lim_{t\to\infty} f(t) &= \lim_{s\to 0}[sF(s)] \end{aligned}$$

Initial Value Theorem

It can be proven that an initial value of a function can be calculated as

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}[sF(s)] \tag{3.1.25}$$

Time Delay

Time delays are phenomena commonly encountered in chemical and food processes and occur in mass transfer processes. Time delays exist implicitly in distributed parameter processes and explicitly in pure mass transport through piping. A typical example are some types of automatic gas analysers that are connected to a process via piping used for transport of analysed media. In this case, *time delay* is defined as time required for transport of analysed media from the process into the analyser.

Consider a function $f(t)$ given for $0 \le t < \infty$, $f(t) \equiv 0$ for $t < 0$. If the Laplace transform of this function is $F(s)$ then

$$\mathcal{L}\{f(t - T_d)\} = e^{-T_d s} F(s) \tag{3.1.26}$$

where T_d is a time delay.

Proof : The relation between functions $f(t)$ and $f(t - T_d)$ is shown in Fig. 3.1.2.

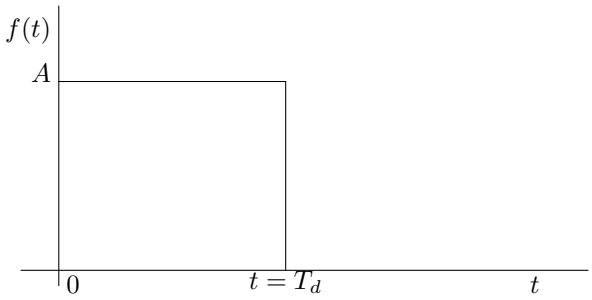

Figure 3.1.3: A rectangular pulse function.

Applying the definition integral to the function $f(t-T_d)$ yields

$$\begin{aligned}\mathcal{L}\{f(t-T_d)\} &= \int_0^\infty f(t-T_d)e^{-st}dt \\ &= e^{-sT_d}\int_0^\infty f(t-T_d)e^{-s(t-T_d)}d(t-T_d)\end{aligned}$$

because $dt = d(t-T_d)$. Denoting $\tau = t - T_d$ follows

$$\begin{aligned}\mathcal{L}\{f(t-T_d)\} &= e^{-sT_d}\int_0^\infty f(\tau)e^{-s\tau}d\tau \\ &= e^{-sT_d}F(s)\end{aligned}$$

Unit Impulse Function

Unit impulse function plays a fundamental role in control analysis and synthesis. Although the derivation of its Laplace transform logically falls into the section dealing with elementary functions, it can be derived only with knowledge of the Laplace transform of delayed function.

Consider a function $f(t) = A1(t) - A1(t-T_d)$ illustrated in Fig. 3.1.3. The Laplace transform of this function is

$$\begin{aligned}\mathcal{L}\{A1(t) - A1(t-T_d)\} &= \frac{A}{s} - \frac{Ae^{-sT_d}}{s} \\ &= \frac{A(1-e^{-sT_d})}{s}\end{aligned}$$

If we substitute in the function $f(t)$ for $A = 1/T_d$ and take the limit case for T_d approaching zero, we obtain a function that is zero except for the point $t = 0$ where its value is infinity. The area of the pulse function in Fig. 3.1.3 is equal to one. This is also the way of defining a function usually denoted by $\delta(t)$ and for which follows

$$\int_{-\infty}^{\infty}\delta(t)dt = 1 \tag{3.1.27}$$

It is called the *unit impulse function* or the *Dirac delta function.*

The Laplace transform of the unit impulse function is

$$\mathcal{L}\{\delta(t)\} = 1 \tag{3.1.28}$$

Proof :

$$\mathcal{L}\{\delta(t)\} = \lim_{T_d\to 0}\frac{1-e^{-sT_d}}{T_d s}$$

The limit in the above equation can easily be found by application of L'Hospital's rule. Taking derivatives with respect to T_d of both numerator and denominator,

$$\mathcal{L}\{\delta(t)\} = \lim_{T_d \to 0} \frac{se^{-sT_d}}{s} = 1$$

The unit impulse function is used as an idealised input variable in investigations of dynamical properties of processes.

3.1.4 Inverse Laplace Transform

When solving differential equations using the Laplace transform technique, the inverse Laplace transform can often be obtained from Table 3.1.1. However, a general function may not exactly match any of the entries in the table. Hence, a more general procedure is required. Every function can be factored as a sum of simpler functions whose Laplace transforms are in the table:

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s) \tag{3.1.29}$$

Then the original solution can be found as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_n(t) \tag{3.1.30}$$

where $f_i(t) = \mathcal{L}^{-1}\{F_i(s)\}, i = 1, \ldots, n$.

The function $F(s)$ is usually given as a rational function

$$F(s) = \frac{M(s)}{N(s)} \tag{3.1.31}$$

where

$M(s) = m_0 + m_1 s + \cdots + m_m s^m$ - numerator polynomial,

$N(s) = n_0 + n_1 s + \cdots + n_n s^n$ - denominator polynomial.

If $M(s)$ is a polynomial of a lower degree than $N(s)$, the function (3.1.31) is called *strictly proper rational function*. Otherwise, it is nonstrictly proper and can be written as a sum of some polynomial $T(s)$ and some strictly proper rational function of the form

$$\frac{M(s)}{N(s)} = T(s) + \frac{Z(s)}{N(s)} \tag{3.1.32}$$

Any strictly proper rational function can be written as a sum of strictly proper rational functions called *partial fractions* and the method of obtaining the partial fractions is called *partial fraction expansion*.

An intermediate step in partial fraction expansion is to find roots of the $N(s)$ polynomial. We can distinguish two cases when $N(s)$ has:

1. n different roots,
2. multiple roots.

Different Roots

If the denominator of (3.1.31) has the roots $s_1, \ldots, s_n$, the the function $F(s)$ can be written as

$$F(s) = \frac{M(s)}{n_n(s - s_1)(s - s_2)\ldots(s - s_n)} \tag{3.1.33}$$

Expansion of $F(s)$ into partial fractions yields

$$F(s) = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \cdots + \frac{K_n}{s - s_n} \tag{3.1.34}$$

and the original $f(t)$ is

$$f(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + \cdots + K_n e^{s_n t} \tag{3.1.35}$$

Note that if $N(s)$ has complex roots $s_{1,2} = a \pm jb$, then for $F(s)$ follows

$$F(s) = \frac{K_1}{s-(a+jb)} + \frac{K_2}{s-(a-jb)} + \cdots \tag{3.1.36}$$

$$= \frac{\beta_0 + \beta_1 s}{\alpha_0 + \alpha_1 s + s^2} + \cdots \tag{3.1.37}$$

The original function corresponding to this term can be found by an inverse Laplace transform using the combination of trigonometric entries in Table 3.1.1 (see example 3.1.3b).

Multiple Roots

If a root s_1 of the polynomial $N(s)$ occurs k-times, then the function $F(s)$ must be factored as

$$F(s) = \frac{K_1}{s-s_1} + \frac{K_2}{(s-s_1)^2} + \cdots + \frac{K_k}{(s-s_1)^k} + \cdots \tag{3.1.38}$$

and the corresponding original $f(t)$ can be found from Table 3.1.1.

3.1.5 Solution of Linear Differential Equations by Laplace Transform Techniques

Linear differential equations are solved by the means of the Laplace transform very simply with the following procedure:

1. Take Laplace transform of the differential equation,
2. solve the resulting algebraic equation,
3. find the inverse of the transformed output variable.

Example 3.1.1: *Solution of the 1st order ODE with zero initial condition*

Consider the heat exchanger shown in Fig. 2.2.3. The state equation is of the form

$$\frac{d\vartheta(t)}{dt} = -\frac{1}{T_1}\vartheta(t) + \frac{Z_1}{T_1}\vartheta_p(t) + \frac{Z_2}{T_1}\vartheta_v(t)$$

The exchanger is in a steady-state if $d\vartheta(t)/dt = 0$. Let the steady-state temperatures be given as $\vartheta_p^s, \vartheta_v^s, \vartheta^s$. Introduce deviation variables

$$\begin{aligned} x_1(t) &= \vartheta(t) - \vartheta^s \\ u_1(t) &= \vartheta_p(t) - \vartheta_p^s \\ r_1(t) &= \vartheta_v(t) - \vartheta_v^s \end{aligned}$$

then the state equation is

$$\frac{dx_1(t)}{dt} = -\frac{1}{T_1}x_1(t) + \frac{Z_1}{T_1}u_1(t) + \frac{Z_2}{T_1}r_1(t)$$

The output equation if temperature ϑ is measured is

$$y_1(t) = x_1(t)$$

so the differential equation describing the heat exchanger is

$$\frac{dy_1(t)}{dt} = -\frac{1}{T_1}y_1(t) + \frac{Z_1}{T_1}u_1(t) + \frac{Z_2}{T_1}r_1(t)$$

Let us assume that the exchanger is up to time t in the steady-state, hence

$$y_1(0) = 0, u_1(0) = 0, r_1(0) = 0 \quad \text{for} \quad t < 0$$

Let us assume that at time $t = 0$ begins the input $u_1(t)$ to change as a function of time $u_1(t) = Z_u e^{-t/T_u}$. The question is the behaviour of $y_1(t), t \geq 0$. From a pure mathematical point of view this is equivalent to the solution of a differential equation

$$T_1 \frac{dy_1(t)}{dt} + y_1(t) = Z_1 Z_u e^{-t/T_u}$$

with initial condition $y_1(0) = 0$. The first step is the Laplace transform of this equation which yields

$$T_1 \mathcal{L}\left\{\frac{dy_1(t)}{dt}\right\} + \mathcal{L}\{y_1(t)\} = Z_1 Z_u \mathcal{L}\left\{e^{-t/T_u}\right\}$$

$$T_1 s Y_1(s) + Y_1(s) = Z_1 Z_u \frac{T_u}{T_u s + 1}$$

Solution of this equation for $Y_1(s)$ is

$$Y_1(s) = \frac{Z_1 Z_u T_u}{(T_1 s + 1)(T_u s + 1)}$$

The right hand side of these equations can be factored as

$$Y_1(s) = \frac{A}{T_1 s + 1} + \frac{B}{T_u s + 1} = \frac{Z_1 Z_u T_u}{T_1 - T_u}\left(\frac{T_1}{T_1 s + 1} - \frac{T_u}{T_u s + 1}\right)$$

The inverse Laplace transform can be calculated using Table 3.1.1 and is given as

$$y_1(t) = \frac{Z_1 Z_u T_u}{T_1 - T_u}\left(e^{-\frac{t}{T_1}} - e^{-\frac{t}{T_u}}\right)$$

Example 3.1.2: *Solution of the 1st order ODE with a nonzero initial condition*

Consider the previous example but with the conditions $y_1(0) = y_{10}$ and $u_1(t) = 0$ for $t \geq 0$. This is mathematically equivalent to the differential equation

$$T_1 \frac{dy_1(t)}{dt} + y_1(t) = 0, \quad y_1(0) = y_{10}$$

Taking the Laplace transform, term by term using Table 3.1.1 :

$$T_1 \mathcal{L}\left\{\frac{dy_1(t)}{dt}\right\} + \mathcal{L}\{y_1(t)\} = 0$$

$$T_1[sY_1(s) - y_1(0)] + Y_1(s) = 0$$

Rearranging and factoring out $Y_1(s)$, we obtain

$$Y_1(s) = \frac{y_{10}}{T_1 s + 1}$$

Now we can take the inverse Laplace transform and obtain

$$y_1(t) = \frac{y_{10}}{T_1} e^{-\frac{t}{T_1}}.$$

Example 3.1.3: *Solution of the 2nd order ODE*

a) Consider a second order differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2u(t)$$

and assume zero initial conditions $y(0) = \dot{y}(0) = 0$. This case frequently occurs for process models with deviation variables that are up to time $t = 0$ in a steady-state. Let us find the solution of this differential equation for unit step function $u(t) = 1(t)$.

After taking the Laplace transform, the differential equation gives

$$(s^2 + 3s + 2)Y(s) = 2\frac{1}{s}$$

$$Y(s) = \frac{2}{s(s^2 + 3s + 2)}$$

$$Y(s) = \frac{2}{s(s + 1)(s + 2)}$$

The denominator roots are all different and partial fraction expansion is of the form

$$\frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

The coefficients K_1, K_2, K_3 can be calculated by multiplying both sides of this equation with the denominator and equating the coefficients of each power of s:

$$\left.\begin{array}{lcl} s^2 & : & K_1 + K_2 + K_3 = 0 \\ s^1 & : & 3K_1 + 2K_2 + K_3 = 0 \\ s^0 & : & 2K_1 = 2 \end{array}\right\} K_1 = 1, K_2 = -2, K_3 = 1$$

The solution of the differential equation can now be read from Table 3.1.1:

$$y(t) = 1 - 2e^{-t} + e^{-2t}$$

b) Consider a second order differential equation

$$\ddot{y}(t) + 2\dot{y}(t) + 5y(t) = 2u(t)$$

and assume zero initial conditions $y(0) = \dot{y}(0) = 0$. Find the solution of this differential equation for unit step function $u(t) = 1(t)$.

Take the Laplace transform

$$(s^2 + 2s + 5)Y(s) = 2\frac{1}{s}$$

$$Y(s) = \frac{2}{s(s^2 + 2s + 5)}$$

The denominator has one real root and two complex conjugate roots, hence the partial fraction expansion is of the form

$$\frac{2}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2s + 5} = \frac{2}{5}\left(\frac{1}{s} - \frac{2 + s}{s^2 + 2s + 5}\right)$$

where the coefficients K_1, K_2, K_3 have been found as in the previous example. The second term on the right side of the previous equation is not in Table 3.1.1 but can be manipulated to obtain a sum of trigonometric terms. Firstly, the denominator is rearranged by completing the squares to $(s + 1)^2 + 4$ and the numerator is then rewritten to match numerators of trigonometric expressions. Hence

$$\begin{aligned} \frac{1 + 2s}{s^2 + 2s + 5} &= \frac{2 + s}{(s + 1)^2 + 4} = \frac{(s + 1) - \frac{1}{2}2}{(s + 1)^2 + 4} \\ &= \frac{s + 1}{(s + 1)^2 + 4} - \frac{1}{2}\frac{2}{(s + 1)^2 + 4} \end{aligned}$$

and $Y(s)$ can be written as

$$Y(s) = \frac{2}{5}\left(\frac{1}{s} - \frac{s + 1}{(s + 1)^2 + 4} - \frac{1}{2}\frac{2}{(s + 1)^2 + 4}\right)$$

Taking the inverse Laplace transform, term by term, yields

$$Y(s) = \frac{2}{5}\left(1 - e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t\right)$$

c) Consider a second order differential equation

$$\ddot{y}(t) + 2\dot{y}(t) + 1y(t) = 2, \quad y(0) = \dot{y}(0) = 0$$

Take the Laplace transform

$$(s^2 + 2s + 1)Y(s) = 2\frac{1}{s}$$

$$Y(s) = \frac{2}{s(s^2 + 2s + 1)}$$

$$Y(s) = \frac{2}{s(s + 1)^2}$$

The denominator has one single root $s_1 = 0$ and one double root $s_{2,3} = -1$. The partial fraction expansion is of the form

$$\frac{2}{s(s+1)^2} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{(s+1)^2}$$

and the solution from Table 3.1.1 reads

$$y(t) = 2 - 2(1-t)e^{-t}$$

3.2 State-Space Process Models

Investigation of processes as dynamical systems is based on *theoretical state-space balance equations*. State-space variables may generally be abstract. If a model of a process is described by state-space equations, we speak about *state-space representation*. This representation includes a description of linear as well as nonlinear models. In this section we introduce the concept of state, solution of state-space equations, canonical representations and transformations, and some properties of systems.

3.2.1 Concept of State

Consider a continuous-time MIMO system with m input variables and r output variables. The relation between input and output variables can be expressed as (see also Section 2.3)

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{3.2.1}$$

$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{3.2.2}$$

where $\boldsymbol{x}(t)$ is a vector of state-space variables, $\boldsymbol{u}(t)$ is a vector of input variables, and $\boldsymbol{y}(t)$ is a vector of output variables.

The *state of a system* at time t_0 is a minimum amount of information which (in the absence of external excitation) is sufficient to determine uniquely the evolution of the system for $t \geq t_0$.

If the vector $\boldsymbol{x}(t_0)$ and the vector of input variables $\boldsymbol{u}(t)$ for $t > t_0$ are known then this knowledge suffices to determine $\boldsymbol{y}(t)$, $t > t_0$, thus

$$\boldsymbol{y}(t_0, t] = \boldsymbol{y}\{\boldsymbol{x}(t_0), \boldsymbol{u}(t_0, t]\} \tag{3.2.3}$$

where $\boldsymbol{u}(t_0, t], \boldsymbol{y}(t_0, t]$ are vectors of input and output variables over the interval $(t_0, t]$ respectively.

The above equation is equivalent to

$$\boldsymbol{x}(t_0, t] = \boldsymbol{x}\{\boldsymbol{x}(t_0), \boldsymbol{u}(t_0, t]\} \tag{3.2.4}$$

Therefore, the knowledge about the states at $t = t_0$ removes the necessity to know the past behavior of the system in order to forecast its future and the future evolution of states is dependent only on its present state and future inputs.

This definition of state will be clearer when we introduce a solution of state-space equation for the general functions of input variables.

3.2.2 Solution of State-Space Equations

Solution of state-space equations will be specified only for linear systems with constant coefficients with the aid of Laplace transform techniques. Firstly, a simple example will be given and then it will be generalised.

Example 3.2.1: *Mixing process - solution of state-space equations*

Consider a process of mixing shown in Fig. 3.2.1 with mathematical model described by the equation

$$V\frac{dc_1}{dt} = qc_0 - qc_1$$

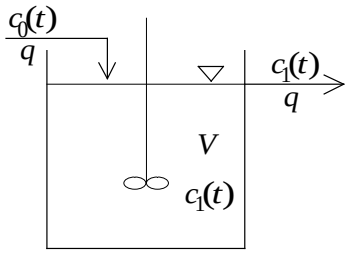

Figure 3.2.1: A mixing process.

where c_0, c_1 are concentrations with dimensions mass/volume, V is a constant volume of the vessel, and q is a constant volumetric flow rate.

In the steady-state holds

$$qc_0^s - qc_1^s = 0$$

Introduce deviation variables

$$\begin{aligned} x &= c_1 - c_1^s \\ u &= c_0 - c_0^s \end{aligned}$$

and define the process output variable $y = x$. Then the process state-space equations are of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + bu \\ y &= cx \end{aligned}$$

where $a = -1/T_1, b = 1/T_1, c = 1$. $T_1 = V/q$ is the process time constant.

Assume that the system is at $t_0 = 0$ in the state $x(0) = x_0$. Then the time solution can be calculated by applying the Laplace transform:

$$sX(s) - x(0) = aX(s) + bU(s)$$

$$X(s) = \frac{1}{s-a}x(0) + \frac{b}{s-a}U(s)$$

The time domain description $x(t)$ can be read from Table 3.1.1 for the first term and from the convolution transformation for the second term and is given as

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

and for $y(t)$

$$y(t) = ce^{at}x(0) + c\int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

After substituting for the constants yields for $y(t)$

$$y(t) = e^{-\frac{q}{V}t}x(0) + \frac{V}{q}\int_0^t e^{-\frac{q}{V}(t-\tau)}u(\tau)d\tau$$

Solution of State-Space Equations for the Multivariable Case

The solution for the multivariable case is analogous as in the previous example. Each state equation is transformed with the Laplace transform applied and transformed back into the time domain. The procedure is simplified if we use matrix notation.

Consider state-space equations

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{3.2.5}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{3.2.6}$$

Taking the Laplace transform yields

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \tag{3.2.7}$$

$$\boldsymbol{X}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{x}_0 + (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s) \tag{3.2.8}$$

and after the inverse transformation for $\boldsymbol{x}(t), \boldsymbol{y}(t)$ hold

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{U}(\tau)d\tau \tag{3.2.9}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \boldsymbol{C}\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{U}(\tau)d\tau \tag{3.2.10}$$

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right\} \tag{3.2.11}$$

The equation (3.2.10) shows some important properties and features. Its solution consists of two parts: initial conditions term (zero-input response) and input term dependent on $\boldsymbol{u}(t)$ (zero-state response).

The solution of (3.2.5) for *free system* ($\boldsymbol{u}(t) = \boldsymbol{0}$) is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) \tag{3.2.12}$$

and the exponential term is defined as

$$e^{\boldsymbol{A}t} = \sum_{i=1}^{\infty} \boldsymbol{A}^i \frac{t^i}{i!} \tag{3.2.13}$$

The matrix

$$\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right\} \tag{3.2.14}$$

is called the *state transition matrix*, (*fundamental matrix, matrix exponential*). The solution of (3.2.5) for $u(t)$ is then

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t - t_0)\boldsymbol{x}(t_0) \tag{3.2.15}$$

The matrix exponential satisfies the following identities:

$$\boldsymbol{x}(t_0) = \boldsymbol{\Phi}(t_0 - t_0)\boldsymbol{x}(t_0) \quad \Rightarrow \quad \boldsymbol{\Phi}(0) = \boldsymbol{I} \tag{3.2.16}$$

$$\boldsymbol{x}(t_2) = \boldsymbol{\Phi}(t_2 - t_1)\boldsymbol{x}(t_1) \tag{3.2.17}$$

$$\boldsymbol{x}(t_2) = \boldsymbol{\Phi}(t_2 - t_1)\boldsymbol{\Phi}(t_1 - t_0)\boldsymbol{x}(t_0) \tag{3.2.18}$$

The equation (3.2.14) shows that the *system matrix* $\boldsymbol{A}$ plays a crucial role in the solution of state-space equations. Elements of this matrix depend on coefficients of mass and heat transfer, activation energies, flow rates, etc. Solution of the state-space equations is therefore influenced by physical and chemical properties of processes.

The solution of state-space equations depends on roots of the *characteristic equation*

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = 0 \tag{3.2.19}$$

This will be clarified from the next example

Example 3.2.2: *Calculation of matrix exponential*

Consider a matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$$

The matrix exponential corresponding to $\boldsymbol{A}$ is defined in equation (3.2.14) as

$$\begin{aligned}
\boldsymbol{\Phi}(t) &= \mathcal{L}^{-1}\left\{\left[s\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}-1&-1\\0&-2\end{pmatrix}\right]^{-1}\right\}\\
&= \mathcal{L}^{-1}\left\{\begin{bmatrix}s+1&1\\0&s+2\end{bmatrix}^{-1}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{\det\begin{pmatrix}s+1&1\\0&s+2\end{pmatrix}}\begin{pmatrix}s+2&-1\\0&s+1\end{pmatrix}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\begin{pmatrix}s+2&-1\\0&s+1\end{pmatrix}\right\}\\
&= \mathcal{L}^{-1}\left\{\begin{pmatrix}\frac{1}{s+1}&\frac{-1}{(s+1)(s+2)}\\0&\frac{1}{s+2}\end{pmatrix}\right\}
\end{aligned}$$

The elements of $\boldsymbol{\Phi}(t)$ are found from Table 3.1.1 as

$$\boldsymbol{\Phi}(t)=\begin{pmatrix}e^{-t}&e^{-2t}-e^{-t}\\0&e^{-2t}\end{pmatrix}$$

3.2.3 Canonical Transformation

Eigenvalues of $\boldsymbol{A}$, $\lambda_1,\ldots,\lambda_n$ are given as solutions of the equation

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I})=0 \tag{3.2.20}$$

If the eigenvalues of $\boldsymbol{A}$ are distinct, then a nonsingular matrix $\boldsymbol{T}$ exists, such that

$$\boldsymbol{\Lambda}=\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} \tag{3.2.21}$$

is an diagonal matrix of the form

$$\boldsymbol{\Lambda}=\begin{pmatrix}\lambda_1&0&\ldots&0\\0&\lambda_2&\ldots&0\\\vdots&&&\vdots\\0&0&\ldots&\lambda_n\end{pmatrix} \tag{3.2.22}$$

The *canonical transformation* (3.2.21) can be used for direct calculation of $e^{-\boldsymbol{A}t}$. Substituting $\boldsymbol{A}$ from (3.2.21) into the equation

$$\frac{d\boldsymbol{x}(t)}{dt}=\boldsymbol{A}\boldsymbol{x}(t),\quad \boldsymbol{x}(0)=\boldsymbol{I} \tag{3.2.23}$$

gives

$$\frac{d(\boldsymbol{T}^{-1}\boldsymbol{x})}{dt}=\boldsymbol{\Lambda}\boldsymbol{T}^{-1}\boldsymbol{x},\quad \boldsymbol{T}^{-1}\boldsymbol{x}(0)=\boldsymbol{T}^{-1} \tag{3.2.24}$$

Solution of the above equation is

$$\boldsymbol{T}^{-1}\boldsymbol{x}=e^{-\boldsymbol{\Lambda}t}\boldsymbol{T}^{-1} \tag{3.2.25}$$

or

$$\boldsymbol{x}=\boldsymbol{T}e^{-\boldsymbol{\Lambda}t}\boldsymbol{T}^{-1} \tag{3.2.26}$$

and therefore

$$\boldsymbol{\Phi}(t)=\boldsymbol{T}e^{-\boldsymbol{\Lambda}t}\boldsymbol{T}^{-1} \tag{3.2.27}$$

where

$$e^{\mathbf{\Lambda}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix} \tag{3.2.28}$$

3.2.4 Stability, Controllability, and Observability of Continuous-Time Systems

Stability, controllability, and observability are basic properties of systems closely related to state-space models. These properties can be utilised for system analysis and synthesis.

Stability of Continuous-Time Systems

An important aspect of system behaviour is *stability*. System can be defined as stable if its response to bounded inputs is also bounded. The concept of stability is of great practical interest as nonstable control systems are unacceptable. Stability can also be determined without an analytical solution of process equations which is important for nonlinear systems.

Consider a system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3.2.29}$$

Such a system is called *forced* as the vector of input variables $\boldsymbol{u}(t)$ appears on the right hand side of the equation. However, stability can be studied on *free (zero-input)* systems given by the equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t), t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3.2.30}$$

$\boldsymbol{u}(t)$ does not appear in the previous equation, which is equivalent to processes with constant inputs. If time t appears explicitly as an argument in process dynamics equations we speak about *nonautonomous* system, otherwise about *autonomous* system.

In our discussion about stability of (3.2.30) we will consider stability of motion of $\boldsymbol{x}^s(t)$ that corresponds to constant values of input variables. Let us for this purpose investigate any solution (motion) of the forced system $\boldsymbol{x}(t)$ that is at $t = 0$ in the neighbourhood of $\boldsymbol{x}^s(t)$. The problem of stability is closely connected to the question if for $t \geq 0$ remains $\boldsymbol{x}(t)$ in the neighbourhood of $\boldsymbol{x}^s(t)$. Let us define deviation

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^s(t) \tag{3.2.31}$$

then,

$$\begin{aligned} \frac{d\tilde{\boldsymbol{x}}(t)}{dt} + \frac{d\boldsymbol{x}^s(t)}{dt} &= \boldsymbol{f}(\tilde{\boldsymbol{x}}(t) + \boldsymbol{x}^s(t), \boldsymbol{u}(t), t) \\ \frac{d\tilde{\boldsymbol{x}}(t)}{dt} &= \boldsymbol{f}(\tilde{\boldsymbol{x}}(t) + \boldsymbol{x}^s(t), \boldsymbol{u}(t), t) - \boldsymbol{f}(\boldsymbol{x}^s(t), t) \\ \frac{d\tilde{\boldsymbol{x}}(t)}{dt} &= \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}(t), \boldsymbol{u}(t), t) \end{aligned} \tag{3.2.32}$$

The solution $\boldsymbol{x}^s(t)$ in (3.2.32) corresponds for all $t > 0$ to relation $\tilde{\boldsymbol{x}}(t) = \boldsymbol{0}$ and $\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{0}$. Therefore the state $\tilde{\boldsymbol{x}}(t) = \boldsymbol{0}$ is called *equilibrium state* of the system described by (3.2.32). This equation can always be constructed and stability of equilibrium point can be interpreted as stability in the beginning of the state-space.

Stability theorems given below are valid for nonautonomous systems. However, such systems are very rare in common processes. In connection to the above ideas about equilibrium point we will restrict our discussion to systems given by

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{3.2.33}$$

The equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ of this system obeys the relation

$$\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0} \tag{3.2.34}$$

as $d\boldsymbol{x}/dt = \boldsymbol{0}$

We assume that the solution of the equation (3.2.33) exists and is unique.

Stability can be intuitively defined as follows: If $\boldsymbol{x}^e = \boldsymbol{0}$ is the equilibrium point of the system (3.2.33), then we may say that $\boldsymbol{x}^e = \boldsymbol{0}$ is the stable equilibrium point if the solution of (3.2.33) $\boldsymbol{x}(t) = \boldsymbol{x}[\boldsymbol{x}(t_0), t]$ that begins in some state $\boldsymbol{x}(t_0)$ "close" to the equilibrium point $\boldsymbol{x}^e = \boldsymbol{0}$ remains in the neighbourhood of $\boldsymbol{x}^e = \boldsymbol{0}$ or the solution approaches this state.

The equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ is unstable if the solution $\boldsymbol{x}(t) = \boldsymbol{x}[\boldsymbol{x}(t_0), t]$ that begins in some state $\boldsymbol{x}(t_0)$ diverges from the neighbourhood of $\boldsymbol{x}^e = \boldsymbol{0}$.

Next, we state the definitions of stability from Lyapunov asymptotic stability and asymptotic stability in large.

Lyapunov stability: The system (3.2.33) is stable in the equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ if for any given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $\boldsymbol{x}(t_0)$ such that $\|\boldsymbol{x}(t_0)\| \leq \delta$ implies $\|\boldsymbol{x}[\boldsymbol{x}(t_0), t]\| \leq \varepsilon$ for all $t \geq 0$.

Asymptotic (internal) stability: The system (3.2.33) is asymptotically stable in the equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ if it is Lyapunov stable and if all $\boldsymbol{x}(t) = \boldsymbol{x}[\boldsymbol{x}(t_0), t]$ that begin sufficiently close to the equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ satisfy the condition $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = 0$.

Asymptotic stability in large: The system (3.2.33) is asymptotically stable in large in the equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ if it is asymptotic stable for all initial states $\boldsymbol{x}(t_0)$.

In the above definitions, the notation $\|\boldsymbol{x}\|$ has been used for the Euclidean norm of a vector $\boldsymbol{x}(t)$ that is defined as the distance of the point given by the coordinates of $\boldsymbol{x}$ from equilibrium point $\boldsymbol{x}^e = \boldsymbol{0}$ and given as $\|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{1/2}$.

Note 3.2.1 Norm of a vector *is some function transforming any vector* $\boldsymbol{x} \in R^n$ *to some real number* $\|\boldsymbol{x}\|$ *with the following properties:*

1. $\|\boldsymbol{x}\| \geq 0$,
2. $\|\boldsymbol{x}\| = 0$ *iff* $\boldsymbol{x} = \boldsymbol{0}$,
3. $\|k\boldsymbol{x}\| = |k|\,\|\boldsymbol{x}\|$ *for any* k,
4. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

Some examples of norms are $\|\boldsymbol{x}\| = (\boldsymbol{x}^T\boldsymbol{x})^{1/2}$, $\|\boldsymbol{x}\| = \sum_{i=1}^{n} |x_i|$, $\|\boldsymbol{x}\| = \max |x_i|$. *It can be proven that all these norms satisfy properties 1-4.*

Example 3.2.3: *Physical interpretation – U-tube*

Consider a U-tube as an example of the second order system. Mathematical model of this system can be derived from Fig. 3.2.2 considering the equilibrium of forces.

We assume that if specific pressure changes, the force with which the liquid flow is inhibited, is proportional to the speed of the liquid. Furthermore, we assume that the second Newton law is applicable. The following equation holds for the equilibrium of forces

$$Fp_v = 2Fg\rho h + kF\frac{dh}{dt} + FL\rho\frac{d^2h}{dt^2}$$

or

$$\frac{d^2h}{dt^2} + \frac{k}{L\rho}\frac{dh}{dt} + \frac{2g}{L}h = \frac{1}{L\rho}p_v$$

where

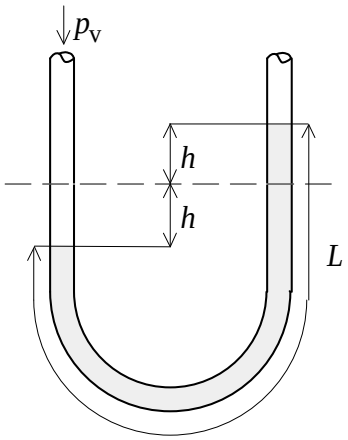

Figure 3.2.2: A U-tube.

F - inner cross-sectional area of tube,

k - coefficient,

p_v - specific pressure,

g - acceleration of gravity,

ρ - density of liquid.

If the input is zero then the mathematical model is of the form

$$\frac{d^2x_1}{dt^2} + a_1\frac{dx_1}{dt} + a_0x_1 = 0$$

where $x_1 = h - h^s, a_0 = 2g/L, a_1 = k/L\rho$. The speed of liquid flow will be denoted by $x_2 = dx_1/dt$. If x_1, x_2 are elements of state vector $\boldsymbol{x}$ then the dynamics of the U-tube is given as

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -a_0x_1 - a_1x_2\end{aligned}$$

If we consider $a_0 = 1, a_1 = 1, \boldsymbol{x}(0) = (1,0)^T$ then the solution of the differential equations is shown in Fig. 3.2.3. At any time instant the total system energy is given as a sum of kinetic and potential energies of liquid

$$V(x_1, x_2) = FL\rho\frac{x_2^2}{2} + \int_0^{x_1} 2Fg\rho x dx$$

Energy V satisfies the following conditions: $V(\boldsymbol{x}) > 0, \boldsymbol{x} \neq \boldsymbol{0}$ and $V(\boldsymbol{0}) = 0$.

These conditions show that the sum of kinetic and potential energies is positive with the exception when liquid is in the equilibrium state $\boldsymbol{x}^e = \boldsymbol{0}$ when $dx_1/dt = dx_2/dt = 0$.

The change of V in time is given as

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt}\\ \frac{dV}{dt} &= 2Fg\rho x_1x_2 + FL\rho x_2\left(-\frac{2g}{L}x_1 - \frac{k}{L\rho}x_2\right)\\ \frac{dV}{dt} &= -Fkx_2^2\end{aligned}$$

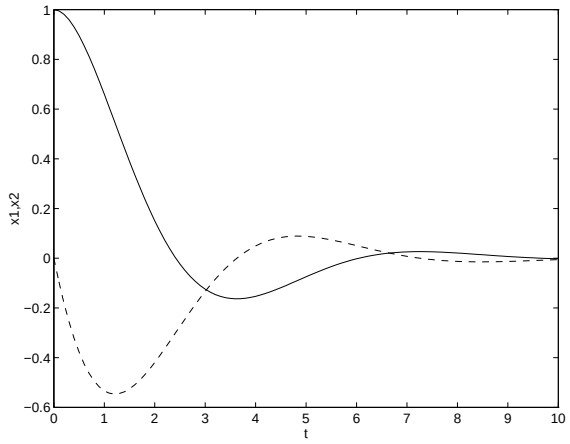

Figure 3.2.3: Time response of the U-tube for initial conditions $(1, 0)^T$.

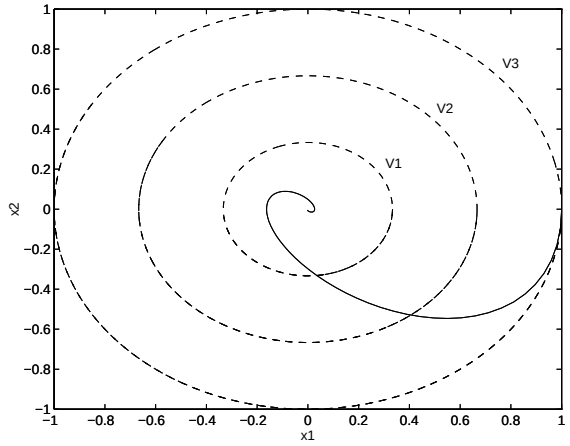

Figure 3.2.4: Constant energy curves and state trajectory of the U-tube in the state plane.

As $k > 0$, time derivative of V is always negative except if $x_2 = 0$ when $dV/dt = 0$ and hence V cannot increase. If $x_2 = 0$ the dynamics of the tube shows that

$$\frac{dx_2}{dt} = -\frac{2g}{L}x_1$$

is nonzero (except $\boldsymbol{x}^e = \mathbf{0}$). The system cannot remain in a nonequilibrium state for which $x_2 = 0$ and always reaches the equilibrium state which is stable. The sum of the energies V is given as

$$\begin{aligned} V(x_1, x_2) &= 2Fg\rho\frac{x_1^2}{2} + FL\rho\frac{x_2^2}{2} \\ V(x_1, x_2) &= \frac{F\rho}{2}(2gx_1^2 + Lx_2^2) \end{aligned}$$

Fig. 3.2.4 shows the state plane with curves of constant energy levels $V_1 < V_2 < V_3$ and state trajectory corresponding to Fig. 3.2.3 where x_1, x_2 are plotted as function of parameter t.

Conclusions about system behaviour and about state trajectory in the state plane can be generalised by general state-space. It is clear that some results about system properties can also be derived without analytical solution of state-space equations.

Stability theory of Lyapunov assumes the existence of the Lyapunov function $V(\boldsymbol{x})$. The continuous function $V(\boldsymbol{x})$ with continuous derivatives is called *positive definite* in some neighbourhood

Δ of state origin if

$$V(\mathbf{0}) = 0 \tag{3.2.35}$$

and

$$V(\boldsymbol{x}) > 0 \tag{3.2.36}$$

for all $\boldsymbol{x} \neq \mathbf{0}$ within Δ. If (3.2.36) is replaced by

$$V(\boldsymbol{x}) \geq 0 \tag{3.2.37}$$

for all $\boldsymbol{x} \in \Delta$ then $V(\boldsymbol{x})$ is *positive semidefinite*. Definitions of *negative definite* and *negative semidefinite* functions follow analogously.

Various definitions of stability for the system $d\boldsymbol{x}(t)/dt = \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ lead to the following theorems:

Stability in Lyapunov sense: If a positive definite function $V(\boldsymbol{x})$ can be chosen such that

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \boldsymbol{f}(\boldsymbol{x}) \leq 0 \tag{3.2.38}$$

then the system is stable in origin in the Lyapunov sense.

The function $V(\boldsymbol{x})$ satisfying this theorem is called the *Lyapunov function*.

Asymptotic stability: If a positive definite function $V(\boldsymbol{x})$ can be chosen such that

$$\frac{dV}{dt} = \left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \boldsymbol{f}(\boldsymbol{x}) < 0, \quad \boldsymbol{x} \neq \mathbf{0} \tag{3.2.39}$$

then the system is asymptotically stable in origin.

Asymptotic stability in large: If the conditions of asymptotic stability are satisfied for all $\boldsymbol{x}$ and if $V(\boldsymbol{x}) \to \infty$ for $\|\boldsymbol{x}\| \to \infty$ then the system is asymptotically stable by large in origin.

There is no general procedure for the construction of the Lyapunov function. If such a function exists then it is not unique. Often it is chosen in the form

$$V(\boldsymbol{x}) = \sum_{k=1}^{n} \sum_{r=1}^{n} K_{rk} x_k x_r \tag{3.2.40}$$

K_{rk} are real constants, $K_{rk} = K_{kr}$ so (3.2.40) can be written as

$$V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{K} \boldsymbol{x} \tag{3.2.41}$$

and $\boldsymbol{K}$ is symmetric matrix. $V(\boldsymbol{x})$ is positive definite if and only if the determinants

$$K_{11}, \begin{vmatrix} K_{11}, & K_{12} \\ K_{21}, & K_{22} \end{vmatrix}, \begin{vmatrix} K_{11}, & K_{12}, & K_{13} \\ K_{21}, & K_{22}, & K_{23} \\ K_{31}, & K_{32}, & K_{33} \end{vmatrix}, \ldots \tag{3.2.42}$$

are greater than zero.

Asymptotic stability of linear systems: Linear system

$$\frac{\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) \tag{3.2.43}$$

is *asymptotically stable* (in large) if and only if one of the following properties is valid:

1. *Lyapunov equation*

$$\boldsymbol{A}^T \boldsymbol{K} + \boldsymbol{K} \boldsymbol{A} = -\boldsymbol{\mu} \tag{3.2.44}$$

where $\boldsymbol{\mu}$ is any symmetric positive definite matrix, has a unique positive definite symmetric solution $\boldsymbol{K}$.

2. all eigenvalues of *system matrix* $\boldsymbol{A}$, i.e. all roots of *characteristic polynomial* $\det(s\boldsymbol{I}-\boldsymbol{A})$ have negative real parts.

Proof: We prove only the sufficient part of 1. Consider the Lyapunov function of the form

$$V(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x} \tag{3.2.45}$$

if $\boldsymbol{K}$ is a positive definite then

$$V(\boldsymbol{x}) > 0 \quad , \quad \boldsymbol{x} \neq \boldsymbol{0} \tag{3.2.46}$$

$$V(\boldsymbol{0}) = 0 \tag{3.2.47}$$

and for dV/dt holds

$$\frac{dV(\boldsymbol{x})}{dt} = \left(\frac{d\boldsymbol{x}}{dt}\right)^T \boldsymbol{K}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{K}\frac{d\boldsymbol{x}}{dt} \tag{3.2.48}$$

Substituting $d\boldsymbol{x}/dt$ from Eq. (3.2.43) yields

$$\frac{dV(\boldsymbol{x})}{dt} = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{K}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{K}\boldsymbol{A}\boldsymbol{x} \tag{3.2.49}$$

$$\frac{dV(\boldsymbol{x})}{dt} = \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{K} + \boldsymbol{K}\boldsymbol{A})\boldsymbol{x} \tag{3.2.50}$$

Applying (3.2.44) we get

$$\frac{dV(\boldsymbol{x})}{dt} = -\boldsymbol{x}^T\boldsymbol{\mu}\boldsymbol{x} \tag{3.2.51}$$

and because $\boldsymbol{\mu}$ is a positive definite matrix then

$$\frac{dV(\boldsymbol{x})}{dt} < 0 \tag{3.2.52}$$

for all $\boldsymbol{x} \neq \boldsymbol{0}$ and the system is asymptotically stable in origin. As the Lyapunov function can be written as

$$V(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 \tag{3.2.53}$$

and therefore

$$V(\boldsymbol{x}) \to \infty \quad \text{for } \|\boldsymbol{x}\| \to \infty \tag{3.2.54}$$

The corresponding norm is defined as $(\boldsymbol{x}^T\boldsymbol{K}\boldsymbol{x})^{1/2}$. It can easily be shown that $\boldsymbol{K}$ exists and all conditions of the theorem on asymptotic stability by large in origin are fulfilled. The second part of the proof - necessity - is much harder to prove.

The choice of $\boldsymbol{\mu}$ for computations is usually

$$\boldsymbol{\mu} = \boldsymbol{I} \tag{3.2.55}$$

Controllability of continuous systems

The concept of controllability together with observability is of fundamental importance in theory of automatic control.

Definition of controllability of linear system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \tag{3.2.56}$$

is as follows: A state $\boldsymbol{x}(t_0) \neq \boldsymbol{0}$ of the system (3.2.56) is *controllable* if the system can be driven from this state to state $\boldsymbol{x}(t_1) = \boldsymbol{0}$ by applying suitable $\boldsymbol{u}(t)$ within finite time $t_1 - t_0, t \in [t_0, t_1]$.

If every state is controllable then the system is *completely controllable*.

Definition of reachable of linear systems: A state $\boldsymbol{x}(t_1)$ of the system (3.2.56) is *reachable* if the system can be driven from the state $\boldsymbol{x}(t_0) = \boldsymbol{0}$ to $\boldsymbol{x}(t_1)$ by applying suitable $\boldsymbol{u}(t)$ within finite time $t_1 - t_0, t \in [t_0, t_1]$.

If every state is reachable then the system is *completely reachable*.

For linear systems with constant coefficients (linear time invariant systems) are all reachable states controllable and it is sufficient to speak about controllability. Often the definitions are simplified and we can speak that the system is completely controllable (shortly controllable) if there exists such $\boldsymbol{u}(t)$ that drives the system from the arbitrary initial state $\boldsymbol{x}(t_0)$ to the final state $\boldsymbol{x}(t_1)$ within a finite time $t_1 - t_0, t \in [t_0, t_1]$.

Theorem (Controllability of linear continuous systems with constant coefficients): The system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{3.2.57}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{3.2.58}$$

is completely controllable if and only if rank of *controllability matrix* $\boldsymbol{Q}_c$ is equal to n. $\boldsymbol{Q}_c[n \times nm]$ is defined as

$$\boldsymbol{Q}_c = (\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \boldsymbol{A}^2\boldsymbol{B} \ldots \boldsymbol{A}^{n-1}\boldsymbol{B}) \tag{3.2.59}$$

where n is the dimension of the vector $\boldsymbol{x}$ and m is the dimension of the vector $\boldsymbol{u}$.

Proof: We prove only the "if" part. Solution of the Eq. (3.2.57) with initial condition $\boldsymbol{x}(t_0)$ is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(t_0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{3.2.60}$$

For $t = t_1$ follows

$$\boldsymbol{x}(t_1) = e^{\boldsymbol{A}t_1}\boldsymbol{x}(t_0) + e^{\boldsymbol{A}t_1}\int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{3.2.61}$$

The function $e^{-\boldsymbol{A}\tau}$ can be rewritten with the aid of the Cayley-Hamilton theorem as

$$e^{-\boldsymbol{A}\tau} = k_0(\tau)\boldsymbol{I} + k_1(\tau)\boldsymbol{A} + k_2(\tau)\boldsymbol{A}^2 + \cdots + k_{n-1}(\tau)\boldsymbol{A}^{n-1} \tag{3.2.62}$$

Substituting for $e^{-\boldsymbol{A}\tau}$ from (3.2.62) into (3.2.61) yields

$$\begin{aligned}\boldsymbol{x}(t_1) &= e^{\boldsymbol{A}t_1}\boldsymbol{x}(t_0) + e^{\boldsymbol{A}t_1}\int_0^{t_1}\big(k_0(\tau)\boldsymbol{B} + k_1(\tau)\boldsymbol{A}\boldsymbol{B} + \\ &\quad + k_2(\tau)\boldsymbol{A}^2\boldsymbol{B} + \cdots + k_{n-1}(\tau)\boldsymbol{A}^{n-1}\boldsymbol{B}\big)\boldsymbol{u}(\tau)d\tau\end{aligned} \tag{3.2.63}$$

or

$$\begin{aligned}\boldsymbol{x}(t_1) &= e^{\boldsymbol{A}t_1}\boldsymbol{x}(t_0) + \\ &\quad + e^{\boldsymbol{A}t_1}\int_0^{t_1}(\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \boldsymbol{A}^2\boldsymbol{B} \ldots \boldsymbol{A}^{n-1}\boldsymbol{B}) \times \\ &\quad \times \begin{pmatrix} k_0(\tau)\boldsymbol{u}(\tau) \\ k_1(\tau)\boldsymbol{u}(\tau) \\ k_2(\tau)\boldsymbol{u}(\tau) \\ \vdots \\ k_{n-1}(\tau)\boldsymbol{u}(\tau) \end{pmatrix} d\tau\end{aligned} \tag{3.2.64}$$

Complete controllability means that for all $\boldsymbol{x}(t_0) \neq \boldsymbol{0}$ there exists a finite time $t_1 - t_0$ and suitable $\boldsymbol{u}(t)$ such that

$$-\boldsymbol{x}(t_0) = (\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \boldsymbol{A}^2\boldsymbol{B} \ldots \boldsymbol{A}^{n-1}\boldsymbol{B})\int_0^{t_1}\begin{pmatrix} k_0(\tau)\boldsymbol{u}(\tau) \\ k_1(\tau)\boldsymbol{u}(\tau) \\ k_2(\tau)\boldsymbol{u}(\tau) \\ \vdots \\ k_{n-1}(\tau)\boldsymbol{u}(\tau) \end{pmatrix} d\tau \tag{3.2.65}$$

From this equation follows that any vector $-\boldsymbol{x}(t_0)$ can be expressed as a linear combination of the columns of $\boldsymbol{Q}_c$. The system is controllable if the integrand in (3.2.64) allows the influence of $\boldsymbol{u}$ to reach all the states $\boldsymbol{x}$. Hence complete controllability is equivalent to the condition of rank of $\boldsymbol{Q}_c$ being equal to n. The controllability theorem enables a simple check of system controllability with regard to $\boldsymbol{x}$. The test with regard to $\boldsymbol{y}$ can be derived analogously and is given below.

Theorem (Output controllability of linear systems with constant coefficients): The system output $\boldsymbol{y}$ of (3.2.57), (3.2.58) is completely controllable if and only if the rank of *controllability matrix* $\boldsymbol{Q}_c^y[r \times nm]$ is equal to r (with r being dimension of the output vector) where

$$\boldsymbol{Q}_c^y = (\boldsymbol{CB}\ \boldsymbol{CAB}\ \boldsymbol{CA}^2\boldsymbol{B} \ldots \boldsymbol{CA}^{n-1}\boldsymbol{B}) \tag{3.2.66}$$

We note that the controllability conditions are also valid for linear systems with time-varying coefficients if $\boldsymbol{A}(t), \boldsymbol{B}(t)$ are known functions of time. The conditions for nonlinear systems are derived only for some special cases. Fortunately, in the majority of practical cases, controllability of nonlinear systems is satisfied if the corresponding linearised system is controllable.

Example 3.2.4: *CSTR - controllability*

Linearised state-space model of CSTR (see Example 2.4.2) is of the form

$$\begin{aligned} \frac{dx_1(t)}{dt} &= a_{11}x_1(t) + a_{12}x_2(t) \\ \frac{dx_2(t)}{dt} &= a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) \end{aligned}$$

or

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u_1(t)$$

where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ b_{21} \end{pmatrix}$$

The controllability matrix $\boldsymbol{Q}_c$ is

$$\boldsymbol{Q}_c = (\boldsymbol{B}|\boldsymbol{AB}) = \begin{pmatrix} 0 & a_{12}b_{21} \\ b_{21} & a_{22}b_{21} \end{pmatrix}$$

and has rank equal to 2 and the system is completely controllable. It is clear that this is valid for all steady-states and hence the corresponding nonlinear model of the reactor is controllable.

Observability

States of a system are in the majority of cases measurable only partially or they are nonmeasurable. Therefore it is not possible to realise a control that assumes knowledge of state variables. In this connection a question arises whether it is possible to determine state vector from output measurements. We speak about observability and reconstructibility. To investigate observability, only a free system can be considered.

Definition of observability: A state $\boldsymbol{x}(t_0)$ of the system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{3.2.67}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) \tag{3.2.68}$$

is observable if it can be determined from knowledge about $\boldsymbol{y}(t)$ within a finite time $t \in [t_0, t_1]$. If every state $\boldsymbol{x}(t_0)$ can be determined from the output vector $\boldsymbol{y}(t)$ within arbitrary finite interval $t \in [t_0, t_1]$ then the system is completely observable.

Definition of reconstructibility : A state of system $\boldsymbol{x}(t_0)$ is reconstructible if it can be determined from knowledge about $\boldsymbol{y}(t)$ within a finite time $t \in [t_{00}, t_0]$. If every state $\boldsymbol{x}(t_0)$ can be determined from the output vector $\boldsymbol{y}(t)$ within arbitrary finite interval $t \in [t_{00}, t_0]$ then the system is completely reconstructible.

Similarly as in the case of controllability and reachability, the terms observability of a system and reconstructibility of a system are used for simplicity. For linear time-invariant systems, both terms are interchangeable.

Theorem: Observability of linear continuous systems with constant coefficients: The system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) \tag{3.2.69}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{3.2.70}$$

is completely observable if and only if rank of *observability matrix* $\boldsymbol{Q}_o$ is equal to n. The matrix $\boldsymbol{Q}_o[nr \times n]$ is given as

$$\boldsymbol{Q_o} = \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA}^2 \\ \vdots \\ \boldsymbol{CA}^{n-1} \end{pmatrix} \tag{3.2.71}$$

Proof: We prove only the "if" part. Solution of the Eq. (3.2.69) is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(t_0) \tag{3.2.72}$$

According to the Cayley-Hamilton theorem, the function $e^{-\boldsymbol{A}t}$ can be written as

$$e^{-\boldsymbol{A}t} = k_0(t)\boldsymbol{I} + k_1(t)\boldsymbol{A} + k_2(t)\boldsymbol{A}^2 + \cdots + k_{n-1}(t)\boldsymbol{A}^{n-1} \tag{3.2.73}$$

Substituting Eq. (3.2.73) into (3.2.72) yields

$$\boldsymbol{x}(t) = [k_0(t)\boldsymbol{I} + k_1(t)\boldsymbol{A} + k_2(t)\boldsymbol{A}^2 + \cdots + k_{n-1}(t)\boldsymbol{A}^{n-1}]\boldsymbol{x}(t_0) \tag{3.2.74}$$

Equation (3.2.70) now gives

$$\boldsymbol{y}(t) = [k_0(t)\boldsymbol{C} + k_1(t)\boldsymbol{CA} + k_2(t)\boldsymbol{CA}^2 + \cdots + k_{n-1}(t)\boldsymbol{CA}^{n-1}]\boldsymbol{x}(t_0) \tag{3.2.75}$$

or

$$\boldsymbol{x}(t_0) = \left[\int_{t_0}^{t_1} (\boldsymbol{k}(t)\boldsymbol{Q}_o)^T(\boldsymbol{k}(t)\boldsymbol{Q}_o)dt\right]^{-1} \int_{t_0}^{t_1} (\boldsymbol{k}(t)\boldsymbol{Q}_o)^T\boldsymbol{y}(t)dt \tag{3.2.76}$$

where $\boldsymbol{k}(t) = [k_0(t), k_1(t), \ldots, k_{n-1}(t)]$.

If the system is observable, it must be possible to determine $\boldsymbol{x}(t_0)$ from (3.2.76). Hence the inverse of $\int_{t_0}^{t_1}(\boldsymbol{k}(t)\boldsymbol{Q}_o)^T(\boldsymbol{k}(t)\boldsymbol{Q}_o)dt$ must exist and the matrix

$$\int_{t_0}^{t_1} (\boldsymbol{k}(t)\boldsymbol{Q}_o)^T(\boldsymbol{k}(t)\boldsymbol{Q}_o)dt = \boldsymbol{Q_o}^T \int_{t_0}^{t_1} (\boldsymbol{k}^T(t)\boldsymbol{k}(t)dt\boldsymbol{Q_o} \tag{3.2.77}$$

must be nonsingular. It can be shown that the matrix $\boldsymbol{k}^T(t)\boldsymbol{k}(t)$ is nonsingular and observability is satisfied if and only if $\text{rank}(\boldsymbol{Q_o}) = n$.

We note that observability and reconstructibility conditions for linear continuous systems with constant coefficients are the same.

Example 3.2.5: *CSTR - observability*

Consider the linearised model of CSTR from Example 2.4.2

$$\frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t)$$

$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t)$$

$$y_1(t) = x_2(t)$$

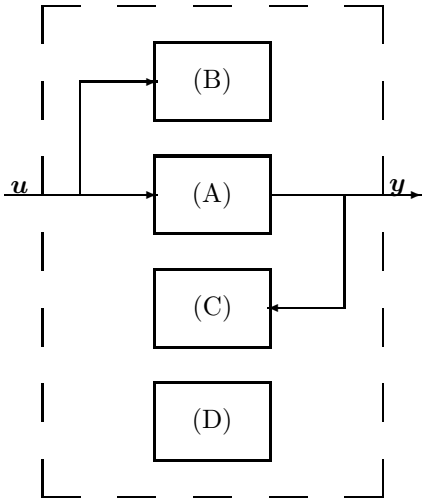

Figure 3.2.5: Canonical decomposition.

The matrices $\boldsymbol{A}, \boldsymbol{C}$ are

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \boldsymbol{C} = (0, 1)$$

and for $\boldsymbol{Q}_o$ yields

$$\boldsymbol{Q}_o = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix}$$

Rank of $\boldsymbol{Q}_o$ is 2 and the system is observable.
(Recall that $a_{21} = (-\Delta H)\dot{r}_{c_A}(c_a^s, \vartheta^s)/\rho c_p$)

3.2.5 Canonical Decomposition

Any continuous linear system with constant coefficients can be transformed into a special state-space form such that four separated subsystems result:

(A) controllable and observable subsystem,

(B) controllable and nonobservable subsystem,

(C) noncontrollable and observable subsystem,

(D) noncontrollable and nonobservable subsystem.

This division is called *canonical decomposition* and is shown in Fig. 3.2.5. Only subsystem A can be calculated from input and output relations.

The system eigenvalues can be also divided into 4 groups:

(A) controllable and observable modes,

(B) controllable and nonobservable modes,

(C) noncontrollable and observable modes,

(D) noncontrollable and nonobservable modes.

State-space model of continuous linear systems with constant coefficients is said to be minimal if it is controllable and observable.

State-space models of processes are more general than I/O models as they can also contain noncontrollable and nonobservable parts that are cancelled in I/O models.

Sometimes the notation *detectability* and *stabilisability* is used. A system is said to be detectable if all nonobservable eigenvalues are asymptotically stable and it is stabilisable if all nonstable eigenvalues are controllable.

3.3 Input-Output Process Models

In this section we focus our attention to transfer properties of processes. We show the relations between state-space and I/O models.

3.3.1 SISO Continuous Systems with Constant Coefficients

Linear continuous SISO (single input, single output) systems with constant coefficients with input $u(t)$ and output $y(t)$ can be described by a differential equation in the form

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_0 u(t) \tag{3.3.1}$$

where we suppose that $u(t)$ and $y(t)$ are deviation variables. After taking the Laplace transform and assuming zero initial conditions we get

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0) Y(s) = (b_m s^m + \cdots + b_0) U(s) \tag{3.3.2}$$

or

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} \tag{3.3.3}$$

where

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0$$

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0$$

$G(s)$ is called a *transfer function* of the system and is defined as the ratio between the Laplace transforms of output and input with zero initial conditions.

Note 3.3.1 *Transfer functions use the variable s of the Laplace transform. Introducing the* derivation operator $p = d/dt$ *then the relation*

$$G(p) = \frac{Y(p)}{U(p)} = \frac{B(p)}{A(p)} \tag{3.3.4}$$

is only another way of writing Eq. (3.3.1).

A transfer function $G(s)$ corresponds to physical reality if

$$n \geq m \tag{3.3.5}$$

Consider the case when this condition is not fulfilled, when $n = 1, m = 0$

$$a_0 y = b_1 \frac{du}{dt} + b_0 u \tag{3.3.6}$$

If $u(t) = 1(t)$ (step change) then the system response is given as a sum of two functions. The first function is an impulse function and the second is a step function. As any real process cannot

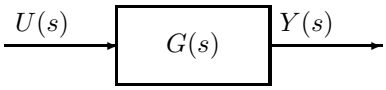

Figure 3.3.1: Block scheme of a system with transfer function $G(s)$.

show on output impulse behaviour, the case $n < m$ does not occur in real systems and the relation (3.3.5) is called the condition of *physical realisability*.

The relation

$$Y(s) = G(s)U(s) \tag{3.3.7}$$

can be illustrated by the block scheme shown in Fig. 3.3.1 where the block corresponds to $G(s)$. The scheme shows that if input to system is $U(s)$ then output is the function $G(s)U(s)$.

Example 3.3.1: *Transfer function of a liquid storage system*

Consider the tank shown in Fig. 2.2.1. State-space equations for this system are of the form (see 2.4.1)

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + b_{11}u \\ y_1 &= x_1 \end{aligned}$$

where $x_1 = h - h^s, u = q_0 - q_0^s$. After taking the Laplace transform and considering the fact that $x_1(0) = 0$ follows

$$\begin{aligned} sX_1(s) &= a_{11}X_1(s) + b_{11}U(s) \\ Y_1(s) &= X_1(s) \end{aligned}$$

and

$$(s - a_{11})Y_1(s) = b_{11}U(s)$$

Hence, the transfer function of this process is

$$G_1(s) = \frac{b_0}{a_1 s + 1} = \frac{Z_1}{T_1 s + 1}$$

where $a_1 = T_1 = (2F\sqrt{h^s})/k_{11}$, $b_0 = Z_1 = (2\sqrt{h^s})/k_{11}$. T_1 is time constant and Z_1 gain of the first order system.

Example 3.3.2: *Two tanks in a series - transfer function*

Consider two tanks shown in Fig. 3.3.2. The level h_1 is not influenced by the level h_2.

The dynamical properties of the first tank can be described as

$$F_1 \frac{dh_1}{dt} = q_0 - q_1$$

and the output equation is of the form

$$q_1 = k_{11}\sqrt{h_1}$$

The dynamical properties can also be written as

$$\frac{dh_1}{dt} = f_1(h_1, q_0)$$

where

$$f_1(h_1, q_0) = -\frac{k_{11}}{F_1}\sqrt{h_1} + \frac{1}{F_1}q_0$$

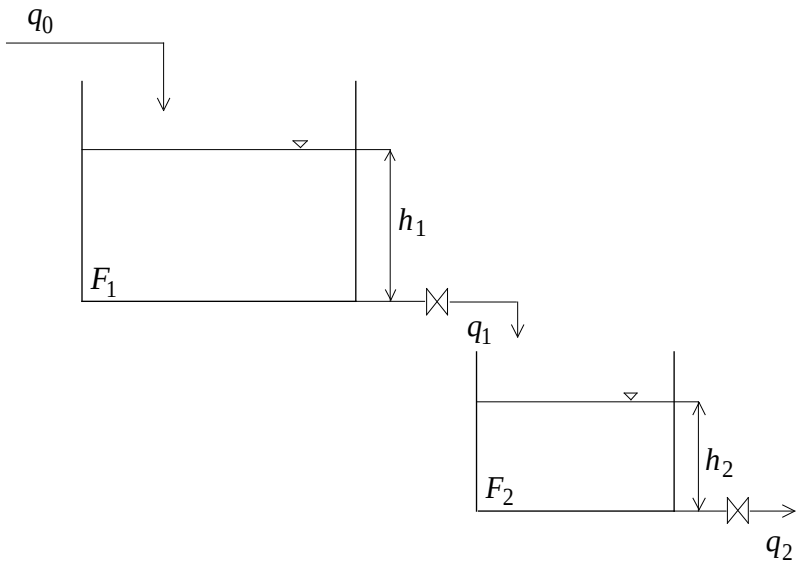

Figure 3.3.2: Two tanks in a series.

For the steady-state follows

$$\begin{aligned} 0 &= -\frac{k_{11}}{F_1}\sqrt{h_1^s} + \frac{1}{F_1}q_0^s \\ q_1^s &= k_{11}\sqrt{h_1^s} \end{aligned}$$

Linearised dynamical properties in the neighbourhood of the steady-state are of the form

$$\frac{dh_1}{dt} = \frac{d(h_1 - h_1^s)}{dt} = -\frac{k_{11}}{2F_1\sqrt{h_1^s}}(h_1 - h_1^s) + \frac{1}{F_1}(q_0 - q_0^s)$$

and linearised output equation is

$$q_1 - q_1^s = \frac{k_{11}}{2\sqrt{h_1^s}}(h_1 - h_1^s)$$

Let us introduce deviation variables

$$\begin{aligned} x_1 &= h_1 - h_1^s \\ u &= q_0 - q_0^s \\ y_1 &= q_1 - q_1^s \end{aligned}$$

Linear state-space model of the first tank is

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + b_{11}u \\ y_1 &= c_{11}x_1 \end{aligned}$$

where

$$a_{11} = -\frac{k_{11}}{2F_1\sqrt{h_1^s}},\ b_{11} = \frac{1}{F_1},\ c_{11} = \frac{k_{11}}{2\sqrt{h_1^s}}.$$

After applying the Laplace transform to these equations and using the fact that initial conditions are zero we obtain

$$\begin{aligned} sX_1(s) &= a_{11}X_1(s) + b_{11}U(s) \\ Y(s) &= c_{11}X_1(s) \end{aligned}$$

or

$$(s - a_{11})Y(s) = c_{11}b_{11}U(s)$$

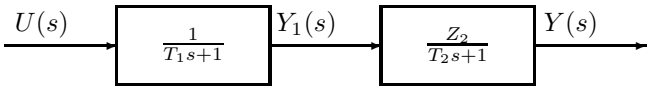

Figure 3.3.3: Block scheme of two tanks in a series.

The first tank transfer function $G_1(s)$ is

$$G_1(s) = \frac{Y_1(s)}{U(s)} = \frac{1}{a_1 s + 1} = \frac{1}{T_1 s + 1}$$

where $a_1 = T_1 = (2F\sqrt{h_1^s})/k_{11}$ and the gain is equal to one.

The transfer function of the second tank $G_2(s)$ can be derived when considering deviation variables $x_2 = y = h_2 - h_2^s$ and the relation $q_2 = k_{22}\sqrt{h_2}$ and is given as

$$G_2(s) = \frac{Y(s)}{Y_1(s)} = \frac{Z_2}{T_2 s + 1}$$

where $T_2 = (2F_2\sqrt{h_2^s})/k_{22}$, $Z_2 = (2\sqrt{h_2^s})/k_{22}$. The output $Y(s)$ can be written as

$$\begin{aligned} Y(s) &= \frac{Z_2}{T_2 s + 1} Y_1(s) \\ Y(s) &= \frac{Z_2}{T_2 s + 1} \frac{1}{T_1 s + 1} U(s) \end{aligned}$$

The overall transfer function of both tanks in a series is then

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Z_2}{T_2 s + 1} \frac{1}{T_1 s + 1}$$

Block scheme of this system is shown in Fig. 3.3.3.

Note 3.3.2 *The example given above shows serial connection of two systems where the second system does not influence the behaviour of the first system. We can speak about "one-way" effect.*

When the systems influence each other, the overall transfer function cannot be obtained as a product of transfer functions of subsystems. This is shown in the next example dealing with the interacting two tanks in a series (See Fig. 2.2.2). Mathematical model of this system described by equations (2.2.17) and (2.2.18) can be linearised in the neighbourhood of the steady-state given by flow rate q_0^s and levels h_1^s, h_2^s as

$$\begin{aligned} \frac{dh_1}{dt} &= \frac{d(h_1 - h_1^s)}{dt} = \frac{1}{F_1}(q_0 - q_0^s) - \frac{k_{11}}{2F_1\sqrt{h_1^s - h_2^s}}[(h_1 - h_1^s) - (h_2 - h_2^s)] \\ \frac{dh_2}{dt} &= \frac{d(h_2 - h_2^s)}{dt} = \frac{k_{11}}{2F_2\sqrt{h_1^s - h_2^s}}[(h_1 - h_1^s) - (h_2 - h_2^s)] \\ &\quad - \frac{k_{22}}{2F_2\sqrt{h_2^s}}(h_2 - h_2^s) \end{aligned}$$

Introducing deviation variables

$$\begin{aligned} x_1 &= h_1 - h_1^s \\ u &= q_0 - q_0^s \\ y = x_2 &= h_2 - h_2^s \end{aligned}$$

then the linear model is given as

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_{11}u\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2\\ y &= x_2\end{aligned}$$

where

$$a_{11} = -\frac{k_{11}}{2F_1\sqrt{h_1^s - h_2^s}}, \quad a_{12} = -a_{11}, \qquad b_{11} = \frac{1}{F_1}$$
$$a_{21} = -\frac{k_{11}}{2F_2\sqrt{h_1^s - h_2^s}}, \quad a_{22} = -a_{21} - \frac{k_{22}}{2F_2\sqrt{h_2^s}},$$

Taking the Laplace transform yields

$$\begin{aligned}sX_1(s) &= a_{11}X_1(s) + a_{12}X_2(s) + b_{11}U(s)\\ sX_2(s) &= a_{21}X_1(s) + a_{22}X_2(s)\\ Y(s) &= X_2(s)\end{aligned}$$

or

$$(s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}))Y(s) = a_{21}b_{11}U(s)$$

and hence the transfer function is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{a_2s^2 + a_1s + 1}$$

where

$$\begin{aligned}b_0 &= \frac{a_{21}b_{11}}{a_{11}a_{22} - a_{12}a_{21}}\\ a_2 &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\\ a_1 &= -\frac{a_{11} + a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\end{aligned}$$

Example 3.3.3: *n tanks in a series - transfer function*

Assume n tanks in a series as shown in Fig. 3.3.4 and the corresponding block scheme in Fig. 3.3.5. The variable $U(s)$ denotes the Laplace transform of $u(t) = q_0(t) - q_0^s$, $Y_i(s)$ are the Laplace transforms of $y_i(t) = q_i(t) - q_0^s,\ i = 1 \ldots n-1$, $Y(s)$ is the Laplace transform of $y(t) = h_n(t) - h_n^s$. $T_1, T_2, \ldots, T_n$ are time constants and Z_n is gain.

Similarly as in the case of the two tanks without interaction, the partial input and output variables are tied up with the following relations

$$\begin{aligned}Y_1(s) &= \frac{1}{T_1s + 1}U(s)\\ Y_2(s) &= \frac{1}{T_2s + 1}Y_1(s)\\ &\vdots\\ Y_i(s) &= \frac{1}{T_is + 1}Y_{i-1}(s)\\ &\vdots\\ Y(s) &= \frac{Z_n}{T_ns + 1}Y_{n-1}(s)\end{aligned}$$

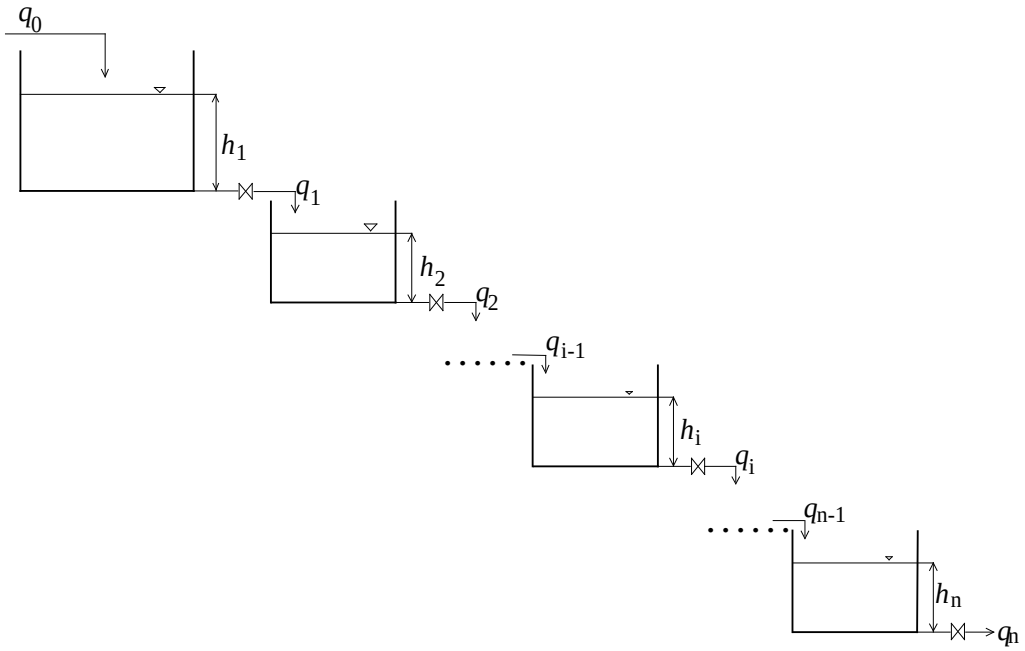

Figure 3.3.4: Serial connection of n tanks.

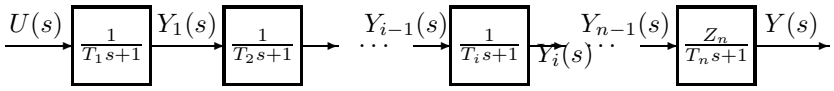

Figure 3.3.5: Block scheme of n tanks in a series.

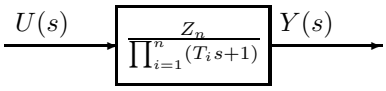

Figure 3.3.6: Simplified block scheme of n tanks in a series.

The overall input variable is $U(s)$ and the overall output variable is $Y(s)$. The overall transfer function is then

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Z_n}{\prod_{i=1}^{n}(T_i s + 1)}$$

Simplified block scheme of this system is shown in Fig. 3.3.6.

Example 3.3.4: *U-tube : transfer function*

Mathematical model of the U-tube shown in Fig. 3.2.2 is of the form

$$\frac{L}{2g}\frac{d^2h}{dt^2} + \frac{k}{2g\rho}\frac{dh}{dt} + h = \frac{1}{2g\rho}p_v$$

Steady-state is determined by level of liquid $h = h^s = 0$. We denote the output deviation variable as $y = h$ and the input deviation variable as $u = h_v = p_v/2g\rho$. Further let us introduce $1/\omega_k^2 = L/2g, 2\zeta/\omega_k = k/2g\rho$ where ω_k is a critical frequency and ζ is a damping coefficient. The terms critical frequency and damping coefficient will become clear from analysis of the solution of the differential equation describing dynamical properties of the U-tube. Mathematical model can be then rewritten as

$$\frac{1}{\omega_k^2}\frac{d^2y}{dt^2} + 2\frac{\zeta}{\omega_k}\frac{dy}{dt} + y = u$$

and the corresponding transfer function as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{T_k^2 s^2 + 2\zeta T_k s + 1}$$

where $T_k = 1/\omega_k$.

Note 3.3.3 *Mathematical model of the U-tube shows that step function on input can result in an oscillatory response. Therefore, U-tube is able of to produce its own oscillations. This is in contrast to other systems of the second order that can be decomposed into two systems of the first order and cannot produce the oscillations.*

Example 3.3.5: *Heat exchanger - transfer function*

Mathematical model of a heat exchanger was developed in the Section 2.2 and was shown to be in the form

$$T_1\frac{dy_1}{dt} + y_1 = Z_1u_1 + Z_2r_1$$

where $y_1 = \vartheta = \vartheta^s$, $u_1 = \vartheta_p - \vartheta_p^s$, $r_1 = \vartheta_v - \vartheta_v^s$ and T_1, T_2, Z_2 are constants. The output variable is the natural state variable $y_1 = x_1$.

To determine the heat exchanger response to the change of inlet temperature ϑ_v it is necessary to set $u_1 = 0$ and analogously if response of the process to the jacket temperature change is desired then $r_1 = 0$. The variable u_1 is usually assumed to be a manipulated variable and r_1 acts as a disturbance.

Taking the Laplace transform and considering zero initial conditions yields

$$(T_1 s + 1)Y_1(s) = Z_1U_1(s) + Z_2R_1(s)$$

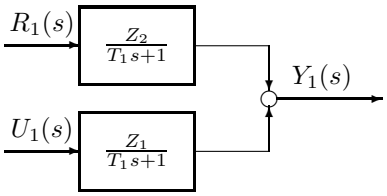

Figure 3.3.7: Block scheme of a heat exchanger.

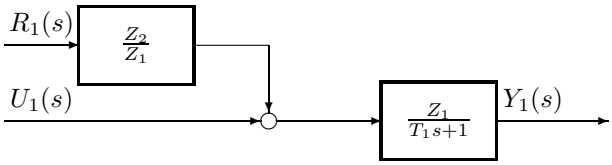

Figure 3.3.8: Modified block scheme of a heat exchanger.

if $R_1(s) = 0$ then

$$G_1(s) = \frac{Y_1(s)}{U_1(s)} = \frac{Z_1}{T_1 s + 1}$$

if $U_1(s) = 0$ then

$$G_2(s) = \frac{Y_1(s)}{R_1(s)} = \frac{Z_2}{T_1 s + 1}$$

$Y_1(s)$ can be written as

$$\begin{aligned} Y_1(s) &= \frac{Z_1}{T_1 s + 1} U_1(s) + \frac{Z_2}{T_1 s + 1} R_1(s) \\ Y_1(s) &= G_1(s) U_1(s) + G_2(s) R_1(s) \end{aligned}$$

Block scheme of this process is shown in Fig. 3.3.7 or in Fig. 3.3.8 where r_1 is moved from output to input to the system. This has an importance in design of control systems because modified block scheme simplifies some considerations.

Example 3.3.6: *CSTR - transfer function*

Consider the CSTR shown in Fig. 2.2.11. Let us introduce deviation variables

$$\begin{aligned} y_1 = x_1 &= c_A - c_A^s \\ y_2 = x_2 &= \vartheta - \vartheta^s \\ u_1 &= c_{Av} - c_{Av}^s \\ u_2 &= \vartheta_c - \vartheta_c^s \end{aligned}$$

The linearised mathematical model is then of the form

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11} x_1 + a_{12} x_2 + b_{11} u_1 \\ \frac{dx_2}{dt} &= a_{21} x_1 + a_{22} x_2 + b_{22} u_2 \end{aligned}$$

Compared to the Example 2.4.2, $b_{11} = h_{11}, b_{22} = b_{21}$ and inlet temperature is assumed to

be constant. We define the following transfer functions

$$G_{11}(s) = \frac{Y_1(s)}{U_1(s)} \quad G_{12}(s) = \frac{Y_1(s)}{U_2(s)}$$
$$G_{21}(s) = \frac{Y_2(s)}{U_1(s)} \quad G_{22}(s) = \frac{Y_2(s)}{U_2(s)}$$

Taking the Laplace transform of linearised mathematical model follows

$$\begin{aligned} sX_1(s) &= a_{11}X_1(s) + a_{12}X_2(s) + b_{11}U_1(s) \\ sX_2(s) &= a_{21}X_1(s) + a_{22}X_2(s) + b_{22}U_2(s) \end{aligned}$$

The transfer function $G_{11}(s)$ can be derived if $U_2(s) = 0$. Analogously, other transfer functions can also be obtained.

$$G_{11}(s) = \frac{b_{11}s - a_{22}b_{11}}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})}$$
$$G_{12}(s) = \frac{a_{12}b_{22}}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})}$$
$$G_{21}(s) = \frac{a_{21}b_{11}}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})}$$
$$G_{22}(s) = \frac{b_{22}s - a_{11}b_{22}}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})}$$

3.3.2 Transfer Functions of Systems with Time Delays

Consider a process described by the differential equation

$$\frac{\partial x_1(\sigma, t)}{\partial t} + v_\sigma \frac{\partial x_1(\sigma, t)}{\partial \sigma} = 0, \; x_1(\sigma, 0) = 0 \tag{3.3.8}$$

This equation is a description of a mathematical model of the double pipe heat exchanger shown in Fig. 2.2.5 with $\alpha = 0$ and x_1 being deviation temperature in the inner pipe. The process input variable is

$$u(t) = x_1(0, t) = x_1^0(t) \tag{3.3.9}$$

and the output variable is defined as

$$y(t) = x_1(L, t) \tag{3.3.10}$$

The system defined with the above equations is called *pure time delay*.

After taking the Laplace transform with argument t we get

$$v_\sigma \frac{\partial X_1(\sigma, s)}{\partial \sigma} + sX_1(\sigma, s) = 0 \tag{3.3.11}$$

where

$$X_1(\sigma, s) = \int_0^\infty x_1(\sigma, t)e^{-st}dt \tag{3.3.12}$$

Applying the Laplace transform with argument σ yields

$$v_\sigma q\bar{X}_1(q, s) - v_\sigma U(s) + s\bar{X}_1(q, s) = 0 \tag{3.3.13}$$

where

$$\bar{X}_1(q, s) = \int_0^\infty X_1(\sigma, s)e^{-q\sigma}d\sigma \tag{3.3.14}$$
$$U(s) = X_1(0, s) \tag{3.3.15}$$

From Eq. (3.3.13) follows

$$\bar{X}_1(q,s) = \frac{1}{q + \frac{s}{v_\sigma}} U(s) \tag{3.3.16}$$

This equation can be transformed back into σ domain

$$X_1(\sigma,s) = e^{-\frac{\sigma}{v_\sigma}s} U(s) \tag{3.3.17}$$

The corresponding transfer function of pure time delay for any $\sigma \in [0, L]$ is

$$G_{d\sigma} = \frac{X_1(\sigma,s)}{U(s)} = e^{-\frac{\sigma}{v_\sigma}s} \tag{3.3.18}$$

and for $\sigma = L$

$$G_d = \frac{Y(s)}{U(s)} = e^{-T_d s} \tag{3.3.19}$$

where $T_d = L/v_\sigma$.

Let us now consider only part of the process of length $\Delta\sigma$ that is perfectly mixed. The equation (3.3.8) can be approximated as

$$\frac{dx_1(\Delta\sigma,t)}{dt} = v_\sigma \frac{-x_1(\Delta\sigma,t) + u(t)}{\Delta\sigma} \tag{3.3.20}$$

This equation after taking the Laplace transform is of the form

$$\frac{X_1(\Delta\sigma,t)}{U(s)} = \frac{1}{1 + \frac{\Delta\sigma}{v_\sigma}s} \tag{3.3.21}$$

Because the term $e^{\frac{\Delta\sigma}{v_\sigma}s}$ can be written as

$$\frac{1}{e^{\frac{\Delta\sigma}{v_\sigma}s}} = \frac{1}{1 + \frac{\Delta\sigma}{v_\sigma}s + \frac{1}{2}\frac{\Delta\sigma^2}{v_\sigma^2}s^2 + \cdots} \tag{3.3.22}$$

then the right hand side term of (3.3.21) can be viewed as the first order approximation of time delay term.

Let us further assume mass balance of the process of length $\Delta\sigma$ of the form

$$\frac{d}{dt}\left(\frac{u(t) + x_1(\Delta\sigma,t)}{2}\right) = v_\sigma \frac{-x_1(\Delta\sigma,t) + u(t)}{\Delta\sigma} \tag{3.3.23}$$

Taking the Laplace transform becomes

$$\frac{X_1(\Delta\sigma,t)}{U(s)} = \frac{1 - \frac{1}{2}\frac{\Delta\sigma}{v_\sigma}s}{1 + \frac{1}{2}\frac{\Delta\sigma}{v_\sigma}s} \tag{3.3.24}$$

This equation can be understood as the first order Pade approximation of time delay term. Similarly for the second order Pade approximation yields

$$\frac{X_1(\Delta\sigma,t)}{U(s)} = \frac{1 - \frac{1}{2}\frac{\Delta\sigma}{v_\sigma}s + \frac{1}{12}\frac{\Delta\sigma^2}{v_\sigma^2}s^2}{1 + \frac{1}{2}\frac{\Delta\sigma}{v_\sigma}s + \frac{1}{12}\frac{\Delta\sigma^2}{v_\sigma^2}s^2} \tag{3.3.25}$$

Example 3.3.7: *Double-pipe heat exchanger - transfer functions*

Consider the heat exchanger shown in Fig. 2.2.5. It can be described by the following differential equation

$$T_1 \frac{\partial\vartheta(\sigma,t)}{\partial t} + v_\sigma T_1 \frac{\partial\vartheta(\sigma,t)}{\partial\sigma} + \vartheta(\sigma,t) = \vartheta_p(t)$$

Assume boundary and initial conditions of the form

$$\begin{aligned}\vartheta(0,t) &= \vartheta^0(t)\\ \vartheta(\sigma,0) &= \vartheta^s(\sigma) = \vartheta_p^s - (\vartheta_p^s - \vartheta_p^{0s})e^{-\frac{\sigma}{v_\sigma T_1}}\\ \vartheta_p(t) &= \vartheta_p^s,\ t<0\\ \vartheta^0(t) &= \vartheta^{0s},\ t<0\end{aligned}$$

and deviation variables

$$\begin{aligned}x_1(\sigma,t) &= \vartheta(\sigma,t) - \vartheta^s(\sigma)\\ u_1(t) &= \vartheta_p(t) - \vartheta_p^s\\ u_2(t) &= \vartheta^0(t) - \vartheta^{0s}\end{aligned}$$

The differential equation of the heat exchanger then becomes

$$T_1\frac{\partial x_1(\sigma,t)}{\partial t} + v_\sigma T_1\frac{\partial x_1(\sigma,t)}{\partial \sigma} + x_1(\sigma,t) = u_1(t)$$

with boundary and initial conditions

$$\begin{aligned}x_1(0,t) &= u_2(t)\\ x_1(\sigma,0) &= 0\\ u_1(t) &= 0,\ t<0\\ u_2(t) &= 0,\ t<0\end{aligned}$$

Taking the Laplace transform with an argument t yields

$$(T_1 s+1)X_1(\sigma,s) + v_\sigma T_1\frac{\partial X_1(\sigma,s)}{\partial \sigma} = U_1(s)$$

where

$$X_1(\sigma,s) = \int_0^\infty x_1(\sigma,t)e^{-st}dt$$

The second Laplace transform gives

$$(T_1 s + v_\sigma T_1 q + 1)\bar{X}_1(q,s) = \frac{1}{q}U_1(s) + v_\sigma T_1 U_2(s)$$

where

$$\begin{aligned}\bar{X}_1(q,s) &= \int_0^\infty X_1(\sigma,s)e^{-q\sigma}d\sigma\\ U_2(s) &= X_1(0,s)\end{aligned}$$

$\bar{X}_1(q,s)$ can be written as

$$\bar{X}_1(q,s) = \frac{1}{T_1 s+1}\frac{a}{q(q+a)}U_1(s) + \frac{1}{q+a}U_2(s)$$

where $a = (T_1 s+1)/v_\sigma T_1$.

The inverse Laplace transform according to σ gives

$$X_1(\sigma,s) = \frac{1}{T_1 s+1}\left(1 - e^{-\frac{T_1 s+1}{v_\sigma T_1}\sigma}\right)U_1(s) + e^{-\frac{T_1 s+1}{v_\sigma T_1}\sigma}U_2(s)$$

which shows that the transfer functions are of the form

$$\begin{aligned}G_{1k} &= \frac{Y_k(s)}{U_1(s)} = \frac{1}{T_1 s+1}\left(1 - e^{-\frac{\sigma_k}{v_\sigma T_1}}e^{-\frac{\sigma_k}{v_\sigma}s}\right)\\ G_{2k} &= \frac{Y_k(s)}{U_2(s)} = e^{-\frac{\sigma_k}{v_\sigma T_1}}e^{-\frac{\sigma_k}{v_\sigma}s}\end{aligned}$$

where

$$Y_k(s) = X_1(\sigma_k,s),\ k=1,2,\ldots,r;\ 0\le\sigma_k\le L$$

Block scheme of the double-pipe heat exchanger is shown in Fig. 3.3.9.

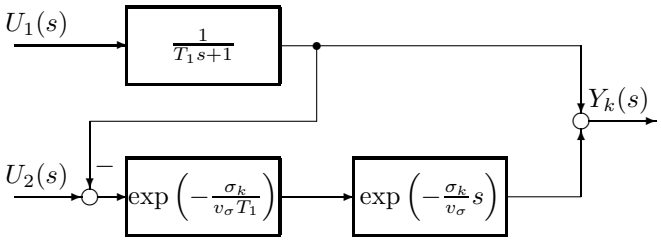

Figure 3.3.9: Block scheme of a double-pipe heat exchanger.

Figure 3.3.10: Serial connection.

3.3.3 Algebra of Transfer Functions for SISO Systems

Investigation of block schemes reveals the fact that all schemes can be decomposed into 3 basic connections: serial, parallel, and feedback. The rules that enable to determine the overall transfer function of a system composed from basic blocks are called *algebra of transfer functions*.

Serial Connection

Serial connection results in a situation when the output variable of the first block is the input variable of the second block (Fig. 3.3.10). The overall transfer function can be written as

$$G(s) = G_1(s)G_2(s) \tag{3.3.26}$$

Generally when n blocks are connected in series, the transfer function is given as a product of partial transfer functions:

$$G(s) = G_1(s)G_2(s)\dots G_n(s) \tag{3.3.27}$$

Parallel Connection

Parallel connection is characterised by one input variable for all systems. Output variable is given as the sum of partial outputs (Fig. 3.3.11). Parallel connection is characterised by the equations

$$Y_1(s) = G_1(s)U(s) \tag{3.3.28}$$
$$Y_2(s) = G_2(s)U(s) \tag{3.3.29}$$
$$Y(s) = Y_1(s) + Y_2(s) \tag{3.3.30}$$

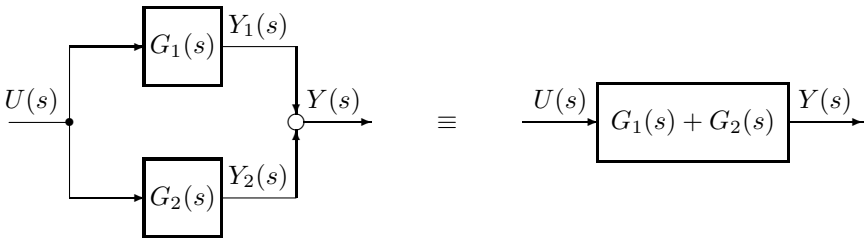

Figure 3.3.11: Parallel connection.

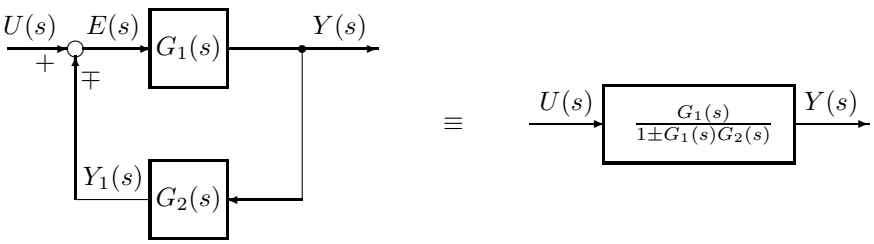

Figure 3.3.12: Feedback connection.

Substituting $Y_1(s)$ from (3.3.28) and $Y_2(s)$ from (3.3.29) into (3.3.30) yields

$$Y(s) = [G_1(s) + G_2(s)]U(s) = G(s)U(s) \tag{3.3.31}$$

and

$$G(s) = G_1(s) + G_2(s) \tag{3.3.32}$$

In general, the overall transfer functions is given as the sum of partial transfer functions

$$G(s) = \sum_{i=1}^{n} G_i(s) \tag{3.3.33}$$

Feedback Connection

Feedback connection of two blocks results when output variables of each block are fed back as the input of the other block (Fig. 3.3.12).

For the feedback connection holds

$$Y(s) = G_1(s)E(s) \tag{3.3.34}$$
$$Y_1(s) = G_2(s)Y(s) \tag{3.3.35}$$
$$E(s) = U(s) \mp Y_1(s) \tag{3.3.36}$$

The minus sign in the Eq. (3.3.36) corresponds to negative feedback and the plus sign to positive feedback. From these equations follow

$$Y_1(s) = G_1(s)G_2(s)E(s) \tag{3.3.37}$$
$$E(s) = \frac{1}{1 \pm G_1(s)G_2(s)}U(s) \tag{3.3.38}$$
$$Y(s) = \frac{G_1(s)}{1 \pm G_1(s)G_2(s)}U(s) = G(s)U(s) \tag{3.3.39}$$

The overall transfer function is then given as

$$G(s) = \frac{G_1(s)}{1 \pm G_1(s)G_2(s)} \tag{3.3.40}$$

The feedback transfer function is a ratio with the numerator given as the transfer function between the input and output signals and with the denominator given as a sum (negative feedback) or difference (positive feedback) of 1 and transfer function of the corresponding open-loop system.

Rule for Moving of the Branching Point

When the branching point is moved against the direction of the previous signal then the moved branch must contain all blocks which are between the original and new branching point (Fig. 3.3.13).

The opposite situation is when the branching point is moved in the direction of the signal flow. In this case the moved branch contains inverses of the relevant blocks (Fig. 3.3.14).

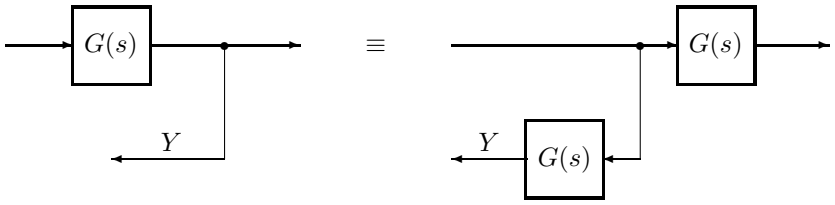

Figure 3.3.13: Moving of the branching point against the direction of signals.

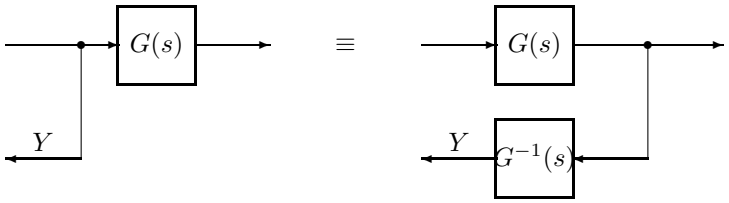

Figure 3.3.14: Moving of the branching point in the direction of signals.

Rule for Moving of the Summation Point

Moving of the summation point is an inverse action to moving of the branching point. The rules are shown in Figs. 3.3.15 and 3.3.16.

3.3.4 Input Output Models of MIMO Systems - Matrix of Transfer Functions

The standard and natural description of MIMO systems is in the form of state-space equations

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{3.3.41}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{3.3.42}$$

where $\boldsymbol{x}[n \times 1]$ is the vector of state variables, $\boldsymbol{u}[m \times 1]$ is the vector of input variables, $\boldsymbol{y}[r \times 1]$ is the vector of output variables, and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ are constant matrices of appropriate dimensions.

When all variables are deviation variables and $\boldsymbol{x}(0) = \boldsymbol{0}$ then the input-output (I/O) properties of this system can be determined from the convolution multiplication

$$\boldsymbol{y}(t) = \int_0^t \boldsymbol{g}(t-\tau)\boldsymbol{u}(\tau)d\tau \tag{3.3.43}$$

where $\boldsymbol{g}(t)$ is *matrix of impulse responses* of the $[r \times m]$ system and is given as

$$\boldsymbol{g}(t) = \begin{cases} \boldsymbol{0} & t < 0 \\ \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B} + \boldsymbol{D}\delta(t) & t \geq 0 \end{cases} \tag{3.3.44}$$

and $\delta(t)$ is the Dirac delta function.

Consider now the system given by Eqs. (3.3.41), (3.3.42). Taking the Laplace transform yields

$$\boldsymbol{Y}(s) = \left(\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right)\boldsymbol{U}(s) \tag{3.3.45}$$

or

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s) \tag{3.3.46}$$

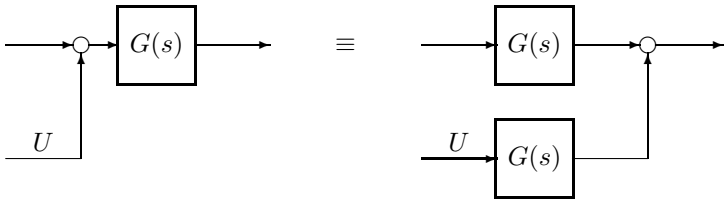

Figure 3.3.15: Moving of the summation point in the direction of signals.

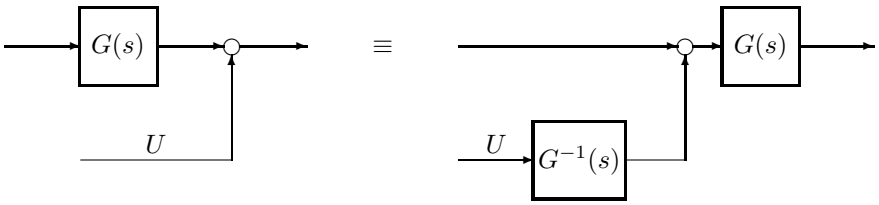

Figure 3.3.16: Moving of the summation point against the direction of signals.

where

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{3.3.47}$$

is $[r \times m]$ *transfer function matrix* of linear continuous system with constant coefficients. This matrix is the Laplace transform of matrix $\boldsymbol{g}(t)$

$$\boldsymbol{G}(s) = \int_0^\infty \boldsymbol{g}(t)e^{-st}dt \tag{3.3.48}$$

The matrix $\boldsymbol{G}(s)$ derived from the original state-space model is the same as the transfer function matrix of controllable and observable part of this system. The noncontrollable and nonobservable modes of the system are cancelled in the process of transformation from state-space models into I/O models. Often, there are tasks of the inverse transformation from I/O to state-space (SS) model. It must be emphasised that one I/O model corresponds to an infinite number of state-space models. We then speak about a *state-space realisation of I/O model*. *Minimum realisation* fulfils the properties of controllability and observability. Hence an unambiguous relation between $\boldsymbol{G}(s)$ and its state-space realisation exists only if the state-space model is minimal.

If we deal with SISO systems we can write

$$G(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{B(s)}{A(s)} \tag{3.3.49}$$

If the state-space model is the minimal realisation of $G(s)$ then

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = A(s) \tag{3.3.50}$$

The degree of characteristic polynomial $A(s)$ is equal to n where n is the number of states of state-space model. We call n as *system order*.

Any transfer function can be one of the following forms:

1. $$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0},\ n \geq m \tag{3.3.51}$$

2. $$G(s) = \frac{b_m(s - s_{N1})(s - s_{N2}) \dots (s - s_{Nm})}{a_n(s - s_1)(s - s_2) \dots (s - s_n)} \tag{3.3.52}$$

   Roots of the characteristic polynomial $s_1, \dots, s_n$ are *system poles*. Roots of numerator polynomial $s_{N1}, \dots, s_{Nn}$ are *system zeros*.

3. $$G(s) = Z_s \frac{(T_{N1}s + 1)(T_{N2}s + 1) \dots (T_{Nm}s + 1)}{(T_1 s + 1)(T_2 s + 1) \dots (T_n s + 1)} \tag{3.3.53}$$

   where $T_1, \dots, T_n, T_{N1}, \dots, T_{Nm}$ are time constants and $Z_s = b_0/a_0$ is the system gain. This expression for transfer function can only be written if all poles and zeros are real. Time constants correspond to negative inverses of poles and zeros.

Transfer function matrix $\boldsymbol{G}(s)$ has dimensions $[r \times m]$. An element of this matrix $G_{kj}(s)$ is the transfer function corresponding to input u_j and output y_k.

$$G_{kj}(s) = \frac{Y_k(s)}{U_j(s)} \tag{3.3.54}$$

The matrix $\boldsymbol{G}(s)$ can also be written as

$$\boldsymbol{G}(s) = \frac{\boldsymbol{C}\text{adj}(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{D}d(s)}{d(s)} \tag{3.3.55}$$

where $d(s) = |s\boldsymbol{I} - \boldsymbol{A}|$.

As all elements of $\text{adj}(s\boldsymbol{I} - \boldsymbol{A})$ are polynomials with a degree less than or equal to $n - 1$ and polynomial $d(s)$ is of degree n then all transfer functions $G_{kj}(s)$ have a degree of numerator less than or equal to the degree of the denominator. $\boldsymbol{G}(s)$ is *proper rational function matrix*. When $\boldsymbol{D} = \boldsymbol{0}$ then all numerator degrees are less than the denominator degrees and $\boldsymbol{G}(s)$ is *strictly proper rational function matrix*.

Definition of proper and strictly proper transfer function matrix $\boldsymbol{G}(s)$: A rational matrix $\boldsymbol{G}(s)[r \times m]$ is proper if all its elements satisfy $\lim_{|s|\to\infty} G_{kj}(s) < \infty$. A rational matrix $\boldsymbol{G}(s)$ is strictly proper if for all its elements hold $\lim_{|s|\to\infty} G_{kj}(s) = 0$. The numerator degree of a *proper SISO system* is smaller or equal to the denominator degree. The numerator degree of a *strictly proper SISO system* is smaller as the denominator degree.

Roots of polynomial $d(s)$ are poles of $\boldsymbol{G}(s)$. If no cancellation of roots occurs during the calculation of $\boldsymbol{G}(s)$ then the matrix poles are the same as *system poles*.

If all poles of $\boldsymbol{G}(s)$ are located in the left half plane of the complex plane then the *frequency transfer function matrix* that is defined as *Fourier transformation* of $\boldsymbol{g}(t)$ exists and can be obtained by the substitution $s = j\omega$, i.e.

$$\boldsymbol{G}(j\omega) = \boldsymbol{C}(j\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{3.3.56}$$

The Fourier transform is defined as

$$\boldsymbol{F}(j\omega) \equiv \int_{-\infty}^{\infty} \boldsymbol{f}(t)e^{-j\omega t}dt \tag{3.3.57}$$

$\boldsymbol{G}(j\omega)$ is called the *frequency transfer function matrix*. The values of $\boldsymbol{G}(j\omega)$ are for any real ω given as values of $\boldsymbol{G}(s)$ for $s = j\omega$.

$$\boldsymbol{G}(j\omega) = \boldsymbol{G}(s)|_{s=j\omega} \tag{3.3.58}$$

This function can be introduced not only for stable but for arbitrary transfer functions. However, if $\boldsymbol{G}(s)$ has a root on imaginary axis $s_i = j\beta_i$ then $\boldsymbol{G}(j\omega)$ has an infinite value for $\omega = \beta_i$.

Example 3.3.8: *CSTR - transfer function matrix*

Consider the CSTR shown in Fig. 2.2.11 and assume the same notation as in the Example 3.3.6. The state-space model matrices are

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \boldsymbol{B} = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, \boldsymbol{C} = I_2$$

From (3.3.47) follows

$$\begin{aligned}
\boldsymbol{G}(s) &= \boldsymbol{I}_2 \left[s\boldsymbol{I}_2 - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \\
&= \begin{pmatrix} s-a_{11} & -a_{12} \\ -a_{21} & s-a_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \\
&= \frac{1}{(s-a_{11})(s-a_{22}) - a_{12}a_{21}} \begin{pmatrix} s-a_{22} & a_{12} \\ a_{21} & s-a_{11} \end{pmatrix} \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix} \\
&= \frac{1}{s^2 - (a_{11}+a_{22})s - (a_{12}a_{21} - a_{11}a_{22})} \begin{pmatrix} b_{11}s - a_{22}b_{11} & a_{12}b_{22} \\ a_{21}b_{11} & b_{22}s - a_{11}b_{22} \end{pmatrix}
\end{aligned}$$

The partial transfer functions of $\boldsymbol{G}(s)$ are the same as in the example 3.3.6.

3.3.5 BIBO Stability

BIBO stability plays an important role among different definitions of stability. The abbreviation stands for *Bounded Input, Bounded Output*. Roughly said, a system is BIBO stable if any bounded input gives a bounded output. This is also the reason why we sometimes speak about BIBO stability as of *external stability*.

Definition of BIBO stability: A linear continuous system with constant coefficients (3.3.41), (3.3.42) with zero initial state $\boldsymbol{x}(t_0)$ is BIBO stable if for all t_0 and for all inputs $\boldsymbol{u}(t)$ that are finite on $[t_0, \infty)$ is output $\boldsymbol{y}(t)$ also finite on $[t_0, \infty)$.

Theorem: BIBO stability. A linear continuous system with constant coefficients (3.3.41), (3.3.42) is BIBO stable if and only if

$$\int_0^\infty \|\boldsymbol{g}(\tau)\| \, d\tau < \infty \tag{3.3.59}$$

where the norm is induced by the norm on u.

An alternate theorem about BIBO stability states: A linear continuous system with constant coefficients (3.3.41), (3.3.42) is BIBO stable if and only if all poles of transfer function matrix $\boldsymbol{G}(s)$ lie in the open left half plane of the complex plane.

Asymptotic stability of linear continuous systems with constant coefficients implies BIBO stability but the opposite case need not be true.

3.3.6 Transformation of I/O Models into State-Space Models

In the previous sections we found out that an input-output model can be transformed into infinitely many state-space models. In this section we show the procedures of this transformation that lead to controllable and observable canonical forms for SISO systems.

Controllable Canonical Form

Consider a system with transfer function in the form

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \; n \geq m \tag{3.3.60}$$

Let us introduce an auxiliary variable $z(t)$ and its Laplace transform $Z(s)$ such that

$$\frac{Y(s)}{Z(s)} = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 \tag{3.3.61}$$

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{3.3.62}$$

Equation (3.3.62) corresponds to the following differential equation

$$\frac{d^n z(t)}{dt^n} + a_{n-1}\frac{d^{n-1} z(t)}{dt^{n-1}} + \cdots + a_1 \dot{z}(t) + a_0 z(t) = u(t) \tag{3.3.63}$$

Now let us define state variables by the following relations

$$\frac{d^i z(t)}{dt^i} = x_{i+1}(t), \quad i = 0, 1, \ldots, n-1 \tag{3.3.64}$$

$$\frac{dx_i(t)}{dt} = x_{i+1}(t), \quad i = 1, 2, \ldots, n-1 \tag{3.3.65}$$

$$\tag{3.3.66}$$

Equation (3.3.63) can now be transformed into n first order differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= x_3(t) \\ &\vdots \\ \frac{dx_{n-1}(t)}{dt} &= x_n(t) \\ \frac{dx_n(t)}{dt} &= \frac{d^n z(t)}{dt^n} = -a_{n-1}x_n - \cdots - a_1 x_2(t) - a_0 x_1(t) + u(t) \end{aligned} \tag{3.3.67}$$

When $n = m$ then Eq. (3.3.61) corresponds to

$$y(t) = b_0 x_1(t) + b_1 x_2(t) + \cdots + b_{n-1} x_n(t) + b_n \dot{x}_n(t) \tag{3.3.68}$$

$\dot{x}_n(t)$ from this equation can be obtained from Eq. (3.3.67) and yields

$$\begin{aligned} y(t) &= (b_0 - a_0 b_n)x_1(t) + (b_1 - a_1 b_n)x_2(t) + \cdots + \\ &\quad + (b_{n-1} - a_{n-1} b_n)x_n(t) + b_n u(t) \end{aligned} \tag{3.3.69}$$

Equations (3.3.67) and (3.3.69) form a general state-space model of the form

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{B}_c \boldsymbol{u}(t) \tag{3.3.70}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}_c \boldsymbol{x}(t) + \boldsymbol{D}_c \boldsymbol{u}(t) \tag{3.3.71}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$,

$$\boldsymbol{A}_c = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}, \quad \boldsymbol{B}_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{C}_c = (b_0 - a_0 b_n \;\; b_1 - a_1 b_n \;\; \ldots \;\; b_{n-1} - a_{n-1} b_n), \quad \boldsymbol{D}_c = b_n$$

We see that if $m < n$ then $\boldsymbol{D} = \boldsymbol{0}$. This system of state-space equations can be shown to be always controllable but it need not be observable. We speak about *controllable canonical form* of a system. The corresponding block scheme is shown in Fig. 3.3.17.

Example 3.3.9: *Controllable canonical form of a second order system*

Consider a system described by the following differential equation

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_1 \dot{u}(t) + b_0 u(t)$$

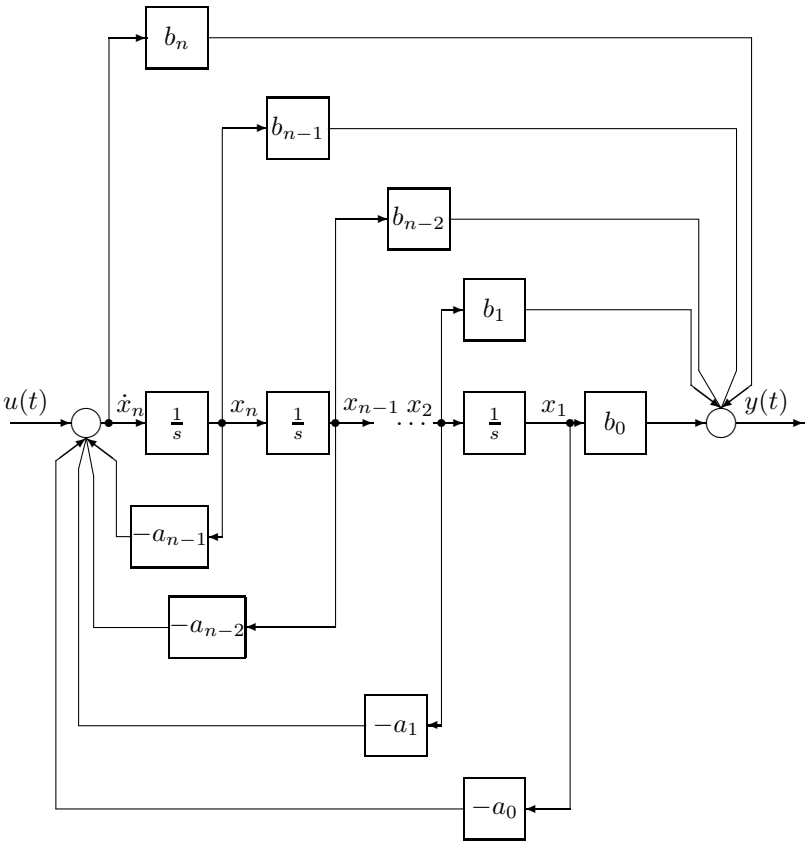

Figure 3.3.17: Block scheme of controllable canonical form of a system.

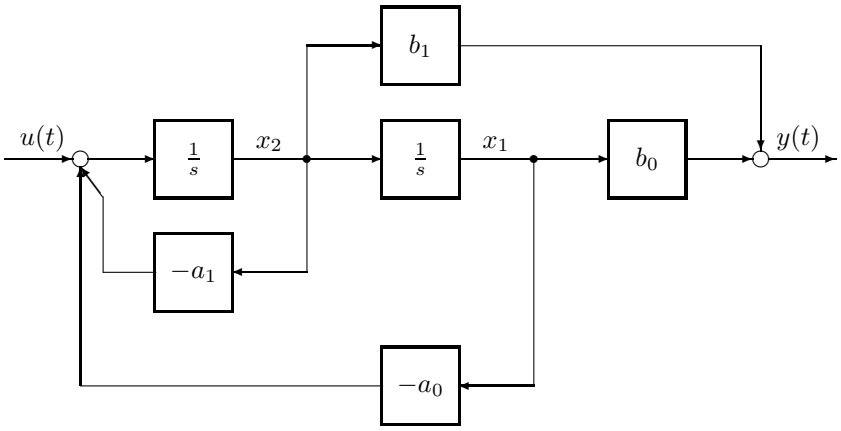

Figure 3.3.18: Block scheme of controllable canonical form of a second order system.

and corresponding transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

We introduce $Z(s)$ such the following equations hold

$$\begin{aligned} \frac{Y(s)}{Z(s)} &= b_1 s + b_0 \\ \frac{Z(s)}{U(s)} &= \frac{1}{s^2 + a_1 s + a_0} \end{aligned}$$

State-space equations can be written as

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2 t)}{dt} &= -a_0 x_1(t) - a_1 x_2(t) + u(t) \\ y(t) &= b_0 x_1(t) + b_1 x_2(t) \end{aligned}$$

and the corresponding block scheme is shown in Fig. 3.3.18.

Observable Canonical Form

Consider again the system with transfer function given by (3.3.60) and assume $n = m$. *Observable canonical form* of this system is given by

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}_o \boldsymbol{x}(t) + \boldsymbol{B}_o \boldsymbol{u}(t) \tag{3.3.72}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}_o \boldsymbol{x}(t) + \boldsymbol{D}_o \boldsymbol{u}(t) \tag{3.3.73}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$,

$$\boldsymbol{A}_o = \begin{pmatrix} -a_{n-1} & 1 & 0 & \ldots & 0 \\ -a_{n-2} & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ -a_1 & 0 & 0 & \ldots & 1 \\ -a_0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \boldsymbol{B}_o = \begin{pmatrix} b_{n-1} - a_{n-1} b_n \\ b_{n-2} - a_{n-2} b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{pmatrix}$$

$$\boldsymbol{C}_o = (1\ 0\ \ldots\ 0), \boldsymbol{D}_o = b_n$$

The corresponding block scheme is shown in Fig. 3.3.19.

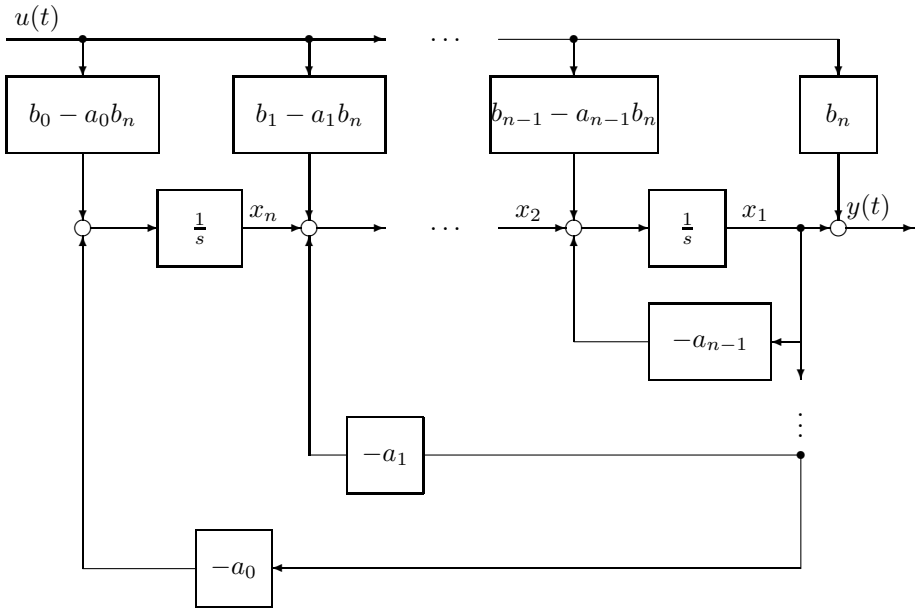

Figure 3.3.19: Block scheme of observable canonical form of a system.

3.3.7 I/O Models of MIMO Systems - Matrix Fraction Descriptions

Transfer function matrices provide a way to describe I/O (*external*) models of MIMO linear continuous systems. An alternate way is to give descriptions of such systems in polynomial matrix fractions that provide a natural generalisation of the singlevariable concept. A fundamental element of such descriptions is the polynomial matrix or matrix polynomial.

Real polynomial matrix is a matrix

$$\boldsymbol{P}(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) & \dots & p_{1m}(s) \\ p_{21}(s) & p_{22}(s) & \dots & p_{2m}(s) \\ \vdots & & & \vdots \\ p_{n1}(s) & p_{n2}(s) & \dots & p_{nm}(s) \end{pmatrix} \tag{3.3.74}$$

with elements being polynomials

$$p_{ij}(s) = p_{ij0} + p_{ij1}s + \cdots + p_{ijd_{ij}}s^{d_{ij}} \tag{3.3.75}$$

where p_{ijk} are real numbers and $i = 1 \dots n$, $j = 1 \dots m$, $k = 0 \dots d_{ij}$.

An element $p_{ij}(s)$ identically equal to zero has according to definition a degree equal to minus one.

Row degree of i-th row of $\boldsymbol{P}$ is denoted by r_i and it is the maximum degree of all polynomials in the i-th row ($r_i = \max_j d_{ij}$). Analogously are defined column degrees c_j as $\max_i d_{ij}$.

Degree of polynomial matrix $\boldsymbol{P}$ is denoted by $\deg \boldsymbol{P}$ and is defined as the maximum degree of all polynomials of $\boldsymbol{P}$ ($\max_{ij} d_{ij}$).

An alternate way of writing (3.3.74) is as *matrix polynomial*

$$\boldsymbol{P}(s) = \boldsymbol{P}_0 + \boldsymbol{P}_1 s + \cdots + \boldsymbol{P}_d s^d \tag{3.3.76}$$

where

$$\begin{aligned} \boldsymbol{P}_k &= \begin{pmatrix} p_{11k} & p_{12k} & \dots & p_{1mk} \\ p_{21k} & p_{22k} & \dots & p_{2mk} \\ \vdots & & & \vdots \\ p_{n1k} & p_{n2k} & \dots & p_{nmk} \end{pmatrix} \quad k = 0, \dots, d \\ d &= \deg \boldsymbol{P} \end{aligned}$$

A square polynomial matrix $\boldsymbol{P}(s)$ is nonsingular if $\det \boldsymbol{P}(s)$ is not identically equal to zero.

Roots of a square polynomial matrix are roots of the determinant of the polynomial matrix. A polynomial matrix is *stable* if all its roots lie in the open left half plane of the complex plane.

An *unimodular* polynomial matrix is a square polynomial matrix with a determinant equal to a nonzero constant. $\boldsymbol{P}(s)$ is unimodular if its inverse is also unimodular.

Rank of polynomial matrix is the highest order of a nonzero minor of this matrix.

For any $\boldsymbol{P}(s)$ there exist unimodular matrices $\boldsymbol{U}(s)$ and $\boldsymbol{V}(s)$ such that

$$\boldsymbol{U}(s)\boldsymbol{P}(s)\boldsymbol{V}(s) = \boldsymbol{\Lambda}(s) \tag{3.3.77}$$

where

$$\boldsymbol{\Lambda}(s) = \begin{pmatrix} \lambda_1(s) & 0 & \dots & 0 & 0 \\ 0 & \lambda_2(s) & \dots & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_r(s) & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

λ_i are unique monic polynomials (i.e. polynomials with a unit first coefficient) for which holds λ_i divides λ_j for $i < j \leq r$ and r is equal to rank $\boldsymbol{P}(s)$. The matrix $\boldsymbol{\Lambda}(s)[n \times m]$ is called the Smith form of polynomial matrix.

Elementary row (column) operations on $\boldsymbol{P}(s)$ are:

1. interchange of any two rows (columns) of $\boldsymbol{P}(s)$,
2. multiplication of any row (column) of $\boldsymbol{P}(s)$ by any nonzero number,
3. addition to any row (column) of a polynomial multiple of any other row (column).

For any polynomial matrix we can find elementary row and column operations transforming the matrix into the Smith form.

A $[r \times m]$ transfer function matrix $\boldsymbol{G}(s)$ can be of the form

$$\boldsymbol{G}(s) = \frac{\boldsymbol{M}(s)}{d(s)} \tag{3.3.78}$$

where $d(s)$ is the least common multiplier of denominators of elements of $\boldsymbol{G}(s)$ and $\boldsymbol{M}(s)$ is a polynomial matrix. The matrix $\boldsymbol{G}(s)$ from Eq. (3.3.78) can also be written as

$$\boldsymbol{G}(s) = \boldsymbol{B}_R(s)\boldsymbol{A}_R^{-1}(s) \tag{3.3.79}$$

where $\boldsymbol{B}_R(s) = \boldsymbol{M}(s)$, $\boldsymbol{A}_R(s) = d(s)\boldsymbol{I}_m$. It is also possible to write

$$\boldsymbol{G}(s) = \boldsymbol{A}_L^{-1}(s)\boldsymbol{B}_L(s) \tag{3.3.80}$$

where $\boldsymbol{B}_L(s) = \boldsymbol{M}(s)$, $\boldsymbol{A}_L(s) = d(s)\boldsymbol{I}_r$.

The description of $\boldsymbol{G}(s)$ given by (3.3.79) is called *right matrix fraction description* (RMFD). The degree of determinant of $\boldsymbol{A}_R(s)$ is

$$\deg \det \boldsymbol{A}_R(s) = d_d m \tag{3.3.81}$$

where d_d is a degree of $d(s)$ and m is a dimension of vector $\boldsymbol{u}$.

Analogously, the description of (3.3.79) is called *left matrix fraction description* (LMFD). The degree of determinant of $\boldsymbol{A}_L(s)$ is

$$\deg \det \boldsymbol{A}_L(s) = d_d r \tag{3.3.82}$$

where r is the dimension of vector $\boldsymbol{y}$.

Given $\boldsymbol{G}(s)$ there are infinitely many LMFD's and RMFD's.

Minimum degree of determinant of any LMFD (RMFD) of $\boldsymbol{G}(s)$ is equal to the minimum order of some realisation of $\boldsymbol{G}(s)$.

If some RMFD of $\boldsymbol{G}(s)$ is of the form

$$\boldsymbol{G}(s) = \boldsymbol{B}_{R1}(s)\boldsymbol{A}_{R1}^{-1}(s) \tag{3.3.83}$$

and some other RMFD of the form

$$\boldsymbol{G}(s) = \boldsymbol{B}_{R}(s)\boldsymbol{A}_{R}^{-1}(s) \tag{3.3.84}$$

then $\boldsymbol{B}_{R1}(s) = \boldsymbol{B}_R(s)\boldsymbol{W}(s)$, $\boldsymbol{A}_{R1}(s) = \boldsymbol{A}_R(s)\boldsymbol{W}(s)$, and $\boldsymbol{W}(s)$ is some polynomial matrix and it is common right divisor of $\boldsymbol{B}_{R1}(s), \boldsymbol{A}_{R1}(s)$. Analogously, the common left divisor can be defined.

Definition of relatively right (left) prime (RRP-RLP) polynomial matrices: Polynomial matrices $\boldsymbol{B}(s)$, $\boldsymbol{A}(s)$ with the same number of columns (rows) are RRP (RLP) if their right (left) common divisors are unimodular matrices.

Matrix fraction description of $\boldsymbol{G}(s)$ given by $\boldsymbol{A}(s), \boldsymbol{B}(s)$ is *right (left) irreducible* if $\boldsymbol{A}(s), \boldsymbol{B}(s)$ are RRP (RLP).

The process of obtaining irreducible MFD is related to greatest common divisors.

Greatest right (left) common divisor (GRCD-GLCD) of polynomial matrices $\boldsymbol{A}(s), \boldsymbol{B}(s)$ with the same number of columns (rows) is a polynomial matrix $\boldsymbol{R}(s)$ that satisfies the following conditions:

- $\boldsymbol{R}(s)$ is common right (left) divisor of $\boldsymbol{A}(s), \boldsymbol{B}(s)$,
- if $\boldsymbol{R}_1(s)$ is any common right (left) divisor of $\boldsymbol{A}(s), \boldsymbol{B}(s)$ then $\boldsymbol{R}_1(s)$ is right (left) divisor of $\boldsymbol{R}(s)$.

Lemma: relatively prime polynomial matrices: Polynomial matrices $\boldsymbol{B}_R(s)$, $\boldsymbol{A}_R(s)$ are RRP if and only if there exist polynomial matrices $\boldsymbol{X}_L(s)$, $\boldsymbol{Y}_L(s)$ such that the following Bezout identity is satisfied

$$\boldsymbol{Y}_L(s)\boldsymbol{B}_R(s) + \boldsymbol{X}_L(s)\boldsymbol{A}_R(s) = \boldsymbol{I}_m \tag{3.3.85}$$

Polynomial matrices $\boldsymbol{B}_L(s), \boldsymbol{A}_L(s)$ are RLP if and only if there exist polynomial matrices $\boldsymbol{X}_R(s)$, $\boldsymbol{Y}_R(s)$ such that the following Bezout identity is satisfied

$$\boldsymbol{B}_L(s)\boldsymbol{Y}_R(s) + \boldsymbol{A}_L(s)\boldsymbol{X}_R(s) = \boldsymbol{I}_r \tag{3.3.86}$$

For any polynomial matrices $\boldsymbol{B}_R(s)[r \times m]$ and $\boldsymbol{A}_R(s)[m \times m]$ an unimodular matrix $\boldsymbol{V}(s)$ exists such that

$$\boldsymbol{V}(s) = \begin{pmatrix} \boldsymbol{V}_{11}(s) & \boldsymbol{V}_{12}(s) \\ \boldsymbol{V}_{21}(s) & \boldsymbol{V}_{22}(s) \end{pmatrix}, \quad \begin{matrix} \boldsymbol{V}_{11}(s) \in [m \times r] & \boldsymbol{V}_{12}(s) \in [m \times m] \\ \boldsymbol{V}_{21}(s) \in [r \times r] & \boldsymbol{V}_{22}(s) \in [r \times m] \end{matrix} \tag{3.3.87}$$

and

$$\boldsymbol{V}(s) \begin{pmatrix} \boldsymbol{B}_R(s) \\ \boldsymbol{A}_R(s) \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}(s) \\ \boldsymbol{0} \end{pmatrix} \tag{3.3.88}$$

$\boldsymbol{R}(s)[m \times m]$ is GRCD$(\boldsymbol{B}_R(s), \boldsymbol{A}_R(s))$. The couples $\boldsymbol{V}_{11}, \boldsymbol{V}_{12}$ and $\boldsymbol{V}_{21}, \boldsymbol{V}_{22}$ are RLP. An analogous property holds for LMFD: For any polynomial matrices $\boldsymbol{B}_L(s)[r \times m]$ and $\boldsymbol{A}_L(s)[r \times r]$ an unimodular matrix $\boldsymbol{U}(s)$ exists such that

$$\boldsymbol{U}(s) = \begin{pmatrix} \boldsymbol{U}_{11}(s) & \boldsymbol{U}_{12}(s) \\ \boldsymbol{U}_{21}(s) & \boldsymbol{U}_{22}(s) \end{pmatrix}, \quad \begin{matrix} \boldsymbol{U}_{11}(s) \in [r \times r] & \boldsymbol{U}_{12}(s) \in [r \times m] \\ \boldsymbol{U}_{21}(s) \in [m \times r] & \boldsymbol{U}_{22}(s) \in [m \times m] \end{matrix} \tag{3.3.89}$$

and

$$(\boldsymbol{A}_L(s)\ \boldsymbol{B}_L(s))\ \boldsymbol{U}(s) = (\boldsymbol{L}(s)\ \boldsymbol{0}) \tag{3.3.90}$$

$\boldsymbol{L}(s)[r \times r]$ is GLCD$(\boldsymbol{B}_L(s), \boldsymbol{A}_L(s))$. The couples $\boldsymbol{U}_{11}, \boldsymbol{U}_{21}$ and $\boldsymbol{U}_{12}, \boldsymbol{U}_{22}$ are RRP.

Equations (3.3.88), (3.3.90) can be used to obtain irreducible MFD of $\boldsymbol{G}(s)$. When assuming RMFD the $\boldsymbol{G}(s)$ is given as

$$\boldsymbol{G}(s) = \boldsymbol{B}_R(s)\boldsymbol{A}_R^{-1}(s) \tag{3.3.91}$$

where $\boldsymbol{B}_R(s) = -\boldsymbol{V}_{12}(s)$ and $\boldsymbol{A}_R(s) = \boldsymbol{V}_{22}(s)$.

Lemma: division algorithm: Let $\boldsymbol{A}(s)[m \times m]$ be a nonsingular polynomial matrix. Then for any $\boldsymbol{B}(s)[r \times m]$ there exist unique polynomial matrices $\boldsymbol{Q}(s)$, $\boldsymbol{R}(s)$ such that

$$\boldsymbol{B}(s) = \boldsymbol{Q}(s)\boldsymbol{A}(s) + \boldsymbol{R}(s) \tag{3.3.92}$$

and $\boldsymbol{R}(s)\boldsymbol{A}^{-1}(s)$ is strictly proper.

The previous lemma deals with right division algorithm. Analogously, the left division can be defined.

This lemma can be used in the process of finding of a strictly proper part of the given R(L)MFD.

Lemma: minimal realisation of MFD: A MFD realisation with a degree equal to the denominator determinant degree is minimal if and only if the MFD is irreducible.

Lemma: BIBO stability: If the matrix transfer function $\boldsymbol{G}(s)$ is given by Eq. (3.3.79) then it is BIBO stable if and only if all roots of $\det \boldsymbol{A}_R(s)$ lie in the open left half plane of the complex plane. (analogously for LMFD).

Spectral factorisation: Consider a real polynomial matrix $\boldsymbol{B}(s)[m \times m]$ such that

$$\boldsymbol{B}^T(-s) = \boldsymbol{B}(s) \tag{3.3.93}$$

$$\boldsymbol{B}(j\omega) > 0 \qquad \forall \omega \in R \tag{3.3.94}$$

Right spectral factor of $\boldsymbol{B}(s)$ is some stable polynomial matrix $\boldsymbol{A}(s)[m \times m]$ that satisfies the following relation

$$\boldsymbol{B}(s) = \boldsymbol{A}^T(-s)\boldsymbol{A}(s) \tag{3.3.95}$$

3.4 References

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3.5 Exercises

Exercise 3.5.1:
Consider the two tanks shown in Fig. 3.3.2. A linearised mathematical model of this process is of the form

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + b_{11}u \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 \\ y &= x_2 \end{aligned}$$

where

$$a_{11} = -\frac{k_{11}}{2F_1\sqrt{h_1^s}}, \quad a_{21} = \frac{k_{11}}{2F_2\sqrt{h_1^s}}$$
$$a_{22} = -\frac{k_{22}}{2F_2\sqrt{h_2^s}}, \quad b_{11} = \frac{1}{F_1}$$

Find:

1. state transition matrix of this system,
2. if $x_1(0) = x_2(0) = 0$ give expressions for functions

$$\begin{aligned} x_1(t) &= f_1(u(t)) \\ x_2(t) &= f_2(u(t)) \\ y(t) &= f_3(u(t)) \end{aligned}$$

Exercise 3.5.2:
Consider CSTR shown in Fig. 2.2.11 and examine its stability. The rate of reaction is given as (see example 2.4.2)

$$r(c_A, \vartheta) = kc_A = k_0 e^{-\frac{E}{R\vartheta}} c_A$$

Suppose that concentration c_{Av} and temperatures ϑ_v, ϑ_c are constant. Perform the following tasks:

1. define steady-state of the reactor and find the model in this steady-state so that $dc_A/dt = d\vartheta/dt = 0$,
2. define deviation variables for reactor concentration and temperature and find a nonlinear model of the reactor with deviation variables,
3. perform linearisation and determine state-space description,
4. determine conditions of stability according to the Lyapunov equation (3.2.44). We assume that if the reactor is asymptotically stable in large in origin then it is asymptotically stable in origin.

Exercise 3.5.3:
Consider the mixing process shown in Fig. 2.7.3. The task is to linearise the process for the input variables q_0, q_1 and output variables h, c_2 and to determine its transfer function matrix.

Exercise 3.5.4:
Consider a SISO system described by the following differential equation

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_1\dot{u}(t) + b_0 u(t)$$

Find an observable canonical form of this system and its block scheme.

Exercise 3.5.5:
Assume 2I/2O system with transfer function matrix given as LMFD (3.3.80) where

$$\boldsymbol{A}_L(s) = \begin{pmatrix} 1 + a_1 s & a_2 s \\ a_3 s & 1 + a_4 s \end{pmatrix}$$
$$\boldsymbol{B}_L(s) = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

By using the method of comparing coefficients, find the corresponding RMFD (3.3.79) where

$$\boldsymbol{A}_R(s) = \begin{pmatrix} a_{1R} + a_{2R} s & a_{3R} + a_{4R} s \\ a_{5R} + a_{6R} s & a_{7R} + a_{8R} s \end{pmatrix}$$
$$\boldsymbol{B}_R(s) = \begin{pmatrix} b_{1R} & b_{2R} \\ b_{3R} & b_{4R} \end{pmatrix}$$

Elements of matrix

$$\boldsymbol{A}_{R0} = \begin{pmatrix} a_{1R} & a_{3R} \\ a_{5R} & a_{7R} \end{pmatrix}$$

can be chosen freely, but $\boldsymbol{A}_{R0}$ must be nonsingular.

Chapter 4

Dynamical Behaviour of Processes

Process responses to various simple types of input variables are valuable for process control design. In this chapter three basic process responses are studied: impulse, step, and frequency responses. These characteristics are usually investigated by means of computer simulations. In this connection we show and explain computer codes that numerically solve systems of differential equations in the programming languages BASIC, C, and MATLAB.

The end of this chapter deals with process responses for the case of stochastic input variables.

4.1 Time Responses of Linear Systems to Unit Impulse and Unit Step

4.1.1 Unit Impulse Response

Consider a system described by a transfer function $G(s)$ and for which holds

$$Y(s) = G(s)U(s) \tag{4.1.1}$$

If the system input variable $u(t)$ is the unit impulse $\delta(t)$ then

$$U(s) = \mathcal{L}\{\delta(t)\} = 1 \tag{4.1.2}$$

and the system response is given as

$$y(t) = g(t) \tag{4.1.3}$$

where $g(t) = \mathcal{L}^{-1}\{G(s)\}$ is *system response to the unit impulse* if the system initial conditions are zero, $g(t)$ is called *impulse response* or *weighting function*.

If we start from the solution of state-space equations (3.2.9)

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \tag{4.1.4}$$

$$y(t) = \boldsymbol{C}\boldsymbol{x}(t) + Du(t) \tag{4.1.5}$$

and replace $u(t)$ with $\delta(t)$ we get

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + e^{\boldsymbol{A}t}\boldsymbol{B} \tag{4.1.6}$$

$$y(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B} + D\delta(t) \tag{4.1.7}$$

For $\boldsymbol{x}(0) = 0$ then follows

$$y(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{B} + D\delta(t) = g(t) \tag{4.1.8}$$

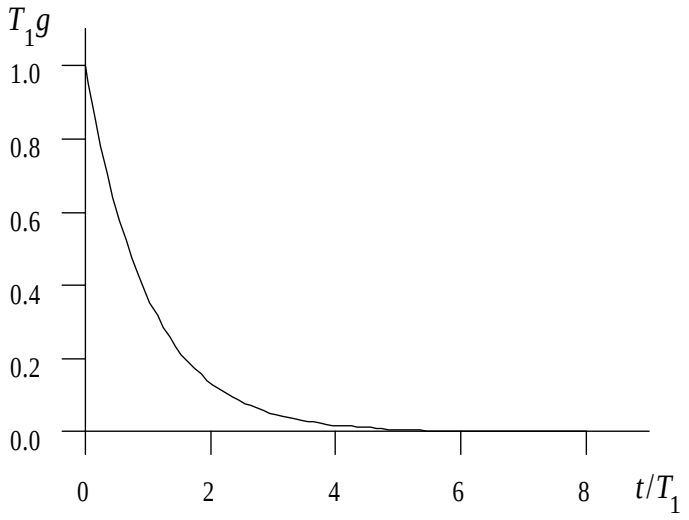

Figure 4.1.1: Impulse response of the first order system.

Consider the transfer function $G(s)$ of the form

$$G(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \tag{4.1.9}$$

The initial value theorem gives

$$g(0) = \lim_{s \to \infty} sG(s) = \begin{cases} \infty, & \text{if } b_n \neq 0 \\ \frac{b_{n-1}}{a_n}, & \text{if } b_n = 0 \\ 0, & \text{if } b_n = b_{n-1} = 0 \end{cases} \tag{4.1.10}$$

and $g(t) = 0$ for $t < 0$.

If for the impulse response holds $g(t) = 0$ for $t < 0$ then we speak about *causal* system.

From the Duhamel integral

$$y(t) = \int_0^t g(t - \tau) u(\tau) d\tau \tag{4.1.11}$$

follows that if the condition

$$\int_0^t |g(t)| dt < \infty \tag{4.1.12}$$

holds then any bounded input to the system results in bounded system output.

Example 4.1.1: *Impulse response of the first order system*

Assume a system with transfer function

$$G(s) = \frac{1}{T_1 s + 1}$$

then the corresponding weighting function is the inverse Laplace transform of $G(s)$

$$g(t) = \frac{1}{T_1} e^{-\frac{t}{T_1}}$$

The graphical representation of this function is shown in Fig. 4.1.1.

4.1.2 Unit Step Response

Step response is a response of a system with zero initial conditions to the unit step function $1(t)$. Consider a system with transfer function $G(s)$ for which holds

$$Y(s) = G(s)U(s) \tag{4.1.13}$$

If the system input variable $u(t)$ is the unit step function

$$u(t) = 1(t) \tag{4.1.14}$$

then the system response (for zero initial conditions) is

$$y(t) = \mathcal{L}^{-1}\left\{G(s)\tfrac{1}{s}\right\} \tag{4.1.15}$$

From this equation it is clear that step response is a time counterpart of the term $G(s)/s$ or equivalently $G(s)/s$ is the Laplace transform of step response. The impulse response is the time derivative of the step response.

Consider again the state-space approach. For $u(t) = 1(t)$ we get from (3.2.9)

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(t)d\tau \tag{4.1.16}$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + e^{\boldsymbol{A}t}(-\boldsymbol{A}^{-1})(e^{-\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{B} \tag{4.1.17}$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + (e^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{A}^{-1}\boldsymbol{B} \tag{4.1.18}$$

$$y(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \boldsymbol{C}(e^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{A}^{-1}\boldsymbol{B} + D \tag{4.1.19}$$

For $\boldsymbol{x}(0) = \boldsymbol{0}$ holds

$$y(t) = \boldsymbol{C}(e^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{A}^{-1}\boldsymbol{B} + D \tag{4.1.20}$$

If all eigenvalues of $\boldsymbol{A}$ have negative real parts, the steady-state value of step response is equal to $G(0)$. This follows from the Final value theorem (see page 61)

$$\lim_{t\to\infty} y(t) = \lim_{s=0} G(s) = -\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} + D = \frac{b_0}{a_0} \tag{4.1.21}$$

The term b_0/a_0 is called *(steady-state) gain* of the system.

Example 4.1.2: *Step response of first order system*

Assume a process that can be described as

$$T_1\frac{dy}{dt} + y = Z_1 u$$

This is an example of the first order system with the transfer function

$$G(s) = \frac{Z_1}{T_1 s + 1}$$

The corresponding step response is given as

$$y(t) = Z_1(1 - e^{-\frac{t}{T_1}})$$

Z_1 the gain and T_1 time constant of this system. Step response of this system is shown in Fig 4.1.2.

Step responses of the first order system with various time constants are shown in Fig 4.1.3. The relation between time constants is $T_1 < T_2 < T_3$.

Example 4.1.3: *Step responses of higher order systems*

Consider two systems with transfer functions of the form

$$G_1(s) = \frac{Z_1}{T_1 s + 1}, \qquad G_2(s) = \frac{Z_2}{T_2 s + 1}$$

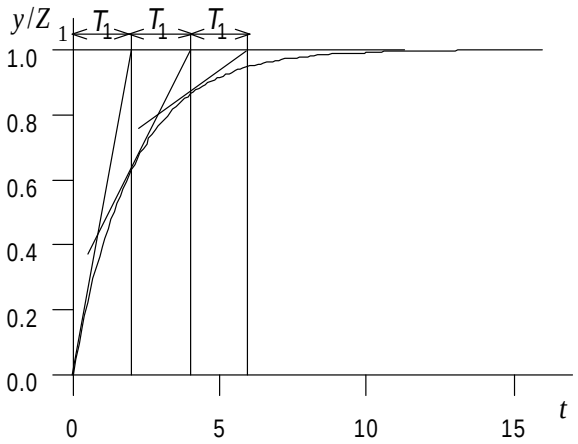

Figure 4.1.2: Step response of a first order system.

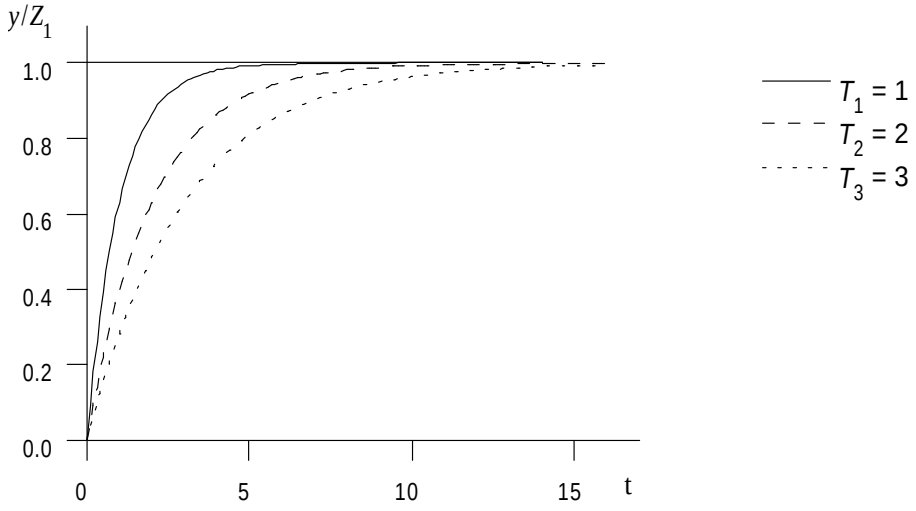

Figure 4.1.3: Step responses of a first order system with time constants T_1, T_2, T_3.

connected in series. The overall transfer function is given as their product

$$\frac{Y(s)}{U(s)} = \frac{Z_1 Z_2}{(T_1 s + 1)(T_2 s + 1)}$$

The corresponding step response function can be calculated as

$$y(t) = Z_1 Z_2 \left[1 - \frac{T_1}{T_1 - T_2} e^{-\frac{t}{T_1}} + \frac{T_2}{T_1 - T_2} e^{-\frac{t}{T_2}} \right]$$

or

$$y(t) = Z_1 Z_2 \left[1 - \frac{T_1 T_2}{T_1 - T_2} \left(\frac{1}{T_2} e^{-\frac{t}{T_1}} - \frac{1}{T_1} e^{-\frac{t}{T_2}} \right) \right]$$

Consider now a second order system with the transfer function given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Z_s}{T_k^2 s^2 + 2\zeta T_k s + 1}$$

As it was shown in the Example 3.3.4, such transfer function can result from the mathematical model of a U-tube.

The characteristic form of the step response depends on the roots of the characteristic equation

$$T_k^2 s^2 + 2\zeta T_k s + 1 = 0$$

If T_k represents the time constant then the dumping factor ζ plays a crucial role in the properties of the step response. In the following analysis the case $\zeta < 0$ will be automatically excluded as that corresponding to an unstable system. We will focus on the following cases of roots:

Case a: $\zeta > 1$ - two different real roots,

Case b: $\zeta = 1$ - double real root,

Case c: $0 < \zeta < 1$ - two complex conjugate roots.

Case a: If $\zeta > 1$ then the characteristic equation can be factorised as follows

$$T_k^2 s^2 + 2\zeta T_k s + 1 = (T_1 s + 1)(T_2 s + 1)$$

where

$$\begin{array}{c} T_k^2 = T_1 T_2 \\ 2\zeta T_k = T_1 + T_2 \end{array} \quad \text{or} \quad \begin{array}{c} T_k = \sqrt{T_1 T_2} \\ \zeta = \dfrac{T_1 + T_2}{2\sqrt{T_1 T_2}} \end{array}$$

Another possibility how to factorise the characteristic equation is

$$T_k^2 s^2 + 2\zeta T_k s + 1 = \left(\frac{T_k}{\zeta - \sqrt{\zeta^2 - 1}} s + 1 \right) \left(\frac{T_k}{\zeta + \sqrt{\zeta^2 - 1}} s + 1 \right)$$

Now the constants T_1, T_2 are of the form

$$T_1 = \frac{T_k}{\zeta - \sqrt{\zeta^2 - 1}}, \quad T_2 = \frac{T_k}{\zeta + \sqrt{\zeta^2 - 1}}$$

Case b: If $\zeta = 1$ then $T_1 = T_k$, $T_2 = T_k$.

Case c: If $0 < \zeta < 1$ then the transfer function can be rewritten as

$$G(s) = \frac{Z_s}{T_k^2 \left(s^2 + \dfrac{2\zeta}{T_k} s + \dfrac{1}{T_k^2} \right)}$$

and the solution of the characteristic equation is given by

$$\begin{aligned} s_{1,2} &= \frac{-\dfrac{2\zeta}{T_k} \pm \sqrt{4\dfrac{\zeta^2}{T_k^2} - 4\dfrac{1}{T_k^2}}}{2} \\ s_{1,2} &= \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{T_k} \end{aligned}$$

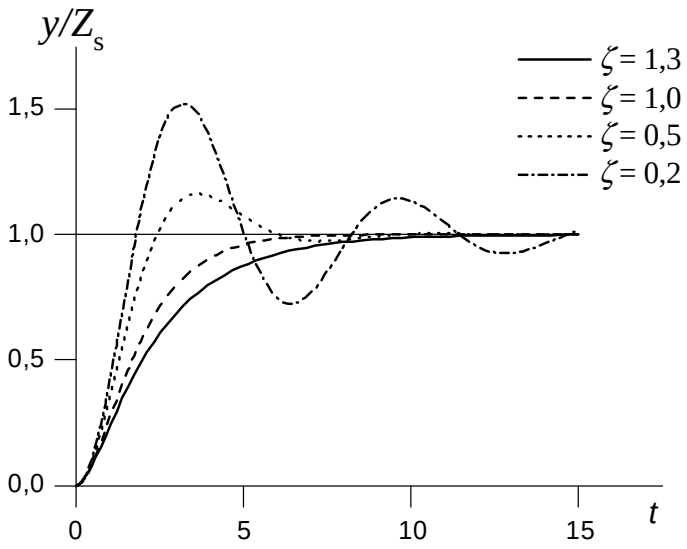

Figure 4.1.4: Step responses of the second order system for the various values of ζ.

The corresponding transfer functions are found from the inverse Laplace transform and are of the form

Case a: $$y(t) = Z_s \left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right)$$

Case b: $$y(t) = Z_s \left[1 - \left(1 - \frac{t}{T_k}\right) e^{-t/T_k}\right]$$

Case c: $$y(t) = Z_s \left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta}{T_k}t} \sin\left(\frac{\sqrt{1-\zeta^2}}{T_k}t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

For the sake of completeness, if $\zeta = 0$ the step response contains only a sinus function. Step responses for various values of ζ are shown in Fig. 4.1.4.

Consider now the system consisting of two first order systems connected in a series. The worst case concerning system inertia occurs if $T_1 = T_2$. In this case the tangent in the inflex point has the largest value. If $T_2 \ll T_1$ then the overall characteristic of the system approaches a first order system with a time constant T_1.

A generalisation of this phenomenon shows that if the system consists of n-in-series connected systems, then the system inertia is the largest if all time constants are equal.

If i-th subsystem is of the form

$$G_i(s) = \frac{Z_i}{T_i s + 1}$$

then for the overall transfer function yields

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\prod_{i=1}^{n} Z_i}{\prod_{i=1}^{n}(T_i s + 1)}$$

If $T_s = T_1 = T_2 = \cdots = T_n$ then the system residence time will be the largest. Consider unit step function on input and $Z_s = Z_1 Z_2 \ldots Z_n$. The step responses for various n are given in Fig. 4.1.5.

Example 4.1.4: *Step response of the n-th order system connected in a series with time delay*

Fig. 4.1.6 shows the block scheme of a system composed of n-th order system and pure time delay connected in a series. The corresponding step response is shown in Fig. 4.1.7 where it is considered that $n = 1$.

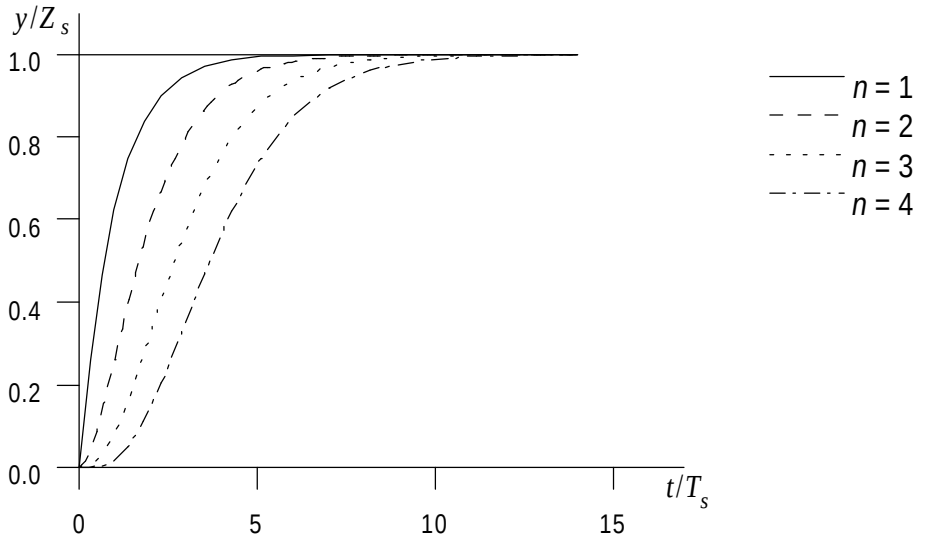

Figure 4.1.5: Step responses of the system with n equal time constants.

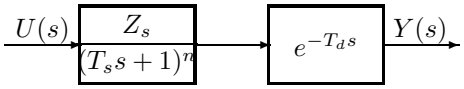

Figure 4.1.6: Block scheme of the n-th order system connected in a series with time delay.

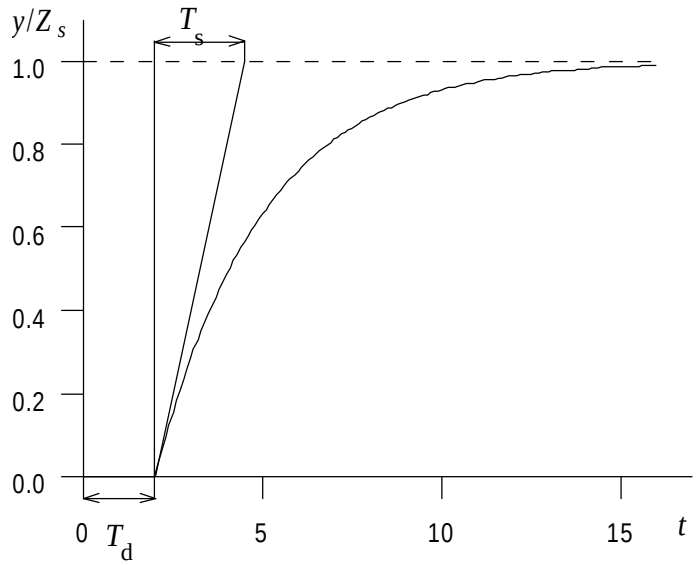

Figure 4.1.7: Step response of the first order system with time delay.

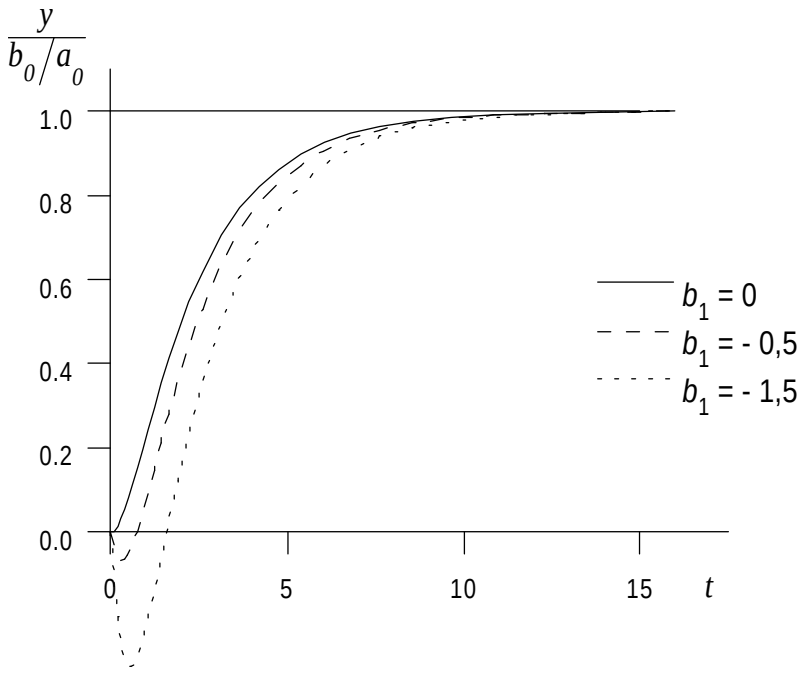

Figure 4.1.8: Step response of the second order system with the numerator $B(s) = b_1 s + 1$.

Example 4.1.5: *Step response of 2nd order system with the numerator* $B(s) = b_0 + b_1 s$

As it was shown in Example 3.3.6, some transfer function numerators of the CSTR can contain first order polynomials. Investigation of such step responses is therefore of practical importance. An especially interesting case is when the numerator polynomial has a positive root.

Consider for example the following system

$$G(s) = \frac{b_1 s + 1}{s^2 + 2.6s + 1}$$

The corresponding step response is illustrated in Fig. 4.1.8.

4.2 Computer Simulations

As it was shown in the previous pages, the investigation of process behaviour requires a solution of differential equations. Analytical solutions can only be found for processes described by linear differential equations with constant coefficients. If the differential equations that describe dynamical behaviour of a process are nonlinear, then it is either very difficult or impossible to find the analytical solution. In such cases it is necessary to utilise numerical methods. These procedures transform the original differential equations into difference equations that can be solved iteratively on a computer. The drawback of this type of solution is a loss of generality as numerical values of the initial conditions, coefficients of the model, and its input functions must be specified. However, in the majority of cases there does not exist any other approach as a numerical solution of differential equations. The use of numerical methods for the determination of process responses is called *simulation*. There is a large number of simulation methods. We will explain Euler and Runge-Kutta methods. The Euler method will be used for the explanation of principles of numerical methods. The Runge-Kutta method is the most versatile approach that is extensively used.

4.2.1 The Euler Method

Consider a process model in the form

$$\frac{dx(t)}{dt} = f(t, x(t)), \qquad x(t_0) = x_0 \tag{4.2.1}$$

At first we transform this equation into its difference equation counterpart. We start from the definition of a derivative of a function

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \tag{4.2.2}$$

if Δt is sufficiently small, the derivative can be approximated as

$$\frac{dx}{dt} \doteq \frac{x(t + \Delta t) - x(t)}{\Delta t} \tag{4.2.3}$$

Now, suppose that the right hand side of (4.2.1) is constant over some interval $(t, t + \Delta t)$ and substitute the left hand side derivative from (4.2.3). Then we can write

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = f(t, x(t)) \tag{4.2.4}$$

or

$$x(t + \Delta t) = x(t) + \Delta t f(t, x(t)) \tag{4.2.5}$$

The assumptions that led to Eq. (4.2.5) are only justified if Δt is sufficiently small. At time $t = t_0$ we can write

$$x(t_0 + \Delta t) = x(t_0) + \Delta t f(t_0, x(t_0)) \tag{4.2.6}$$

and at time $t_1 = t_0 + \Delta t$

$$x(t_1 + \Delta t) = x(t_1) + \Delta t f(t_1, x(t_1)) \tag{4.2.7}$$

In general, for $t = t_k$, $t_{k+1} = t_k + \Delta t$ Eq. (4.2.5) yields

$$x(t_{k+1}) = x(t_k) + \Delta t f(t_k, x(t_k)) \tag{4.2.8}$$

Consider now the following differential equation

$$\frac{dx(t)}{dt} = f(t, x(t), u(t)), \qquad x(t_0) = x_0 \tag{4.2.9}$$

We assume again that the right hand side is constant over the interval (t_k, t_{k+1}) and is equal to $f(t_k, x(t_k), u(t_k))$. Applying the approximation (4.2.3) yields

$$x(t_{k+1}) = x(t_k) + \Delta t f(t_k, x(t_k), u(t_k)) \tag{4.2.10}$$

In this equation we can notice that the continuous-time variables $x(t), u(t)$ have been replaced by discrete variables $x(t_k), u(t_k)$. Let us denote

$$x(t_k) \equiv x_k \tag{4.2.11}$$

$$u(t_k) \equiv u_k \tag{4.2.12}$$

and we obtain a recursive relation called *difference equation*

$$x_{k+1} = x_k + \Delta t f(t_k, x_k, u_k) \tag{4.2.13}$$

that can be solved recursively for $k = 0, 1, 2, \ldots$ for the given initial value x_0.

Equation (4.2.13) constitutes *the Euler method* of solving the differential equation (4.2.9) and it is easily programmable on a computer. The difference $h = t_{k+1} - t_k$ is usually called *integration step*.

As the basic Euler method is only very crude and inaccurate, the following modification of *modified Euler method* was introduced

$$x_{k+1} = x_k + \frac{h}{2}(f_k + f_{k+1}) \tag{4.2.14}$$

where

$t_k = t_0 + kh,\ k = 0, 1, 2, \ldots$

$f_k = f(t_k, x(t_k), u(t_k))$

$f_{k+1} = f[t_{k+1}, x(t_k) + hf(t_k, x(t_k), u(t_k)), u(t_{k+1})]$

4.2.2 The Runge-Kutta method

This method is based on the Taylor expansion of a function. The Taylor expansion helps to express the solution $x(t+\Delta t)$ of a differential equation in relation to $x(t)$ and its time derivatives as follows

$$x(t + \Delta t) = x(t) + \Delta t\dot{x}(t) + \frac{1}{2}(\Delta t)^2\ddot{x}(t) + \cdots \tag{4.2.15}$$

If the solved differential equation is of the form

$$\frac{dx(t)}{dt} = f(t, x(t)) \tag{4.2.16}$$

then the time derivatives can be expressed as

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)) \\ \ddot{x}(t) &= \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}f \\ &\vdots \end{aligned} \tag{4.2.17}$$

Substituting (4.2.17) into (4.2.15) yields

$$x(t + \Delta t) = x(t) + \Delta tf + \frac{1}{2}(\Delta t)^2(f_t + f_x f) + \cdots \tag{4.2.18}$$

where $f = f(t, x(t))$, $f_t = \partial f/\partial t$, $f_x = \partial f/\partial x$.

The solution $x(t + \Delta t)$ in Eq. (4.2.18) depends on the knowledge of derivatives of the function f. However, the higher order derivatives of f are difficult to obtain. Therefore only some first terms of (4.2.18) are assumed to be significant and others are neglected. The Taylor expansion is truncated and forms the basis for *Runge-Kutta methods*. The number of terms determines order of the Runge-Kutta method.

Assume that the integration step is given as

$$t_{k+1} = t_k + h \tag{4.2.19}$$

The second order Runge-Kutta method is based on the difference equation

$$x(t_{k+1}) = x(t_k) + h\dot{x}(t_k) + \frac{1}{2}h^2\ddot{x}(t_k) \tag{4.2.20}$$

or

$$x_{k+1} = x_k + hf_k + \frac{1}{2}h^2(f_t + f_x f)_k \tag{4.2.21}$$

The recursive relation suitable for numerical solution is then given by

$$x_{k+1} = x_k + \gamma_1 k_1 + \gamma_2 k_2 \tag{4.2.22}$$

where γ_1, γ_2 are weighting constants and

$$k_1 = hf(t_k, x_k) \tag{4.2.23}$$

$$k_2 = hf(t_k + \alpha_1 h, x_k + \beta_1 k_1) \tag{4.2.24}$$

and α_1, α_2 are some constants. The proof that (4.2.22) is a recursive solution following from the second order Runge-Kutta method can be shown as follows. Allow at first perform the Taylor expansion for k_2

$$k_2 = h[f_k + (f_t)_k\alpha_1 h + (f_x)_k\beta_1 hf_k + \cdots] \tag{4.2.25}$$

and neglect all terms not explicitly given. Substituting k_1 from (4.2.23) and k_2 from (4.2.25) into (4.2.22) gives

$$x_{k+1} = x_k + h(\gamma_1 f_k + \gamma_2 f_k) + h^2[\gamma_2\alpha_1 (f_t)_k + \gamma_2\beta_1 (f_x)_k f_k] \tag{4.2.26}$$

Comparison of (4.2.20) and (4.2.26) gives

$$\gamma_1 + \gamma_2 = 1 \tag{4.2.27}$$

$$\gamma_2\alpha_1 = \frac{1}{2} \tag{4.2.28}$$

$$\gamma_2\beta_1 = \frac{1}{2} \tag{4.2.29}$$

This showed that (4.2.22) is a recursive solution that follows from the second order Runge-Kutta method.

The best known recursive equations suitable for a numerical solution of differential equations is the fourth order Runge-Kutta method that is of the form

$$x_{k+1} = x_k + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \tag{4.2.30}$$

where

$$\begin{array}{rcl} k_1 &=& hf(t_k, x_k) \\ k_2 &=& hf(t_k + \frac{1}{2}h, x_k + \frac{1}{2}k_1) \\ k_3 &=& hf(t_k + \frac{1}{2}h, x_k + \frac{1}{2}k_2) \\ k_4 &=& hf(t_k + h, x_k + k_3) \end{array} \tag{4.2.31}$$

4.2.3 Runge-Kutta method for a System of Differential Equations

The Runge-Kutta method can be used for the solution of a system of differential equations

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(t, \boldsymbol{x}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{4.2.32}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^T$.

Vectorised equivalents of equations (4.2.20), (4.2.22), (4.2.23), (4.2.24) are as follows

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{x}(t_k) + h\dot{\boldsymbol{x}}(t_k) + \frac{1}{2}h^2\ddot{\boldsymbol{x}}(t_k) \tag{4.2.33}$$

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \gamma_1\boldsymbol{k}_1 + \gamma_2\boldsymbol{k}_2 \tag{4.2.34}$$

$$\boldsymbol{k}_1 = h\boldsymbol{f}(t_k, \boldsymbol{x}_k) \tag{4.2.35}$$

$$\boldsymbol{k}_2 = h\boldsymbol{f}(t_k + \alpha_1 h, \boldsymbol{x}_k + \beta_1\boldsymbol{k}_1) \tag{4.2.36}$$

The programs for implementation of the fourth order Runge-Kutta method in various computer languages are given in the next example.

Example 4.2.1: *Programs for the solution of state-space equations*

We will explain the use of the fourth order Runge-Kutta method applied to the following second order differential equation

$$T_1T_2\frac{d^2y_2}{dt^2} + (T_1 + T_2)\frac{dy_2}{dt} + y_2 = Z_1u$$

with zero initial conditions and for $u(t) = 1(t)$. T_1, T_2 are time constants and Z_1 gain of this system. At first we transform this differential equation into state-space so that two differential equations of the first order result

$$\begin{array}{rcl} \dfrac{dx_1}{dt} &=& \dfrac{Z_1u - x_2 - (T_1 + T_2)x_1}{T_1T_2} \\ \dfrac{dx_2}{dt} &=& x_1 \end{array}$$

The program written in GW-BASIC is given in Program 4.2.1. The state-space differential equations are defined on lines 550, 560. The solution $y_1(t) = x_1(t)$, $y_2(t) = x_2(t)$ calculated with this program is given in Table 4.2.1. The values of variable $y_2(t)$ represent the step response of the system with transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Z_1}{(T_1 s + 1)(T_2 s + 1)}$$

Program 4.2.2 is written in C. The example of the solution in the simulation environment MATLAB/Simulink is given in Program 4.2.3. This represents `m-file` that can be introduced as S-function into Simulink block scheme shown in Fig. 4.2.1. The graphical solution is then shown in Fig. 4.2.2 and it is the same as in Tab. 4.2.1.

Program 4.2.1 (Simulation program in GW-BASIC)

```
5 REM ruku_4.bas
10 REM solution of the ODE system
20 REM  n  number of equations
30 REM  h  integration step
50 REM y(1),y(2),...,y(n)  initial conditions
55 DATA 2: READ n
58 DIM y(n), x(n), f(n), k(4, n)
60 DATA  .5,       0, 0
70 READ h
80 FOR i = 1 TO n: READ y(i): NEXT i
140 PRINT "t", "y1", "y2"
160 PRINT t, y(1), y(2)
200 FOR k = 0 TO 19
205   FOR i = 1 TO n: x(i) = y(i): NEXT i: GOSUB 470
240   FOR i = 1 TO n
242     k(1, i) = h * f(i) : x(i) = y(i) + k(1, i) / 2
244   NEXT i: GOSUB 470
290   FOR i = 1 TO n
292     k(2, i) = h * f(i): x(i) = y(i) + k(2, i) / 2
294   NEXT i: GOSUB 470
340   FOR i = 1 TO n
342     k(3, i) = h * f(i): x(i) = y(i) + k(3, i)
344   NEXT i: GOSUB 470
390   FOR i = 1 TO n
392     k(4, i) = h * f(i)
410     y(i) = y(i) + (k(1,i) + 2*k(2,i) + 2*k(3,i) + k(4,i)) / 6
420   NEXT i
430   t = t + h
440   PRINT t, y(1), y(2)
450 NEXT k
460 END
470 REM assignments
480 z1 = 1: te1 = 1: te2 = 2
510 u = 1
520 x1 = x(1): x2 = x(2)
540 REM funkcie
550 f(1) = (z1 * u - x2 - (te1 + te2) * x1) / (te1 * te2)
560 f(2) = x1
570 RETURN
```

Program 4.2.2 (Simulation program in C)

```
#include <stdio.h>

void rk45 (double *u,double *y, double *f, double dt);
void fun(double y[], double f[], double u[]);

#define N 2 /* number of ODEs */

int main(void)
{
  double t=0, tend=10, dt=0.5;
  double y[N], u[1];
  double f[N];

  u[0]=1;y[0]=0;y[1]=0;
  printf("%f  %f   %f\n",t, y[0],y[1]);
  do{
    rk45 (u, y, f, dt);
    t+=dt;
    printf("%f  %f   %f\n",t, y[0],y[1]);
  }while (t<tend);
  return 0;
}

void fun(double y[], double f[], double u[])
{
  static double te1=1, te2=2, z=1;
  f[0]=(z*u[0]-y[1]-(te1+te2)*y[0])/(te1*te2);
  f[1]=y[0];
}

void rk45 (double *u, double *y, double *f, double dt)
{
  int i,j;
  double yold[N], fh[4*N];

  for (i=0 ; i<N ; i++)
    yold[i]=y[i];
  for(i=0; i<4; i++){
    fun(y, f, u);
    for(j=0;j<N; j++){
      fh[i*N+j]=dt*f[j];
      if(i<2) y[j]=yold[j]+0.5*fh[i*N+j];
      if(i==2) y[j]=yold[j]+fh[i*N+j];
    }
  }
  for(i=0; i<N; i++)
    y[i]=yold[i]+(fh[i]+2.0*(fh[N+i]+fh[2*N+i])+fh[3*N+i])/6.0;

}
```

Program 4.2.3 (Source code in MATLAB)

```
function [sys, x0] = nfs1m(t,x,u,flag,a)
% System transfer function;
```

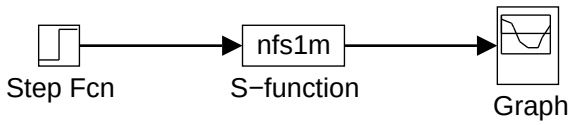

Figure 4.2.1: Simulink block scheme.

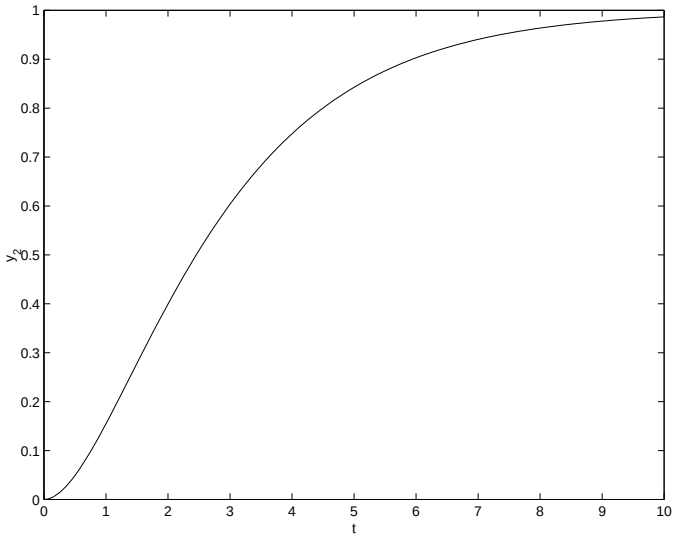

Figure 4.2.2: Results from simulation.

```
%              b0+b1*s+b2*s^2      ;
%      F(s)=  ----------------     ;
%              a0+a1*s+a2*s^2      ;

% a0=a(1), a1=a(2), a2=a(3), b0=a(4), b1=a(5), b2=a(6);

if flag ==0
        sys = [2,0,1,1,0,0];
        x0 = [0,0];
elseif abs(flag) == 1

% Right hand sides of ODEs;

        sys(1)=-(a(2)/a(3))*x(1)+x(2)+(a(5)-(a(2)/a(3))*a(6))*u(1);
        sys(2)=-(a(1)/a(3))*x(1)     +(a(4)-(a(1)/a(3))*a(6))*u(1);
elseif flag == 3

% Output signal;

        sys(1)=(1/a(3))*x(1)+(a(6)/a(3))*u(1);
else
        sys = [];
end
```

Table 4.2.1: Solution of the second order differential equation

t	y1	y2
0	0	0
.5	.1720378	.0491536
1	.238372	.1550852
1.5	.2489854	.2786338
2	.2323444	.3997614
2.5	.2042715	.5092093
3	.1732381	.6036183
3.5	.1435049	.6827089
4	.1169724	.7476815
4.5	.0926008	.8003312
5	.07532854	.8425783
5.5	.05983029	.8762348
6	.04730249	.9029045
6.5	.03726806	.9239527
7	.02928466	.9405137
7.5	.02296489	.9535139
8	.01798097	.9637005
8.5	.01406181	.9716715
9	.01098670	.9779023
9.5	.00857792	.9827687
10	.00669353	.9865671

4.2.4 Time Responses of Liquid Storage Systems

Consider the liquid storage system shown in Fig. 2.2.1. Assume its mathematical model in the form

$$F_1 \frac{dh_1}{dt} + c_1 h_1 + c_2 \sqrt{h_1} = q_0 \tag{4.2.37}$$

where c_1 and c_2 are constants obtained from measurements on a real process. The steady-state is given by the following equation

$$c_1 h_1^s + g(h_1^s) = q_0^s, \quad \text{where} \quad g(h_1^s) = c_2 \sqrt{h_1^s} \tag{4.2.38}$$

Let us introduce the deviation variables

$$\begin{aligned} x_1 &= h_1 - h_1^s \\ u_1 &= q_0 - q_0^s \end{aligned} \tag{4.2.39}$$

The mathematical model can then be rewritten as

$$F_1 \frac{dx_1}{dt} + c_1 x_1 + c_1 h_1^s + c_2 \sqrt{x_1 + h_1^s} = u_1 + q_0^s \tag{4.2.40}$$

Substracting (4.2.37) from (4.2.40) yields

$$F_1 \frac{dx_1}{dt} + c_1 x_1 + c_2 \sqrt{x_1 + h_1^s} - c_2 \sqrt{h_1^s} = u_1 \tag{4.2.41}$$

Let us introduce the function

$$G(x_1) = g(x_1 + h_1^s) - g(h_1^s) \tag{4.2.42}$$

then the mathematical model is finally given with deviation variables as

$$F_1 \frac{dx_1}{dt} + c_1 x_1 + G(x_1) = u_1 \tag{4.2.43}$$

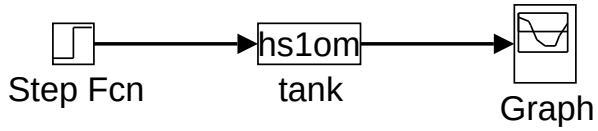

Figure 4.2.3: Simulink block scheme for the liquid storage system.

The Simulink block scheme that uses the MATLAB file `hs1om.m` (Program 4.2.4) as the S-function for solution of Eq. (4.2.43) is shown in Fig.4.2.3.

Program 4.2.4 (MATLAB file hs1om.m)

```
function [sys, x0] = hs10m(t,x,u,flag)
% Deviation model of the level tank;

% F1*(dx1/dt) + c1*x1 + c2*(x1+h1s)^(1/2) - c2*(h1s)^(1/2) = u1 ;
% h1s =1.5 dm, q0s= 0.006359 dm^3/s    ;
% c1= 0.00153322 dm^2/s ,  c2 = 0.00331442 (dm^5/2)/s      ;

% F1 = 1.44dm^2,  Step change q00s = new value for t>=0;
% u1 is constrained as <-0.006359, 0.004161 dm^3/s>;

if flag ==0
        sys = [1,0,1,1,0,0];
        x0 = [0];
elseif abs(flag) == 1
        c1 = 0.00153322;
        f1 = 1.44;
        a1 = -(c1/f1);
        c2 = 0.00331442;
        a2 = -(c2/f1);
        b1 = 1/f1;
        h1s = 1.5;
% Right hand sides;

        sys(1)=a1*x(1)+a2*((x(1)+h1s)^(1/2))-a2*(h1s)^(1/2)+b1*u(1) ;

elseif flag == 3

% Output variable;

        sys(1)=x(1);
else
        sys = [];
end
```

Linearised mathematical model in the neighbourhood of the steady-state $x_1 = 0$ is

$$\frac{dx_1}{dt} = -\frac{c_1}{F_1}x_1 - \frac{1}{F_1}\frac{\partial G(0)}{\partial x_1}x_1 + \frac{1}{F_1}u_1 \tag{4.2.44}$$

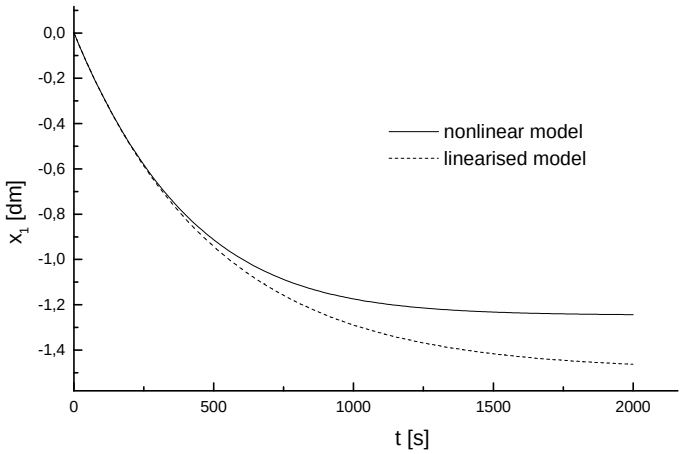

Figure 4.2.4: Response of the tank to step change of q_0.

where

$$\frac{\partial G}{\partial x_1} = \frac{c_2}{2}\frac{1}{\sqrt{x_1 + h_1^s}} \quad , \quad \frac{\partial G(0)}{\partial x_1} = \frac{c_2}{2}\frac{1}{\sqrt{h_1^s}}$$

and finally

$$\frac{dx_1}{dt} = \left(-\frac{c_1}{F_1} - \frac{c_2}{2F_1\sqrt{h_1^s}}\right)x_1 + \frac{1}{F_1}u_1 \tag{4.2.45}$$

Fig. 4.2.4 shows the response of the tank to step change of the flow q_0 equal to $-0.0043\text{dm}^3\text{s}^{-1}$. The steady-state before the change was characterised by the flow $q_0^s = 0.006359\text{dm}^3\text{s}^{-1}$ and the tank level $h_1^s = 1.5\text{dm}$. The coefficients c_1, c_2 and the crossover area F_1 corresponding to the real liquid storage tank are

$$\begin{aligned} c_1 &= 1.53322.10^{-3}\text{dm}^2\text{s}^{-1} \\ c_2 &= 3.31142.10^{-3}\text{dm}^{2.5}\text{s}^{-1} \\ F_1 &= 1.44\text{dm}^2 \end{aligned}$$

Fig. 4.2.4 also shows the step response of the linearised model (4.2.45). Both curves can serve for analysis of the error resulting from linearisation of the mathematical model of the tank.

4.2.5 Time Responses of CSTR

Consider a CSTR with a cooling jacket. In the reactor, an irreversible exothermic reaction takes place. We assume that its mathematical model is in the form

$$\frac{dc_A}{dt'} = \frac{q}{V}c_{Av} - c_A\left(\frac{q}{V} + k\right) \tag{4.2.46}$$

$$\frac{d\vartheta}{dt'} = \frac{q}{V}\vartheta_v - \frac{q}{V}\vartheta - \frac{\alpha F}{V\rho c_p}(\vartheta - \vartheta_c) + \frac{kH_r}{\rho c_p}c_A \tag{4.2.47}$$

$$\frac{d\vartheta_c}{dt'} = \frac{q_c}{V_c}\vartheta_{cv} - \frac{q_c}{V_c}\vartheta_c + \frac{\alpha F}{V_c\rho_c c_{pc}}(\vartheta - \vartheta_c) \tag{4.2.48}$$

$$k = k_0 \exp\left(-\frac{g}{\vartheta}\right) \tag{4.2.49}$$

where

t' - time variable,

c_A - molar concentration of A (mole/volume) in the outlet stream,

c_{Av} - molar concentration of A (mole/volume) in the inlet stream,

V - reactor volume,

q - volumetric flow rate,

ϑ - temperature of reaction mixture,

ϑ_v - temperature in the inlet stream,

F - heat transfer area,

α - overall heat transfer coefficient,

ρ - liquid density,

c_p - liquid specific heat capacity,

H_r - heat of reaction,

ϑ_c - cooling temperature,

ϑ_{cv} - cooling temperature in the inlet cooling stream,

q_c - volumetric flow rate of coolant,

V_c - coolant volume in the jacket,

ρ_c - coolant density,

c_{pc} - coolant specific heat capacity,

k_0 - frequency factor,

g - activation energy divided by the gas constant.

Compared to the CSTR model from page 37 we included the equation (4.2.48) describing the behaviour of cooling temperature ϑ_c. The state-space variables are $c_A, \vartheta, \vartheta_c$, the input variable is q_c, and the output variable is ϑ. We assume that the variables $c_{Av}, \vartheta_v, \vartheta_{cv}$ are held constant and the other parameters of the reactor are also constant.

The reactor model can be rewritten as

$$\frac{dx'_1}{dt} = 1 - x'_1 - f'_1(x'_1, x'_2) \tag{4.2.50}$$

$$\frac{dx'_2}{dt} = b'_1 - x'_2 - f'_2(x'_2) + f'_3(x'_3) + f'_1(x'_1, x'_2) \tag{4.2.51}$$

$$\frac{dx'_3}{dt} = f'_4(x'_2) - f'_5(x'_3) - f'_6(u', x'_3) + f'_7(u') \tag{4.2.52}$$

$$k = k_0 \exp(-\frac{g\rho c_p}{c^s_{Av} H_r x'_2}) \tag{4.2.53}$$

where

$$\begin{aligned}
x_1' &= \frac{c_A}{c_{Av}^s} & , \quad x_2' &= \frac{\rho c_p \vartheta}{c_{Av}^s H_r} \\
x_3' &= \frac{\rho_c c_{pc} \vartheta_c}{c_{Av}^s H_r} & , \quad t &= \frac{q}{V} t' \\
u' &= \frac{q_c}{q_c^s} & , \quad b_1' &= \frac{\rho c_p \vartheta_v^s}{c_{Av}^s H_r} \\
f_1'(x_1', x_2') &= \frac{V}{q} x_1' k_0 \exp(-\frac{g \rho c_p}{c_{Av}^s H_r x_2'}) & , \quad f_2'(x_2') &= \frac{\alpha F}{V \rho c_p} x_2' \\
f_3'(x_3') &= \frac{\alpha F}{V \rho_c c_{pc}} x_3' & , \quad f_4'(x_2') &= \frac{\alpha F V}{q V_c \rho c_p} x_2' \\
f_5'(x_3') &= \frac{\alpha F V}{q V_c \rho_c c_p} x_3' & , \quad f_6'(u', x_3') &= \frac{V q_c^s}{q V_c} u' x_3' \\
f_7'(u') &= \frac{\rho_c c_{pc} V q_c^s \vartheta_{cv}^s}{c_{Av}^s H_r q V_c}
\end{aligned}$$

and variables with superscript ^{s} denote steady-state values.

In the steady-state for state variables x_1', x_2', x_3' holds

$$0 = 1 - x_1'^s - f_1'(x_1'^s, x_2'^s) \tag{4.2.54}$$

$$0 = b_1' - x_2'^s - f_2'(x_2'^s) + f_3'(x_3'^s) + f_1'(x_1'^s, x_2'^s) \tag{4.2.55}$$

$$0 = f_4'(x_2'^s) - f_5'(x_3'^s) - f_6'(u'^s, x_3'^s) + f_7'(u'^s) \tag{4.2.56}$$

We define deviation variables and functions

$$x_1 = x_1' - x_1'^s \tag{4.2.57}$$

$$x_2 = x_2' - x_2'^s \tag{4.2.58}$$

$$x_3 = x_3' - x_3'^s \tag{4.2.59}$$

$$u = u' - u'^s \tag{4.2.60}$$

$$f_1(x_1, x_2) = f_1'(x_1 + x_1'^s, x_2 + x_2'^s) - f_1'(x_1'^s, x_2'^s) \tag{4.2.61}$$

$$f_2(x_2) = f_2'(x_2 + x_2'^s) - f_2'(x_2'^s) \tag{4.2.62}$$

$$f_3(x_3) = f_3'(x_3 + x_3'^s) - f_3'(x_3'^s) \tag{4.2.63}$$

$$f_4(x_2) = f_4'(x_2 + x_2'^s) - f_4'(x_2'^s) \tag{4.2.64}$$

$$f_5(x_3) = f_5'(x_3 + x_3'^s) - f_5'(x_3'^s) \tag{4.2.65}$$

$$f_6(u, x_3) = f_6'(u + u_1'^s, x_3 + x_3'^s) - f_6'(u'^s, x_3'^s) \tag{4.2.66}$$

$$f_7(u) = f_7'(u + u'^s) - f_7'(u'^s) \tag{4.2.67}$$

The original mathematical model of the reactor given by (4.2.46) - (4.2.48) can finally be rewritten as

$$\frac{dx_1}{dt} = -x_1 - f_1(x_1, x_2) \tag{4.2.68}$$

$$\frac{dx_2}{dt} = -x_2 - f_2(x_2) + f_3(x_3) + f_1(x_1, x_2) \tag{4.2.69}$$

$$\frac{dx_3}{dt} = f_4(x_2) - f_5(x_3) - f_6(u, x_3) + f_7(u) \tag{4.2.70}$$

Figure 4.2.5 shows the Simulink block-scheme that uses the program `rea7m.m` (Program 4.2.5) as its S-function for the solution of the differential equations (4.2.68) - (4.2.70).

Program 4.2.5 (MATLAB file rea7m.m)

```
function [sys,x0] = rea7m(t,x,u,flag)
% Non-linear deviation model of the CSTR;
% 3 equations, Reaction of the type A ----> B;
```

```
%   k0 = a1        [1/min];
%   g = a2         [K];
%   ro = a3        [kg/m3];
%   cp = a4        [kJ/kg.K];
%   V = a5         [m3];
%   Hr = a6        [kJ/kmol];
%   F = a7         [m2];
%   al = a8        [kJ/m2.min.K];
%   roc = a9       [kg/m3];
%   cpc = a10      [kJ/kg.K];
%   Vc = a11       [m3];
%   qs = a12       [m3/min];
%   Thvs = a13     [K];
%   cavs = a14     [kmol/m3];
%   Thcvs = a15    [K];
%   qcs = a16      [m3/min];
%   cas = a17      [kmol/m3];
%   Ths = a18      [K];
%   Thcs = a19     [K];

%   x(1) - concentration of A, dimensionless, deviation;
%   x(2) - temperature of the mixture, dimensionless, deviation;
%   x(3) - jacket temperature, dimensionless, deviation;
%   u(1) - cooling flowrate, dimensionless, deviation;
%   u=qc/qcs ;
%   u(1)=<-0.2,   0.2> = cca<-0.001m3/min,0.001m3/min>;

% Calculation of f1c, f2c,...,f7c in the steady-state;
  a1=10000000000000;
  a2=11078;
  a3=970;
  a4=4.05;
  a5=1.8;
  a6=149280;
  a7=5.04;
  a8=130;
  a9=998;
  a10=4.18;
  a11=0.7;
  a12=0.25;
  a13=370;
  a14=0.88;
  a15=285;
  a16=0.05;
  a17=0.0345;
  a18=385.877;
  a19=361.51753;


  x1cs = a17/a14;
  x2cs = (a3*a4*a18)/(a14*a6);
  x3cs = (a9*a10*a19)/(a14*a6);
  u1cs = a16/a16;
```

```
  f1cs = ((a5/a12)*x1cs*a1)*(exp((-a2*a3*a4)/(a14*a6*x2cs)));
  f2cs = ((a7*a8)/(a12*a3*a4))*x2cs;
  f3cs = ((a7*a8)/(a12*a9*a10))*x3cs;
  f4cs = ((a7*a8*a5)/(a12*a11*a3*a4))*x2cs;
  f5cs = ((a5*a7*a8)/(a12*a11*a9*a10))*x3cs;
  f6cs = ((a5*a16)/(a12*a11))*u1cs*x3cs;
  f7cs = ((a9*a10*a5*a16*a15)/(a14*a6*a12*a11))*u1cs;

if flag ==0
        sys = [3,0,6,1,0,0];
x0 = [0,0,0];

elseif abs(flag) == 1
% Deviation definitions;
  x1c = x(1) + x1cs;
  x2c = x(2) + x2cs;
  x3c = x(3) + x3cs;
  u1c = u(1) + u1cs;
  f1c = ((a5/a12)*x1c*a1)*(exp((-a2*a3*a4)/(a14*a6*x2c)));
  f2c = ((a7*a8)/(a12*a3*a4))*x2c;
  f3c = ((a7*a8)/(a12*a9*a10))*x3c;
  f4c = ((a7*a8*a5)/(a12*a11*a3*a4))*x2c;
  f5c = ((a5*a7*a8)/(a12*a11*a9*a10))*x3c;
  f6c = ((a5*a16)/(a12*a11))*u1c*x3c;
  f7c = ((a9*a10*a5*a16*a15)/(a14*a6*a12*a11))*u1c;
  f1 = f1c - f1cs;
  f2 = f2c - f2cs;
  f3 = f3c - f3cs;
  f4 = f4c - f4cs;
  f5 = f5c - f5cs;
  f6 = f6c - f6cs;
  f7 = f7c - f7cs;

% Right hand sides of ODEs;
  sys(1)=-x(1) - f1;
  sys(2)=-x(2) - f2 + f3 + f1;
  sys(3)=f4 - f5 - f6 + f7;
elseif flag == 3
% Output variables;
  sys(1)=x(1);
  sys(4)=a14*x(1);
  sys(2)=x(2);
  sys(5)=(a14*a6*x(2))/(a3*a4);
  sys(3)=x(3);
  sys(6)=(a14*a6*x(3))/(a9*a10);
else
  sys = [];
end
```

Linearised mathematical model of the CSTR with steady-state $x_1 = x_2 = x_3 = 0$ is of the form

$$\frac{dx_1}{dt} = -x_1 - \frac{\partial f_1(0,0)}{\partial x_1}x_1 - \frac{\partial f_1(0,0)}{\partial x_2}x_2 \tag{4.2.71}$$

$$\frac{dx_2}{dt} = -x_2 - \frac{\partial f_2(0)}{\partial x_2}x_2 + \frac{\partial f_3(0)}{\partial x_3}x_3 + \frac{\partial f_1(0,0)}{\partial x_1}x_1 + \frac{\partial f_1(0,0)}{\partial x_2}x_2 \tag{4.2.72}$$

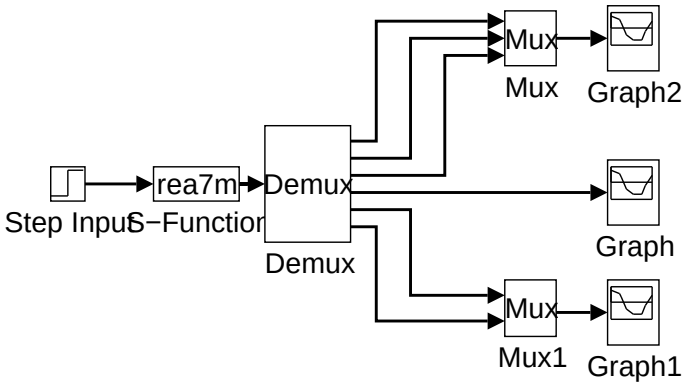

Figure 4.2.5: Simulink block scheme for the nonlinear CSTR model.

$$\begin{aligned}\frac{dx_3}{dt} &= \frac{\partial f_4(0)}{\partial x_2}x_2 - \frac{\partial f_5(0)}{\partial x_3}x_3 \\ &\quad -\frac{\partial f_6(0,0)}{\partial x_3}x_3 - \frac{\partial f_6(0,0)}{\partial u}u + \frac{\partial f_7(0)}{\partial u}u \end{aligned} \tag{4.2.73}$$

or

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \tag{4.2.74}$$

where

$$\begin{aligned}\boldsymbol{x} &= (x_1, x_2, x_3)^T \\ \boldsymbol{A} &= \begin{pmatrix} a_{s11}, a_{s12}, a_{s13} \\ a_{s21}, a_{s22}, a_{s23} \\ a_{s31}, a_{s32}, a_{s33} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ b_{s31} \end{pmatrix}\end{aligned}$$

$$\begin{aligned} a_{s11} &= -1 - \frac{\partial f_1(0,0)}{\partial x_1} & , \quad a_{s12} &= -\frac{\partial f_1(0,0)}{\partial x_2} \\ a_{s13} &= 0 & , \quad a_{s21} &= \frac{\partial f_1(0,0)}{\partial x_1} \\ a_{s22} &= -1 - \frac{\partial f_2(0)}{\partial x_2} + \frac{\partial f_1(0,0)}{\partial x_2} & , \quad a_{s23} &= \frac{\partial f_3(0)}{\partial x_3} \\ a_{s31} &= 0 & , \quad a_{s32} &= \frac{\partial f_4(0)}{\partial x_2} \\ a_{s33} &= -\frac{\partial f_5(0)}{\partial x_3} + \frac{\partial f_6(0,0)}{\partial x_3} & , \quad b_{s31} &= \frac{\partial f_7(0)}{\partial u} - \frac{\partial f_6(0,0)}{\partial u} \end{aligned}$$

Figures 4.2.6, 4.2.7, 4.2.8 show responses of the CSTR to step change of q_c ($\Delta u = 10$). Numerical values of all variables are given in Program 4.2.6.

File `kolire8.m` (Program 4.2.6) calculates for the given steady-state of the CSTR parameters of the corresponding linearised mathematical model.

Program 4.2.6 (MATLAB file kolire8.m)

```
% Linearised model of the CSTR, 3 equations;
% state-space equations, Reaction A ----> B;

%   k0 = a1        [1/min];
%   g = a2         [K];
%   ro = a3        [kg/m3];
%   cp = a4        [kJ/kg.K];
```

```
%   V = a5         [m3];
%   Hr = a6        [kJ/kmol];
%   F = a7         [m2];
%   al = a8        [kJ/m2.min.K];
%   roc = a9       [kg/m3];
%   cpc = a10      [kJ/kg.K];
%   Vc = a11       [m3];
%   qs = a12       [m3/min];
%   Thvs = a13     [K];
%   cavs = a14     [kmol/m3];
%   Thcvs = a15    [K];
%   qcs = a16      [m3/min];
%   cas = a17      [kmol/m3];
%   Ths = a18      [K];
%   Thcs = a19     [K];

%   x(1) - concentration of A, dimensionless, deviation;
%   x(2) - temperature of the mixture, dimensionless, deviation;
%   x(3) - jacket temperature, dimensionless, deviation;
%   u(1) - cooling flowrate, dimensionless, deviation;
%   u=qc/qcs ;
%   u(1)=<-0.2,   0.2> = cca<-0.001m3/min,0.001m3/min>;

a1=10000000000000; a2=11078;
a3=970; a4=4.05;
a5=1.8; a6=149280;
a7=5.04; a8=130;
a9=998; a10=4.18;
a11=0.7; a12=0.25;
a13=370; a14=0.88;
a15=285; a16=0.05;
a17=0.0345; a18=385.877;
a19=361.51753;

% Calculation of the state-space coefficients;
as11=-1-((a5*a1)/a12)*(exp(-a2/a18));
as12=-((a5*a1*a6*a2*a17)/(a12*a3*a4*((a18)^2)))*(exp(-a2/a18));
as13=0;
as21=((a5*a1)/a12)*(exp(-a2/a18));
as22=-1-((a7*a8)/(a12*a3*a4))
as22=as22 +((a5*a1*a6*a2*a17)/(a12*a3*a4*((a18)^2)))*(exp(-a2/a18));
as23=(a7*a8)/(a12*a9*a10);
as31=0;
as32=(a7*a8*a5)/(a12*a11*a3*a4);
as33=-(a7*a8*a5)/(a12*a11*a9*a10)-(a5*a16)/(a12*a11);
bs11=0;
bs21=0;
bs31=-(a5*a16*a9*a10*a19)/(a12*a11*a14*a6);
bs31=bs31+(a9*a10*a5*a16*a15)/(a14*a6*a12*a11);
```

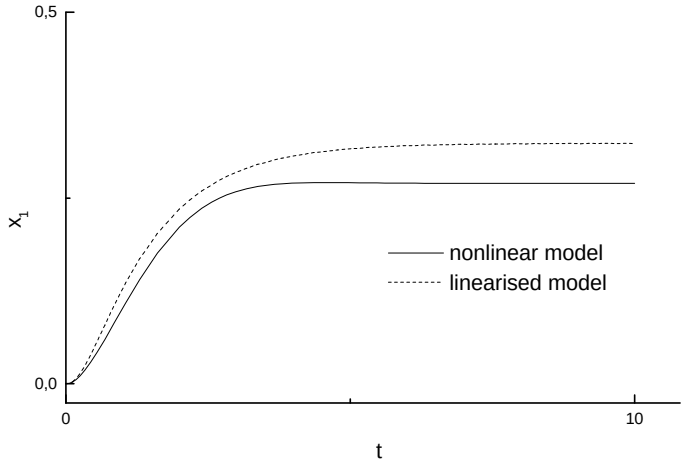

Figure 4.2.6: Responses of dimensionless deviation output concentration x_1 to step change of q_c.

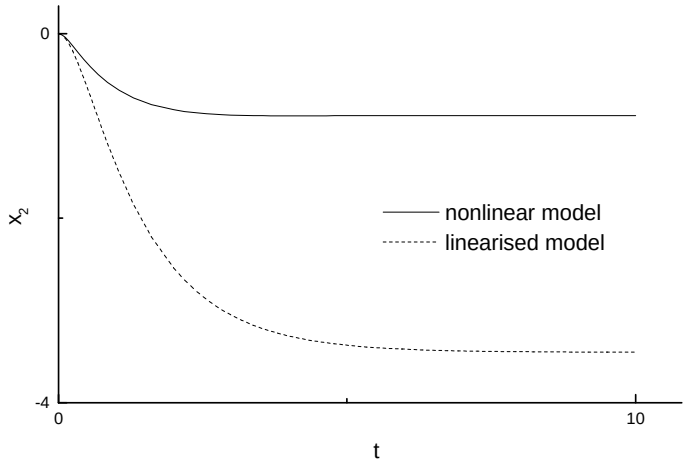

Figure 4.2.7: Responses of dimensionless deviation output temperature x_2 to step change of q_c.

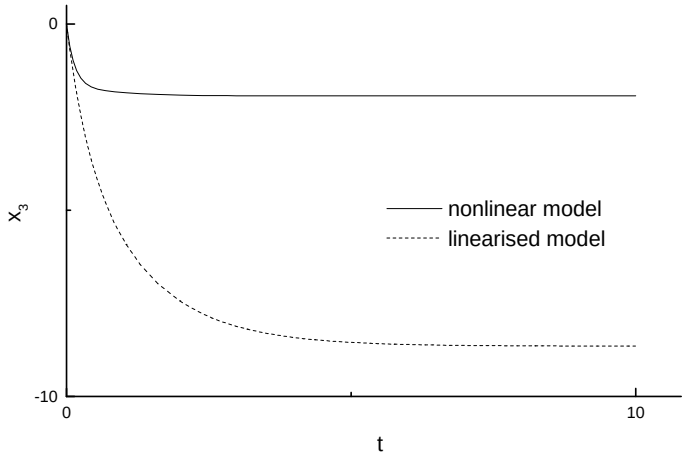

Figure 4.2.8: Responses of dimensionless deviation cooling temperature x_3 to step change of q_c.

4.3 Frequency Analysis

4.3.1 Response of the Heat Exchanger to Sinusoidal Input Signal

Consider a jacketed heat exchanger described by the differential equation (see Example 3.3.5)

$$T_1\frac{dy}{dt}+y=Z_1u \tag{4.3.1}$$

where $y=\vartheta-\vartheta^s$ is the deviation of the liquid temperature from its steady state value, $u=\vartheta_p-\vartheta_p^s$ is the deviation of the jacket temperature from its steady state value, T_1 is the process time constant, and Z_1 is the process steady-state gain.

The process transfer function of (4.3.1) is given as

$$G(s)=\frac{Y(s)}{U(s)}=\frac{Z_1}{T_1s+1} \tag{4.3.2}$$

Let the following sinusoidal signal with amplitude A_1 and frequency ω influence the heat exchanger:

$$u(t)=A_1\sin\omega t\ 1(t) \tag{4.3.3}$$

Then the Laplace transform of the process output can be written as

$$Y(s) = \frac{Z_1}{T_1s+1}\frac{A_1\omega}{s^2+\omega^2} \tag{4.3.4}$$

$$Y(s) = \frac{Z_1A_1\omega}{T_1s^3+s^2+T_1\omega^2s+\omega^2} \tag{4.3.5}$$

$$Y(s) = \frac{\frac{Z_1A_1\omega}{T_1}}{s^3+\frac{1}{T_1}s^2+\omega^2s+\frac{\omega^2}{T_1}} \tag{4.3.6}$$

The denominator roots are:

$$s_1=-j\omega,\ s_2=j\omega,\ s_3=-1/T_1$$

After taking the inverse Laplace transform the output of the process is described by

$$\begin{aligned} y(t) = {} & \frac{\frac{Z_1A_1}{T_1}\omega(-j\omega)}{-j\omega[3(-j\omega)^2+\frac{2}{T_1}(-j\omega)+\omega^2]}e^{-j\omega t}+ \\ & +\frac{\frac{Z_1A_1}{T_1}\omega(j\omega)}{j\omega[3(j\omega)^2+\frac{2}{T_1}(j\omega)+\omega^2]}e^{j\omega t}+ \\ & +\frac{\frac{Z_1A_1}{T_1}\omega(-\frac{1}{T_1})}{-\frac{1}{T_1}[2(-\frac{1}{T_1})^2+\frac{2}{T_1}(-\frac{1}{T_1})+\omega^2]}e^{-\frac{t}{T_1}} \end{aligned} \tag{4.3.7}$$

$$y(t) = \frac{Z_1A_1}{-2\omega T_1-2j}e^{-j\omega t}+\frac{Z_1A_1}{-2\omega T_1+2j}e^{j\omega t}+Ke^{-\frac{t}{T_1}} \tag{4.3.8}$$

where $K=(2\omega T_1A_1)/(2T_1-2+\omega^2T_1^2)$.

If the sine wave is continued for a long time, the exponential term disappears and the remaining terms can be further manipulated to yield

$$y(t) = Z_1A_1\left[\frac{1}{-2\omega T_1-2j}\frac{-2\omega T_1+2j}{-2\omega T_1+2j}e^{-j\omega t}+\frac{1}{-2\omega T_1+2j}\frac{-2\omega T_1-2j}{-2\omega T_1-2j}e^{j\omega t}\right] \tag{4.3.9}$$

$$y(t) = Z_1A_1\left[\frac{-\omega T_1+j}{2(\omega^2T_1^2+1)}e^{-j\omega t}+\frac{-\omega T_1-j}{2(\omega^2T_1^2+1)}e^{j\omega t}\right] \tag{4.3.10}$$

$$y(t) = Z_1A_1\left[\frac{-\omega T_1}{(\omega^2T_1^2+1)}\frac{e^{-j\omega t}+e^{j\omega t}}{2}+\frac{1}{(\omega^2T_1^2+1)}\frac{e^{j\omega t}-e^{-j\omega t}}{2}\right] \tag{4.3.11}$$

Applying the Euler identities (3.1.15) yields

$$y(t) = Z_1 A_1 \left[-\frac{\omega T_1}{\omega^2 T_1^2 + 1} \cos \omega t + \frac{1}{\omega^2 T_1^2 + 1} \sin \omega t \right] \tag{4.3.12}$$

Finally, using the trigonometric identity

$$\sin(\omega t + \varphi) = \sin \varphi \cos \omega t + \cos \varphi \sin \omega t$$

gives

$$y(t) = Z_1 A_1 \left[\frac{\sqrt{\omega^2 T_1^2 + 1}}{\omega^2 T_1^2 + 1} \sin(\omega t + \varphi) \right] \tag{4.3.13}$$

where $\varphi = -\arctan \omega T_1$.

If we set in (4.3.2) $s = j\omega$, then

$$G(j\omega) = \frac{Z_1}{T_1 j\omega + 1} \tag{4.3.14}$$

$$|G(j\omega)| = \frac{Z_1}{\sqrt{\omega^2 T_1^2 + 1}} \tag{4.3.15}$$

which is the same as the amplitude in (4.3.13) divided by A_1. Thus $y(t)$ can also be written as

$$y(t) = A_1 |G(j\omega)| \sin(\omega t + \varphi) \tag{4.3.16}$$

It follows from (4.3.13) that the output amplitude is a function of the input amplitude A_1, input frequency ω, and the system properties. Thus,

$$A_1 |G(j\omega)| = A_1 f(\omega, Z_1, T_1). \tag{4.3.17}$$

For the given system with the constants Z_1 and T_1, it is clear that increasing input frequency results in decreasing output amplitude. The phase lag is given as

$$\varphi = -\arctan T_1 \omega \tag{4.3.18}$$

and the influence of the input frequency ω to φ is opposite to amplitude.

The simulation of $u(t)$ and $y(t)$ from the equations (4.3.3) and (4.3.13) for $Z_1 = 0.4$, $T_1 = 5.2$ min, $A_1 = 5^o C$, and $\omega = 0.2$rad/min is shown in Fig. 4.3.1.

4.3.2 Definition of Frequency Responses

A time periodic function $f(t)$ with the period T_f satisfying the Dirichlet conditions can be expanded into the Fourier expansion

$$f(t) = \frac{2}{T_f} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_f t + b_n \sin n\omega_f t) \right] \tag{4.3.19}$$

where $\omega_f = 2\pi/T_f$ is the basic circular frequency. The coefficients a_0, a_n, b_n $(n = 1, 2, \ldots)$ are given as

$$a_0 = \int_{-T_f/2}^{T_f/2} f(\tau) d\tau$$

$$a_n = \int_{-T_f/2}^{T_f/2} f(\tau) \cos \frac{2\pi n \tau}{T_f} d\tau$$

$$b_n = \int_{-T_f/2}^{T_f/2} f(\tau) \sin \frac{2\pi n \tau}{T_f} d\tau$$

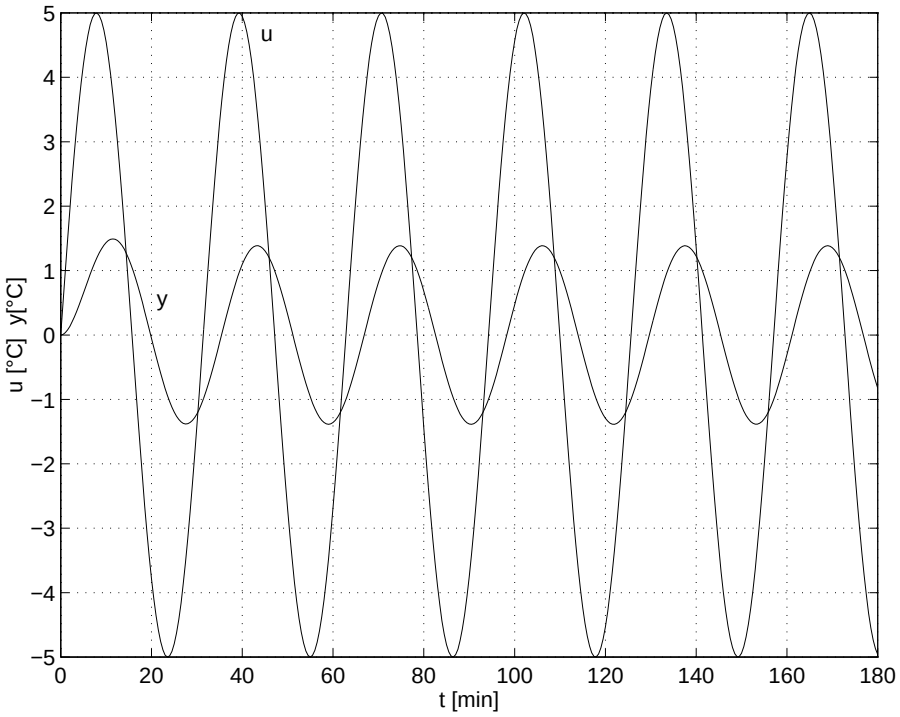

Figure 4.3.1: Ultimate response of the heat exchanger to sinusoidal input.

Using the Euler identity, the Fourier expansion can be rewritten as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_f t} \tag{4.3.20}$$

where

$$c_n = \frac{1}{T_f} \int_{-T_f/2}^{T_f/2} f(\tau) e^{-jn\omega_f \tau} d\tau$$

Any nonperiodic time function can be assumed as periodic with the period approaching infinity. If we define $\omega = n\omega_f$ then

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left\{ \int_{-T_f/2}^{T_f/2} f(\tau) e^{-j\omega\tau} d\tau \right\} e^{j\omega t} [(n+1)\omega_f - n\omega_f] \tag{4.3.21}$$

If $T_f \to \infty$ and $[(n+1)\omega_f - n\omega_f] = \Delta\omega_f \to 0$ then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \tag{4.3.22}$$

or

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \tag{4.3.23}$$

The equations (4.3.22), (4.3.23) are *the Fourier integral equations* that describe the influence between the original time function and its frequency transform $F(j\omega)$. Compared to the Fourier expansion they describe an expansion to the continuous spectrum with the infinitesimally small difference $d\omega$ of two neighbouring harmonic signals.

The integral (4.3.23) expresses a transformation (or operator) that assigns to the function $f(t)$ the function $F(j\omega)$. This transformation is called *the Fourier transform* of $f(t)$. If we know the transformed $F(j\omega)$, the original function $f(t)$ can be found from (4.3.22) by the inverse Fourier transform. The important condition of the Fourier transform is the existence of the integrals (4.3.22), (4.3.23).

Complex transfer function, or *frequency transfer function* $G(j\omega)$ is the Fourier transform corresponding to the transfer function $G(s)$. For $G(j\omega)$ holds

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} \tag{4.3.24}$$

If the frequency transfer function exists, it can be easily obtained from the system transfer function by formal exchange of the argument s,

$$G(j\omega) = G(s)_{s=j\omega} \tag{4.3.25}$$

or

$$G(j\omega) = \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_0}{a_n(j\omega)^n + b_{n-1}(j\omega)^{n-1} + \cdots + a_0} \tag{4.3.26}$$

The frequency transfer function of a singlevariable system can be obtained from

$$G(j\omega) \equiv \int_0^\infty g(t)e^{-j\omega t}dt. \tag{4.3.27}$$

Analogously, for multivariable systems follows

$$\boldsymbol{G}(j\omega) \equiv \int_0^\infty \boldsymbol{g}(t)e^{-j\omega t}dt. \tag{4.3.28}$$

If the transfer function matrix $\boldsymbol{G}(s)$ is stable, then *frequency transfer function matrix* exists as the Fourier transform of the impulse response matrix and can be calculated from (see equations (3.2.8), (4.1.8))

$$\boldsymbol{G}(j\omega) = \boldsymbol{C}(j\omega\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} \tag{4.3.29}$$

Frequency transfer function is a complex variable corresponding to a real variable ω that characterises the forced oscillations of the output $y(t)$ for the harmonic input $u(t)$ with frequency ω. Harmonic functions can be mathematically described as

$$\bar{u} = A_1 e^{j(\omega t+\varphi_1)} \tag{4.3.30}$$

$$\bar{y} = A_2 e^{j(\omega t+\varphi_2)}. \tag{4.3.31}$$

The ratio of these functions is a complex variable $G(j\omega)$ defined by (4.3.27). Thus it can be written as

$$\frac{\bar{y}}{\bar{u}} = \frac{A_2}{A_1}e^{j(\varphi_2-\varphi_1)} = G(j\omega). \tag{4.3.32}$$

The *magnitude* of $G(j\omega)$

$$|G(j\omega)| = A(\omega) \tag{4.3.33}$$

is given as the ratio A_2/A_1 of output and input variable magnitudes. The *phase angle*

$$\varphi(\omega) = \varphi_2 - \varphi_1 \tag{4.3.34}$$

is determined as the phase shift between the output and input variable and is given in units of radians as a function of ω. $G(j\omega)$ can be written in the polar form

$$G(j\omega) = A(\omega)e^{j\varphi(\omega)} \tag{4.3.35}$$

The graph of $G(j\omega)$

$$G(j\omega) = |G(j\omega)|e^{j \arg G(j\omega)} = \Re[G(j\omega)] + j\Im[G(j\omega)] \tag{4.3.36}$$

in the complex plane is called the *Nyquist diagram*. The magnitude and phase angle can be expressed as follows:

$$|G(j\omega)| = \sqrt{\{\Re[G(j\omega)]\}^2 + \{\Im[G(j\omega)]\}^2} \tag{4.3.37}$$

$$|G(j\omega)| = \sqrt{G(j\omega)G(-j\omega)} \tag{4.3.38}$$

$$\tan\varphi = \frac{\Im[G(j\omega)]}{\Re[G(j\omega)]} \tag{4.3.39}$$

$$\varphi = \arctan\frac{\Im[G(j\omega)]}{\Re[G(j\omega)]} \tag{4.3.40}$$

Essentially, the Nyquist diagram is a polar plot of $G(j\omega)$ in which frequency ω appears as an implicit parameter.

The function $A = A(\omega)$ is called *magnitude frequency response* and the function $\varphi = \varphi(\omega)$ *phase angle frequency response*. Their plots are usually given with logarithmic axes for frequency and magnitude and are referred to as Bode plots.

Let us investigate the logarithm of $A(\omega)\exp[j\varphi(\omega)]$

$$\ln G(j\omega) = \ln A(\omega) + j\varphi(\omega) \tag{4.3.41}$$

The function

$$\ln A(\omega) = f_1(\log\omega) \tag{4.3.42}$$

defines the *magnitude logarithmic amplitude frequency response* and is shown in the graphs as

$$L(\omega) = 20\log A(\omega) = 20\ 0.434\ln A(\omega). \tag{4.3.43}$$

L is given in decibels (dB) which is the unit that comes from the acoustic theory and merely rescales the amplitude ratio portion of a Bode diagram.

Logarithmic phase angle frequency response is defined as

$$\varphi(\omega) = f_2(\log\omega) \tag{4.3.44}$$

Example 4.3.1: *Nyquist and Bode diagrams for the heat exchanger as the first order system*

The process transfer function of the heat exchanger was given in (4.3.2). $G(j\omega)$ is given as

$$\begin{aligned} G(j\omega) &= \frac{Z_1}{T_1 j\omega + 1} = \frac{Z_1(T_1 j\omega - 1)}{(T_1 j\omega)^2 + 1} \\ &= \frac{Z_1}{(T_1\omega)^2 + 1} - j\frac{Z_1 T_1 \omega}{(T_1\omega)^2 + 1} \\ &= \frac{Z_1}{\sqrt{(T_1\omega)^2 + 1}} e^{-j \arctan T_1\omega} \end{aligned}$$

The magnitude and phase angle are of the form

$$A(\omega) = \frac{Z_1}{\sqrt{(T_1\omega)^2 + 1}}$$

$$\varphi(\omega) = -\arctan T_1\omega$$

Nyquist and Bode diagrams of the heat exchanger for $Z_1 = 0.4$, $T_1 = 5.2$ min are shown in Figs. 4.3.2, 4.3.3, respectively.

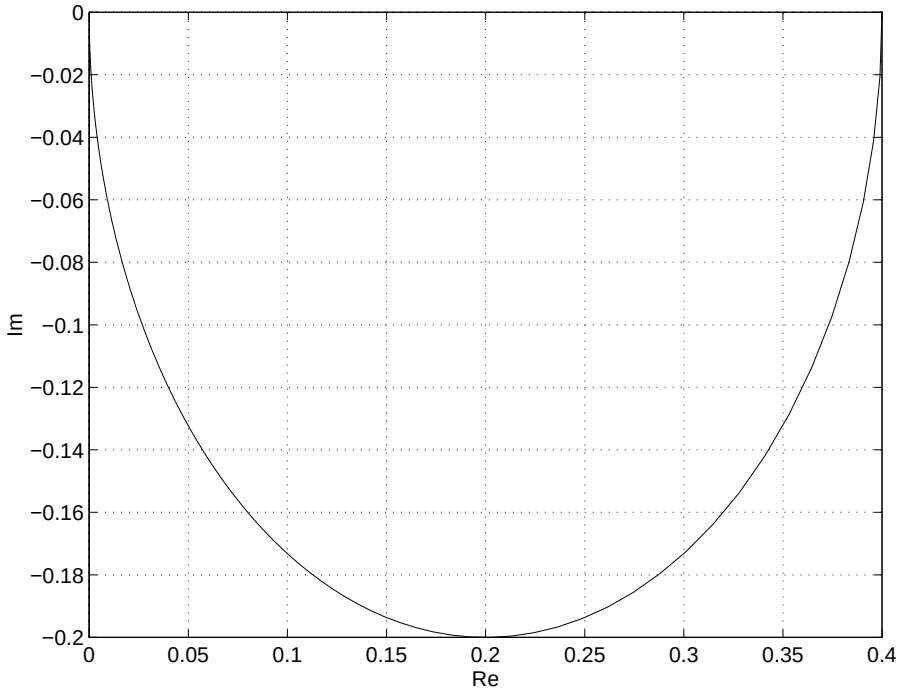

Figure 4.3.2: The Nyquist diagram for the heat exchanger.

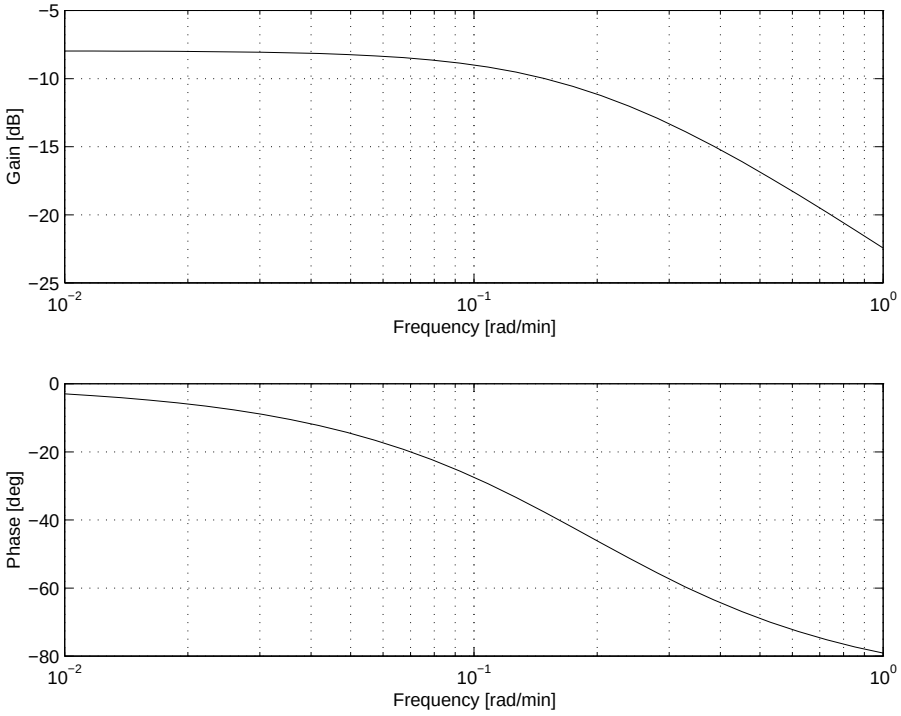

Figure 4.3.3: The Bode diagram for the heat exchanger.

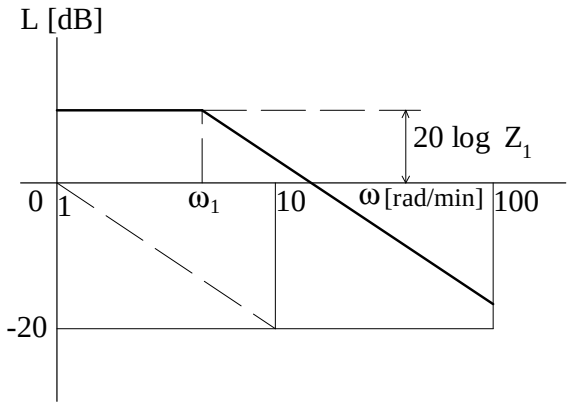

Figure 4.3.4: Asymptotes of the magnitude plot for a first order system.

4.3.3 Frequency Characteristics of a First Order System

In general, the dependency $\Im[G(j\omega)]$ on $\Re[G(j\omega)]$ for a first order system described by (4.3.2) can easily be found from the equations

$$u = \Re[G(j\omega)] = \frac{Z_1}{(T_1\omega)^2+1} \tag{4.3.45}$$

$$v = \Im[G(j\omega)] = -\frac{Z_1 T_1\omega}{(T_1\omega)^2+1} \tag{4.3.46}$$

Equating the terms $T_1\omega$ in both equations yields

$$(v-0)^2 - \left(u - \frac{Z_1}{2}\right) = \left(\frac{Z_1}{2}\right)^2 \tag{4.3.47}$$

which is the equation of a circle.

Let us denote $\omega_1 = 1/T_1$. The transfer function (4.3.2) can be written as

$$G(s) = \frac{\omega_1 Z_1}{s+\omega_1}. \tag{4.3.48}$$

The magnitude is given as

$$A(\omega) = \frac{Z_1\omega_1}{\sqrt{\omega_1^2+\omega^2}} \tag{4.3.49}$$

and its logarithm as

$$L = 20\log Z_1 + 20\log\omega_1 - 20\log\sqrt{\omega_1^2+\omega^2}. \tag{4.3.50}$$

This curve can easily be sketched by finding its asymptotes. If ω approaches zero, then

$$L \to 20\log Z_1 \tag{4.3.51}$$

and if it approaches infinity, then

$$\sqrt{\omega_1^2+\omega^2} \to \sqrt{\omega^2} \tag{4.3.52}$$

$$L \to 20\log Z_1 + 20\log\omega_1 - 20\log\omega. \tag{4.3.53}$$

This is the equation of an asymptote that for $\omega = \omega_1$ is equal to $20\log Z_1$. The slope of this asymptote is -20 dB/decade (Fig 4.3.4).

Table 4.3.1: The errors of the magnitude plot resulting from the use of asymptotes.

ω	$\frac{1}{5}\omega_1$	$\frac{1}{4}\omega_1$	$\frac{1}{2}\omega_1$	ω_1	$2\omega_1$	$4\omega_1$	$5\omega_1$
δ(dB)	0.17	-0.3	-1	-3	-1	-0.3	-0.17

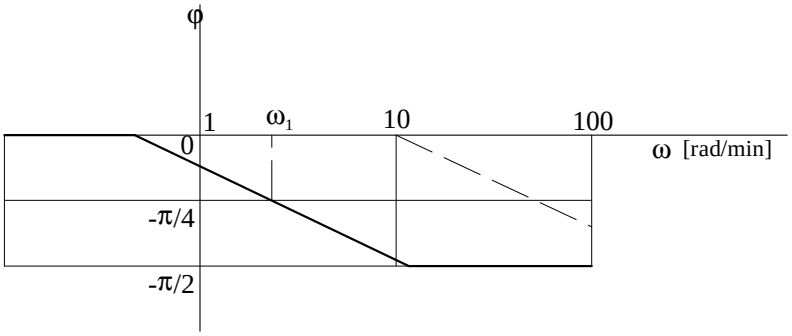

Figure 4.3.5: Asymptotes of phase angle plot for a first order system.

The asymptotes (4.3.51) and (4.3.53) introduce an error δ for $\omega < \omega_1$:

$$\delta = 20 \log \omega_1 - 20 \log \sqrt{\omega_1^2 + \omega^2} \tag{4.3.54}$$

and for $\omega > \omega_1$:

$$\delta = 20 \log \omega_1 - 20 \log \sqrt{\omega_1^2 + \omega^2} - [20 \log \omega_1 - 20 \log \omega] \tag{4.3.55}$$

The tabulation of errors for various ω is given in Table 4.3.1.

A phase angle plot can also be sketched using asymptotes and tangent in its inflex point (Fig 4.3.5).

We can easily verify the following characteristics of the phase angle plot:

If $\omega = 0$, then $\varphi = 0$,

If $\omega = \infty$, then $\varphi = -\pi/2$,

If $\omega = 1/T_1$, then $\varphi = -\pi/4$,

and it can be shown that the curve has an inflex point at $\omega = \omega_1 = 1/T_1$. This frequency is called the *corner frequency*. The slope of the tangent can be calculated if we substitute for $\omega = 10^z$ ($\log \omega = z$) into $\varphi = \arctan(-T_1\omega)$:

$$\dot{\varphi} = \frac{1}{1+x^2}, \quad x = -T_1\omega$$

$$\begin{aligned} \frac{d\varphi}{dz} &= \frac{-2.3}{1+(-T_1 10^z)^2} T_1 10^z \\ \frac{d\varphi}{d\log\omega} &= \frac{-2.3}{1+(-T_1\omega)^2} T_1\omega \end{aligned}$$

for $\omega = 1/T_1$

$$\frac{d\varphi}{d\log\omega} = -1.15 \text{ rad/decade}$$

-1.15 rad corresponds to -66^o. The tangent crosses the asymptotes $\varphi = 0$ and $\varphi = -\pi/2$ with error of $11^o40'$.

4.3.4 Frequency Characteristics of a Second Order System

Consider an underdamped system of the second order with the transfer function

$$G(s) = \frac{1}{T_k^2 s^2 + 2\zeta T_k s + 1}, \quad \zeta < 1. \tag{4.3.56}$$

Its frequency transfer function is given as

$$G(j\omega) = \frac{\frac{1}{T_k^2}}{\sqrt{\left(\frac{1}{T_k^2} - \omega^2\right)^2 + \left(\frac{2\zeta}{T_k}\right)^2 \omega^2}} \exp\left(j \arctan \frac{-\frac{2\zeta}{T_k}\omega}{\frac{1}{T_k^2} - \omega^2}\right) \tag{4.3.57}$$

$$A(\omega) = \frac{\frac{1}{T_k^2}}{\sqrt{\left(\frac{1}{T_k^2} - \omega^2\right)^2 + \left(\frac{2\zeta}{T_k}\right)^2 \omega^2}} \tag{4.3.58}$$

$$\varphi(\omega) = \arctan \frac{-\frac{2\zeta}{T_k}\omega}{\frac{1}{T_k^2} - \omega^2}. \tag{4.3.59}$$

The magnitude plot has a maximum for $\omega = \omega_k$ where $T_k = 1/\omega_k$ (resonant frequency). If $\omega = \infty$, $A = 0$. The expression

$$M = \frac{A(\omega_k)}{A(0)} = \frac{A_{\max}}{A(0)} \tag{4.3.60}$$

is called the *resonance coefficient*.

If the system gain is Z_1, then

$$L(\omega) = 20 \log |G(j\omega)| = 20 \log \left| \frac{Z_1}{\frac{(j\omega)^2}{\omega_k^2} + 2\frac{\zeta}{\omega_k} j\omega + 1} \right| \tag{4.3.61}$$

$$L(\omega) = 20 \log \left| \frac{Z_1}{T_k^2 (j\omega)^2 + 2\zeta T_k j\omega + 1} \right| \tag{4.3.62}$$

$$L(\omega) = 20 \log Z_1 - 20 \log \sqrt{(1 - T_k^2\omega^2)^2 + (2\zeta T_k \omega)^2} \tag{4.3.63}$$

It follows from (4.3.63) that the curve $L(\omega)$ for $Z_1 \neq 1$ is given by summation of $20 \log Z_1$ to normalised L for $Z_1 = 1$. Let us therefore investigate only the case $Z_1 = 1$. From (4.3.63) follows

$$L(\omega) = -20 \log \sqrt{(1 - T_k^2\omega^2)^2 + (2\zeta T_k \omega)^2}. \tag{4.3.64}$$

In the range of low frequencies ($\omega \ll 1/T_k$) holds approximately

$$L(\omega) \approx -20 \log \sqrt{1} = 0. \tag{4.3.65}$$

For high frequencies ($\omega \gg 1/T_k$) and $T_k^2\omega^2 \gg 1$ and $(2\zeta T_k \omega)^2 \ll (T_k^2\omega^2)^2$ holds

$$L(\omega) \approx -20 \log (T_k \omega)^2 = -2 \; 20 \log T_k \omega = -40 \log T_k \omega. \tag{4.3.66}$$

Thus, the magnitude frequency response can be approximated by the curve shown in Fig 4.3.6. Exact curves deviate with an error δ from this approximation. For $0.38 \leq \zeta \leq 0.7$ the values of δ are maximally $\pm$3dB.

The phase angle plot is described by the equation

$$\varphi(\omega) = -\arctan \frac{2\zeta T_k}{1 - T_k^2 \omega^2}. \tag{4.3.67}$$

At the corner frequency $\omega_k = 1/T_k$ this gives $\varphi(\omega) = -90^o$.

Bode diagrams of the second order systems with $Z_1 = 1$ and $T_k = 1$ min are shown in Fig. 4.3.7.

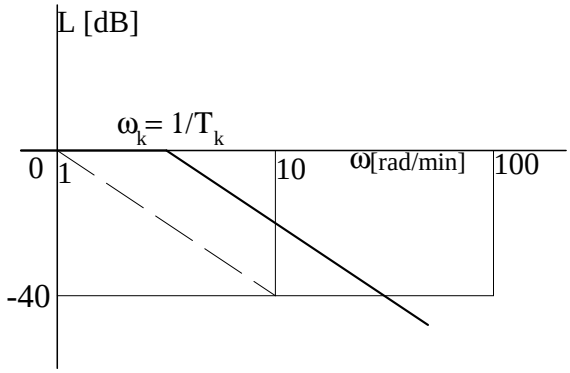

Figure 4.3.6: Asymptotes of magnitude plot for a second order system.

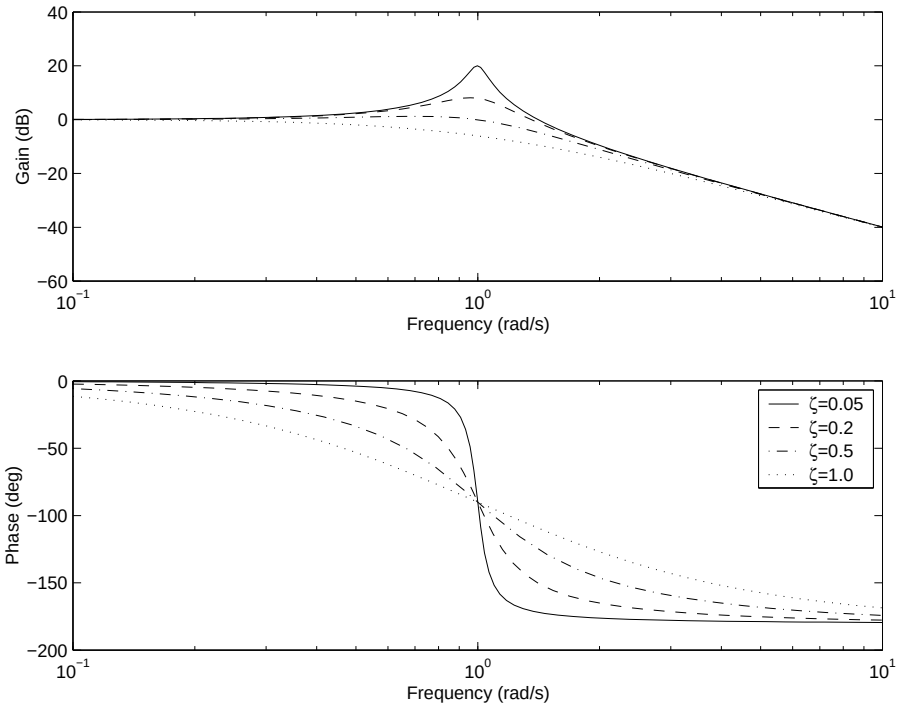

Figure 4.3.7: Bode diagrams of an underdamped second order system ($Z_1 = 1$, $T_k = 1$).

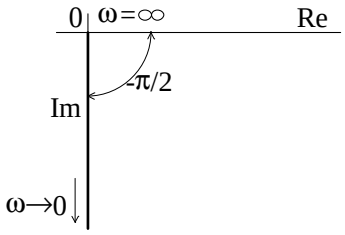

Figure 4.3.8: The Nyquist diagram of an integrator.

4.3.5 Frequency Characteristics of an Integrator

The transfer function of an integrator is

$$G(s) = \frac{Z_1}{s} \tag{4.3.68}$$

where $Z_1 = 1/T_I$ and T_I is the time constant of the integrator.

We note, that integrator is an astatic system. Substitution for $s = j\omega$ yields

$$G(j\omega) = \frac{Z_1}{j\omega} = -j\frac{Z_1}{\omega} = \frac{Z_1}{\omega}e^{-j\frac{\pi}{2}}. \tag{4.3.69}$$

From this follows

$$A(\omega) = \frac{Z_1}{\omega}, \tag{4.3.70}$$

$$\varphi(\omega) = -\frac{\pi}{2}. \tag{4.3.71}$$

The corresponding Nyquist diagram is shown in Fig. 4.3.8. The curve coincides with the negative imaginary axis. The magnitude is for increasing ω decreasing. The phase angle is independent of frequency. Thus, output variable is always delayed to input variable for 90^o.

Magnitude curve is given by the expression

$$L(\omega) = 20\log A(\omega) = 20\log\frac{Z_1}{\omega} \tag{4.3.72}$$

$$L(\omega) = 20\log Z_1 - 20\log\omega. \tag{4.3.73}$$

The phase angle is constant and given by (4.3.71).

If $\omega_2 = 10\omega_1$, then

$$20\log\omega_2 = 20\log 10\omega_1 = 20 + 20\log\omega_1, \tag{4.3.74}$$

thus the slope of magnitude plot is -20dB/decade.

Fig. 4.3.9 shows Bode diagram of the integrator with $T_I = 5$ min. The values of $L(\omega)$ are given by the summation of two terms as given by (4.3.73).

4.3.6 Frequency Characteristics of Systems in a Series

Consider a system with the transfer function

$$G(s) = G_1(s)G_2(s)\ldots G_n(s). \tag{4.3.75}$$

Its frequency response is given as

$$G(j\omega) = \prod_{i=1}^{n} A_i(\omega)e^{j\varphi_i(\omega)} \tag{4.3.76}$$

$$G(j\omega) = \exp\left(j\sum_{i=1}^{n}\varphi_i(\omega)\right)\prod_{i=1}^{n}A_i(\omega), \tag{4.3.77}$$

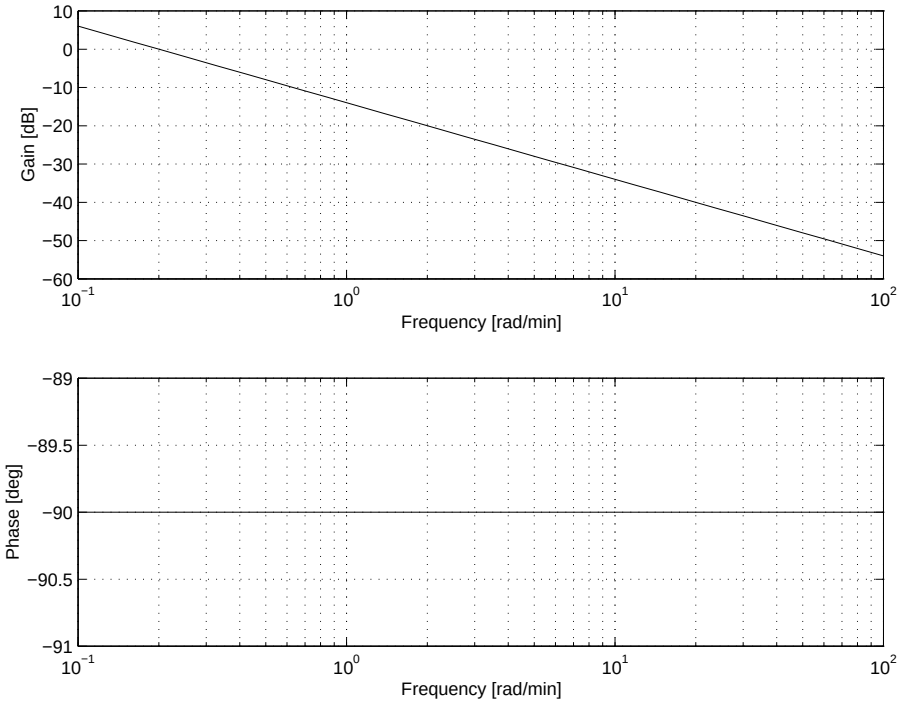

Figure 4.3.9: Bode diagram of an integrator.

and

$$20 \log A(\omega) = \sum_{i=1}^{n} 20 \log A_i(\omega), \tag{4.3.78}$$

$$\varphi(\omega) = \sum_{i=1}^{n} \varphi_i(\omega). \tag{4.3.79}$$

It is clear from the last equations that magnitude and phase angle plots are obtained as the sum of individual functions of systems in series.

Example 4.3.2: *Nyquist and Bode diagrams for a third order system*

Consider a system with the transfer function

$$G(s) = \frac{Z_3}{s(T_1 s + 1)(T_2 s + 1)}.$$

The function $G(j\omega)$ is then given as

$$G(j\omega) = \frac{Z_3}{j\omega(T_1 j\omega + 1)(T_2 j\omega + 1)}.$$

Consequently, for magnitude follows

$$L(\omega) = 20 \log \frac{Z_3}{\omega \sqrt{(T_1\omega)^2 + 1}\sqrt{(T_2\omega)^2 + 1}}$$

$$L(\omega) = 20 \log Z_3 - 20 \log \omega - 20 \log \sqrt{(T_1\omega)^2 + 1} - 20 \log \sqrt{(T_2\omega)^2 + 1}$$

and for phase angle:

$$\varphi(\omega) = -\frac{\pi}{2} - \arctan(T_1\omega) - \arctan(T_2\omega)$$

Nyquist and Bode diagrams for the system with $Z_3 = 0.5$, $T_1 = 2$ min, and $T_2 = 3$ min are given in Figs. 4.3.10 and 4.3.11.

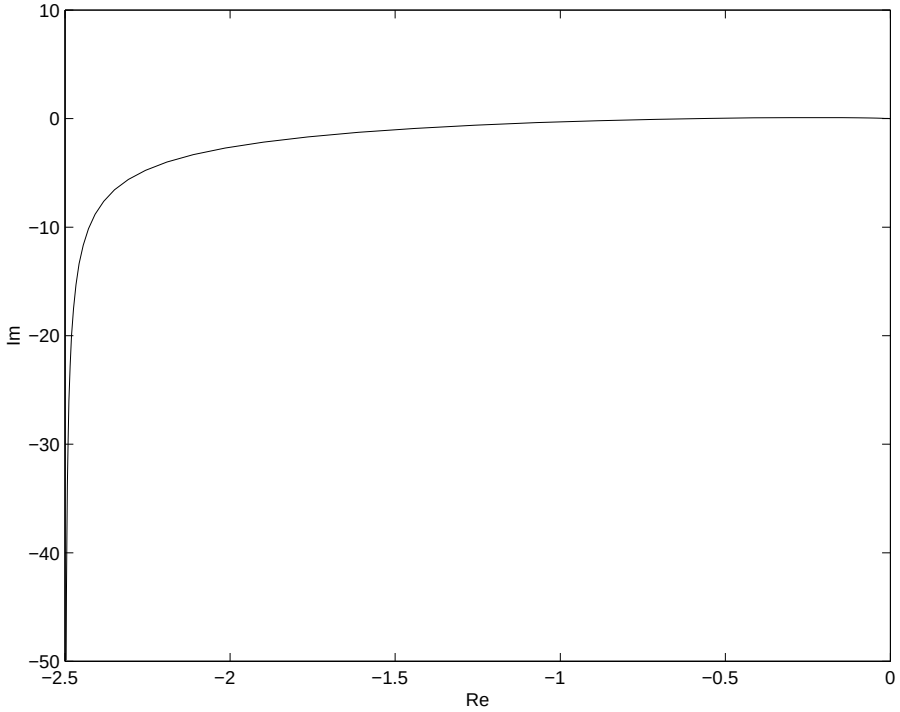

Figure 4.3.10: The Nyquist diagram for the third order system.

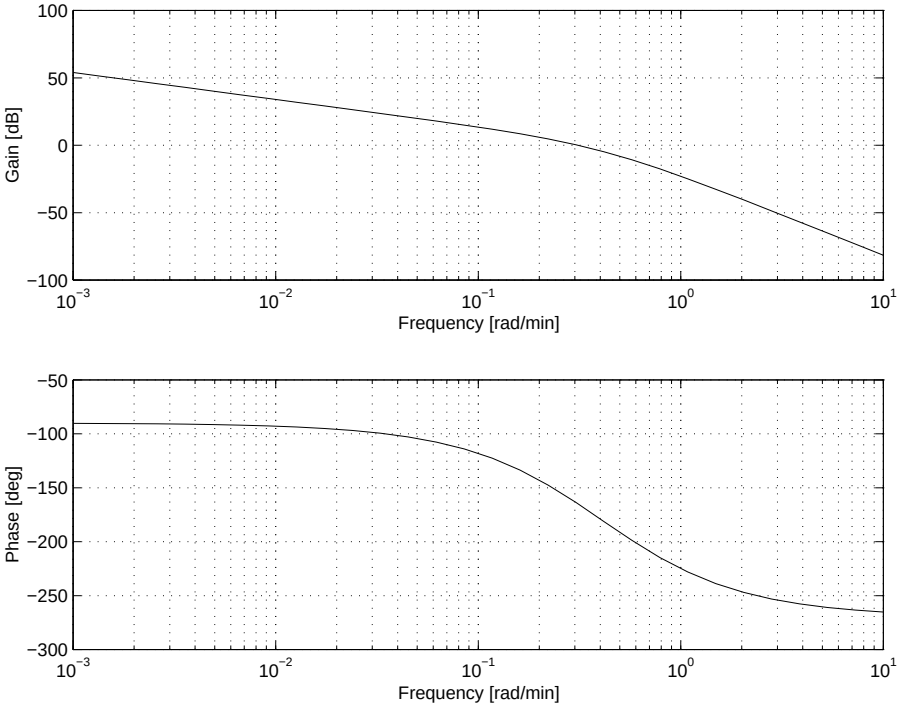

Figure 4.3.11: Bode diagram for the third order system.

4.4 Statistical Characteristics of Dynamic Systems

Dynamic systems are quite often subject to input variables that are not functions exactly specified by time as opposed to step, impulse, harmonic or other *standard* functions. A *concrete (deterministic)* time function has at any time a completely determined value.

Input variables may take different random values through time. In these cases, the only characteristics that can be determined is probability of its influence at certain time. This does not imply from the fact that the input influence cannot be foreseen, but from the fact that a large number of variables and their changes influence the process simultaneously.

The variables that at any time are assigned to a real number by some statement from a space of possible values are called *random*.

Before investigating the behaviour of dynamic systems with random inputs let us now recall some facts about random variables, stochastic processes, and their probability characteristics.

4.4.1 Fundamentals of Probability Theory

Let us investigate an *event* that is characterised by some conditions of existence and it is known that this event may or may not be realised within these conditions. This *random* event is characterised by its *probability*. Let us assume that we make N experiments and that in m cases, the event A has been realised. The fraction m/N is called *the relative occurrence*. It is the experimental characteristics of the event. Performing different number of experiments, it may be observed, that different values are obtained. However, with the increasing number of experiments, the ratio approaches some constant value. This value is called *probability* of the random event A and is denoted by $P(A)$.

There may exist events with probability equal to one (sure events) and to zero (impossible events). For all other, the following inequality holds

$$0 < P(A) < 1 \tag{4.4.1}$$

Events A and B are called *disjoint* if they are mutually exclusive within the same conditions. Their probability is given as

$$P(A \cup B) = P(A) + P(B) \tag{4.4.2}$$

An event A is independent from an event B if $P(A)$ is not influenced when B has or has not occurred. When this does not hold and A is *dependent* on B then $P(A)$ changes if B occurred or not. Such a probability is called *conditional probability* and is denoted by $P(A|B)$.

When two events A, B are independent, then for the probability holds

$$P(A|B) = P(A) \tag{4.4.3}$$

Let us consider two events A and B where $P(B) > 0$. Then for the conditional probability $P(A|B)$ of the event A when B has occurred holds

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{4.4.4}$$

For independent events we may also write

$$P(A \cap B) = P(A)P(B) \tag{4.4.5}$$

4.4.2 Random Variables

Let us consider discrete random variables. Any random variable can be assigned to any real value from a given set of possible outcomes. A *discrete random variable* ξ is assigned a real value from

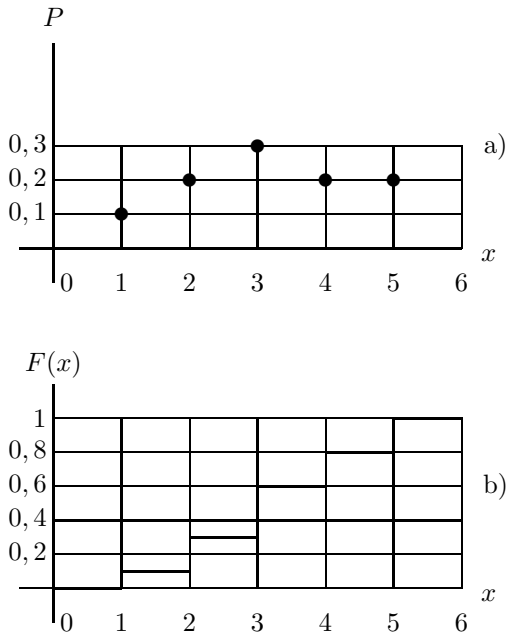

Figure 4.4.1: Graphical representation of the law of distribution of a random variable and of the associated distribution function

a finite set of values $x_1, x_2, \ldots, x_n$. A discrete random variable is determined by the set of finite values and their corresponding probabilities P_i $(i = 1, 2, \ldots, n)$ of their occurrences. The table

$$\xi = \begin{cases} x_1, x_2, \ldots, x_n \\ P_1, P_2, \ldots, P_n \end{cases} \tag{4.4.6}$$

represents the *law of distribution* of a random variable. An example of the graphical representation for some random variable is shown in Fig. 4.4.1a. Here, the values of probabilities are assigned to outcomes of some random variable with values x_i. The random variable can attain any value of x_i $(i = 1, 2, \ldots, n)$. It is easily confirmed that

$$\sum_{i=1}^{n} P_i = 1 \tag{4.4.7}$$

Aside from the law of distribution, we may use another variable that characterises the probability of a random variable. It is denoted by $F(x)$ and defined as

$$F(x) = \sum_{x_i \le x} P_i \tag{4.4.8}$$

and called *cumulative distribution function*, or simply distribution function of a variable ξ. This function completely determines the distribution of all real values of x. The symbol $x_i \le x$ takes into account all values of x_i less or equal to x. $F(x)$ is a probability of event $\xi \le x$ written as

$$F(x) = P(\xi \le x) \tag{4.4.9}$$

Further, $F(x)$ satisfies the inequality

$$0 \le F(x) \le 1 \tag{4.4.10}$$

When the set of all possible outcomes of a random variable ξ is reordered such that $x_1 \le x_2 \le \cdots \le x_n$, the from the probability definition follows that $F(x) = 0$ for any $x < x_1$. Similarly,

$F(x) = 1$ for any $x > x_n$. Graphical representation of the distribution function for a random variable in Fig. 4.4.1a is shown in Fig. 4.4.1b.

Although the distribution function characterises a completely random variable, for practical reasons there are also defined other characteristics given by some non-random values. Among the possible, its expected value, variance, and standard deviation play an important role.

The *expected value* of a discrete random variable is given as

$$\mu = E[\xi] = \sum_{i=1}^{n} x_i P_i \tag{4.4.11}$$

In the case of uniform distribution law the expected value (4.4.11) can be written as

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{4.4.12}$$

For a play-cube tossing yields

$$\mu = \sum_{i=1}^{6} x_i P_i = (1\frac{1}{6} + 2\frac{1}{6} + 3\frac{1}{6} + 4\frac{1}{6} + 5\frac{1}{6} + 6\frac{1}{6}) = 3.5$$

The properties of the expected value are the following:

Constant Z

$$E[Z] = Z \tag{4.4.13}$$

Multiplication by a constant Z

$$E[Z\xi] = ZE[\xi] \tag{4.4.14}$$

Summation

$$E[\xi + \eta] = E[\xi] + E[\eta] \tag{4.4.15}$$

Multiplication of independent random variables

$$E[\xi\eta] = E[\xi]E[\eta] \tag{4.4.16}$$

If ξ is a random variable and μ is its expected value then the variable $(\xi - \mu)$ that denotes the deviation of a random variable from its expected value is also a random variable.

Variance of a random variable ξ is the expected value of the squared deviation $(\xi - \mu)$

$$\sigma^2 = D[\xi] = E\left[(\xi - \mu)^2\right] = \sum_{i=1}^{n} (x_i - \mu)^2 P_i \tag{4.4.17}$$

Whereas the expected value is a parameter in the neighbourhood of which all values of a random variable are located, variance characterises the distance of the values from μ. If the variance is small, then the values far from the expected value are less probable.

Variance can easily be computed from the properties of expected value:

$$\sigma^2 = E\left[\xi^2 - 2\xi E[\xi] + (E[\xi])^2\right] = E[\xi^2] - (E[\xi])^2, \tag{4.4.18}$$

i.e. variance is given as the difference between the expected value of the squared random variable and squared expected value of random variable. Because the following holds always

$$E[\xi^2] \geq (E[\xi])^2, \tag{4.4.19}$$

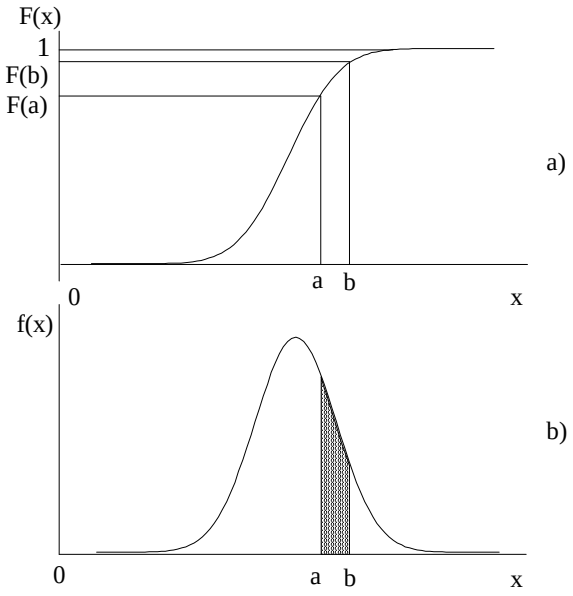

Figure 4.4.2: Distribution function and corresponding probability density function of a continuous random variable

variance is always positive, i.e.

$$\sigma^2 \geq 0 \tag{4.4.20}$$

The square root of the variance is called *standard deviation* of a random variable

$$\sigma = \sqrt{D[\xi]} = \sqrt{E[\xi^2] - (E[\xi])^2} \tag{4.4.21}$$

A *continuous random variable* can be assigned to any real value within some interval if its distribution function $F(x)$ is continuous on this interval. The *distribution function of a continuous random variable* ξ

$$F(x) = P(\xi < x) \tag{4.4.22}$$

is the probability the random variable is less than x. A typical plot of such a distribution function is shown in Fig. 4.4.2a. The following hold for $F(x)$:

$$\lim_{x \to \infty} F(x) = 1 \tag{4.4.23}$$

$$\lim_{x \to -\infty} F(x) = 0 \tag{4.4.24}$$

The probability that a random variable attains some specified value is infinitesimally small. On the contrary, the probability of random variable lying in a some interval (a, b) is finite and can be calculated as

$$P(a \leq \xi < b) = F(b) - F(a) \tag{4.4.25}$$

The probability that a continuous random variable is between x and $x + dx$ is given as

$$P(x \leq \xi < x + dx) = \frac{dF(x)}{dx} dx \tag{4.4.26}$$

where the variable

$$f(x) = \frac{dF(x)}{dx} \tag{4.4.27}$$

is called *probability density*. Figure 4.4.2b shows an example of $f(x)$. Thus, the distribution function $F(x)$ may be written as

$$F(x) = \int_{-\infty}^{x} f(x)dx \tag{4.4.28}$$

Because $F(x)$ is non-decreasing, the probability density function must be positive

$$f(x) \geq 0 \tag{4.4.29}$$

The probability that a random variable is within an interval (a, b) calculated from its density function is given as the surface under curve $f(x)$ within the given interval. Thus, we can write

$$P(a \leq \xi < b) = \int_{a}^{b} f(x)dx \tag{4.4.30}$$

Correspondingly, when the interval comprises of all real values, yields

$$\int_{-\infty}^{\infty} f(x)dx = 1 \tag{4.4.31}$$

Expected value of a continuous random variable is determined as

$$\mu = E[\xi] = \int_{-\infty}^{\infty} xf(x)dx \tag{4.4.32}$$

A random variable can be characterised by the equation

$$E[\xi^m] = \int_{-\infty}^{\infty} x^m f(x)dx \tag{4.4.33}$$

which determines *m-th moment* of a random variable ξ. The first moment is the expected value. The second moment is given as

$$E[\xi^2] = \int_{-\infty}^{\infty} x^2 f(x)dx \tag{4.4.34}$$

Central m-th moment is of the form

$$E[(\xi - \mu)^m] = \int_{-\infty}^{\infty} (x - \mu)^m f(x)dx \tag{4.4.35}$$

Variance of a continuous random variable ξ can be expressed as follows

$$\sigma^2 = E[(\xi - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx \tag{4.4.36}$$

$$\sigma^2 = E[\xi^2] - (E[\xi])^2 \tag{4.4.37}$$

The *standard deviation* is its square root

$$\sigma = \sqrt{E[\xi^2] - (E[\xi])^2} \tag{4.4.38}$$

Normal distribution for a continuous random variable is given by the following density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{4.4.39}$$

Let us now consider two independent continuous random variables ξ_1, ξ_2 defined in the same probability space. Their *joint density* function is given by the product

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \tag{4.4.40}$$

where $f_1(x_1), f_2(x_2)$ are density functions of the variables ξ_1, ξ_2.

Similarly as for one random variable, we can introduce the moments (if they exist) also for two random variables, for example by

$$E[\xi_1^r, \xi_2^s] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^r x_2^s f(x_1, x_2)dx_1dx_2 \tag{4.4.41}$$

Correspondingly, the central moments are defined as

$$E[(\xi_1 - \mu_1)^r(\xi_2 - \mu_2)^s] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_1)^r(x_2 - \mu_2)^s f(x_1, x_2)dx_1dx_2 \tag{4.4.42}$$

where $\mu_1 = E(\xi_1), \mu_2 = E(\xi_2)$.

Another important property characterising two random variables is their *covariance* defined as

$$\begin{aligned} \mathrm{Cov}(\xi_1, \xi_2) &= E[(\xi_1 - \mu_1)(\xi_2 - \mu_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_1)(x_2 - \mu_2)f(x_1, x_2)dx_1dx_2 \end{aligned} \tag{4.4.43}$$

If ξ_1, ξ_2 have finite variances, then the number

$$r(\xi_1, \xi_2) = \frac{\mathrm{Cov}(\xi_1, \xi_2)}{\sigma_1\sigma_2} \tag{4.4.44}$$

is called the *correlation coefficient* ($\sigma_1 = \sqrt{D[\xi_1]}, \sigma_2 = \sqrt{D[\xi_2]}$).

Random variables ξ_1, ξ_2 are uncorrelated if

$$\mathrm{Cov}(\xi_1, \xi_2) = 0 \tag{4.4.45}$$

Integrable random variables ξ_1, ξ_2 with integrable term $\xi_1\xi_2$ are uncorrelated if and only if

$$E[\xi_1, \xi_2] = E[\xi_1]E[\xi_2] \tag{4.4.46}$$

This follows from the fact that $\mathrm{Cov}(\xi_1, \xi_2) = E[\xi_1, \xi_2] - E[\xi_1]E[\xi_2]$. Thus the multiplicative property of probabilities extends to expectations.

If ξ_1, ξ_2 are independent integrable random variables then they are uncorrelated.

A vector of random variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^T$ is usually computationally characterised only by its expected value $E[\boldsymbol{\xi}]$ and the covariance matrix $\mathrm{Cov}(\boldsymbol{\xi})$.

The expected value $E[\boldsymbol{\xi}]$ of a vector $\boldsymbol{\xi}$ is given as the vector of expected values of the elements ξ_i.

The *covariance matrix* $\mathrm{Cov}(\boldsymbol{\xi})$ of a vector $\boldsymbol{\xi}$ with the expected value $E[\boldsymbol{\xi}]$ is the expected value of the matrix $(\boldsymbol{\xi} - E[\boldsymbol{\xi}])(\boldsymbol{\xi} - E[\boldsymbol{\xi}])^T$, hence

$$\mathrm{Cov}(\boldsymbol{\xi}) = E[(\boldsymbol{\xi} - E[\boldsymbol{\xi}])(\boldsymbol{\xi} - E[\boldsymbol{\xi}])^T] \tag{4.4.47}$$

A covariance matrix is a symmetric positive (semi-)definite matrix that contains in i-th row and j-th column covariances of the random variables ξ_i, ξ_j:

$$\mathrm{Cov}(\xi_i, \xi_j) = E\left[(\xi_i - E[\xi_i])(\xi_j - E[\xi_j])\right] \tag{4.4.48}$$

Elements of a covariance matrix determine a degree of correlation between random variables where

$$\mathrm{Cov}(\xi_i, \xi_j) = \mathrm{Cov}(\xi_j, \xi_i) \tag{4.4.49}$$

The main diagonal of a covariance matrix contains variances of random variables ξ_i :

$$\mathrm{Cov}(\xi_i, \xi_i) = E\left[(\xi_i - E[\xi_i])(\xi_i - E[\xi_i])\right] = \sigma_i^2 \tag{4.4.50}$$

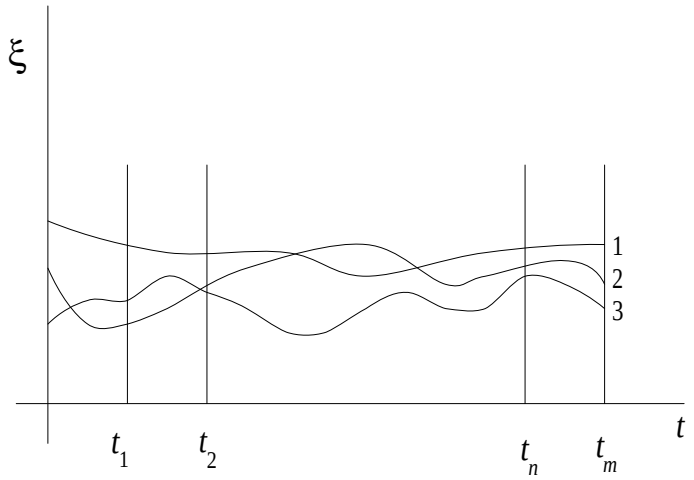

Figure 4.4.3: Realisations of a stochastic process.

4.4.3 Stochastic Processes

When dealing with dynamic systems, some phenomenon can be observed as a function of time. When some physical variable (temperature in a CSTR) is measured under the same conditions in the same time t_m several times, the results may resemble trajectories shown in Fig. 4.4.3. The trajectories 1,2,3 are all different. It is impossible to determine the trajectory 2 from the trajectory 1 and from 2 we are unable to predict the trajectory 3. This is the reason that it is not interesting to investigate time functions independently but rather their large sets. If their number approaches infinity, we speak about a *stochastic (random) process*.

A *stochastic process* is given as a set of time-dependent random variables $\xi(t)$. Thus, the concept of a random variable ξ is broadened to a random function $\xi(t)$. It might be said that a stochastic process is such a function of time whose values are at any time instant random variables. A random variable in a stochastic process yields random values not only as an outcome of an experiment but also as a function of time. A random variable corresponding to some experimental conditions and changing in time that belongs to the set of random variables $\xi(t)$ is called the *realisation of a stochastic process*.

A stochastic process in some fixed time instants $t_1, t_2, \ldots, t_n$ depends only on the outcome of the experiment and changes to a corresponding random variable with a given density function. From this follows that a stochastic process can be determined by a set of density functions that corresponds to random variables $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ in the time instants $t_1, t_2, \ldots, t_n$. The density function is a function of time and is denoted by $f(x,t)$. For any time t_i $(i = 1, 2, \ldots, n)$ exists the corresponding density function $f(x_i, t_i)$.

Consider a time t_1 and the corresponding random variable $\xi(t_1)$. The probability that $\xi(t_1)$ will be between x_1 and $x_1 + dx_1$ is given as

$$P(x_1 \leq \xi(t_1) < x_1 + dx_1) = f_1(x_1, t_1)dx_1 \tag{4.4.51}$$

where $f_1(x_1, t_1)$ is the density function in time t_1 (*one-dimensional density function*).

Now consider two time instants t_1 and t_2. The probability that a random variable $\xi(t_1)$ will be in time t_1 between x_1 and $x_1 + dx_1$ and in time t_2 between x_2 and $x_2 + dx_2$ can be calculated as

$$P(x_1 \leq \xi(t_1) < x_1 + dx_1; x_2 \leq \xi(t_2) < x_2 + dx_2) = f_2(x_1, t_1; x_2, t_2)dx_1 dx_2 \tag{4.4.52}$$

where $f_2(x_1, t_1; x_2, t_2)$ is *two-dimensional density function* and determines the relationship between the values of a stochastic process $\xi(t)$ in the time instants t_1 and t_2.

Sometimes, also *n-dimensional density function* $f_2(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)$ is introduced and is analogously defined as a probability that a process $\xi(t)$ passes through n points with deviation not greater than $dx_1, dx_2, \ldots, dx_n$.

A stochastic process is statistically completely determined by the density functions $f_1, \ldots, f_n$ and the relationships among them.

The simplest stochastic process is a totally independent stochastic process (*white noise*). For this process, any random variables at any time instants are mutually independent. For this process holds

$$f_2(x_1, t_1; x_2, t_2) = f(x_1, t_1) f(x_2, t_2) \tag{4.4.53}$$

as well as

$$f_n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = f(x_1, t_1) f(x_2, t_2) \ldots f(x_n, t_n) \tag{4.4.54}$$

Based on the one-dimensional density function, the expected value of a stochastic process is given by

$$\mu(t) = E[\xi(t)] = \int_{-\infty}^{\infty} x f_1(x, t) dx \tag{4.4.55}$$

In (4.4.55) the index of variables of f_1 is not given as it can be arbitrary.

Variance of a stochastic process can be written as

$$D[\xi(t)] = \int_{-\infty}^{\infty} [x - \mu(t)]^2 f_1(x, t) dx \tag{4.4.56}$$

$$D[\xi(t)] = E[\xi^2(t)] - (E[\xi(t)])^2 \tag{4.4.57}$$

Expected value of a stochastic process $\mu(t)$ is a function of time and it is the mean value of all realisations of a stochastic process. The variance $D[\xi(t)]$ gives information about dispersion of realisations with respect to the mean value $\mu(t)$.

Based on the information given by the two-dimensional density function, it is possible to find an influence between the values of a stochastic process at times t_1 and t_2. This is given by the *auto-correlation function* of the form

$$R_\xi(t_1, t_2) = E[\xi(t_1)\xi(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, t_1; x_2, t_2) dx_1 dx_2 \tag{4.4.58}$$

The *auto-covariance function* is given as

$$\begin{aligned} \mathrm{Cov}_\xi(t_1, t_2) &= E\left[(\xi(t_1) - \mu(t_1))(\xi(t_2) - \mu(t_2))\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [x_1 - \mu(t_1)][x_2 - \mu(t_2)] f_2(x_1, t_1; x_2, t_2) dx_1 dx_2 \end{aligned} \tag{4.4.59}$$

For the auto-correlation function follows

$$R_\xi(t_1, t_2) = \mathrm{Cov}_\xi(t_1, t_2) - \mu(t_1)\mu(t_2) \tag{4.4.60}$$

Similarly, for two stochastic processes $\xi(t)$ and $\eta(t)$, we can define the *correlation function*

$$R_{\xi\eta}(t_1, t_2) = E[\xi(t_1)\eta(t_2)] \tag{4.4.61}$$

and the *covariance function*

$$\mathrm{Cov}_{\xi\eta}(t_1, t_2) = E[(\xi(t_1) - \mu(t_1))(\eta(t_2) - \mu_\eta(t_2))] \tag{4.4.62}$$

If a stochastic process with normal distribution is to be characterised, it usually suffices to specify its mean value and the correlation function. However, this does not hold in the majority of cases.

When replacing the arguments t_1, t_2 in Equations (4.4.58), (4.4.60) by t and τ then

$$R_\xi(t,\tau) = E[\xi(t)\xi(\tau)] \tag{4.4.63}$$

and

$$\mathrm{Cov}_\xi(t,\tau) = E[(\xi(t)-\mu(t))(\xi(\tau)-\mu(\tau))] \tag{4.4.64}$$

If $t = \tau$ then

$$\mathrm{Cov}_\xi(t,t) = E[(\xi(t)-\mu(t))^2] \tag{4.4.65}$$

where $\mathrm{Cov}_\xi(t,t)$ is equal to the variance of the random variable ξ. The abbreviated form $\mathrm{Cov}_\xi(t) = \mathrm{Cov}_\xi(t,t)$ is also often used.

Consider now mutually dependent stochastic processes $\xi_1(t), \xi_2(t), \ldots \xi_n(t)$ that are elements of *stochastic process vector* $\boldsymbol{\xi}(t)$. In this case, the mean values and auto-covariance function are often sufficient characteristics of the process.

The *vector mean value* of the vector $\boldsymbol{\xi}(t)$ is given as

$$\boldsymbol{\mu}(t) = E[\boldsymbol{\xi}(t)] \tag{4.4.66}$$

The expression

$$\mathrm{Cov}_{\boldsymbol{\xi}}(t_1,t_2) = E\left[(\boldsymbol{\xi}(t_1)-\boldsymbol{\mu}(t_1))(\boldsymbol{\xi}(t_2)-\boldsymbol{\mu}(t_2))^T\right] \tag{4.4.67}$$

or

$$\mathrm{Cov}_{\boldsymbol{\xi}}(t,\tau) = E[(\boldsymbol{\xi}(t)-\boldsymbol{\mu}(t))(\boldsymbol{\xi}(\tau)-\boldsymbol{\mu}(\tau))^T] \tag{4.4.68}$$

is the corresponding *auto-covariance matrix* of the stochastic process vector $\boldsymbol{\xi}(t)$.

The auto-covariance matrix is symmetric, thus

$$\mathrm{Cov}_{\boldsymbol{\xi}}(\tau,t) = \mathrm{Cov}^T_{\boldsymbol{\xi}}(t,\tau) \tag{4.4.69}$$

If a stochastic process is normally distributed then the knowledge about its mean value and covariance is sufficient for obtaining any other process characteristics.

For the investigation of stochastic processes, the following expression is often used

$$\bar{\mu} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(t)dt \tag{4.4.70}$$

$\bar{\mu}$ is not time dependent and follows from observations of the stochastic process in a sufficiently large time interval and $\xi(t)$ is any realisation of the stochastic process. In general, the following expression is used

$$\bar{\mu^m} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\xi(t)]^m dt \tag{4.4.71}$$

For $m = 2$ this expression gives $\bar{\mu^2}$.

Stochastic processes are divided into *stationary* and *non-stationary*. In the case of a stationary stochastic process, all probability densities $f_1, f_2, \ldots f_n$ do not depend on the start of observations and onedimensional probability density is not a function of time t. Hence, the mean value (4.4.55) and the variance (4.4.56) are not time dependent as well.

Many stationary processes are *ergodic*, i.e. the following holds with probability equal to one

$$\mu = \int_{-\infty}^{\infty} x f_1(x)dx = \bar{\mu} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(t)dt \tag{4.4.72}$$

$$\mu^2 = \bar{\mu^2}, \qquad \mu^m = \bar{\mu^m} \tag{4.4.73}$$

The usual assumption in practice is that stochastic processes are stationary and ergodic.

The properties (4.4.72) and (4.4.73) show that for the investigation of statistical properties of a stationary and ergodic process, it is only necessary to observe its one realisation in a sufficiently large time interval.

Stationary stochastic processes have a two-dimensional density function f_2 independent of the time instants t_1, t_2, but dependent on $\tau = t_2 - t_1$ that separates the two random variables $\xi(t_1)$, $\xi(t_2)$. As a result, the auto-correlation function (4.4.58) can be written as

$$R_\xi(\tau) = E\{\xi(t_1)\xi(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, x_2, \tau) dx_1 dx_2 \tag{4.4.74}$$

For a stationary and ergodic process hold the equations (4.4.72), (4.4.73) and the expression $E\{\xi(t)\xi(t+\tau)\}$ can be written as

$$\begin{aligned} E\{\xi(t)\xi(t+\tau)\} &= \overline{\xi(t)\xi(t+\tau)} \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\xi(t+\tau)dt \end{aligned} \tag{4.4.75}$$

Hence, the auto-correlation function of a stationary ergodic process is in the form

$$R_\xi(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\xi(t+\tau)dt \tag{4.4.76}$$

Auto-correlation function of a process determines the influence of a random variable between the times $t+\tau$ and t. If a stationary ergodic stochastic process is concerned, its auto-correlation function can be determined from any of its realisations.

The auto-correlation function $R_\xi(\tau)$ is symmetric

$$R_\xi(-\tau) = R_\xi(\tau) \tag{4.4.77}$$

For $\tau = 0$ the auto-correlation function is determined by the expected value of the square of the random variable

$$R_\xi(0) = E[\xi^2(t)] = \overline{\xi(t)\xi(t)} \tag{4.4.78}$$

For $\tau \to \infty$ the auto-correlation function is given as the square of the expected value. This can easily be proved.

$$R_\xi(\tau) = \overline{\xi(t)\xi(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_2(x_1, x_2, \tau) dx_1 dx_2 \tag{4.4.79}$$

For $\tau \to \infty$, $\xi(t)$ and $\xi(t+\tau)$ are mutually independent. Using (4.4.53) that can be applied to a stochastic process yields

$$R_\xi(\infty) = \int_{-\infty}^{\infty} x_1 f(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f(x_2) dx_2 = \mu^2 = (\bar{\mu})^2 \tag{4.4.80}$$

The value of the auto-correlation function for $\tau = 0$ is in its maximum and holds

$$R_\xi(0) \geq R_\xi(\tau) \tag{4.4.81}$$

The *cross-correlation function* of two *mutually ergodic* stochastic processes $\xi(t)$, $\eta(t)$ can be given as

$$E[\xi(t)\eta(t+\tau)] = \overline{\xi(t)\eta(t+\tau)} \tag{4.4.82}$$

or

$$\begin{aligned} R_{\xi\eta}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_2(x_1, y_2, \tau) dx_1 dy_2 \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\eta(t+\tau)dt \end{aligned} \tag{4.4.83}$$

Consider now a stationary ergodic stochastic process with corresponding auto-correlation function $R_\xi(\tau)$. This auto-correlation function provides information about the stochastic process in the time domain. The same information can be obtained in the frequency domain by taking the Fourier transform of the auto-correlation function. The Fourier transform $S_\xi(\omega)$ of $R_\xi(\tau)$ is given as

$$S_\xi(\omega) = \int_{-\infty}^{\infty} R_\xi(\tau) e^{-j\omega\tau} d\tau \tag{4.4.84}$$

Correspondingly, the auto-correlation function $R_\xi(\tau)$ can be obtained if $S_\xi(\omega)$ is known using the inverse Fourier transform

$$R_\xi(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\xi(\omega) e^{j\omega\tau} d\omega \tag{4.4.85}$$

$R_\xi(\tau)$ and $S_\xi(\omega)$ are non-random characteristics of stochastic processes. $S_\xi(\omega)$ is called *power spectral density* of a stochastic process. This function has large importance for investigation of transformations of stochastic signals entering linear dynamical systems.

The power spectral density is an even function of ω:

$$S_\xi(-\omega) = S_\xi(\omega) \tag{4.4.86}$$

For its determination, the following relations can be used.

$$S_\xi(\omega) = 2 \int_0^{\infty} R_\xi(\tau) \cos\omega\tau d\tau \tag{4.4.87}$$

$$R_\xi(\tau) = \frac{1}{\pi} \int_0^{\infty} S_\xi(\omega) \cos\omega\tau d\omega \tag{4.4.88}$$

The *cross-power spectral density* $S_{\xi\eta}(\omega)$ of two mutually ergodic stochastic processes $\xi(t), \eta(t)$ with zero means is the Fourier transform of the associated cross-correlation function $R_{\xi\eta}(\tau)$:

$$S_{\xi\eta}(\omega) = \int_{-\infty}^{\infty} R_{\xi\eta}(\tau) e^{-j\omega\tau} d\tau \tag{4.4.89}$$

The inverse relation for the cross-correlation function $R_{\xi\eta}(\tau)$ if $S_{\xi\eta}(\omega)$ is known, is given as

$$R_{\xi\eta}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\xi\eta}(\omega) e^{j\omega\tau} d\omega \tag{4.4.90}$$

If we substitute in (4.4.75), (4.4.85) for $\tau = 0$ then the following relations can be obtained

$$E\left\{\xi(t)^2\right\} = R_\xi(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi^2(t) dt \tag{4.4.91}$$

$$E\left\{\xi(t)^2\right\} = R_\xi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_\xi(\omega) d\omega = \frac{1}{\pi} \int_0^{\infty} S_\xi(\omega) d\omega \tag{4.4.92}$$

The equation (4.4.91) describes energetical characteristics of a process. The right hand side of the equation can be interpreted as the average power of the process. The equation (4.4.92) determines the power as well but expressed in terms of power spectral density. The average power is given by the area under the spectral density curve and $S_\xi(\omega)$ characterises power distribution of the signal according to the frequency. For $S_\xi(\omega)$ holds

$$S_\xi(\omega) \geq 0 \tag{4.4.93}$$

4.4.4 White Noise

Consider a stationary stochastic process with a constant power spectral density for all frequencies

$$S_\xi(\omega) = V \tag{4.4.94}$$

This process has a "white" spectrum and it is called *white noise*. Its power spectral density is shown in Fig. 4.4.4a. From (4.4.92) follows that the average power of white noise is indefinitely large, as

$$E\left\{\xi(t)^2\right\} = \frac{1}{\pi} V \int_0^\infty d\omega \tag{4.4.95}$$

Therefore such a process does not exit in real conditions.

The auto-correlation function of white noise can be determined from (4.4.88)

$$R_\xi(\tau) = \frac{1}{\pi} \int_0^\infty V \cos\omega\tau d\omega = V\delta(\tau) \tag{4.4.96}$$

where

$$\delta(\tau) = \frac{1}{\pi} \int_0^\infty \cos\omega\tau d\omega \tag{4.4.97}$$

because the Fourier transform of the delta function $F_\delta(j\omega)$ is equal to one and the inverse Fourier transform is of the form

$$\begin{aligned}
\delta(\tau) &= \frac{1}{2\pi} \int_{-\infty}^\infty F_\delta(j\omega) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \cos\omega\tau d\omega + j\frac{1}{2\pi} \int_{-\infty}^\infty \sin\omega\tau d\omega \\
&= \frac{1}{\pi} \int_0^\infty \cos\omega\tau d\omega
\end{aligned} \tag{4.4.98}$$

The auto-correlation function of white noise (Fig. 4.4.4b) is determined by the delta function and is equal to zero for any non-zero values of τ. White noise is an example of a stochastic process where $\xi(t)$ and $\xi(t+\tau)$ are independent.

A physically realisable white noise can be introduced if its power spectral density is constrained

$$\begin{aligned}
S_\xi(\omega) &= V \quad \text{for } |\omega| < \omega_1 \\
S_\xi(\omega) &= 0 \quad \text{for } |\omega| > \omega_1
\end{aligned} \tag{4.4.99}$$

The associated auto-correlation function can be given as

$$R_\xi(\tau) = \frac{V}{\pi} \int_0^{\omega_1} \cos\omega\tau d\omega = \frac{V}{\pi\tau} \sin\omega_1\tau \tag{4.4.100}$$

The following relation also holds

$$\bar{\mu^2} = D = \frac{V}{2\pi} \int_{-\omega_1}^{\omega_1} d\omega = \frac{V\omega_1}{\pi} \tag{4.4.101}$$

Sometimes, the relation (4.4.94) is approximated by a continuous function. Often, the following relation can be used

$$S_\xi(\omega) = \frac{2aD}{\omega^2 + a^2} \tag{4.4.102}$$

The associated auto-correlation function is of the form

$$R_\xi(\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{2aD}{\omega^2 + a^2} e^{j\omega\tau} d\omega = De^{-a|\tau|} \tag{4.4.103}$$

The figure 4.4.5 depicts power spectral density and auto-correlation function of this process. The equations (4.4.102), (4.4.103) describe many stochastic processes well. For example, if $a \gg 1$, the approximation is usually "very" good.

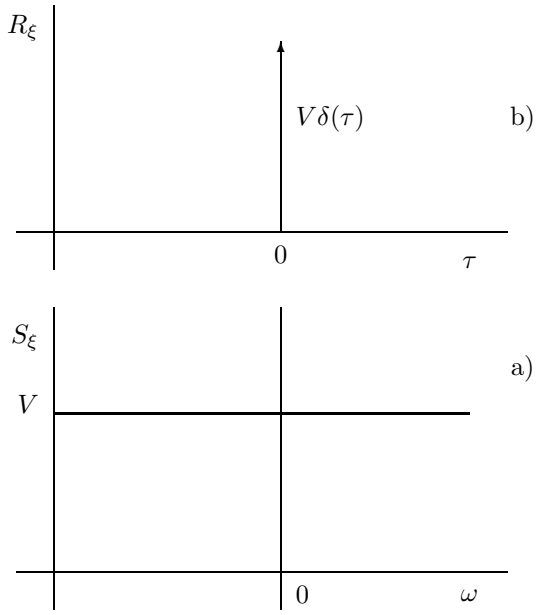

Figure 4.4.4: Power spectral density and auto-correlation function of white noise

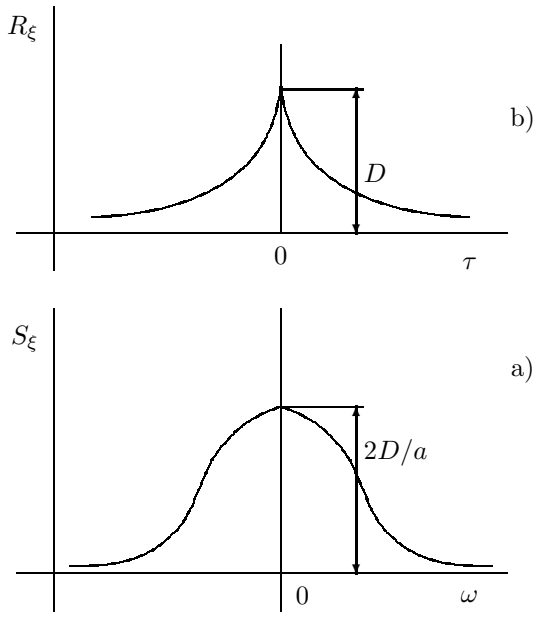

Figure 4.4.5: Power spectral density and auto-correlation function of the process given by (4.4.102) and (4.4.103)

4.4.5 Response of a Linear System to Stochastic Input

Consider a continuous linear system with constant coefficients

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{\xi}(t) \tag{4.4.104}$$

$$\boldsymbol{x}(0) = \boldsymbol{\xi}_0 \tag{4.4.105}$$

where $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the state vector, $\boldsymbol{\xi}(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_m(t)]^T$ is a stochastic process vector entering the system. $\boldsymbol{A}, \boldsymbol{B}$ are $n \times n, n \times m$ constant matrices, respectively. The initial condition $\boldsymbol{\xi}_0$ is a vector of random variables.

Suppose that the expectation $E[\boldsymbol{\xi}_0]$ and the covariance matrix $\mathrm{Cov}(\boldsymbol{\xi}_0)$ are known and given as

$$E[\boldsymbol{\xi}_0] = \boldsymbol{x}_0 \tag{4.4.106}$$

$$E[(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)^T] = \mathrm{Cov}(\boldsymbol{\xi}_0) = \mathrm{Cov}_{\mathbf{0}} \tag{4.4.107}$$

Further, suppose that $\boldsymbol{\xi}(t)$ is independent on the initial condition vector $\boldsymbol{\xi}_0$ and that its mean value $\boldsymbol{\mu}(t)$ and its auto-covariance function $\mathrm{Cov}_{\boldsymbol{\xi}}(t, \tau)$ are known and holds

$$E[\boldsymbol{\xi}(t)] = \boldsymbol{\mu}(t), \qquad \text{for } t \geq 0 \tag{4.4.108}$$

$$E[(\boldsymbol{\xi}(t) - \boldsymbol{\mu}(t))(\boldsymbol{\xi}(\tau) - \boldsymbol{\mu}(\tau))^T] = \mathrm{Cov}_{\boldsymbol{\xi}}(t, \tau), \quad \text{for } t \geq 0, \tau \geq 0 \tag{4.4.109}$$

$$E[(\boldsymbol{\xi}(t) - \boldsymbol{\mu}(t))(\boldsymbol{\xi}_0 - \boldsymbol{\mu}_0)^T] \equiv 0, \quad \text{for } t \geq 0 \tag{4.4.110}$$

As $\boldsymbol{\xi}_0$ is a vector of random variables and $\boldsymbol{\xi}(t)$ is a vector of stochastic processes then $\boldsymbol{x}(t)$ is a vector of stochastic processes as well. We would like to determine its mean value $E[\boldsymbol{x}(t)]$, covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t) = \mathrm{Cov}_{\boldsymbol{x}}(t, t)$, and auto-covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t, \tau)$ for given $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}(t)$.

Any stochastic state trajectory can be determined for given initial conditions and stochastic inputs as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{\xi}_0 + \int_0^t \boldsymbol{\Phi}(t - \alpha)\boldsymbol{B}\boldsymbol{\xi}(\alpha)d\alpha \tag{4.4.111}$$

where $\boldsymbol{\Phi}(t) = e^{\boldsymbol{A}t}$ is the system transition matrix.

Denoting

$$E[\boldsymbol{x}(t)] = \bar{\boldsymbol{x}}(t) \tag{4.4.112}$$

then the following holds

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}_0 + \int_0^t \boldsymbol{\Phi}(t - \alpha)\boldsymbol{B}\boldsymbol{\mu}(\alpha)d\alpha \tag{4.4.113}$$

This corresponds with the solution of the differential equation

$$\frac{d\bar{\boldsymbol{x}}(t)}{dt} = \boldsymbol{A}\bar{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{\mu}(t) \tag{4.4.114}$$

with initial condition

$$\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0 \tag{4.4.115}$$

To find the covariance matrix and auto-correlation function, consider at first the deviation $\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)$:

$$\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t) = \boldsymbol{\Phi}(t)[\boldsymbol{\xi}_0 - \bar{\boldsymbol{x}}_0] + \int_0^t \boldsymbol{\Phi}(t - \alpha)\boldsymbol{B}[\boldsymbol{\xi}(\alpha) - \boldsymbol{\mu}(\alpha)]d\alpha \tag{4.4.116}$$

It is obvious that $\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)$ is the solution of the following differential equation

$$\frac{d\boldsymbol{x}(t)}{dt} - \frac{d\bar{\boldsymbol{x}}(t)}{dt} = \boldsymbol{A}[\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t)] + \boldsymbol{B}[\boldsymbol{\xi}(t) - \boldsymbol{\mu}(t)] \tag{4.4.117}$$

with initial condition

$$\boldsymbol{x}(0) - \bar{\boldsymbol{x}}(0) = \boldsymbol{\xi}_0 - \boldsymbol{x}_0 \tag{4.4.118}$$

From the equation (4.4.116) for $\mathrm{Cov}_{\boldsymbol{x}}(t)$ follows

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{x}}(t) &= E[(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t))(\boldsymbol{x}(t) - \bar{\boldsymbol{x}}(t))^T] \\ &= E\left[\left\{\boldsymbol{\Phi}(t)[\boldsymbol{\xi}_0 - \boldsymbol{x}_0] + \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}[\boldsymbol{\xi}(\alpha) - \boldsymbol{\mu}(\alpha)]d\alpha\right\} \times \right. \\ &\quad \left. \times \left\{\boldsymbol{\Phi}(t)[\boldsymbol{\xi}_0 - \boldsymbol{x}_0] + \int_0^t \boldsymbol{\Phi}(t-\beta)\boldsymbol{B}[\boldsymbol{\xi}(\beta) - \boldsymbol{\mu}(\beta)]d\beta\right\}^T\right]\end{aligned} \tag{4.4.119}$$

and after some manipulations,

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{x}}(t) &= \boldsymbol{\Phi}(t)E[(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)^T]\boldsymbol{\Phi}^T(t) \\ &+ \int_0^t \boldsymbol{\Phi}(t)E[(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)(\boldsymbol{\xi}(\beta) - \boldsymbol{\mu}(\beta))^T]\boldsymbol{B}^T\boldsymbol{\Phi}^T(t-\beta)d\beta \\ &+ \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}E[(\boldsymbol{\xi}(\alpha) - \boldsymbol{\mu}(\alpha))(\boldsymbol{\xi}_0 - \boldsymbol{x}_0)^T]\boldsymbol{\Phi}^T(t)d\alpha \\ &+ \int_0^t \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}E[(\boldsymbol{\xi}(\alpha) - \boldsymbol{\mu}(\alpha))(\boldsymbol{\xi}(\beta) - \boldsymbol{\mu}(\beta))^T]\boldsymbol{B}^T\boldsymbol{\Phi}^T(t-\beta)d\beta d\alpha\end{aligned} \tag{4.4.120}$$

Finally, using equations (4.4.107), (4.4.109), (4.4.110) yields

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{x}}(t) &= \boldsymbol{\Phi}(t)\mathrm{Cov}_{\boldsymbol{0}}\boldsymbol{\Phi}^T(t) \\ &\quad + \int_0^t \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}\mathrm{Cov}_{\boldsymbol{\xi}}(\alpha,\beta)\boldsymbol{B}^T\boldsymbol{\Phi}^T(t-\beta)d\beta d\alpha\end{aligned} \tag{4.4.121}$$

Analogously, for $\mathrm{Cov}_{\boldsymbol{x}}(t,\tau)$ holds

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{x}}(t,\tau) &= \boldsymbol{\Phi}(t)\mathrm{Cov}_{\boldsymbol{0}}\boldsymbol{\Phi}^T(t) \\ &\quad + \int_0^t \int_0^\tau \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}\mathrm{Cov}_{\boldsymbol{\xi}}(\alpha,\beta)\boldsymbol{B}^T\boldsymbol{\Phi}^T(\tau-\beta)d\beta d\alpha\end{aligned} \tag{4.4.122}$$

Consider now a particular case when the system input is a white noise vector, characterised by

$$\begin{aligned}E[(\boldsymbol{\xi}(t) - \boldsymbol{\mu}(t))(\boldsymbol{\xi}(\tau) - \boldsymbol{\mu}(\tau))^T] &= \boldsymbol{V}(t)\delta(t-\tau) \\ &\quad \text{for } t \geq 0, \tau \geq 0, \boldsymbol{V}(t) = \boldsymbol{V}^T(t) \geq 0\end{aligned} \tag{4.4.123}$$

The state covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t)$ can be determined if the auto-covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t,\tau)$ of the vector white noise $\boldsymbol{\xi}(t)$

$$\mathrm{Cov}_{\boldsymbol{\xi}}(\alpha,\beta) = \boldsymbol{V}(\alpha)\delta(\alpha-\beta) \tag{4.4.124}$$

is used in the equation (4.4.121) that yields

$$\begin{aligned}\mathrm{Cov}_{\boldsymbol{x}}(t) &= \boldsymbol{\Phi}(t)\mathrm{Cov}_{\boldsymbol{0}}\boldsymbol{\Phi}^T(t) \\ &\quad + \int_0^t \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}\boldsymbol{V}(\alpha)\delta(\alpha-\beta)\boldsymbol{B}^T\boldsymbol{\Phi}^T(t-\beta)d\beta d\alpha\end{aligned} \tag{4.4.125}$$

$$\begin{aligned}&= \boldsymbol{\Phi}(t)\mathrm{Cov}_{\boldsymbol{0}}\boldsymbol{\Phi}^T(t) \\ &\quad + \int_0^t \boldsymbol{\Phi}(t-\alpha)\boldsymbol{B}\boldsymbol{V}(\alpha)\boldsymbol{B}^T\boldsymbol{\Phi}^T(t-\alpha)d\alpha\end{aligned} \tag{4.4.126}$$

The covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t)$ of the state vector $\boldsymbol{x}(t)$ is the solution of the matrix differential equation

$$\frac{d\mathrm{Cov}_{\boldsymbol{x}}(t)}{dt} = \boldsymbol{A}\mathrm{Cov}_{\boldsymbol{x}}(t) + \mathrm{Cov}_{\boldsymbol{x}}(t)\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{V}(t)\boldsymbol{B}^T \tag{4.4.127}$$

with initial condition

$$\mathrm{Cov}_{\boldsymbol{x}}(0) = \mathrm{Cov}_{\boldsymbol{0}} \tag{4.4.128}$$

The auto-covariance matrix $\mathrm{Cov}_{\boldsymbol{x}}(t,\tau)$ of the state vector $\boldsymbol{x}(t)$ is given by applying (4.4.124) to (4.4.122). After some manipulations follows

$$\begin{array}{lcll} \mathrm{Cov}_{\boldsymbol{x}}(t,\tau) & = & \boldsymbol{\Phi}(t-\tau)\mathrm{Cov}_{\boldsymbol{x}}(t) & \text{for } t > \tau \\ \mathrm{Cov}_{\boldsymbol{x}}(t,\tau) & = & \mathrm{Cov}_{\boldsymbol{x}}(t)\boldsymbol{\Phi}(\tau-t) & \text{for } \tau > t \end{array} \tag{4.4.129}$$

If a linear continuous system with constant coefficients is asymptotically stable and it is observed from time $-\infty$ and if the system input is a stationary white noise vector, then $\boldsymbol{x}(t)$ is a stationary stochastic process.

The mean value

$$E[\boldsymbol{x}(t)] = \bar{\boldsymbol{x}} \tag{4.4.130}$$

is the solution of the equation

$$\boldsymbol{0} = \boldsymbol{A}\bar{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{\mu} \tag{4.4.131}$$

where $\boldsymbol{\mu}$ is a vector of constant mean values of stationary white noises at the system input.

The covariance matrix

$$E[(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})(\boldsymbol{x}(t) - \bar{\boldsymbol{x}})^T] = \mathrm{Cov}_{\boldsymbol{x}} \tag{4.4.132}$$

is a constant matrix and is given as the solution of

$$\boldsymbol{0} = \boldsymbol{A}\mathrm{Cov}_{\boldsymbol{x}} + \mathrm{Cov}_{\boldsymbol{x}}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{V}\boldsymbol{B}^T \tag{4.4.133}$$

where $\boldsymbol{V}$ is a symmetric positive-definite constant matrix defined as

$$E[(\boldsymbol{\xi}(t) - \boldsymbol{\mu})(\boldsymbol{\xi}(t) - \boldsymbol{\mu})^T] = \boldsymbol{V}\delta(t-\tau) \tag{4.4.134}$$

The auto-covariance matrix

$$E[(\boldsymbol{x}(t_1) - \bar{\boldsymbol{x}})(\boldsymbol{x}(t_2) - \bar{\boldsymbol{x}})^T] = \mathrm{Cov}_{\boldsymbol{x}}(t_1,t_2) \equiv \mathrm{Cov}_{\boldsymbol{x}}(t_1 - t_2, 0) \tag{4.4.135}$$

is in the case of stationary processes dependent only on $\tau = t_1 - t_2$ and can be determined as

$$\begin{array}{lcll} \mathrm{Cov}_{\boldsymbol{x}}(\tau,0) & = & e^{\boldsymbol{A}\tau}\mathrm{Cov}_{\boldsymbol{x}} & \text{for } \tau > 0 \\ \mathrm{Cov}_{\boldsymbol{x}}(\tau,0) & = & \mathrm{Cov}_{\boldsymbol{x}}e^{-\boldsymbol{A}^T\tau} & \text{for } \tau < 0 \end{array} \tag{4.4.136}$$

Example 4.4.1: *Analysis of a first order system*

Consider the mixing process example from page 67 given by the state equation

$$\frac{dx(t)}{dt} = ax(t) + b\xi(t) \tag{4.4.137}$$

where $x(t)$ is the output concentration, $\xi(t)$ is a stochastic input concentration, $a = -1/T_1$, $b = 1/T_1$, and $T_1 = V/q$ is the time constant defined as the ratio of the constant tank volume V and constant volumetric flow q.

Suppose that

$$x(0) = \xi_0 \tag{4.4.138}$$

where ξ_0 is a random variable.

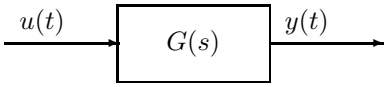

Figure 4.4.6: Block-scheme of a system with transfer function $G(s)$.

Further assume that the following probability characteristics are known

$$\begin{aligned} &E[\xi_0] = x_0 \\ &E[(\xi_0 - x_0)^2] = \mathrm{Cov}_0 \\ &E[\xi(t)] = \mu \qquad \text{for } t \geq 0 \\ &E[(\xi(t) - \mu)(\xi(\tau) - \mu)] = V\delta(t - \tau) \quad \text{for } t, \tau \geq 0 \\ &E[(\xi(t) - \mu)(\xi_0 - x_0)] \equiv 0 \quad \text{for } t \geq 0 \end{aligned} \tag{4.4.139}$$

The task is to determine the mean value $E[x(t)]$, variance $\mathrm{Cov}_x(t)$, and auto-covariance function in the stationary case $\mathrm{Cov}_x(\tau, 0)$.

The mean value $E[x(t)]$ is given as

$$\bar{x} = e^{at}x_0 - \frac{b}{a}\left(1 - e^{at}\right)\mu$$

As $a < 0$, the output concentration for $t \to \infty$ is an asymptotically stationary stochastic process with the mean value

$$\bar{x}_\infty = -\frac{b}{a}\mu$$

The output concentration variance is determined from (4.4.126) as

$$\mathrm{Cov}_x(t) = e^{2at}\mathrm{Cov}_0 - \frac{b^2}{2a}\left(1 - e^{2at}\right)V$$

Again, for $t \to \infty$ the variance is given as

$$\lim_{t\to\infty} \mathrm{Cov}_x(t) = -\frac{b^2 V}{2a}$$

The auto-covariance function in the stationary case can be written as

$$\mathrm{Cov}_x(\tau, 0) = -e^{a|\tau|}\frac{b^2 V}{2a}$$

4.4.6 Frequency Domain Analysis of a Linear System with Stochastic Input

Consider a continuous linear system with constant coefficients (Fig. 4.4.6). The system response to a stochastic input signal is a stochastic process determined by its auto-correlation function and power spectral density. The probability characteristics of the stochastic output signal can be found if the process input and system characteristics are known.

Let $u(t)$ be any realisation of a stationary stochastic process in the system input and $y(t)$ be the associated system response

$$y(t) = \int_{-\infty}^{\infty} g(\tau_1)u(t - \tau_1)d\tau_1 \tag{4.4.140}$$

where $g(t)$ is the impulse response. The mean value of $y(t)$ can be determined in the same way as

$$E[y(t)] = \int_{-\infty}^{\infty} g(\tau_1)E[u(t - \tau_1)]d\tau_1 \tag{4.4.141}$$

Analogously to (4.4.140) which determines the system output in time t, in another time $t+\tau$ holds

$$y(t+\tau) = \int_{-\infty}^{\infty} g(\tau_2) E[u(t+\tau-\tau_2)] d\tau_2 \tag{4.4.142}$$

The auto-correlation function of the output signal is thus given as

$$\begin{aligned} R_{yy}(\tau) &= E[y(t)y(y+\tau)] \\ &= E\left[\left\{\int_{-\infty}^{\infty} g(\tau_1)u(t-\tau_1)d\tau_1\right\}\left\{\int_{-\infty}^{\infty} g(\tau_2)u(t+\tau-\tau_2)d\tau_2\right\}\right] \end{aligned} \tag{4.4.143}$$

or

$$R_{yy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1)g(\tau_2)E[u(t-\tau_1)u(t+\tau-\tau_2)]d\tau_1 d\tau_2 \tag{4.4.144}$$

As the following holds

$$E[u(t-\tau_1)u(t+\tau-\tau_2)] = E\left[u(t-\tau_1)u\{(t-\tau_1)+(\tau+\tau_1-\tau_2)\}\right] \tag{4.4.145}$$

then it follows, that

$$R_{yy}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1)g(\tau_2)R_{uu}(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2 \tag{4.4.146}$$

where $R_{uu}(\tau+\tau_1-\tau_2)$ is the input auto-correlation function with the argument $(\tau+\tau_1-\tau_2)$.

The mean value of the squared output signal is given as

$$\overline{y^2(t)} = R_{yy}(0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1)g(\tau_2)R_{uu}(\tau_1-\tau_2)d\tau_1 d\tau_2 \tag{4.4.147}$$

The output power spectral density is given as the Fourier transform of the associated auto-correlation function as

$$\begin{aligned} S_{yy}(\omega) &= \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-j\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau_1)g(\tau_2)R_{uu}[\tau+(\tau_1-\tau_2)]e^{-j\omega\tau}d\tau_1 d\tau_2 d\tau \end{aligned} \tag{4.4.148}$$

Multiplying the subintegral term in the above equation by $(e^{j\omega\tau_1}e^{-j\omega\tau_2})(e^{-j\omega\tau_1}e^{j\omega\tau_2}) = 1$ yields

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} g(\tau_1)e^{j\omega\tau_1}d\tau_1 \int_{-\infty}^{\infty} g(\tau_2)e^{-j\omega\tau_2}d\tau_2 \int_{-\infty}^{\infty} R_{uu}[\tau+(\tau_1-\tau_2)]e^{-j\omega(\tau+\tau_1-\tau_2)}d\tau \tag{4.4.149}$$

Now we introduce a new variable $\tau' = \tau+\tau_1-\tau_2$, yielding

$$S_{yy}(\omega) = \left[\int_{-\infty}^{\infty} g(\tau_1)e^{j\omega\tau_1}d\tau_1\right]\left[\int_{-\infty}^{\infty} g(\tau_2)e^{-j\omega\tau_2}d\tau_2\right]\left[\int_{-\infty}^{\infty} R_{uu}(\tau')e^{-j\omega\tau'}d\tau'\right] \tag{4.4.150}$$

The last integral is the input power spectral density

$$S_{uu}(\omega) = \int_{-\infty}^{\infty} R_{uu}(\tau')e^{-j\omega\tau'}d\tau' \tag{4.4.151}$$

The second integral is the Fourier transform of the impulse function $g(t)$, i.e. it is the frequency transfer function of the system.

$$G(j\omega) = \int_{-\infty}^{\infty} g(\tau_2)e^{-j\omega\tau_2}d\tau_2 \tag{4.4.152}$$

Finally, the following holds for the first integral

$$G(-j\omega) = \int_{-\infty}^{\infty} g(\tau_1)e^{j\omega\tau_1}d\tau_1 \tag{4.4.153}$$

Hence, from (4.4.149)–(4.4.153) follows

$$S_{yy}(\omega) = |G(j\omega)|^2 S_{uu}(\omega) \tag{4.4.154}$$

with

$$|G(j\omega)|^2 = G(-j\omega)G(j\omega).$$

If the power spectral density $S_{yy}(\omega)$ is known then the mean value of the squared output variable is in the form

$$\overline{y^2(t)} = \frac{1}{\pi}\int_0^\infty S_{yy}(\omega)d\omega = \frac{1}{\pi}\int_0^\infty |G(j\omega)|^2 S_{uu}(\omega)d\omega \tag{4.4.155}$$

4.5 References

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4.6 Exercises

Exercise 4.6.1:
Consider a system with the transfer function given as

$$G(s) = \frac{0.6671s + 3.0610}{s^2 + 4.0406s + 5.4345}$$

Find:

1. response of the system to the unit impulse,
2. response of the system to the unit step.

Exercise 4.6.2:
Consider the system from the previous exercise. Plot:

1. the Nyquist diagram,
2. the Bode diagram.

Exercise 4.6.3:
Consider a system with the transfer function given as

$$C(s) = \frac{-0.0887s + 1.774}{1.25s}$$

Find:

1. response of the system to the unit step,
2. the Nyquist diagram,

3. the Bode diagram.

Exercise 4.6.4:
Consider a system with the transfer function given as

$$L(s) = G(s)C(s)$$

where $G(s)$ is the first exercise and $C(s)$ is in the third exercise. Find:

1. the Nyquist diagram,
2. the Bode diagram.

Exercise 4.6.5:
Consider a system with the transfer function given as

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

where $G(s)$ is in the first exercise and $C(s)$ is in the third exercise. Find:

1. response of the system to the unit step,
2. the Bode diagram.

Exercise 4.6.6:
Consider a system with the transfer function given as

$$S(s) = \frac{1}{1 + G(s)C(s)}$$

where $G(s)$ is in the first exercise and $C(s)$ is in the third exercise. Find its Bode diagram.

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