A Visual Introduction to Linear Algebra

From Vectors to Data Reduction

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What is Linear Algebra?

The Study of Grids and Transformations

Linear Algebra is the study of **vectors**, **vector spaces**, and **linear transformations**.

At its core, it's about what happens when you transform space in a way that keeps grid lines parallel and evenly spaced.

Why is it important?

It's the language of data science, computer graphics, machine learning, physics simulations, and much more.

Core Concept: Vector Spaces

Definition: Vector Space

A vector space is a collection of objects called **vectors** that can be added together and multiplied by **scalars** (numbers), and the result stays within the collection (this is called **closure**).

Example (The 2D Plane: $\mathbb{R}^2$)

The familiar 2D Cartesian plane is a vector space.

- **Addition:** You can add any two vectors, and the result is still a 2D vector.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

- **Scalar Multiplication:** You can scale any vector, and the result is still a 2D vector.

$$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Core Concept: Linear Transformations

Definition: Linear Transformation

A transformation (or function) T is **linear** if it preserves vector operations:

1. **Additivity:** $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. **Homogeneity:** $T(c\vec{v}) = cT(\vec{v})$

Key Idea: The Matrix Representation

A linear transformation can be fully described by a matrix. The columns of the matrix show where the basis vectors land.

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}$$

This matrix transforms $\hat{\imath}$ to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\hat{\jmath}$ to $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Core Concept: Eigenvalues & Eigenvectors

Definition

An **eigenvector** of a square matrix A is a non-zero vector $\vec{v}$ that, when transformed by A, only changes in scale, not direction. The scaling factor λ is the **eigenvalue**.

$$A\vec{v} = \lambda\vec{v}$$

Example (Intuition)

Eigenvectors are the "axes of transformation." They are the vectors that stay on their own span during the transformation, only being stretched, shrunk, or flipped.

Core Concept: Singular Value Decomposition (SVD)

Theorem (SVD)

Any $m \times n$ matrix A can be factored into the product of three matrices:

$$A = U\Sigma V^T$$

where:

- *U is an $m \times m$ orthogonal matrix (a rotation/reflection).*
- *Σ is an $m \times n$ diagonal matrix of* **singular values** *(scaling).*
- *V^T is the transpose of an $n \times n$ orthogonal matrix (another rotation/reflection).*

Intuition

Any linear transformation is just a combination of a rotation, a scaling, and another rotation.

Engineering Application: PCA

Principal Component Analysis (PCA)

PCA is a data reduction technique that transforms a high-dimensional dataset into a lower-dimensional one while preserving as much variance as possible.

Example (How it Works)

1. Compute the **covariance matrix** of the data.
2. Find the **eigenvectors and eigenvalues** of this matrix.
3. The eigenvector with the largest eigenvalue is the **first principal component**—the direction of maximum variance.
4. Reduce dimensionality by projecting the data onto the first few principal components.

Worked Problem: Finding Eigenvalues

Problem

Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$.

1. **Find eigenvalues (λ):** Solve $\det(A - \lambda I) = 0$.

$$\det \begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) - (-2)$$
$$0 = \lambda^2 - 5\lambda + 6$$
$$0 = (\lambda - 2)(\lambda - 3)$$

   The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

2. **Find eigenvectors ($\vec{v}$):** For each λ, solve $(A - \lambda I)\vec{v} = \vec{0}$.
   - For $\lambda_1 = 2$, we get $2x - 2y = 0 \implies x = y$. Eigenvector: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   - For $\lambda_2 = 3$, we get $x - 2y = 0 \implies x = 2y$. Eigenvector: $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.