Appendix

Mathematical background

1. Norms

General remarks

It is useful to have a single number which gives an overall measure of the size of a vector, a matrix, a signal, or a system. For this purpose we use functions which are called norms.

Def. A1.1 A norm of e (which may be a vector, matrix, signal or system) is a real number, denoted $\|e\|$, that satisfies the following properties:

1. Non-negative: $\|e\| \geq 0$.
2. Positive: $\|e\| = 0 \Leftrightarrow e = 0$ (for semi-norms we have $\|e\| = 0 \Leftarrow e = 0$)
3. Homogeneous: $\|\alpha \cdot e\| = |\alpha| \cdot \|e\|$ for all complex scalars α.
4. Triangle inequality: $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$ (A1.1)

In this lecture we consider norms of four different objects:

1. e is a constant vector.
2. e is a constant matrix.
3. e is a time-dependent signal, $e(t)$, which at each fixed t is a constant scalar or vector.
4. e is a "system", a transfer function $G(s)$ or impulse response $g(t)$, which at each fixed t is a constant scalar or vector.

Cases 1 and 2 are finite-dimensional norms, while cases 3 and 4 are infinite-dimensional.

The definition of norm can be derived more precisely using function spaces which are not considered here.

Vector norms

We will consider a column vector a with m complex elements a_i, $i = 1\ldots m$, and introduce three norms which are special cases of the vector p-norm

$$\|a\|_p = \left(\sum_i |a_i|^p \right)^{1/p} \tag{A1.2}$$

where we must have $p \geq 1$ to satisfy the triangle inequality.

Vector 1-norm (or sum norm) $\|a\|_1$. We have

$$\|a\|_1 = \sum_i |a_i| \tag{A1.3}$$

Vector 2-norm (Euclidean norm) $\|a\|_2$. This is the most common vector norm and defined as

$$\|a\|_2 = \sqrt{\sum_i |a_i|^2} \tag{A1.4}$$

Vector ∞-norm (or max norm) $\|a\|_\infty$. This is the largest-element magnitude in the vector

$$\|a\|_\infty = \max_i |a_i| \tag{A1.5}$$

Matrix norms

We will consider a $l \times m$ matrix A with complex elements.

Def. A1.2 A norm on a matrix $\|A\|$ is a matrix norm, if, in addition to the four norm properties in Def. A1.1, it also satisfies the multiplicative property (consistency condition):

$$\|AB\| \le \|A\| \cdot \|B\| \tag{A1.6}$$

We first examine three norms which are direct extensions of the considered vector norms.

Sum matrix norm $\|A\|_{\text{sum}}$. This is the sum of the element magnitudes

$$\|A\|_{\text{sum}} = \sum_{i,j} |a_{ij}| \tag{A1.7}$$

Frobenius matrix norm (or Euclidean norm) $\|A\|_{\text{F}}$. This is the square root of the sum of the squared element magnitudes

$$\|A\|_{\text{F}} = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\text{tr}(A^{\text{T}*} A)} \tag{A1.8}$$

The trace tr is the sum of the diagonal elements, and $A^{\text{T}*}$ is the complex conjugate transpose of A.

Max element norm $\|A\|_{\text{max}}$. This is the largest-element magnitude (not a matrix norm!)

$$\|A\|_{\infty} = \max_{i,j} |a_{ij}| \tag{A1.9}$$

Induced matrix norms

Induced matrix norms are important because of their close relationship to signal amplification in systems. Consider the equation

$$y = Au \tag{A1.10}$$

We may think of u as the input vector and y as the output vector and consider the "gain" of the matrix A as defined by the ratio $||y||/||u||$. The maximum gain of all possible input directions is given by the induced norm defined as

$$\|A\|_{ip} = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p} \tag{A1.11}$$

where $||u||_p = (\Sigma_i |u_i|^p)^{1/p}$ denotes the vector p-norm.

For the induced 1-, 2-, and ∞-norms the following identities hold:

Maximum column sum $$\|A\|_{i1} = \max_j \left(\sum_i |a_{ij}|\right) \tag{A1.12}$$

Maximum row sum $$\|A\|_{i\infty} = \max_i \left(\sum_j |a_{ij}|\right) \tag{A1.13}$$

Maximum singular value $$\|A\|_{i2} = \sigma_{\max}(A) = \sqrt{\rho(A^{T_*} A)} \tag{A1.14}$$

where the spectral radius $\rho(A) = \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix A. the induced 2-norm of a matrix is equal to the (largest) singular value and is often called the spectral norm.

The multiplicative property

Theorem A1.1 All induced norms $\|A\|_{ip}$ are matrix norms and thus satisfy the multiplicative property

$$\|AB\|_{ip} \leq \|A\|_{ip} \cdot \|B\|_{ip} \tag{A1.15}$$

The multiplicative property also holds if we choose A and B as vectors. For example

$$\|Aa\| \leq \|A\| \cdot \|a\| \tag{A1.16}$$

From (A.16) we also get for any matrix norm that

$$\|A\| \geq \max_{a \neq 0} \frac{\|Aa\|}{\|a\|} \tag{A1.17}$$

Note that the induced norms are defined such that we always have equality in (A1.17). Choosing both $A = a^{\mathrm{T}*}$ and $B = b$ as vectors, and using the induced 2-norm (singular value) in (A1.16) we obtain the Cauchy-Schwarz inequality

$$\left|a^{\mathrm{T}*} b\right| \leq \|a\|_2 \cdot \|b\|_2 \tag{A1.18}$$

where a and b are column vectors of the same dimension and $a^{\mathrm{T}*}b$ is the Euclidean inner product between the vectors a and b.

The spectral radius

The spectral radius $\rho(A)$ is the magnitude of the largest eigenvalue of the matrix A.

$$\rho(A) = \max_i \left|\lambda_i(A)\right| \tag{A1.19}$$

It is *not* a norm, as it does not satisfy norm properties 2 and 4 in Def. A1.1. Although the spectral radius is not a norm, it provides a lower bound on any matrix norm, which can be very useful.

Theorem A1.2 For any matrix norm (and in particular for any induced norm) $\|A\|$ we have

$$\rho(A) \le \|A\| \tag{A1.20}$$

A simple physical interpretation of (A1.20) is that the eigenvalue measures the gain of the matrix only in certain directions (given by the eigenvectors), and must therefore be less than that for a matrix norm which allows any direction and yields the maximum gain, recall (A1.17).

There is an important relationship between the Frobenius norm and the singular values, $\sigma_i(A)$, of a matrix A, namely

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_i \sigma_i^2(A)} \tag{A1.21}$$

Signal norms (1)

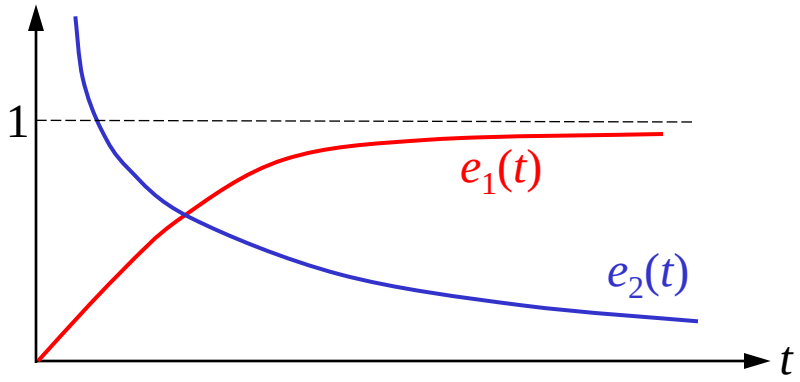
We will now consider the temporal norm of a time-varying (or frequency-varying) signal, $e(t)$. There is a big difference between the spatial norms (vector and matrix norms) and the temporal norms, since the signal can change strongly during time. For example, for the signal $e_1(t)$, shown in the figure below, the ∞-norm (peak value) is 1, i.e. $\|e_1(t)\|_\infty = 1$, whereas since the signal does not "die out" the 2-norm is infinite, i.e. $\|e_1(t)\|_2 = \infty$. For $e_2(t)$, the opposite is true.
For signals, we may compute the norm in two steps:

- "Sum up" the channels at a given time or frequency using a vector norm (for a scalar signal we simply take the absolute value).
- "Sum up" in time or frequency using a temporal norm.

Generally, we use the same p-norm for both the vector and signal, and define the temporal p-norm, $\|e(t)\|_p$, of a time-varying vector as

L_p-norm: $$\left\|e(t)\right\|_p = \left(\int_{-\infty}^{\infty} \sum_i \left|e_i(\tau)\right|^p \mathrm{d}\tau \right)^{1/p} \qquad \text{(A1.22)}$$

Commonly used are the following temporal norms of signals:



Signal norms (2)

1-norm in time (integral absolute value) $\|e(t)\|_1$

$$\left\|e(t)\right\|_1 = \int_{-\infty}^{\infty} \sum_i \left|e_i(\tau)\right| \mathrm{d}\tau \tag{A1.23}$$

2-norm in time (quadratic norm, integral square value, "energy" of signal) $\|e(t)\|_2$

$$\left\|e(t)\right\|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_i \left|e_i(\tau)\right|^2 \mathrm{d}\tau} \tag{A1.24}$$

∞-norm in time (peak value in time) $\|a\|_\infty$. This is the largest-element magnitude in the vector

$$\left\|e(t)\right\|_\infty = \max_t \left(\max_i \left|e_i(t)\right| \right) \tag{A1.25}$$

In addition, we also consider the power norm or rms (root-mean-square) norm (which is actually only a semi-norm since it does not satisfy norm property 2)

$$\left\|e(t)\right\|_{\mathrm{pow}} = \lim_{T\to\infty} \sqrt{\frac{1}{2T} \int_{-T}^{T} \sum_i \left|e_i(\tau)\right|^2 \mathrm{d}\tau} \tag{A1.26}$$

Remark:
To be mathematically correct we should have used $\sup_t$ (least upper bound) rather than $\max_t$ in (A1.25), since the maximum value may not actually be reached (e.g. if it occurs for $t = \infty$).

The terms $\mathcal{H}_\infty$ and $\mathcal{H}_2$

The terms $\mathcal{H}_\infty$ and $\mathcal{H}_2$, which are purely mathematical, have established themselves in the control community. They are, used in control context, induced signal norms of transfer functions. The $\mathcal{H}_\infty$ norm of a stable scalar transfer function $G(s)$ is simply the peak value (or strictly speaking the, supremum, the least lower bound) of $|G(\mathrm{j}\omega)|$ as a function of frequency, i.e.,

$$\|G(s)\|_\infty = \sup_\omega |G(\mathrm{j}\omega)| \qquad \text{(A1.27)}$$

The symbol ∞ comes from the fact that the peak value over frequency may be written as

$$\sup_\omega |G(\mathrm{j}\omega)| = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |G(\mathrm{j}\omega)|^p \, \mathrm{d}\omega \right)^{1/p}$$

The symbol $\mathcal{H}$ stands for "Hardy space" and $\mathcal{H}_\infty$ in our context is the set of transfer functions with bounded ∞–norm, which is simply the set of *stable and proper* transfer functions.

Similarly, the symbol $\mathcal{H}_2$ stands for the Hardy space of transfer functions with bounded 2-norm, which is the set of stable and strictly proper transfer functions. The norm $\mathcal{H}_2$ is then defined as

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\mathrm{j}\omega)|^2 \, \mathrm{d}\omega} \qquad \text{(A1.28)}$$

the $\mathcal{H}_2$ norm of a semi-proper transfer function is infinite, whereas its $\mathcal{H}_\infty$ norm is finite, for example: the sensitivity function $S = (\mathrm{I} + GK)^{-1}$.

2. Singular value decomposition (SVD)

Singular value decomposition (1)

Def. A2.1 Unitary matrix. A (complex) matrix $\underline{\mathbf{V}}$ is unitary if

$$\underline{\mathbf{V}}^{\mathrm{T}_*} = \underline{\mathbf{V}}^{-1} \tag{A2.1}$$

All the eigenvalues of a unitary matrix have absolute value equal to 1, and all its singular values (as defined below) are equal to 1.

For a non-square matrix it is not possible to define eigenvalues. An extension of the eigenvalue concept can be obtained by singular values. Given a (complex) matrix $\underline{\mathbf{A}}$ of dimensions $(l \times m)$, where the matrix $\underline{\mathbf{A}}^{\mathrm{T}_*}\underline{\mathbf{A}}$ has k, $k = \min(l,m)$, non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$ (ordered from the largest to the smallest) which can be expressed in the form

$$\sigma_i(\underline{\mathbf{A}}) = \sqrt{\lambda_i(\underline{\mathbf{A}}^{\mathrm{T}_*}\underline{\mathbf{A}})} = \sqrt{\lambda_i(\underline{\mathbf{A}}\underline{\mathbf{A}}^{\mathrm{T}_*})}\,, \quad \lambda_i = \sigma_i^2, \qquad \sigma_i \geq 0 \tag{A2.2}$$

The scalars $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \geq 0$ are said to be the *singular values* of matrix $\underline{\mathbf{A}}$. the *singular value decomposition* (SVD) of matrix $\underline{\mathbf{A}}$ is then given by

$$\underline{\mathbf{A}} = \underline{\mathbf{U}}\underline{\mathbf{\Sigma}}\underline{\mathbf{V}}^{\mathrm{T}_*} \tag{A2.3}$$

where $\underline{\mathbf{U}}$ is an $(l \times l)$ orthogonal unitary matrix

$$\underline{\mathbf{U}} = \left[\underline{\boldsymbol{u}}_1 \;\; \underline{\boldsymbol{u}}_2 \;\; \cdots \;\; \underline{\boldsymbol{u}}_m\right] \tag{A2.4}$$

Singular value decomposition (2)

$\underline{\mathbf{V}}$ is an $(m \times m)$ orthogonal unitary matrix

$$\underline{\mathbf{V}} = \left[\underline{\boldsymbol{v}}_1 \;\; \underline{\boldsymbol{v}}_2 \;\; \cdots \;\; \underline{\boldsymbol{v}}_n\right] \tag{A2.5}$$

and $\underline{\boldsymbol{\Sigma}}$ is an $(l \times m)$ matrix of the form

$$\underline{\boldsymbol{\Sigma}} = \begin{bmatrix} \underline{\boldsymbol{D}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix} \qquad \underline{\boldsymbol{D}} = diag\left\{\sigma_1, \sigma_2, \ldots, \sigma_r\right\} \tag{A2.6}$$

where the number of non-zero singular values σ_i is equal to the rank r of matrix $\underline{\mathbf{A}}$.

The columns of $\underline{\mathbf{U}}$ are the eigenvectors of the matrix $\underline{\mathbf{A}}\underline{\mathbf{A}}^{\mathrm{T}*}$, whereas the columns of $\underline{\mathbf{V}}$ are the eigenvectors of the matrix $\underline{\mathbf{A}}^{\mathrm{T}*}\underline{\mathbf{A}}$. In view of the partitions of $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$ in (A2.4) and (A2.5), we have

$$\underline{\mathbf{A}}\underline{\boldsymbol{v}}_i = \sigma_i \underline{\boldsymbol{u}}_i \,, \qquad i = 1, \ldots, r \tag{A2.7}$$

and

$$\underline{\mathbf{A}}\underline{\boldsymbol{v}}_i = \underline{\mathbf{0}} \,, \qquad i = r+1, \ldots, k \tag{A2.8}$$

$\underline{\boldsymbol{v}}_1$ corresponds to the input direction with largest amplification, and $\underline{\boldsymbol{u}}_1$ is the corresponding output direction in which the inputs are most effective. The directions involving both $\underline{\boldsymbol{v}}_1$ and $\underline{\boldsymbol{u}}_1$ are sometimes referred to as the “high gain” directions.

Singular value decomposition (3)

In general, the singular values must be computed numerically. For (2×2) matrices, however, an analytical solution is easily derived. We introduce

$$b = \mathrm{tr}(\underline{\mathbf{A}}^{\mathrm{T}_*}\underline{\mathbf{A}}) = \sum_{i,j}\left|a_{ij}\right|^2, \quad c = \det(\underline{\mathbf{A}}^{\mathrm{T}_*}\underline{\mathbf{A}})$$

Solving now a quadratic equation we obtain the singular values

$$\sigma_{\max}(\underline{\mathbf{A}}) = \sqrt{\frac{b+\sqrt{b^2-4c}}{2}}\,; \quad \sigma_{\min}(\underline{\mathbf{A}}) = \sqrt{\frac{b-\sqrt{b^2-4c}}{2}} \tag{A2.9}$$

Because of (A2.3) we get, provided the $(m \times m)$ matrix $\underline{\mathbf{A}}$ is non-singular, that

$$\underline{\mathbf{A}}^{-1} = \underline{\mathbf{V}}\underline{\mathbf{\Sigma}}^{-1}\underline{\mathbf{U}}^{\mathrm{T}_*} \tag{A2.10}$$

This is the SVD of $\underline{\mathbf{A}}^{-1}$ but with the order of the singular values reversed. It follows from (A2.10) for $j = m\text{-}i\text{+}1$ that

$$\sigma_i(\underline{\mathbf{A}}^{-1}) = 1/\sigma_j(\underline{\mathbf{A}}), \quad u_i(\underline{\mathbf{A}}^{-1}) = v_j(\underline{\mathbf{A}}), \quad v_i(\underline{\mathbf{A}}^{-1}) = u_j(\underline{\mathbf{A}}) \tag{A2.11}$$

and in particular

$$\sigma_{\max}(\underline{\mathbf{A}}^{-1}) = 1/\sigma_{\min}(\underline{\mathbf{A}}) \tag{A2.12}$$

3. Poles and zeros of MIMO systems

Poles from a transfer function matrix (TFM)

We now consider a system described by a transfer function matrix (TFM) $G(s)$

$$\begin{bmatrix} y_1(s) \\ \vdots \\ y_l(s) \end{bmatrix} = y(s) = G(s)u(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ \vdots & \ddots & \vdots \\ g_{l1}(s) & \cdots & g_{lm}(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ \vdots \\ u_m(s) \end{bmatrix} \tag{A3-1}$$

The poles of $G(s)$ may be interpreted as the frequencies $s = p_i$ where $G(p_i)$ has a singularity (is infinite). A practical way for calculating the poles is given by the following theorem.

Theorem A3-1

The pole polynomial $\Delta(s)$ corresponding to a minimal realization of a system with transfer function matrix $G(s)$ is the least common denominator of all non-identically-zero minors of all orders of $G(s)$. A minor of a matrix is the determinant of the square matrix obtained by deleting certain rows and/or columns of the matrix. The poles are the roots of the pole polynomial.

Essentially, poles of a TFM are those of its elements. However by looking at only the elements it is not possible to determine the multiplicity of the poles (and the order of the system!).

Poles from a state-space realization

The poles of a MIMO system in state-space description

$$\dot{x} = \mathrm{A}x + \mathrm{B}u\,, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$
$$y = \mathrm{C}x + \mathrm{D}u\,, \quad y \in \mathbb{R}^l \qquad \text{(A3-2)}$$

are generally eigenvalues $\lambda_i(\mathrm{A})$, $i = 1, \ldots n$ of the system matrix A. The pole polynomial or characteristic polynomial $\Delta(s)$ is defined as $\det(s\mathrm{I}\text{-}\mathrm{A})$. Thus the system's poles are the roots of the characteristic polynomial

$$\Delta(s) = \det(s\mathrm{I} - \mathrm{A}) = 0 \qquad \text{(A3-3)}$$

If (A3-2) is a minimal realization (controllable and observable system) of (A3-1), the definition of the pole polynomial from Theorem 3.1 and in (A3-3) are identical. However, if $(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D})$ does not correspond to a minimal realization, then the poles by this definition will include the poles (eigenvalues) corresponding to uncontrollable and/or unobservable states. For controllable and observable realizations the poles of a state-space model are the same as those of the TFM:

$$G(s) = \mathrm{C}(s\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D} \qquad \text{(A3-4)}$$

Zeros derived from a transfer function matrix

Def. A3.1 The normal rank of a TFM $G(s)$ is equal to r if $\text{rank}\{G(s)\} = r$ for almost all values of s. It is clear that $r \leq \min(l, m)$.

Suppose $m \leq l$ (more outputs than inputs). Then if $r < m$, it follows that the m columns of $G(s)$ are linearly dependent. This may be interpreted as the inputs to $G(s)$ being redundant.

Suppose $m \geq l$ (more inputs than outputs). Then if $r < l$, it follows that the l rows of $G(s)$ are linearly dependent. This may be interpreted as the outputs to $G(s)$ being redundant.

We will only consider non-redundant systems in the following ($r = \min(l, m)$).

Def. A3.2 $s = z_0$ is a zero of $G(s)$ if the rank of $G(s)$ is less than its normal rank ($G(s)$ loses rank at z_0).

Note that this definition of zeros is based on the transfer function matrix corresponding to a minimal realization of a system. These zeros are sometimes called transmission zeros.

Theorem A3-2

The zero polynomial $\zeta(s)$ corresponding to a minimal realization of the system $G(s)$ is the greatest common divisor of all the numerators of all order-r minors of $G(s)$ where r is the normal rank of $G(s)$ provided that these minors have been adjusted in such a way as to have the pole polynomial $\Delta(s)$ as their denominators. The transmission zeros are the roots of the zero polynomial.

Zeros derived from a state space realization (1)

Consider a minimal realization (A3-2). The zeros of the system can be also defined based on the Transmission Blocking Property:

Transmission zeros are associated with modes of behavior wherein for some frequency z_0 the input and states of a system are nonzero, yet the output equals zero, similar as in SISO cases.

To see this property we consider a non-zero input $u(t) = u_0 e^{z_0 t}$ and the corresponding state $x(t) = x_0 e^{z_0 t}$ and prove the condition for a vanishing output $y(t) \equiv 0$ for $t \geq 0$.

$$\dot{x} = \mathrm{A}x + \mathrm{B}u \;\Rightarrow\; x_0 z_0 e^{z_0 t} = \mathrm{A}x_0 e^{z_0 t} + \mathrm{B}u_0 e^{z_0 t} \;\Rightarrow\; \begin{bmatrix} z_0\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

Also,

$$y = \mathrm{C}x + \mathrm{D}u = 0 \;\Rightarrow\; \mathrm{C}x_0 e^{z_0 t} + \mathrm{D}u_0 e^{z_0 t} = 0 \;\Rightarrow\; \begin{bmatrix} \mathrm{C} & \mathrm{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

Leading to a generalized eigenvalue problem (GEP)

$$\begin{bmatrix} z_0\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(A3-5)}$$

Zeros derived from a state space realization (2)

Especially, if $m = l$, we have

$$\det \begin{vmatrix} s_0 \mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{vmatrix} = 0 \qquad \text{(A3-6)}$$

The solution to the generalized eigenvalue problem is the transmission zeros of the system.
In general, the matrix

$$P(s) = \begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} \qquad \text{(A3-7)}$$

is called the Rosenbrock system matrix.

We also say that u_0 is the *input zero direction* and x_0 is the *state zero direction* of the system. Essentially, it means that if the system has a transmission zero and these directions are chosen for the input and initial state, the output is constantly zero for the frequency represented by the transmission zero (transmission blocked).

Naturally, the transmission zeros derived from a transfer function matrix must be the same as those derived from its minimal state-space realization. To see this, we look at the equality

Zeros derived from a state space realization (2)

$$\begin{bmatrix} \mathrm{I} & 0 \\ -C(s_0\mathrm{I}\text{-}\mathrm{A})^{-1} & \mathrm{I} \end{bmatrix}\begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} = \begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ 0 & G(s) \end{bmatrix} \tag{A3-8}$$

As far as z_0 is not an eigenvalue of A, we have

$$\begin{aligned} &normal\ rank\begin{bmatrix} \mathrm{I} & 0 \\ -C(z_0\mathrm{I}\text{-}\mathrm{A})^{-1} & \mathrm{I} \end{bmatrix}\begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} = normal\ rank\begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{bmatrix} \\ &= normal\ rank\begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \\ 0 & G(s) \end{bmatrix} = n + normal\ rank\{G(s)\} \end{aligned} \tag{A3-9}$$

It is therefore clear that $P(s)$ will lose rank if $G(s)$ loses rank, and vice versa. Thus, the transmission zeros of $P(s)$ (minimal state-space realization) are exactly also whose of $G(s)$ (TFM).

Zeros derived from a state space realization (3)

If z_0 is not an eigenvalue of A, then $x_0 = (z_0\,\mathrm{I}\text{-}\mathrm{A})^{-1}\mathrm{B}u_0$ which gives

$$\left[\mathrm{C}(z_0\mathrm{I}\text{-}\mathrm{A})^{-1}\mathrm{B}+\mathrm{D}\right]u_0 = G(z_0)u_0 = 0 \qquad \text{(A3-10)}$$

which implies that $G(s)$ loses rank at z_0.

In cases $G(s)$ is square, one can (mostly) find the zero frequencies by directly solving

$$\det G(s) = 0 \qquad \text{(A3-11)}$$

However, if any of the resulting roots are also eigenvalues of A, then one needs to re-check the generalized eigenvalue matrix condition (A3-5).

If $(\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D})$ does not correspond to a minimal realization, the condition (A3-5) may lead to additional zeros making Rosenbrock system matrix losing rank. These zeros are usually called invariant zeros. They usually correspond to the uncontrollable or unobservable modes of the system and lead to pole/zero cancellations in the transfer function matrix. Note that because MIMO systems always have directions, such a cancellation basically means that both the location and direction of a pole (represented by the eigenvalue of A and the corresponding eigenvector) coincide with the location and the direction of the respective invariant zero (represented by the “eigenvalue” and “eigenvector” of the associated generalized eigenvalue problem (A3-5)). Especially, if $P(s)$ loses

Zeros derived from a state space realization (4)

rank due to the rank deficiency of its upper part

$$rank\begin{bmatrix} z_0\mathrm{I}\text{-}\mathrm{A} & -\mathrm{B} \end{bmatrix} < n \qquad \text{(A3-12)}$$

then the corresponding invariant zero z_0 is also named an *input decoupling zero* (associated with an uncontrollable mode) and leads to a pole/zero cancellation in the transfer function matrix calculation using (A3-4). On the other hand, if $P(s)$ loses rank due to the rank deficiency of its left part

$$rank\begin{bmatrix} s\mathrm{I}\text{-}\mathrm{A} \\ \mathrm{C} \end{bmatrix} < n \qquad \text{(A3-13)}$$

then the corresponding invariant zero z_0 is also named an *output decoupling zero* (associated with an unobservable mode) and leads also to a pole/zero cancellation in the transfer function matrix.

4. Linear fractional transformation (LFT)

Linear fractional transformation (1)

Def. A4.1 A (complex) function $F : Z \to S$ of the form $S = (\alpha+\beta Z)(\gamma+\delta Z)^{-1}$ is called a *linear fractional transformation (LFT)*.

An important application of LFTs is in robust control theory. As they are used there, the parameters α, β, γ, δ and the variables S and Z belong to the set of transfer function matrices.

Consider a matrix P of dimension $(n_1 + n_2) \times (m_1 + m_2)$ and partition it as follows

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{A4.1}$$

Let the matrices Δ and K (pls. ref. also to Chapter 3) have dimensions $m_1 \times n_1$ and $m_2 \times n_2$, respectively (compatible with the upper and lower partitions of P). We adopt the following notation for the lower and upper linear fractional transformations:

$$F_l(P,K) = P_{11} + P_{12}K(\mathrm{I} - P_{22}K)^{-1}P_{21} \tag{A4.2}$$

$$F_u(P,\Delta) = P_{22} + P_{21}\Delta(\mathrm{I} - P_{11}\Delta)^{-1}P_{12} \tag{A4.3}$$

where subscript l denotes lower and subscript u upper. Note that $F_l(\cdot,\cdot)$ and $F_u(\cdot,\cdot)$ are special cases of the general LFT defined above.

Linear fractional transformation (2)

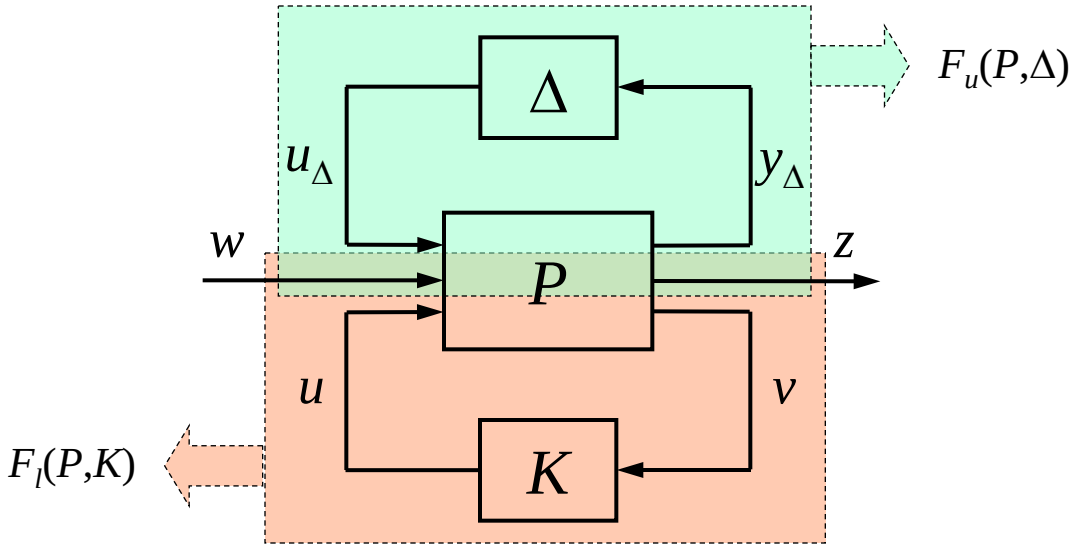
The lower fractional transformation $F_l(P,K)$ is the transfer function R_l resulting from wrapping feedback K around the lower part of P as illustrated in the following figure. We have in this case

$$z = P_{11}w + P_{12}u\,, \quad v = P_{21}w + P_{22}u\,, \quad u = Kv \tag{A4.4}$$

and

$$z = F_l(P,K)w = [P_{11} + P_{12}K(\mathrm{I} - P_{22}K)^{-1}P_{21}]w \tag{A4.5}$$

Similarly, the upper LFT of P, $F_u(P,\Delta)$, is obtained by wrapping feedback Δ around the upper part of P.



Linear fractional transformation (3)

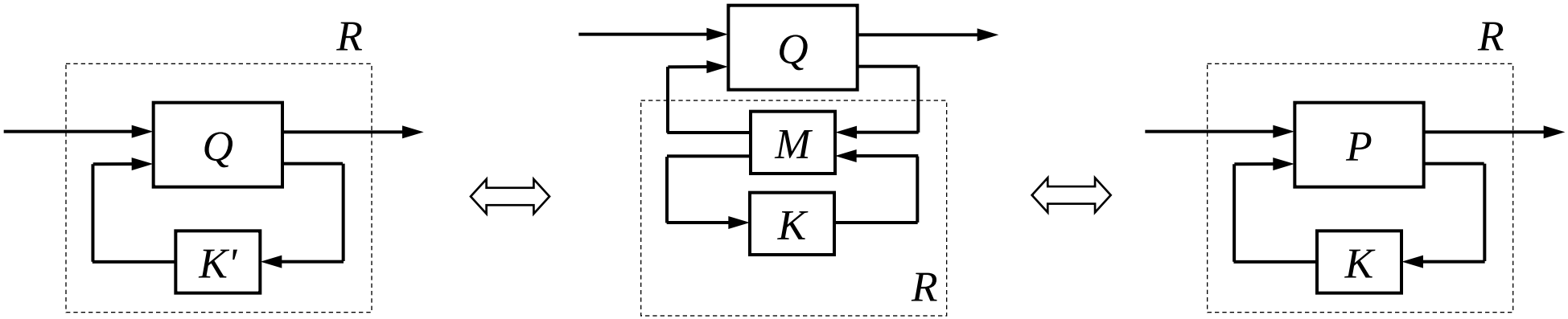
An important property of LFT is that any interconnection of LFTs is again an LFTT. Consider the figure below where R is written in terms of a lower LFT of K', which again is a lower LFT of K, and we want to express R directly as an LFT of K. It is

$$R = F_l(Q, K') \quad \text{where} \quad K' = F_l(M, K) \tag{A4.6}$$

and we want to calculate P (in terms of Q and M) such that $R = F_l(P,K)$. We find

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} + Q_{12}M_{11}(\mathrm{I} - Q_{22}M_{11})^{-1}Q_{21} & Q_{12}(\mathrm{I} - M_{11}Q_{22})^{-1}M_{12} \\ M_{21}(\mathrm{I} - Q_{22}M_{11})^{-1}Q_{21} & M_{22} + M_{21}Q_{22}(\mathrm{I} - M_{11}Q_{22})^{-1}M_{12} \end{bmatrix} \tag{A4.7}$$

Similar expressions can be obtained when we use upper LFTs.



Linear fractional transformation (4)

F_l and F_u are closely related. If we know $R = F_l(M,K)$, then we can directly obtain R in terms of an upper LFT of K by reordering M, that is

$$F_u(\tilde{M},K) = F_l(M,K) \tag{A4.8}$$

where

$$\tilde{M} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{A4.9}$$

Using the matrix inversion lemma

$$(A_1 + A_2 A_3 A_4)^{-1} = A_1^{-1} - A_1^{-1} A_2 (A_4 A_1^{-1} A_2 + A_3^{-1})^{-1} A_4 A_1^{-1} \tag{A4.10}$$

we can easily derive the following relationship for the inverse of LFTs, on the assumption that all relevant matrix inverses exist

$$(F_i(M,K))^{-1} = F_l(\tilde{M},K) \tag{A4.11}$$

Where

$$\tilde{M} = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1} M_{12} \\ M_{21} M_{11}^{-1} & M_{22} - M_{21} M_{11}^{-1} M_{12} \end{bmatrix} \tag{A4.12}$$