Algebra of Subspaces

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Dr K Madhavi

Govt. Degree College, Kuppam.

Theorem : *The intersection of two subspaces W_1 and W_2 of a vector space V (F) is also a subspace.*

Proof : Given, W_1 and W_2 are two subspaces of a vector space V (F).

To prove that $W_1 \cap W_2$ is also a subspace of V (F).

For this we need to prove that a, b ∈F and $\alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\,\beta \in W_1 \cap W_2$

Since W_1 and W_2 are two subspaces, $\overline{0} \in W_1$ and $\overline{0} \in W_2$.

$$\Rightarrow \overline{0} \in W_1 \cap W_2$$

$$\therefore W_1 \cap W_2 \neq \phi.$$

Let a, b ∈F and $\alpha, \beta \in W_1 \cap W_2$

$\therefore \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$

Since a, b ∈F, $\alpha, \beta \in W_1$ and W_1 is a subspace $\Rightarrow a\alpha + b\,\beta \in W_1$

Again a, b ∈F, $\alpha, \beta \in W_2$ and W_2 is a subspace $\Rightarrow a\alpha + b\,\beta \in W_2$

$\therefore$ a, b ∈F and $\alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\,\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of V (F).

Note : 1. The intersection of any family of subspaces of a vector space is also a subspace.

2. The union of two subspaces may not be a subspace of V (F).

Example : Let W_1 and W_2 be two subspaces of a vector space V_3 (R)

Where $W_1 = \{ (0,y,0) \mid y \in R \}$

$W_2 = \{ (0,0,z) \mid z \in R \}$

Then $W_1 \cup W_2 = \{ (0,y,0) \cup (0,0,z) \mid y, z \in R \}$

Now (0,y,0) + (0,0,z) = (0,y,z)

But $(0,y,z) \notin W_1$ and $(0,y,z) \notin W_2$

$\therefore (0,y,z) \notin W_1 \cup W_2$

$\Rightarrow W_1 \cup W_2$ is not closed under vector addition.

$\therefore W_1 \cup W_2$ is not a subspace of V (F).

Theorem : *The union of two subspaces is a subspace if and only if one is contained in the other.*

Proof : Let W_1 and W_2 be two subspaces of a vector space V (F).

Part - I : The condition is necessary.

Let one subspace is contained in the other.

i.e. let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

To prove that $W_1 \cup W_2$ is a subspace of V (F).

Suppose $W_1 \subseteq W_2$ then $W_1 \cup W_2 = W_2$ and since W_2 is a subspace, $W_1 \cup W_2$ is also a subspace.

Similarly if $W_2 \subseteq W_1$ then $W_1 \cup W_2 = W_1$ and since W_1 is a subspace, $W_1 \cup W_2$ is also a subspace.

Part – II : *The condition is sufficient.*

Let $W_1 \cup W_2$ is a subspace.

To prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If possible let $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

Now $W_1 \not\subset W_2 \Rightarrow \exists\, x \in W_1$ and $x \notin W_2$ …… (1)

Again $W_2 \not\subset W_1 \Rightarrow \exists\, y \in W_2$ and $y \notin W_1$ …… (2)

$\therefore x \in W_1 \cup W_2$ and $y \in W_1 \cup W_2$

Since $W_1 \cup W_2$ is a subspace, by closure property $(x + y) \in W_1 \cup W_2$ …… (3)

Then $(x + y) \in W_1$ or $(x + y) \in W_2$

Suppose $(x + y) \in W_1$

Then $x \in W_1$, $(x + y) \in W_1$ and W_1 is a subspace $\Rightarrow 1(x+y) + (-1)\, x \in W_1$

$\Rightarrow y \in W_1$

But this is absurd (From (2)). $\therefore$ $(x + y) \notin W_1$ …… (4)

Similarly suppose $(x + y) \in W_2$

Then $y \in W_2$, $(x + y) \in W_2$ and W_2 is a subspace $\Rightarrow 1(x+y) + (-1)\, y \in W_2$

$\Rightarrow x \in W_2$

But this is also absurd (From (1)). $\therefore$ $(x + y) \notin W_2$ …… (5)

Thus (4) & (5) contradict (3).

$\therefore$ Our assumption that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$ is wrong.

And hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$\therefore$ The union of two subspaces is a subspace if and only if one is contained in the other.

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