Technische Universität Kaiserslautern

Chapter 3

System models and characteristic properties of linear time-invariant systems in the time-domain

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Differential equation as model of LTI systems

A *LTI-System* is a **l**inear **t**ime-**i**nvariant dynamic system and can be modeled in SISO case using a linear ordinary differential equation with constant coefficients, which defines the relationship between the system input $u(t)$ and system output $y(t)$

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_p\frac{\mathrm{d}^p u}{\mathrm{d}t^p} + b_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \ldots + b_1\frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u \qquad (3.1)$$

It is always $n \geq p$ (only those systems are physically realizable!) and the highest number n of differentiations of the system output y is called the *order* of the system.

Eq. (3.1) represents the input-output relation of the dynamic system and is one of the standard mathematical models used for system modeling.

The model equation (3.1) can describe a physical process (plant), a controller or a closed-loop control as well.

Which system properties can we recognize from this standard form of system model? To answer the question, we first consider its simplest variant, the first-order dynamic LTI systems.

Notice:

An exact definition of linearity and time-invariance of dynamic systems will be given later.

First-order dynamic LTI systems

We know from the theory of differential equations that the first-order homogeneous equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = 0 \tag{3.2}$$

has the solution

$$y(t) = Ce^{-at} \tag{3.3}$$

where $C = y(0)$ is an initial condition depending constant. The solution of the inhomogeneous equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = bu(t) \tag{3.4}$$

with u as system input is known as

$$y(t) = Ce^{-at} + b\int_0^t e^{-a(t-\tau)}u(\tau)\mathrm{d}\tau \tag{3.5}$$

and consists of the *free response* (the first Term, depending on initial condition only) and the *forced response* (the second Term, depending on system input only).

Note:

-a is the solution to the equation $s+a=0$.

Active excise

Stop and think!

Which system parameter characterizes the system behavior?

...

Parameter a!

The "system stability" (whether the system output goes to infinity) depends only on a!

Second-order dynamic LTI systems (1)

The solution of the second-order homogeneous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = 0 \tag{3.6}$$

can be given as

$$y(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \tag{3.7}$$

where α_1, α_2 are roots of the so-called characteristic equation

$$s^2 + a_1 s + a_0 = (s - \alpha_1)(s - \alpha_2) = 0 \tag{3.8}$$

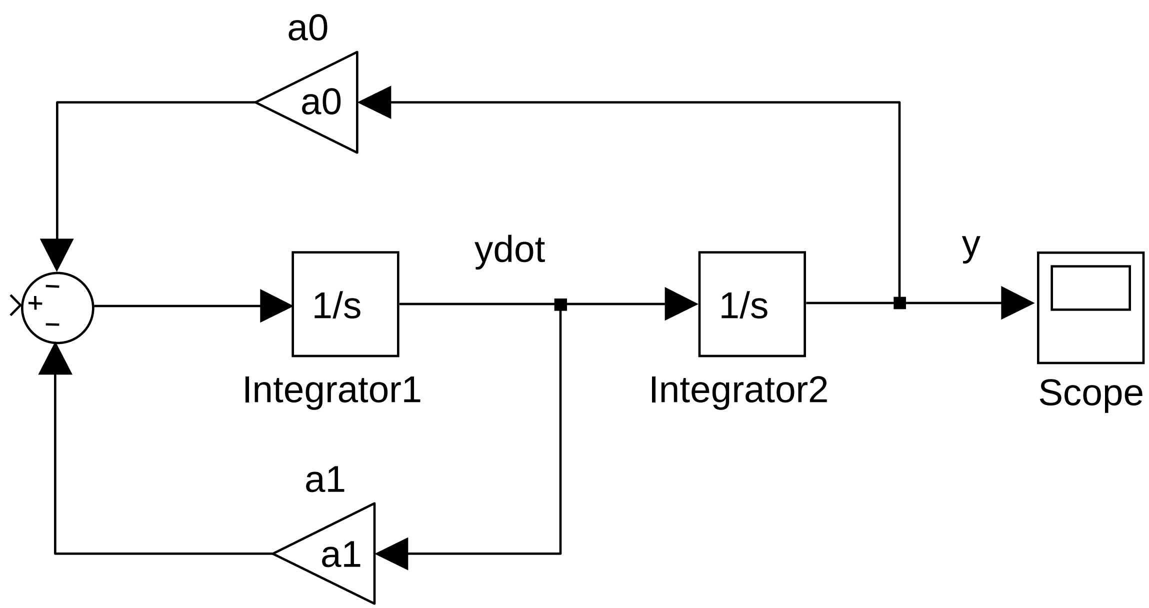
and C_1, C_2 are constants to be determined using initial conditions. The solution of the inhomogeneous LTI equation, consisting again of the free and the forced response, is for the case $\alpha_1 \neq \alpha_2$:

$$y(t) = \underbrace{C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}}_{\text{free response}} + \underbrace{(\alpha_2 - \alpha_1)^{-1} \int_0^t \left(e^{\alpha_2 (t-\tau)} - e^{\alpha_1 (t-\tau)}\right) u(\tau)\mathrm{d}\tau}_{\text{forced response}} \tag{3.9}$$

The system behavior for different system inputs u can be conveniently studied by computer simulations.

Second-order dynamic LTI systems (2)

Basically, the system response, i.e., the solutions of the differential equation, depends on the type (real, imaginary, or complex) of the roots α_1, α_2 of the characteristic equation as well as on their signs. See also appropriate mathematical text books for more details.



MATLAB/SIMULINK model for simulation studies

Active excise

We consider again the closed-loop dynamic of our cruise control design example with a PI controller

$$\ddot{e} + (0.02 + k_P)\dot{e} + k_I e = 10\dot{\theta} \qquad \text{resp.} \qquad \ddot{e} + 2d\omega_0\dot{e} + \omega_0^2 e = 0$$

The solution is greatly influenced by the parameters *damping factor* d and *undamped natural frequency* ω_0

Stop and think!

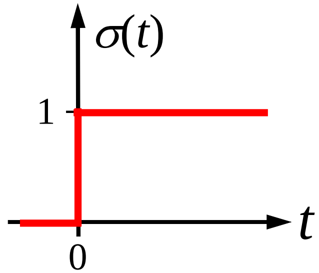
What happens for $\begin{cases} d < -1 \\ -1 \le d < 0 \\ d = 0 \\ 0 < d \le 1 \\ d > 1 \end{cases}$?

How is the solution influenced by ω_0?

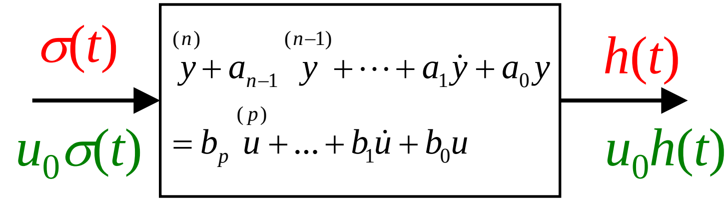
Step response and impulse response

Unit step function (Heaviside Function):

$$\sigma(t)=\begin{cases}0 & \forall t<0\\ 1 & \forall t\geq 0\end{cases}$$

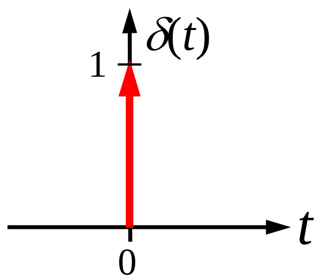


Step response bzw. *Sprungantwort*:

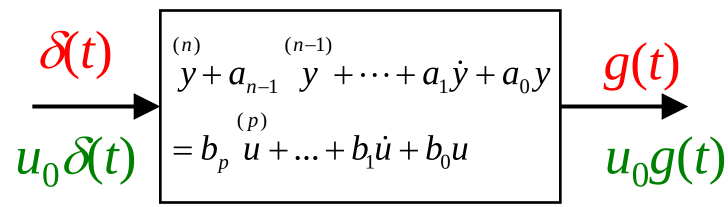


Unit impulse function (δ impulse, Dirac impuls):

$$\begin{cases}\delta(t)=0 \quad \forall t\neq 0\\ \int_{-\infty}^{\infty}\delta(t)\mathrm{d}t=1\end{cases}$$



Impulse response bzw. *Impulsantwort*:



Sifting property of unit impulse: $\int_{-\infty}^{\infty}\delta(t-\tau)\cdot f(\tau)\mathrm{d}\tau = f(t)$

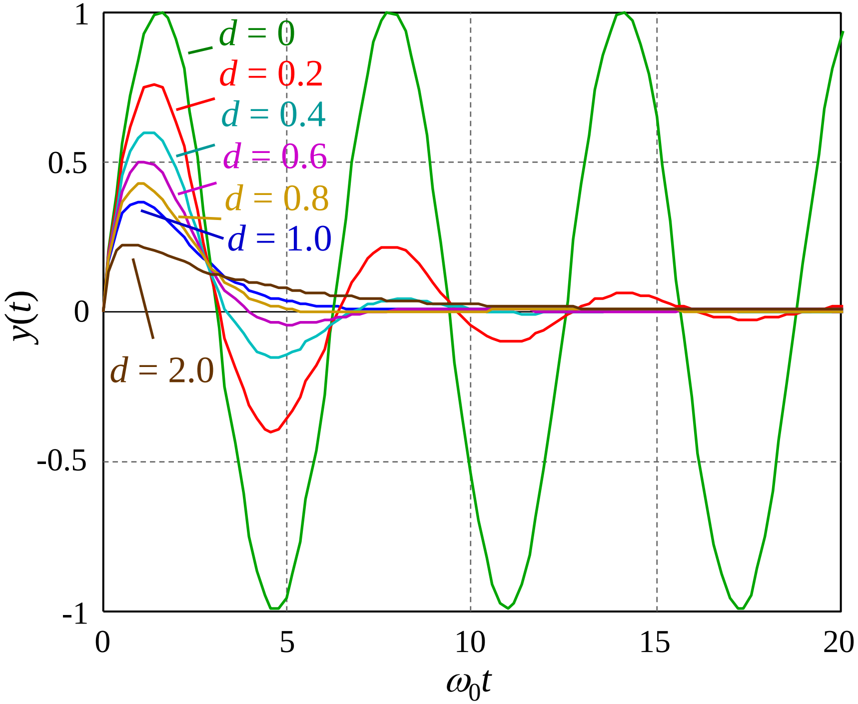
Relation between σ and δ: $\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t)=\delta(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}h(t)=g(t)$

Second-order dynamic LTI systems (3)

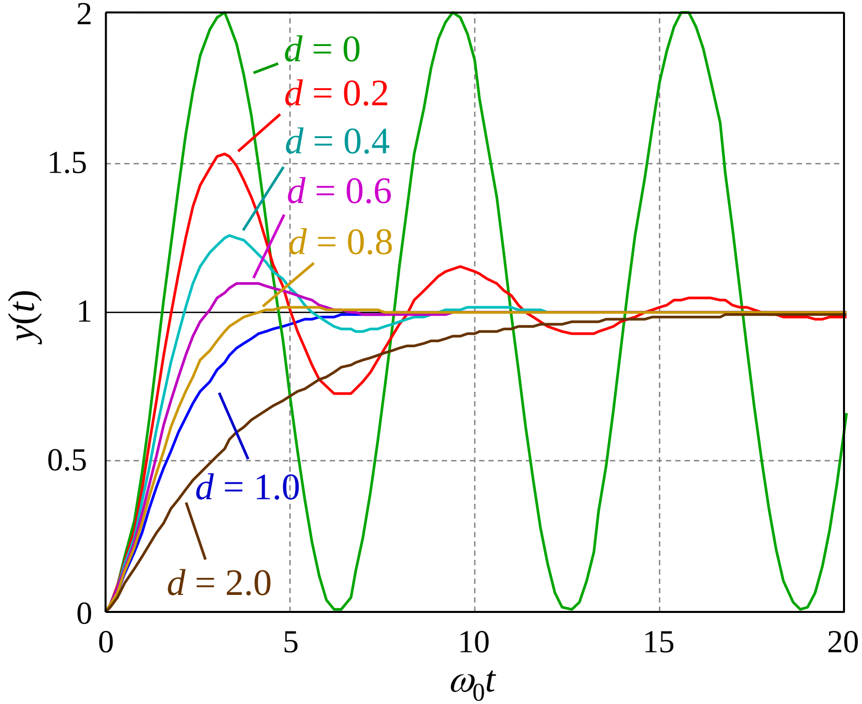
$$\ddot{y}(t) + 2d\omega_0\dot{y}(t) + \omega_0^2 y(t) = \omega_0^2 u(t)$$

The corresponding characteristic equation $\quad (s + d\omega_0 + \mathrm{j}\omega_0\sqrt{1-d^2})(s + d\omega_0 - \mathrm{j}\omega_0\sqrt{1-d^2}) = 0$

Impulse response



Step response



High-order dynamic LTI systems (1)

The nth-order homogeneous equation (no input, free response) is given by

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = 0 \qquad (3.10)$$

The corresponding *characteristic equation* with the *characteristic polynomial* $\Delta(s)$ is then

$$\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (3.11)$$

If $\Delta(\alpha) = 0$, then $y(t) = \mathrm{e}^{\alpha t}$ is a solution of (3.10)! For the case that the roots of the characteristic equation α_k, $k = 1,\ldots, n$, are completely distinct, the general solution of the homogeneous equation (3.10) can be stated as a linear combination of the single solutions

$$y(t) = \sum_{k=1}^{n} C_k e^{-\alpha_k t} \qquad (3.12)$$

<u>Notes:</u>

1) The coefficients C_k can be given as linear combination of the initial conditions $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$.
2) α_k can be real or complex. The complex roots are always conjugate complex pairs. By combining the corresponding conjugate complex solutions one obtains finally exponential oscillation terms $\mathrm{e}^{\sigma t}\sin\omega t$ and $\mathrm{e}^{\sigma t}\sin\omega t$ with $\sigma = \mathrm{Re}\{\alpha_k\}$.

High-order dynamic LTI systems (2)

The dynamic of a high-order system is the sum of first-order and second-order dynamics. It is essentially determined by the roots of the characteristic equation!

Special case: multiple roots of the characteristic equation

If α is a solution of multiplicity k to the characteristic equation, then a (k-1)-degree polynomial $C(t)$ exists so that $y(t) = C(t)e^{\alpha t}$ is a solution to the homogeneous differential equation. Thus, the general solution to equation (3.10) can be given as

$$y(t) = \sum_{k=1}^{n_k} C_{k-1}(t) e^{-a_k t} \tag{3.13}$$

where a_k, $k = 1,\ldots n_k$ are roots of the characteristic equation.

Note:

Generally there are n solutions to the characteristic equation with some of them possibly in multiplicity. The corresponding polynomial $C_{k\text{-}1}(t)$ to a root of multiplicity k is of degree k-1. In case of distinct roots the polynomials $C_{k\text{-}1}(t)$ reduce to constant coefficients.!

High-order dynamic LTI systems (3)

Before starting to discuss the solution for the general system model (3.1) we consider first the following special case

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = u \tag{3.14}$$

The solution to above equation is known as

$$y(t) = \sum_{k=1}^{n} C_{k-1}(t) e^{a_k t} + \int_0^t g_h(t-\tau)u(\tau)\mathrm{d}\tau \tag{3.15}$$

where $h(t)$ is the solution to the homogeneous equation

$$\frac{\mathrm{d}^n g_h}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} g_h}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}g_h}{\mathrm{d}t} + a_0 g_h = 0 \tag{3.16}$$

with following special initial conditions

$$g_h(0) = 0, \quad \dot{g}_h(0) = 0, \quad \ldots, \quad \overset{(n-2)}{g_h}(0) = 0, \quad \overset{(n-1)}{g_h}(0) = 1 \tag{3.17}$$

Based on this solution the general problem can be now tackled.

High-order dynamic LTI systems (4)

The solution to the general system model of n-th order

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_p\frac{\mathrm{d}^p u}{\mathrm{d}t^p} + b_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \ldots + b_1\frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u \tag{3.18}$$

comprises again two terms

$$y(t) = \underbrace{\sum_{k=1}^{n} C_{k-1}(t)e^{-a_k t}}_{\text{free response}} + \underbrace{\int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau}_{\text{forced response}} \tag{3.19}$$

where $g(t)$ is referred to as the *impulse response*

$$g(t) = b_p \overset{(p)}{g_h}(t) + b_{p-1} \overset{(p-1)}{g_h}(t) + \ldots + b_1 \dot{g}_h(t) + b_0 g_h(t) \tag{3.20}$$

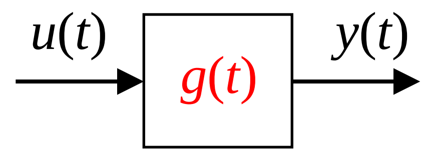
with $g_h(t)$ being the solution to (3.16).

From now on we assume generally zero initial conditions (no free response), unless mentioned otherwise, to drop our attention on the system's input-output behavior.

System response and convolution integral

The system response $y(t)$ to an arbitrary input signal $u(t)$ on zero initial conditions is given by the integral

$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau = g(t) * u(t)$$



which is known as *convolution integral* or shortly *convolution*.

The system output is the integral of the system input, weighted by the impulse response. The current input is more strongly weighted as the past.

Some properties of convolution integral

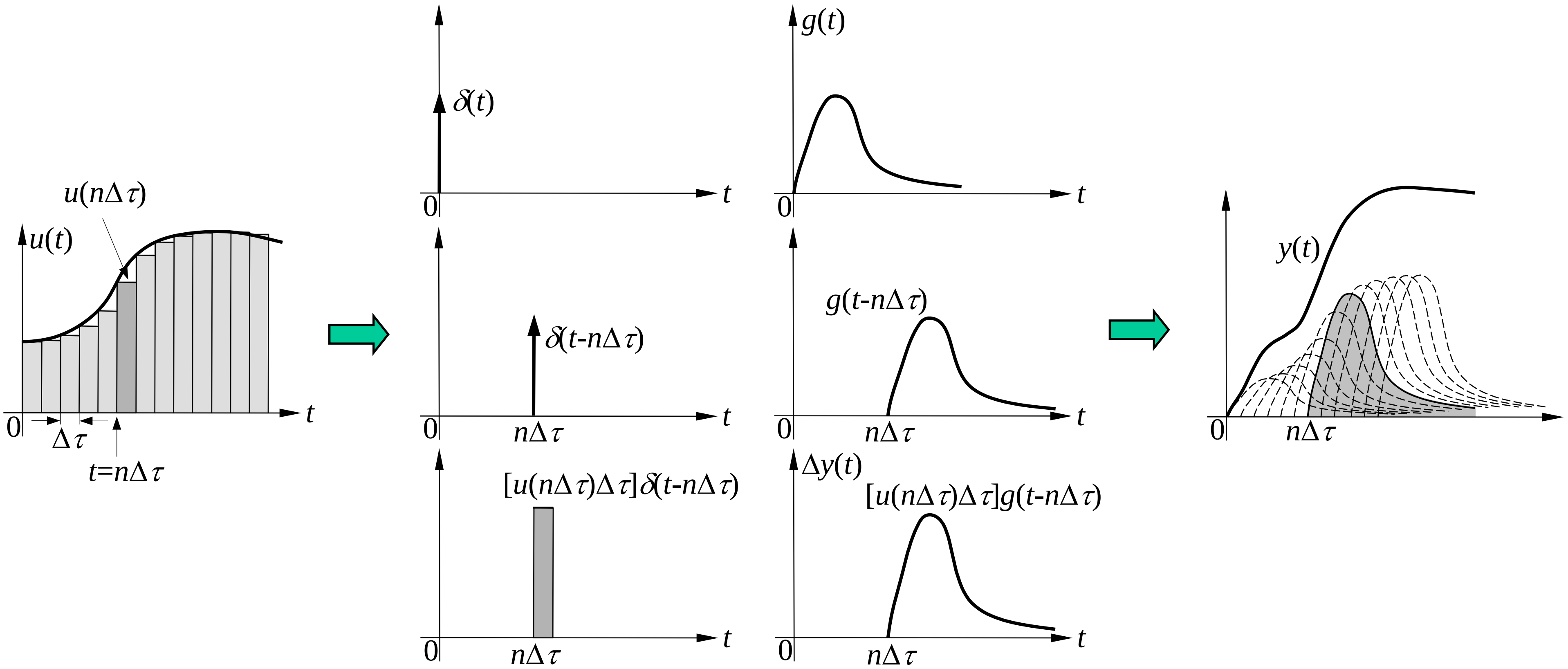
commutative: $f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\mathrm{d}\tau = \int_0^t f_2(\tau) f_1(t-\tau)\mathrm{d}\tau = f_2(t) * f_1(t)$

distributive: $f_1(t) * \left[f_2(t) + f_3(t)\right] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

associative: $f_1(t) * \left[f_2(t) * f_3(t)\right] = \left[f_1(t) * f_2(t)\right] * f_3(t)$

Shift property: $f_1(t) * f_2(t) = f(t) \;\Rightarrow\; f_1(t-\tau_1) * f_2(t-\tau_2) = f(t-\tau_1-\tau_2)$

Convolution integral: a graphical interpretation



Active excise

Stop and think!

For the purpose of controller design it is usually too complicated to solve the differential equation for the consideration of system behavior

Is there an alternative (and easier) way?

Can we just "read off" the most important properties from a more concise system description?

System description in the Laplace-domain

By transforming the total system model on *zero initial conditions* into the frequency-domain (also known as Laplace-domain) using Laplace transform, one obtains

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = (b_p s^p + b_{p-1}s^{p-1} + \ldots + b_1 s + b_0)U(s) \tag{3.21}$$

where s is the Laplace variable. By defining the denominator and numerator polynomials

$$\begin{aligned} N(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ Z(s) &= b_p s^p + b_{p-1}s^{p-1} + \ldots + b_1 s + b_0 \end{aligned} \tag{3.22}$$

the solution to the system equation can be easily given in the Laplace-domain

$$\begin{aligned} Y(s) = G(s)U(s) = \frac{Z(S)}{N(s)}U(s) &= \frac{b_p s^p + b_{p-1}s^{p-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(S) \\ &= \mathrm{K}\frac{(s-\beta_1)(s-\beta_2)\cdots(s-\beta_p)}{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n)}U(S) \end{aligned} \tag{3.23}$$

The system function $G(s)$ is called the *transfer function*. The zeros α_i of the denominator polynomial $N(s)$ are the *poles* of the dynamic system (or the transfer function) and those of the numerator polynomial $Z(s)$, β_i, are known as the system *zeros* (or zeros of the transfer function).

Transfer function and impulse response

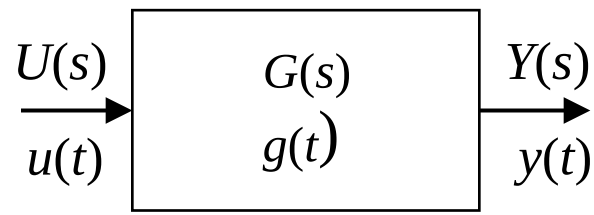
Comparing the definitions for the convolution integral and the transfer function and taking into account the convolution properties of the Laplace transform (s. appendix at the end of this chapter), it is easy to see that the transfer function (Laplace domain) is simply the Laplace transform of the impulse response (time-domain).

Thus, the general behavior of a dynamic system on zero initial conditions can be characterized equivalently either in the Laplace-domain by the transfer function $G(s)$ or in the time-domain by the impulse response $g(t)$. Because of this fundamental property, the impulse response is also called *natural response* of a system.

$$Y(s) = G(s)U(s)$$

$$y(t) = g(t) * u(t)$$

$$y(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau = \mathcal{L}^{-1}\{G(s)U(s)\}$$



Poles and system stability

The system poles determine the components of the time function building up the solution of the differential equation (i.e., the system response to initial excitation or system input).

A dynamic LTI system is said to be bibo-stable (bounded-input bounded-output), if the system output is always bounded for a bounded input signal.

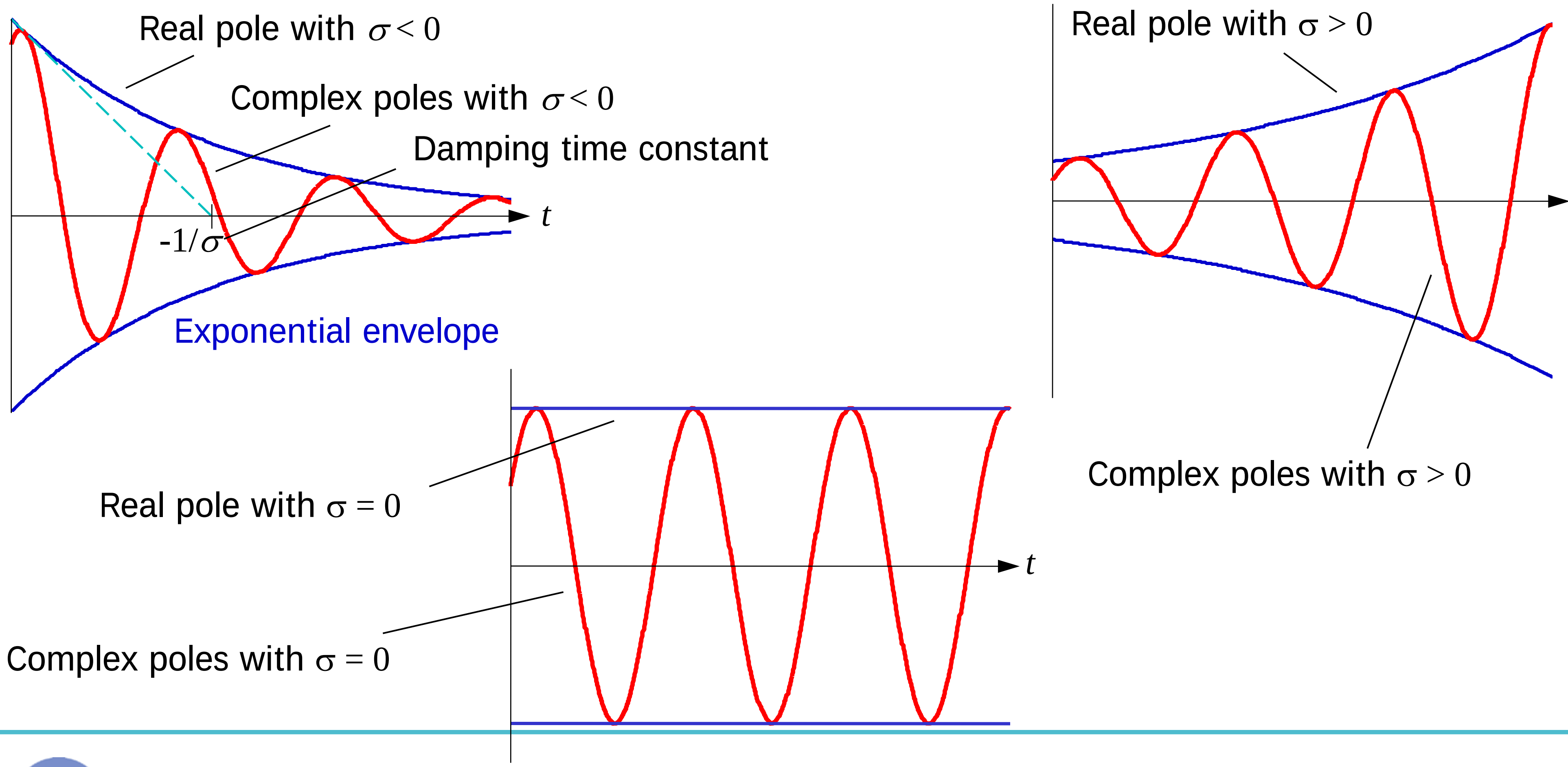
Therefore, a dynamic LTI system is bibo-stable, if and only if *all* the system poles have negative real part, i.e., they are located in the left half of the complex s-plane (LHP = left half-plane).

In order to design a stable control loop, the controller parameters are to be chosen in the way that the closed-loop transfer function only has stable poles.

Stop and think!

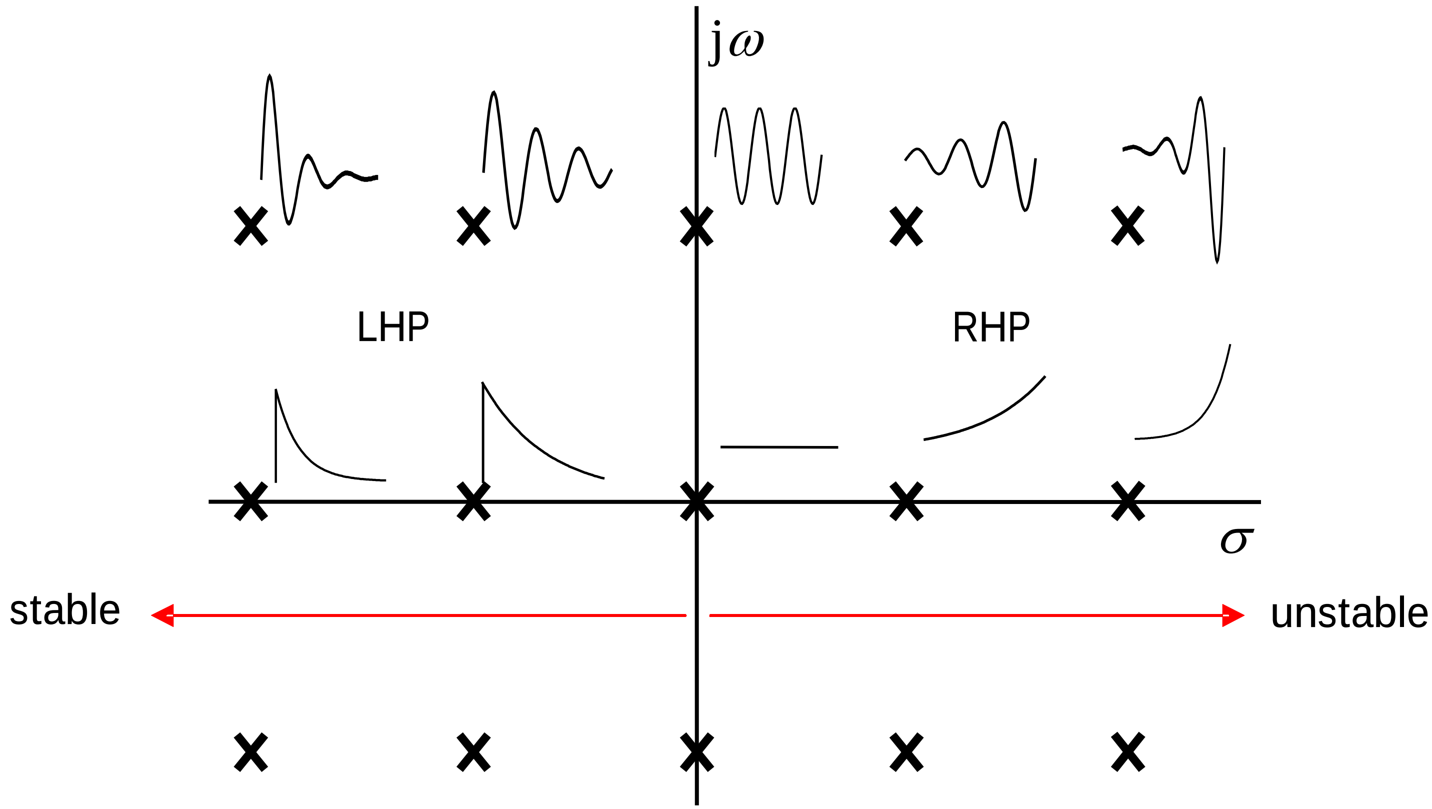
What happens, if one or several poles are located in the right half-plane (RHP) or on the imaginary axis?

Pole location (σ+jω) and free response



Pole location and dynamic behavior

The pole location, marked as "**x**" in the zero-pole-diagram, determines the fundamental behavior of the system response.



Interpretation of zeros

If $s = \beta$ is a zero of the transfer function

$$Z(\beta) = b_p \beta^p + b_{p-1}\beta^{p-1} + \ldots + b_1\beta + b_0 = 0 \tag{3.24}$$

and $u(t) = C(t)e^{\beta t}$, it follows that for the right-hand side of equation (3.18)

$$b_p \frac{\mathrm{d}^p u}{\mathrm{d}t^p} + b_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \ldots + b_1 \frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u = Z(\beta)Ce^{\beta t} = 0 \tag{3.25}$$

A zero $s = \beta$ blocks the transmission of the signal (i.e. eliminates the effect of the input excitation) $u(t) = C(t)e^{\beta t}$.

Stop and think!

How can the influence of a constant disturbance be eliminated?

...

e.g. by using a control making the closed-loop transfer function with respect to the disturbance to have a zero at $s = 0$!

System static gain

We consider again the general input-output system model

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_p\frac{\mathrm{d}^p u}{\mathrm{d}t^p} + b_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \ldots + b_1\frac{\mathrm{d}u}{\mathrm{d}t} + b_0 u$$

For constant input and output, i.e., $u(t) = u_0$ and $y(t) = y_0$, it obviously yields

$$a_0 y_0 = b_0 u_0$$

The ratio

$$V = \frac{b_0}{a_0} \tag{3.26}$$

is called *static gain* of the system (different from K from (3.23)!).

Active excise:

Calculate the static gain of an integrator!

Theory review

What have we learned from the theory in this chapter?

- A dynamic LTI system can be described by a linear ordinary differential equation (ODE) with constant coefficients.
- The solution of the ODE can be analyzed to determine the system behavior.
- The solution of first and second order ODE is easy to get, but difficult for high-order systems (convolution integral for the forced response!). The free response, however, is just the additive sum of first and second order free responses.
- Basic element of the solution is the exponential function, real or complex (exponential function with sinusoidal oscillation). The real part of the roots of characteristic equation is completely decisive on the stability of the solution!
- Making use of the Laplace transform (and transfer function) a more concise system analysis is possible. The poles determine essentially the dynamic performance (stability, damping, natural frequency). What is the influence of the system zeros on dynamic performance? We have some idea, but still need more insight.
- Step and impulse response are often used as "benchmarking" signals for controller design. How can they be determined by experiments?

Conclusions for practical control design

How can this theoretical knowledge help in practical control design?

- The roots of the characteristic equation (= poles of the dynamic system) of the closed-loop control must be placed in the LHP to ensure the stability of the control (indication for the proper choice of controller structure and/or parameters!)
- The closed-loop static gain should be chosen to unity in order to eliminate steady-state control error (indication for the proper choice of controller structure and/or parameters!)
- The dynamic performance (= transient behavior) of the control is essentially influenced by the closed-loop pole location. A real pole results in an exponential convergence and a conjugate complex pair of poles causes exponential oscillation, which, in case of small damping, leads to overshoot in transient situations (indication for the proper choice of controller structure and/or parameters!)
- Simulation of step and impulse response is useful to "tune" the control performance

What else?:

- Transfer function is an efficient and very useful form of system description. Important for controller design: how can plant (process), controller and control loop be linked together using transfer functions? How can a systematic design make use of such linked structures?
- How can the controller be realized, if designed in form of a transfer function?

Design example: once again cruise control (1)

Simplified process model: $$\frac{dv}{dt} + 0.02v = u - 10\theta$$

PI controller: $$u = k_P(v_r - v) + k_I \int_0^t (v_r - v)d\tau$$

The error equation: $$e = v_r - v$$

Taking Laplace transform yields

$$(s + 0.02)V(s) = U(s) - 10\theta(s)$$

$$U(s) = k_P E(s) + \frac{k_I}{s} E(s)$$

$$E(s) = V_r(s) - V(s)$$

Manipulating the equations one obtains (cp. also page 3-7)

$$E(s) = G_V(s)V_r(s) + G_\theta(s)\theta(s) = \underbrace{\frac{s(s+0.02)}{s^2 + (0.02 + k_P)s + k_I}}_{\text{Behavior on reference change}} V_r(s) + \underbrace{\frac{10s}{s^2 + (0.02 + k_P)s + k_I}}_{\text{Disturbance (road slope) behavior}} \theta(s)$$

Design example: once again cruise control (2)

Both $G_V(s)$ and $G_\theta(s)$ have a zero at $s = 0$. Therefore, a step change on reference speed or road slope will, independently of controller parameters, cause no steady-state control error. In other words, the control error will be eliminated by the controller (effect of controller integral action)!

The denominator polynomial is for both transfer functions the same! For determining the transient behavior we choose e.g. (cp. page 3-9) $d = 0.8$ (almost no overshoot). The corresponding curves of step and impulse response show that dynamic changeover is almost completed after ca. $\omega_0 t = 5$. if we for instance specify this transition with a real time of $2.5\,[\mathrm{s}]$, then it is $\omega_0 = 5/2.5 = 2\,[1/\mathrm{s}]$. From the values for d and ω_0 we can finally calculate the parameters of the controller as

$$k_P = 3.18 \quad k_I = 4$$

To study the disturbance behavior in more detail we assume that $V_r = \mathrm{const.}$. It is then

$$E(s) = \frac{10s}{s^2 + (0.02 + k_P)s + k_I}\theta(s)$$

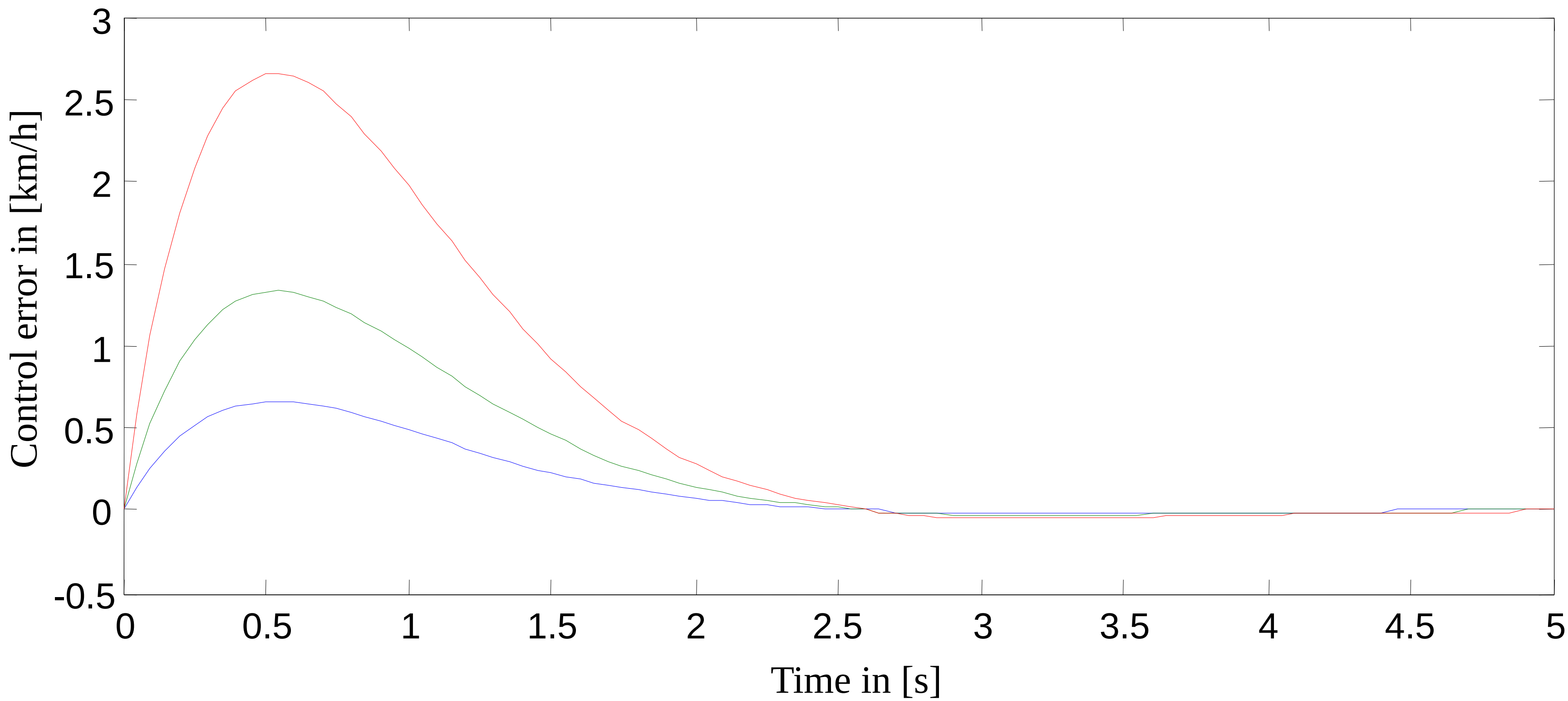
The solution to a step change of θ in the Laplace-domain is $(\theta(s) = \Delta\theta/s)$

$$E(s) = \frac{10s}{s^2 + 3.2s + 4} \cdot \frac{\Delta\theta}{s}$$

Design example: once again cruise control (3)

Converting this result to a time function we find the expression for the control error:

$$e(t) = \frac{25}{3} \Delta\theta \cdot e^{-1.6t} \sin(1.2t)\sigma(t)$$

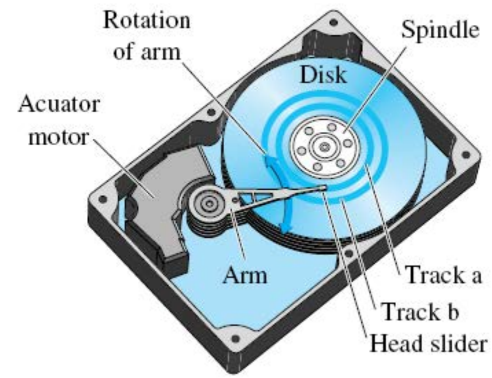


Sequential design example: Disk drive read system (1)

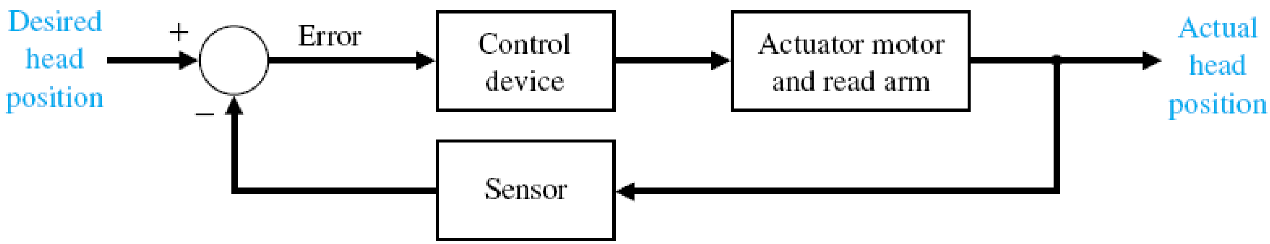
[2]

Positioning of the reader head

- Fast motion
- High accuracy
- No mechanical oscillations
- Microcontroller-based control



Head above the disk: < 100 nm



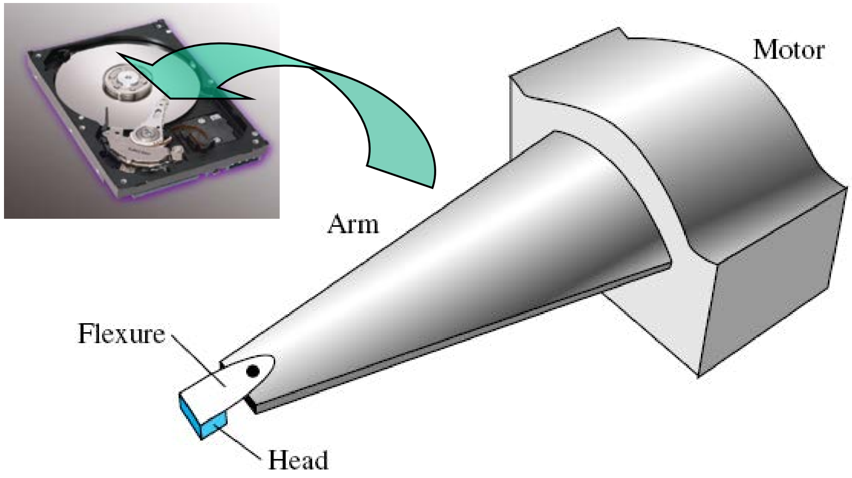
Measurement: Position angle of the reader head

Actuation: Electronic amplifier, DC motor, mechanical arm

Computing: Microcontroller in real-time operation mode

Results: Accurate positioning (tolerance: 1 µm) and high dynamic (track-to-track change: 10 ms)

Sequential design example: Disk drive read system (2)



Plant modeling

System input u (motor voltage)

System output θ (position angle of read head)

Input-output differential equation:

$$u = \mathrm{R}i + \mathrm{L}\dot{i} + u_q = \mathrm{R}i + \mathrm{L}\dot{i} + c\phi\omega$$

$$\mathrm{J}\dot{\omega} = c\phi i - b\omega$$

$$\Downarrow$$

$$u = \frac{\mathrm{LJ}}{c\phi}\dddot{\theta} + \frac{\mathrm{RJ}+b\mathrm{L}}{c\phi}\ddot{\theta} + \left(\frac{\mathrm{R}b}{c\phi} + c\phi\right)\dot{\theta}$$

Red marked terms can be neglected in case of appropriate feedforward compensation or for approximative consideration ("Mechanic is much slower than electronic")

Typical parameters of the read system:

Inertia of arm and read head	$\mathrm{J} = 1\ \mathrm{Nms}^2$
Friction constant	$b = 20\ \mathrm{kgm}^2/\mathrm{s}$
Motor armature resistance	$\mathrm{R} = 1\ \Omega$
Motor armature inductance	$\mathrm{L} = 1\ \mathrm{mH}$
Motor constant	$c\phi = 5\ \mathrm{Nm/A}$

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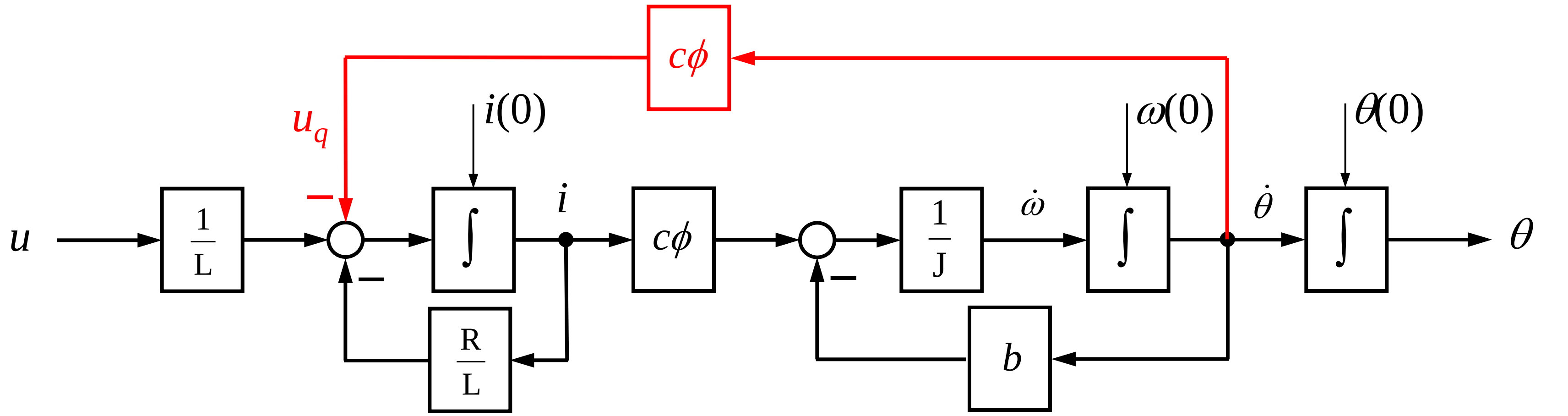
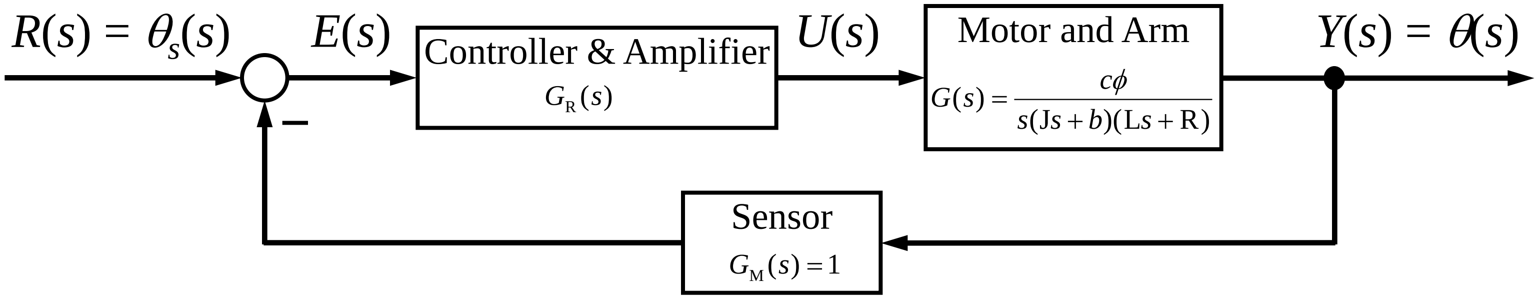
Lehrstuhl für Regelungssysteme
Technische Universität Kaiserslautern

Sequential design example: Disk drive read system (3)

Block diagram of the plant



Sequential design example: Disk drive read system (4)



$$G(s) = \frac{c\phi}{s(Js+b)(Ls+R)} = \frac{5000}{s(s+20)(s+1000)}$$

A 3rd order system! Further simplification possible (electronic is much faster than mechanic)?

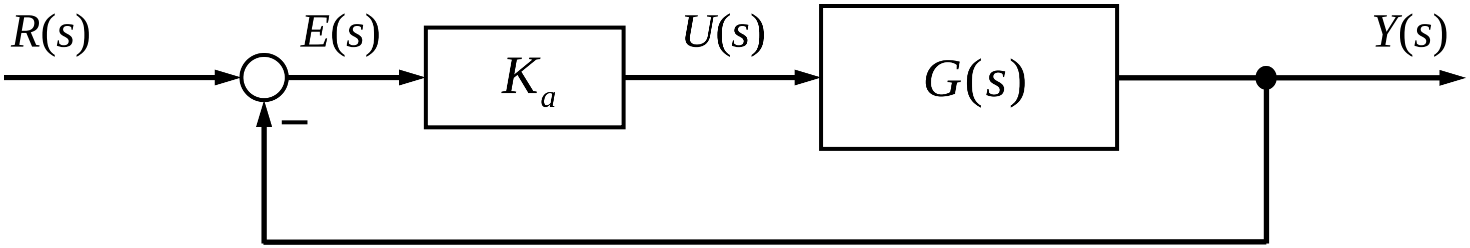
$$G(s) = \frac{c\phi/bR}{s(\tau_L s+1)(\tau s+1)} \approx \frac{c\phi/bR}{s(\tau_L s+1)} = \frac{0.25}{s(0.05s+1)}$$

or

$$G(s) = \frac{5}{s(s+20)}$$

The plant is unstable (one pole at $s=0$)!

Sequential design example: Disk drive read system (5)



We choose a simple P-type controller to stabilize the closed-loop control. The gain of the P controller incl. amplifier is assumed to be K_a. Then the closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a G(s)} = \frac{5K_a}{s^2 + 20s + 5K_a}$$

It is obviously that a) the steady-state control error is zero independent of K_a (why?), and b) we can influence the damping and natural frequency (but not independently from each other) by tuning on K_a

$$\omega_0^2 = 5K_a\,, \quad d = \frac{20}{2\omega_0}$$

In order to avoid a too large overshoot, we choose $d = 1/\sqrt{2}$, thus $K_a = 40$. The control performance can be now verified by simulating the step response.

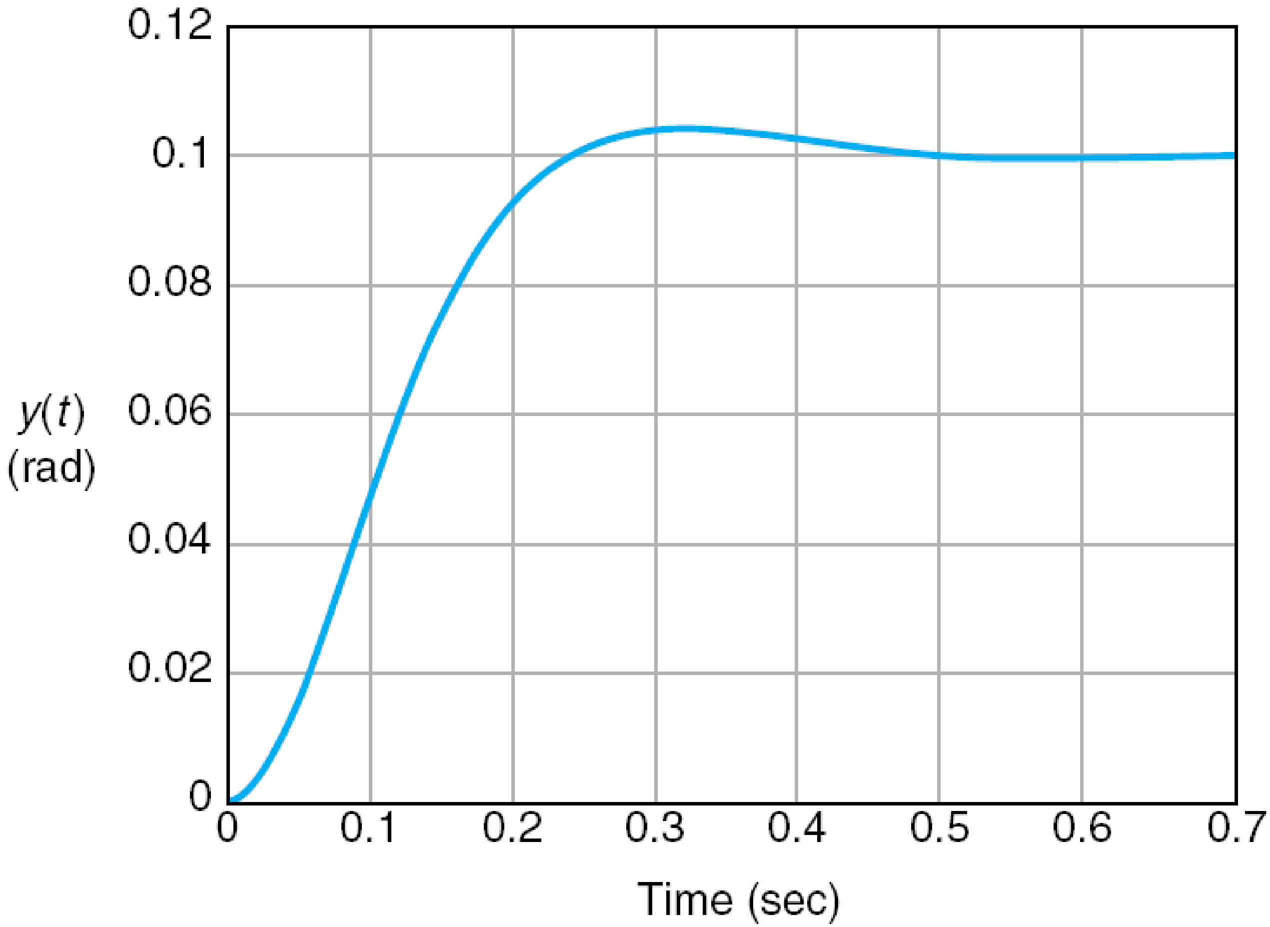
Sequential design example: Disk drive read system (6)

$$Y(s) = \frac{200}{s^2 + 20s + 200} R(s)$$

Simulation of control performance for a step change of the reference value from 0 to 0.1 rad:

MATLAB script:

```
Gs=tf([200],[1 20 200])
Step(0.1*Gs,0.7)
```



Appendix 3A: Laplace transform

The unilateral (or one-sided) Laplace transform (for causal systems):

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-0}^{\infty} f(t) \mathrm{e}^{-st} \mathrm{d}t \quad \text{with Laplace variable} \quad s = \sigma + \mathrm{j}\omega$$

or as linear mapping:

$$f(t) \quad \mapsto \quad F(s)$$

time-domain (real) o—● Laplace-domain (complex)

Inverse Laplace transform:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi \mathrm{j}} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) \mathrm{e}^{st} \mathrm{d}s$$

Properties of Laplace transforms (1)

$$f(t) \quad \circ\!-\!\bullet \quad F(s); \quad g(t) \quad \circ\!-\!\bullet \quad G(s)$$

Linearity:

$$\Rightarrow \quad a_1 f(t) + a_2 g(t) \;\circ\!-\!\bullet\; a_1 F(s) + a_2 G(s)$$

a_1, a_2 arbitrary complex constants

Time and frequency shift:

$$f(t-\tau) \quad \circ\!-\!\bullet \quad e^{-\tau s} F(s)$$

τ arbitrary real number

$$F(s-\alpha) \quad \bullet\!-\!\circ \quad e^{\alpha t} f(t)$$

α arbitrary complex number

Differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \quad \circ\!-\!\bullet \quad sF(s) - f(-0)$$

$$f^{(\mathrm{n})}(t) \quad \circ\!-\!\bullet \quad s^{\mathrm{n}} F(s) - s^{\mathrm{n}-1} f(-0) - \cdots - f^{(\mathrm{n}-1)}(-0)$$

Properties of Laplace transforms (2)

Integration:

$$\int_{-0}^{t} f(\tau)\mathrm{d}\tau \quad \circ\!-\!\bullet \quad \frac{1}{s}F(s)$$

Convolution:

$$f(t)*g(t) \quad \circ\!-\!\bullet \quad F(s)\cdot G(s)$$ Convolution in time-domain

$$F(s)*G(s) \quad \circ\!-\!\bullet \quad f(t)\cdot g(t)$$ Convolution in frequency-domain

Initial and final value theorem:

$$\lim_{t\to+0} f(t) = \lim_{s\to\infty}\left(sF(s)\right)$$ Initial value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left(sF(s)\right)$$ Final value theorem

The final value theorem is only valid, if such a value exists!

Correspondence table of Laplace transform

No.	$f(t)$	$F(s)$	No.	$f(t)$	$F(s)$
1	$\delta(t)$	1	7	$t^2 \cdot e^{-at}$	$\frac{2}{(s+a)^3}$
2	$\sigma(t)$	$\frac{1}{s}$	8	$1 - e^{-at}$	$\frac{a}{s \cdot (s+a)}$
3	t	$\frac{1}{s^2}$	9	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
4	$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$	10	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
5	e^{-at}	$\frac{1}{s+a}$	11	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
6	$t \cdot e^{-at}$	$\frac{1}{(s+a)^2}$	12	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$

Solving differential equations using Laplace transform

Exp. 3A1: solve the second-order linear differential equation

$$\ddot{y} + 5\dot{y} + 6y = \dot{u} + u$$

if the initial conditions are $y(0) = 2$, $\dot{y}(0) = 1$, and the input $u(t) = e^{-4t}\sigma(t)$.

Taking the Laplace transform of the differential equation including the initial values we have

$$s^2Y(s) - sy(0) - \dot{y}(0) + 5[sY(s) - y(0)] + 6Y(s) = sU(s) - u(0) + U(s)$$

Collecting all the terms of $Y(s)$ we easily can solve the algebraic equation in the Laplace-domain

$$Y(s) = \frac{s+1}{s^2+5s+6}U(s) + \frac{2s+10}{s^2+5s+6}$$

By substituting $U(s) = 1/(s+4)$ and expanding the right-hand side into partial fractions we have

$$Y(s) = \left(\frac{-1/2}{s+2} + \frac{2}{s+3} - \frac{3/2}{s+4}\right) + \left(\frac{6}{s+2} - \frac{4}{s+3}\right)$$

The time-domain solution can be obtained via inverse Laplace transform

$$y(t) = \underbrace{(-\tfrac{1}{2}e^{-2t} + 2e^{-3t} - \tfrac{3}{2}e^{-4t})\sigma(t)}_{\text{forced response}} + \underbrace{(6e^{-2t} - 4e^{-3t})\sigma(t)}_{\text{free response}} = (\tfrac{11}{2}e^{-2t} - 2e^{-3t} - \tfrac{3}{2}e^{-4t})\sigma(t)$$