The Essence of Calculus

Understanding the Mathematics of Change

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What is Calculus?

The Study of Continuous Change

Calculus is built on two foundational ideas that are inverses of each other:

1. Differential Calculus

Slices a problem into infinitely small pieces to find the **instantaneous rate of change** (the slope of a curve).

2. Integral Calculus

Sums up these infinitely small pieces to find the **total accumulated quantity** (the area under a curve).

Core Concept: Limits

Definition: Limit

A limit describes the value that a function $f(x)$ "approaches" as its input x "approaches" some value c. We write this as:

$$\lim_{x \to c} f(x) = L$$

Example (An Illustrative Example)

The function $f(x) = \frac{x^2-1}{x-1}$ is undefined at $x = 1$. But the limit as x approaches 1 is 2.

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2$$

Core Concept: Derivatives

Definition: Derivative

The derivative of a function $f(x)$, denoted $f'(x)$ or $\frac{dy}{dx}$, measures the **instantaneous rate of change** of the function. It is the slope of the line tangent to the function at a given point, formally defined by a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

If $f(x) = x^2$, its derivative is $f'(x) = 2x$. This new function tells you the slope of the original function at any point.

Core Concept: Integrals

Definition: Integral

An integral is the continuous analogue of a sum, often interpreted as the **area under the curve** of a function.

Definite Integral

Calculates area between a and b:

$$\int_a^b f(x)\,dx$$

Indefinite Integral

Finds the antiderivative:

$$\int f(x)\,dx = F(x) + C$$

Key Theorem: The Fundamental Theorem of Calculus

Part 1: The Derivative of an Integral

The derivative of an area function gives you back the original function.

$$\text{If } F(x) = \int_a^x f(t)\,dt, \text{ then } F'(x) = f(x)$$

Theorem (Part 2: The Integral of a Derivative)

To find the area under $f(x)$, find its antiderivative $F(x)$ and compute the change in F.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Core Concept: Multivariate Gradients

Partial Derivatives and the Gradient

For a function with multiple inputs, like $f(x, y)$, the **partial derivative** $\frac{\partial f}{\partial x}$ is the derivative with respect to one variable, holding others constant. The

gradient, ∇f, is the vector of all partial derivatives:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

Intuition

The gradient vector points in the direction of the **steepest ascent** on a surface. This is key for optimization algorithms.

Application: Numerical Methods with NumPy

Numerical Integration (Area of $y = x^2$)

```
import numpy as np
x = np.linspace(0, 1, 1001)
y = x**2
area = np.trapz(y, x)
# area is approx 0.333...
```

Numerical Differentiation (Slope of $y = x^2$)

```
# Using the same x and y from above
grad = np.gradient(y, x)
# grad at x=1 will be approx 2.0
```

Worked Problem: Finding an Integral

Problem

Calculate the definite integral of $f(x) = 3x^2 + 4x + 2$ from $x = 0$ to $x = 2$.

1. **Find the Antiderivative $F(x)$:**

$$\begin{aligned} F(x) &= \int (3x^2 + 4x + 2)\, dx \\ &= 3\left(\frac{x^3}{3}\right) + 4\left(\frac{x^2}{2}\right) + 2x \\ &= x^3 + 2x^2 + 2x \end{aligned}$$

2. **Evaluate $F(b) - F(a)$:**

$$\begin{aligned} F(2) &= (2)^3 + 2(2)^2 + 2(2) = 8 + 8 + 4 = 20 \\ F(0) &= 0 \end{aligned}$$

$$\int_0^2 f(x)\, dx = F(2) - F(0) = 20$$