Internship Report on

A Study on Estimation of Scale Parameter of Selected Uniform Distribution

by

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ABSTRACT

In Statistics, Estimation theory deals with the problem of making a decision about the value or values of unknown parameter or parameters of a population based on the information obtained from one or more sample/s. In this study, k independent Uniform populations are considered. The best population is selected based on a general class of selection rules. The scaled-squared loss function, the entropy loss function and the asymmetric loss function are considered in this study. Under these loss functions, various estimators of the scale parameter of the selected population are studied. A comparative study of risk values of the estimators are reported through simulation in this study. Finally, two seperate data are analyzed.

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1 Introduction

Ranking and selection and related estimation problems from selected subset of populations have received a great attention of statisticians and researchers in recent times. It is important in various field of applications. For example, a consumer would like to buy electric lamps of best quality from lamps of various company. Then he would also like to know an estimate of failure time of best quality bulb. Selection problems mainly deal with selecting the best (worst) population from k available populations, where quality is define in terms of an unknown parameter associated with the population. After the selection of best population, the obvious interest is to estimate the unknown parameters of the selected population. This problem is referred to as "Estimation after Selection" problem.

Let π_1, π_2, ..., π_k be independent uniform populations over the intervals $(0,\ \theta_1)$, $(0,\ \theta_2)$, ..., $(0,\ \theta_k)$ respectively, where θ_i >0 are unknown scale parameters. A sample of size n_i $(X_{i1}, X_{i2}, ..., X_{in_i})$ is drawn from the population π_i, i=1(1)k. Let X_i=max$\{X_{i1}, X_{i2}, ..., X_{in_i}\}$. Then the random variables $X_1, X_2, ..., X_n$ are independent and has the probability density function

$$f_i(x|\theta_i) = \begin{cases} \frac{n_i x^{n_i-1}}{\theta_i^{n_i}} & if\ 0 < x < \theta_i \\ 0 & otherwise. \end{cases} \tag{1}$$

Now $\mathbf{X}$=$(X_1, X_2, ..., X_k)$ is a complete and sufficient statistic for estimating $\boldsymbol{\theta}$=$(\theta_1, \theta_2, ..., \theta_k) \in \boldsymbol{\Theta}(= \mathbb{R}_k^+)$, where $\boldsymbol{\Theta}$ is the parametric space and $\mathbb{R}_k^+$ is the positive part of k-dimensional Euclidean space (see Casella and Berger, 2002, page 286). So inference procedures which depend on samples only through the complete sufficient statistic $\mathbf{X}$=$(X_1, X_2, ..., X_k)$ will be considered for further selection and estimation techniques.

In the next section, some literature studies, related to estimation and selection problems, will be discussed.

2 Literature Study

The problems of estimating parameters of a selected population have received a great attention in last five decades. A brief history of the study of "Estimation after Selection" problems are discussed here.

Cohen and Sackrowitz (1982) worked on estimating the mean of selected population. Sackrowitz and Samuel-Cahn (1984) worked on estimation of mean after selection from negetive exponential population. Jeyaratnam and Panchapakesan (1986) provided some results on estimation after selection from exponential populations. Sackrowitz and Samuel-Cahn (1986) evaluated the chosen population considering a Bayes and minimax approach. Vellaisamy et al. (1988) considered the problem of estimating the mean after selecting from uniform population. Song (1992) continued the study of Vellaisamy et al. (1988) by extending some results. Misra and Singh (1993) studied on the uniformly minimum risk unbiased estimators of parameter of selected exponential population. Parsian and Farsipour (1999) provided estimation of mean of selected population under asymmetric loss function. Kumar and Kar (2001) estimated quantiles of selected exponential population. Kumar and Tripathi (2003) studied estimation of moments of a selected uniform population. Misra and Meulen (2003) worked on estimating mean of selected normal population under LINEX loss function. Kumar and Gangopadhyay (2005) considered the problem of estimation after selection from Pareto population. Misra et al. (2006a) considered the problem of estimating the scale parameter of selected gamma population under the scale invariant squared error loss function. Nematollahi and Motamed-Shariati (2009) considered similar works under the entropy loss function. Nematollahi and Motamed-Shariati (2012) extended the works of Vellaisamy et al. (1988) and Song (1992) under equal sample size and entropy loss function. Nematollahi and Jozani (2016) worked on risk unbiased estimation after selection.

Most of the works on selection and estimation after selection problems discussed here are carried out under the assumption of equal nuisance parameter and/or equal sample sizes. The study under the setup where nuisance parameter and/or sample sizes are unequal is comparatively complex. Risko (1985) worked on selecting Binomial population with unequal sample sizes. Abughalous and Miescke (1989) worked on selecting the largest probability with unequal sample sizes. Vellaisamy (1996) considered the problem of estimating selected scale parameters. Misra and Arshad (2014) made selection of the best of two gamma population. Arshad and Misra (2015a) worked on estimation after selection from gamma paopulation. Al-Mosawi and Vellaisamy (2015) worked on estimation of the parameters of selected Binomial population. Arshad and Misra (2015b, 2016) worked on selection and estimation problems from exponential population. Arshad and Misra (2015a) considered the problem of estimation of scale-parameter of selected uniform population under scaled-squared error loss function. Arshad and Misra (2017) extended the works of Song (1992) and Nematollahi and Motamed-Shariati (2012) under unequal sample sizes. Meena et al. (2018) considered the problem of estimating the scale-parameter of selected uniform population under squared log error function. Arshad and Abdalghani (2019) extended similar problem under the asymmetric loss function. Arshad and Abdalghani (2020) extended the study of Arshad and Misra (2016) considering LINEX loss function. Recently, Azhad et al. (2021) worked on the problem of common location parameter of several heterogeneous exponential populations based on generalised order statistics. Recently, Arshad et al. (2022) considered the problem of estimation after selection from bivariate normal population.

3 Definitions, Concepts and Terms

Before going to the procedures of selecting best population and then estimating the scale parameter of that selected population, some definitions, concepts and terms, which will be important for the later stages of the study, are recalled.

3.1 The Uniform Distribution

Suppose X be a random variable and X$\sim$Uniform(a, b), where $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Then the probability density function of X is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & if\ a < x < b \\ 0 & otherwise. \end{cases} \tag{2}$$

The mean and variance of X are respectively $E(X) = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$. If a=0 and b=1, then that is called the **Standard Uniform Distribution**.

3.2 Loss Function and Risk Function

Let δ be an estimator of an unknown parameter θ. In Decision Theory, a loss function gives a notion of distance between the estimated value and true value of the parameter θ, if estimated by δ. Generally a loss function L(θ, δ) is a non-negative function that increases as the distance between θ and δ increases. L(θ, δ)=0 if $\theta = \delta$ i.e., if θ is correctly estimated by δ. If θ is real-valued, two commonly used loss functions are

$$absolute\ error\ loss, L(\theta, \delta) = |\delta - \theta| \tag{3}$$

and

$$squared\ error\ loss, L(\theta, \delta) = (\delta - \theta)^2. \tag{4}$$

Some special types of loss functions will be discussed in the later sections.

In Decision Theory, quality of an estimator is measured by the risk function. At given θ, the average loss, if δ is used to estimate θ, is the risk function, R(θ, δ).

$$R(\theta, \delta) = E_\theta L(\theta, \delta).$$

As the true value of θ is unknown, one should find an estimator of θ so that the average expected loss is small regardless the actual value of θ. So the basic objective is to find an estimator which has a small risk value for all θ.

3.3 Scaled-squared Error Loss Function

The scaled-squared error loss function is considered in this study.

$$L(\theta, \varphi) = \left(\frac{\varphi}{\theta} - 1\right)^2 \tag{5}$$

where the estimator φ is used to estimate the parameter θ.

Scaled-squared error loss function penalizes overestimation and underestimation equally.

3.4 Entropy Loss Function

Another important loss function which is considered in this study is entropy loss function.

$$L(\theta, \varphi) = \frac{\theta}{\varphi} - \ln\left(\frac{\theta}{\varphi}\right) - 1 \tag{6}$$

where the estimator φ is used to estimate the parameter θ.

Entropy loss function penalizes underestimation more than over estimation. But gross underestimation is penalized just as heavily as gross overestimation.

3.5 Asymmetric Loss Function

Another special type of loss function is asymmetric loss function.

$$L(\theta, \varphi) = \left(\sqrt{\frac{\varphi}{\theta}} - \sqrt{\frac{\theta}{\varphi}}\right)^2, \theta > 0 \tag{7}$$

where the estimator φ is used to estimate the parameter θ.

Underestimation is more penalized than overestimation by asymmetric loss function.

3.6 Uniformly Minimum Risk Unbiased (UMRU) Estimator

An estimator δ of θ is said to be risk unbiased if it satisfies

$$R(\theta, \delta(\mathbf{X})) \leq R(\theta', \delta) \qquad \forall \;\; \theta' \neq \theta.$$

An estimator δ of θ is said to be uniformly minimum risk unbiased if it minimises the risk function $R(\theta, \delta) = E_\theta L(\theta, \delta)$.

3.7 Bayes Estimator

In Bayesian approach, the unknown parameter θ is supposed to have a **prior distribution**, π. Let the **posterior distribution** of θ is π^*. Then the posterior risk for estimating θ by $\delta(\mathbf{X})$ is

$$r(\delta(\mathbf{X}), \mathbf{X}) = E_{\pi^*}(L(\theta, \delta)|\mathbf{X}).$$

An estimator $\delta(\mathbf{X})$ of θ is defined to be the Bayes estimator if it minimises the posterior risk. For example, if squared error loss (4) is considered as loss function, $\mathrm{E}(\theta|\mathbf{X})$ becomes the Bayes estimator of θ.

Sometimes the functions, which are not actually a probability distribution, are taken as the prior distribution, π. Such a prior is called a non-informative prior. In this case, the estimator which minimises the posterior risk is called the **Generalised Bayes estimator**.

3.8 Minimax Estimator

An estimator δ^M of θ is said to be minimax estimator if it minimises the maximum risk i.e., an estimator δ^M of θ is said to be minimax estimator if

$$sup_\theta R(\theta, \delta^M) = inf_\delta sup_\theta R(\theta, \delta).$$

A minimax estimator should be a Bayes estimator. The Bayes risk (see Casella and Berger, 2002, page 352) is

$$r_\pi = \int_\Theta R(\theta, \delta)\pi(\theta)\, d\theta.$$

A prior distribution, π is **least favourable** if $r_\pi \geq r_{\pi'}$, for all prior distribution $\pi^{'}$ i.e., for which prior average risk is maximum.

Theorem: If $r_\pi = \int_\Theta R(\theta, \delta_\pi)\pi(\theta)\, d\theta = sup_\theta R(\theta, \delta_\pi)$, then

a. δ_π is minimax.

b. If δ_π is unique Bayes estimator, then it is unique minimax estimator.

c. π is least favourable.

3.9 Admissibility of An Estimator

An estimator δ is said to be inadmissible if there exists another estimator $\delta^{'}$ such that $R(\theta, \delta^{'}) \leq R(\theta, \delta)$, for all θ, with strict inequality at least for some θ. Then it is said that $\delta^{'}$ dominates δ. An estimator δ is said to be admissible if no such $\delta^{'}$ exists.

Arshad and Misra (2015a), Arshad and Misra (2017), and Arshad and Abdalghani (2019) provided sufficient conditions for inadmissibity of scale-invariant estimators of scale-parameter of selected uniform population under respectively the scaled-squared error loss function, the entropy loss function and the asymmetric loss function.

4 Selection Rule

Consider the problem of selecting best population from uniform populations introduced in Section (1). Let $\theta_{(1)}, \theta_{(2)}, ..., \theta_{(k)}$ be the ordered values of $\theta_1, \theta_2, ..., \theta_k$. Let $\pi_{(i)}$ denotes the unknown population corresponding to the i^{th} ordered scale-parameter $\theta_{(i)}$. This $\pi_{(k)}$ is called the best population. In case of ties for the best population, the population $\pi_{(i)}$ having the largest subscript i among tied populations is considered as best population. Since X_i is the maximum likelihood estimator of $\theta_i, i = 1(1)k$, to select the best population, a natural selection rule $\boldsymbol{\delta^N}$ is to select the population corresponding to $X_{(k)} = max\{X_1, X_2, ..., X_k\}$. Consider the indicator function

$$I(A, B) = \begin{cases} 1 & if\ A \geq B \\ 0 & otherwise. \end{cases}$$

Then the scale parameter of the selected population is $\theta_S = \sum_{i=1}^{k} \theta_i \{\prod_{j \neq i} I(X_i, X_j)\}$.

But this selection rule is not appropriate in the case of unequal sample sizes (see Misra and Dhariyal, 1994). For selecting a fixed selection rule $\boldsymbol{\delta^a} \in \mathcal{D}_1 = \{\boldsymbol{\delta^a} : \boldsymbol{\delta^a} = (\delta_1^{\boldsymbol{a}}, \delta_2^{\boldsymbol{a}}, ..., \delta_k^{\boldsymbol{a}}, \boldsymbol{a} \in \mathbb{R}_+^k)\}$, where

$$\delta_i^{\boldsymbol{a}} = \begin{cases} 1 & if\ a_i X_i > max_{j \neq i} a_j X_j \\ 0 & otherwise, \end{cases} \tag{8}$$

is considered. Then the scale parameter of the selected population is

$$\theta_S = \sum_{i=1}^{k} \theta_i \delta_i^{\boldsymbol{a}}(\mathbf{X}).$$

Let $\mathcal{X}(= \mathbb{R}_+^k)$ denotes the sample space and let $A_i = \{\mathbf{x} \in \mathcal{X} : a_i x_i > a_j x_j, \forall j \neq i, j = 1(1)k\}, i = 1(1)k$. Then the selected scale parameter

$$\theta_S = \sum_{i=1}^{k} \theta_i I_{A_i}(\mathbf{X}). \tag{9}$$

For k=2 and unequal sample sizes, this rule becomes $\boldsymbol{\delta^{a^*}} = (\delta_1^{a^*}, \delta_2^{a^*})$, where

$$\delta_1^{a^*}(\mathbf{X}) = \begin{cases} 1 & if\ X_1 > a^* X_2 \\ 0 & if\ X_1 \leq a^* X_2 \end{cases} \quad \delta_2^{a^*}(\mathbf{X}) = 1 - \delta_1^{a^*}(\mathbf{X}) \tag{10}$$

where

$$a^* \equiv a^*(n_1, n_2) = \begin{cases} \left(\frac{n_1+n_2}{2n_2}\right)^{\frac{1}{n_1}} & if\ n_1 \leq n_2 \\ \left(\frac{2n_1}{n_1+n_2}\right)^{\frac{1}{n_2}} & if\ n_1 > n_2. \end{cases} \tag{11}$$

This selection rule $\boldsymbol{\delta}^{\boldsymbol{a}^*}$ is minimax rule under 0-1 loss function and is generalised Bayes rule with respect to non-informative prior (see Arshad and Misra, 2015b).

5 Estimation under Scaled-squared Error Loss Function

Arshad and Misra (2015a) proposed six various estimators of θ_S under the scaled-squared error loss function (5).

5.1 Some Natural Estimators

For $1 = 1(1)k$, X_i, $\frac{n_i+1}{n_i}X_i$ and $\frac{n_i+2}{n_i+1}X_i$ are respectively the maximum likelihood estimator, the uniformly minimum variance unbiased estimator and uniformly minimum risk unbiased estimator of θ_i in component estimation problem. Based on these estimators, following are three natural estimators.

$$\varphi_{N,1}(\mathbf{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\mathbf{X}), \tag{12}$$

$$\varphi_{N,2}(\mathbf{X}) = \sum_{i=1}^{k} \left(\frac{n_i+1}{n_i}\right) X_i I_{A_i}(\mathbf{X}) \tag{13}$$

and

$$\varphi_{N,3}(\mathbf{X}) = \sum_{i=1}^{k} \left(\frac{n_i+2}{n_i+1}\right) X_i I_{A_i}(\mathbf{X}). \tag{14}$$

5.2 The UMVU Estimator

The UMVUE of θ_S is given by

$$\varphi_U(\mathbf{X}) = \sum_{i=1}^{k} \frac{X_i}{n_i} \left\{ (n_i+1) - \left(max_{j\neq i} \frac{a_j X_j}{a_i X_i} \right) \right\} I_{A_i}(\mathbf{X}). \tag{15}$$

5.3 Inadmissibity Results

Now some quantities are to be defined. Let $T_j = \frac{X_j}{X_1}, j = 2(1)k$.

Let $B_1 = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_j < \frac{a_1}{a_j}, j = 2(1)k\}$ and $B_l = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_l > max(\frac{a_1}{a_l}, max_{2\leq j\leq k, j\neq l}\frac{a_j t_j}{a_l}), j = 2(1)k\}, l = 2(1)k$, so that $\{B_1, B_2, ..., B_k\}$ forms a partition of $\mathbb{R}_+^{k-1}$.

Define $\phi(\boldsymbol{t}, \boldsymbol{\theta}) = \sum_{i=1}^{k} \theta_i \frac{E_{\boldsymbol{\theta}}(X_1|\mathbf{T}=\mathbf{t})}{E_{\boldsymbol{\theta}}(X_1^2|\mathbf{T}=\mathbf{t})} I_{B_i}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Define

$$\phi_*(\mathbf{t}) = inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} = \begin{cases} \left(\frac{\sum_{j=1}^{k} n_j+2}{\sum_{j=1}^{k} n_j+1}\right) & if\ \mathbf{t}\in B_1 \\ \left(\frac{\sum_{j=1}^{k} n_j+2}{\sum_{j=1}^{k} n_j+1}\right)t_l & if\ \mathbf{t}\in B_l, l=2(1)k. \end{cases}$$

Define

$$\psi_U(\mathbf{t}) = \begin{cases} \frac{1}{n_1}\{(n_1+1)-\left(\frac{max_{j\neq 1}(a_j t_j)}{a_1}\right)^{n_1}\} & if\ \mathbf{t}\in B_1 \\ \frac{t_l}{n_1}\{(n_1+1)-\left(\frac{max\{a_1, max_{j\neq l, j\neq 1}(a_j t_j)\}}{a_l t_l}\right)^{n_l}\} & if\ \mathbf{t}\in B_l, l=2(1)k. \end{cases}$$

Under the scaled-squared error loss function (5), the UMVU estimator $\varphi_U(\mathbf{X})$ and the natural estimator $\varphi_{N,1}(\mathbf{X})$ are inadmissible and dominated by respectively

$$\varphi_U^*(\mathbf{X}) = X_1 max\{\psi_U(\mathbf{t}), \phi_*(\mathbf{t})\} \tag{16}$$

and

$$\varphi_{N,1}^*(\mathbf{X}) = \left(\frac{\sum_{j=1}^{k} n_j+2}{\sum_{j=1}^{k} n_j+1}\right)\varphi_{N,1}(\mathbf{X}). \tag{17}$$

The estimator $\varphi_{N,2}$ is inadmissible and is dominated by $\varphi_{N,3}$.

5.4 Generalised Bayes Estimator

Consider the non-informative prior for $\boldsymbol{\theta}=(\theta_1,\theta_2,...,\theta_k)$

$$\Pi(\theta_1,\theta_2,...,\theta_k) = \begin{cases} \frac{1}{\theta_1\theta_2...\theta_k} & if\ \boldsymbol{\theta}\in\boldsymbol{\Theta} \\ 0 & otherwise. \end{cases} \tag{18}$$

With respect to the non-informative prior (18), the natural estimator $\varphi_{N,3}$ is the generalised Bayes estimator of θ_S, under the scaled-squared error loss function (5).

5.5 Risk Comparison of Various Competing Estimators

For $k=2$, the risk values of various estimators against various values of $\theta=\frac{\theta_2}{\theta_1}$ are plotted. As the selection rule (10) depends on the sample sizes, the sample sizes are defined before. The sample sizes considered here are $(n_1,n_2)\in\{(3, 6), (6, 3), (7, 9), (9, 7)\}$.

From the Figure (1), Figure (2), Figure (3) and Figure (4), the natural estimator $\varphi_{N,3}$ is recommended.

Figure 1: Comparison of Various Estimators under Scaled-squared Error Loss Function for (n_1, n_2)=(3, 6)

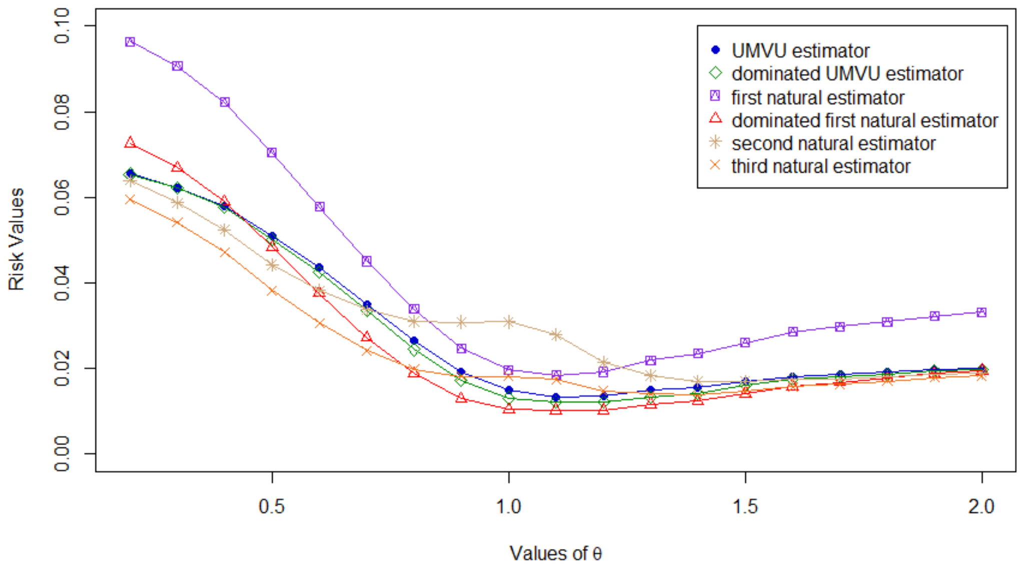


Figure 2: Comparison of Various Estimators under Scaled-squared Error Loss Function for (n_1, n_2)=(6, 3)

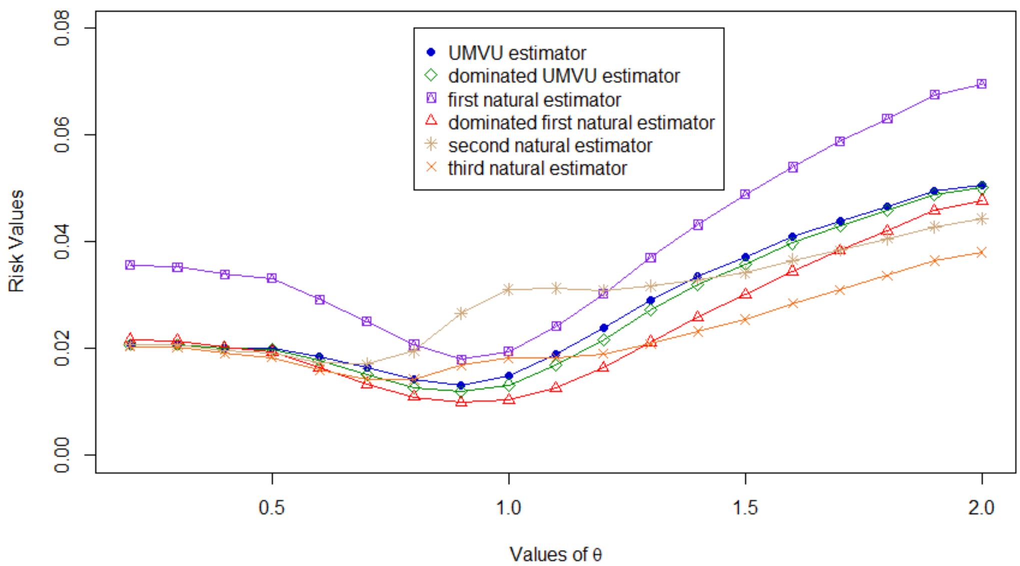


Figure 3: Comparison of Various Estimators under Scaled-squared Error Loss Function for (n_1, n_2)=(7, 9)

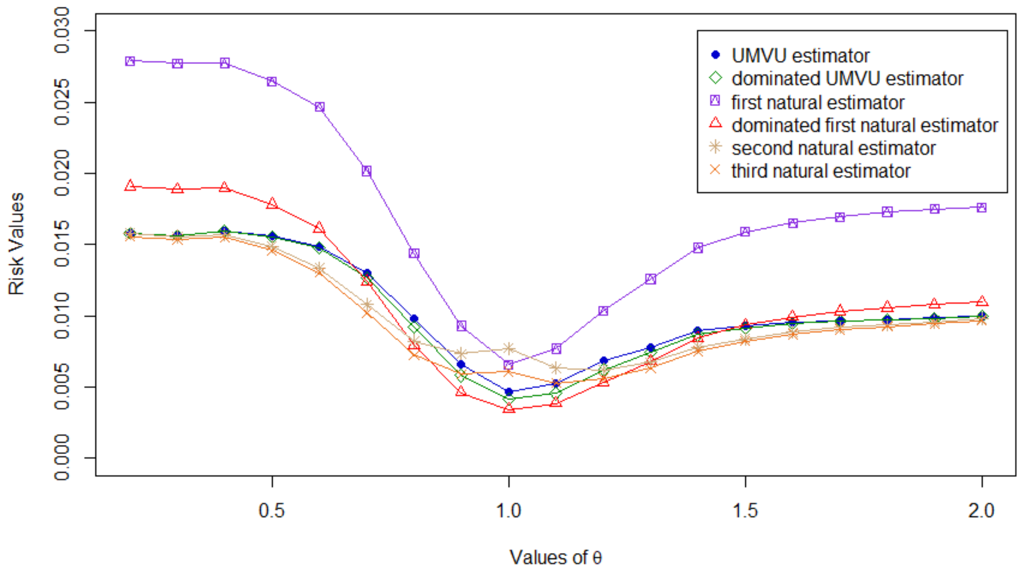
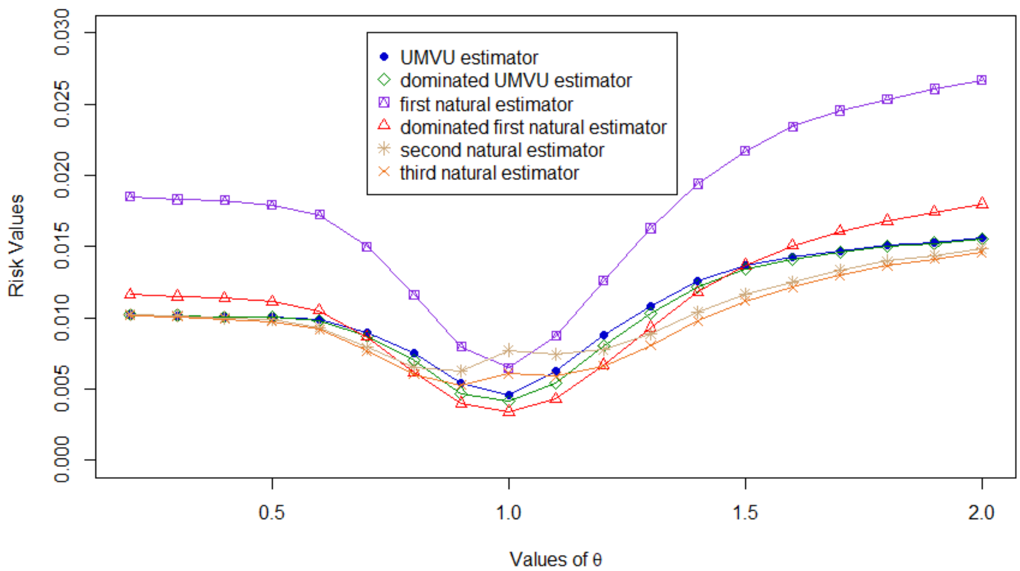


Figure 4: Comparison of Various Estimators under Scaled-squared Error Loss Function for (n_1, n_2)=(9, 7)



6 Estimation under Entropy Loss Function

Arshad and Misra (2017) proposed five various estimators of θ_S under the entropy loss function (6).

6.1 Some Natural Estimators

For $1 = 1(1)k$, X_i and $\frac{n_i}{n_i-1}X_i$ are respectively the maximum likelihood estimator and uniformly minimum risk unbiased estimator of θ_i under entropy loss function (6) in component estimation problem. Based on these estimators, following are two natural estimators.

$$\varphi_{N,1}(\mathbf{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\mathbf{X}) \tag{19}$$

and

$$\varphi_{N,2}(\mathbf{X}) = \sum_{i=1}^{k} \left(\frac{n_i}{n_i - 1}\right) X_i I_{A_i}(\mathbf{X}). \tag{20}$$

6.2 The UMRU Estimator

The UMRU estimator of θ_S is given by

$$\varphi_U(\mathbf{X}) = \sum_{i=1}^{k} \frac{n_i X_i}{\{(n_i - 1) + (\frac{max_{j\neq i} a_j X_j}{a_i X_i})^{n_i}\}} I_{A_i}(\mathbf{X}). \tag{21}$$

6.3 Inadmissibity Results

Now some quantities are to be defined. Let $T_j = \frac{X_j}{X_1}, j = 2(1)k$.

Let $B_1 = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_j < \frac{a_1}{a_j}, j = 2(1)k\}$ and $B_l = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_l > max(\frac{a_1}{a_l}, max_{2\leq j\leq k, j\neq l} \frac{a_j t_j}{a_l}), j = 2(1)k\}, l = 2(1)k$, so that $\{B_1, B_2, ..., B_k\}$ forms a partition of $\mathbb{R}_+^{k-1}$.

Define $\phi(\boldsymbol{t}, \boldsymbol{\theta}) = \sum_{i=1}^{k} \theta_i E_{\boldsymbol{\theta}}(\frac{1}{X_i}|\mathbf{T}{=}\mathbf{t}) I_{B_i}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Define

$$\phi_*(\mathbf{t}) = inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} = \begin{cases} \left(\frac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right) & if\ \mathbf{t} \in B_1 \\ \left(\frac{\sum_{j=1}^{k} n_j}{\sum_{j=1}^{k} n_j - 1}\right) t_l & if\ \mathbf{t} \in B_l, l = 2(1)k. \end{cases}$$

Define

$$\psi_U(\mathbf{t}) = \begin{cases} \frac{n_1}{(n_1-1)+\left(\frac{max_{j\neq 1}a_jt_j}{a_1}\right)^{n_1}} & if\ \mathbf{t}\in B_1 \\ \frac{n_lt_l}{(n_l-1)+\left(\frac{max\{a_1,max_{j\neq l,j\neq i}a_jt_j\}}{a_lt_l}\right)^{n_l}} & if\ \mathbf{t}\in B_l, l=2(1)k. \end{cases}$$

Under the entropy loss function (6), the UMRU estimator $\varphi_U(\mathbf{X})$ and the natural estimator $\varphi_{N,1}(\mathbf{X})$ are inadmissible and dominated by respectively

$$\varphi_U^*(\mathbf{X}) = X_1 max\{\psi_U(\mathbf{t}), \phi_*(\mathbf{t})\} \tag{22}$$

and

$$\varphi_{N,1}^*(\mathbf{X}) = \left(\frac{\sum_{j=1}^k n_j}{\sum_{j=1}^k n_j - 1}\right)\varphi_{N,1}(\mathbf{X}). \tag{23}$$

6.4 Generalised Bayes Estimator

Consider the non-informative prior for $\boldsymbol{\theta}$=$(\theta_1,\theta_2,...,\theta_k)$ defined in (18). Also distribution of X_i is defined in (1). Then the joint distribution of $\mathbf{X}$=$(X_1,X_2,...,X_k)$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} \prod_{i=1}^k \frac{n_ix_i^{n_i-1}}{\theta_i^{n_i}} & if\ 0<x_i<\theta_i \forall i=1(1)k \\ 0 & otherwise. \end{cases} \tag{24}$$

Then posterior probability density function of $\boldsymbol{\theta}$=$(\theta_1,\theta_2,...,\theta_k)$ given $\mathbf{X}$=$\mathbf{x}$ is

$$\Pi^*(\boldsymbol{\theta}|\mathbf{x}) = \begin{cases} \frac{n_ix^{n_i}}{\theta_i^{n_i+1}} & if\ \theta_i>x_i, i=1(1)k \\ 0 & otherwise. \end{cases} \tag{25}$$

It is easy to verify that under entropy loss function (6), the generalised Bayes estimator of θ_S is given by

$$\varphi_G(\mathbf{X}) = E_{\Pi^*}(\theta_S) = \sum_{i=1}^k E_{\Pi^*}(\theta_i)I_{A_i}(\mathbf{X}).$$

Now using the posterior distribution (25), for i=1(1)k

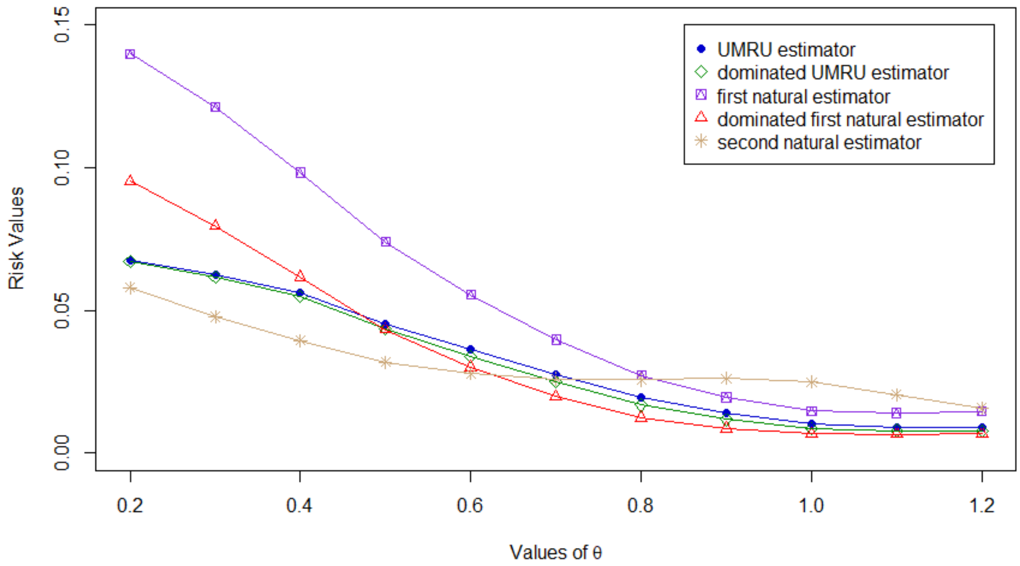
$$E_{\Pi^*}(\theta_i) = \frac{n_i}{n_i-1}x_i$$

So the natural estimator $\varphi_{N,2}$ is the generalised Bayes estimator of θ_S under the entropy loss function (6).

6.5 Risk Comparison of Various Competing Estimators

For $k = 2$, the risk values of various estimators against various values of $\theta = \frac{\theta_2}{\theta_1}$ are plotted. As the selection rule (10) depends on the sample sizes, the sample sizes are defined before. The sample sizes considered here are $(n_1, n_2) \in \{(3, 6), (6, 3), (7, 9), (9, 7)\}$.

Figure 5: Comparison of Various Estimators under Entropy Loss Function for (n_1, n_2)=(3, 6)



From the Figure (5), Figure (6), Figure (7) and Figure (8), the estimator $\varphi^*_{N,1}$ is recommended.

Figure 6: Comparison of Various Estimators under Entropy Loss Function for (n_1, n_2)=(6, 3)

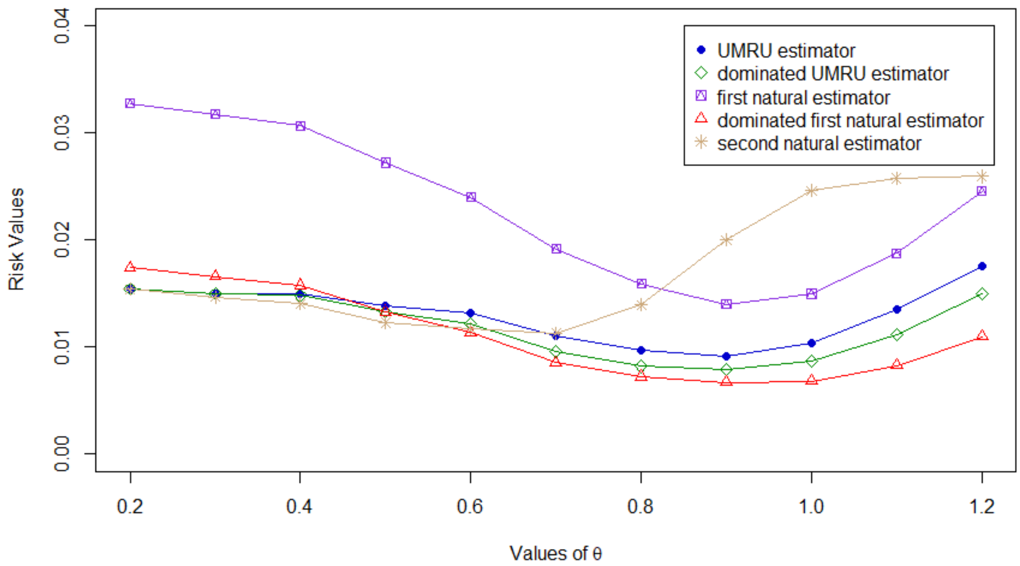


Figure 7: Comparison of Various Estimators under Entropy Loss Function for (n_1, n_2)=(7, 9)

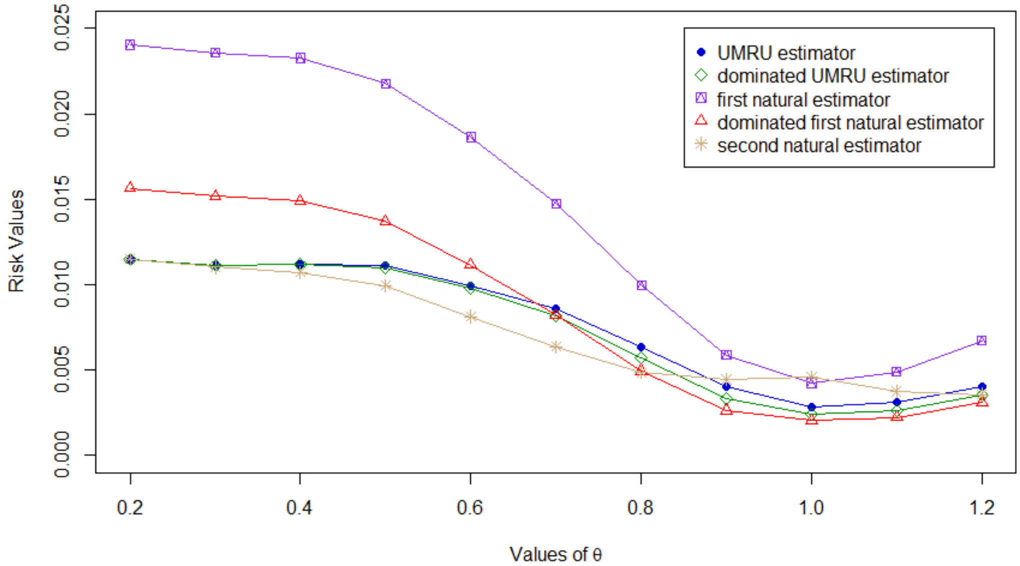
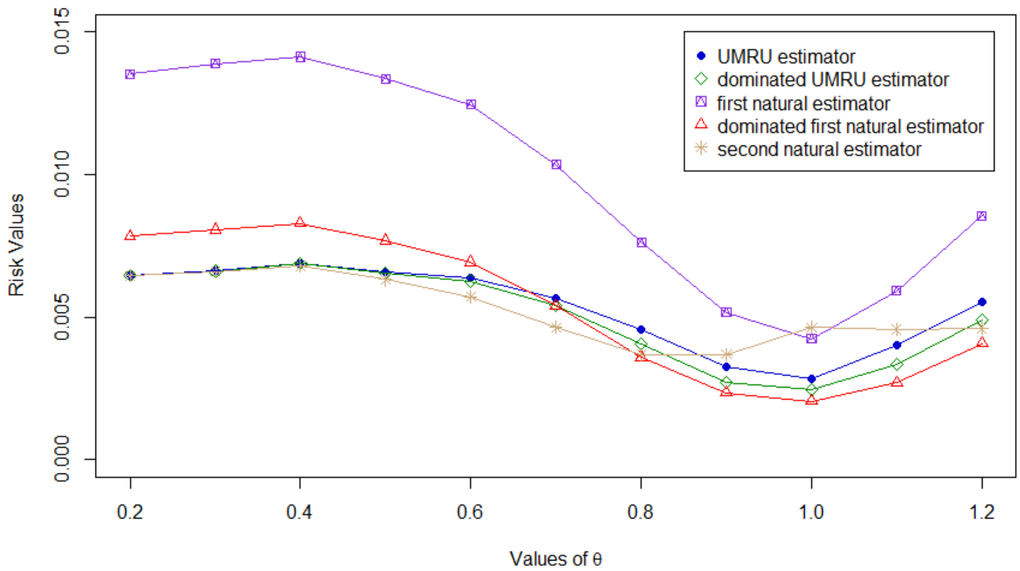


Figure 8: Comparison of Various Estimators under Entropy Loss Function for (n_1, n_2)=(9, 7)



7 Estimation under Asymmetric Loss Function

Arshad and Abdalghani (2019) proposed four various estimators of θ_S under the asymmetric loss function (7).

7.1 Some Natural Estimators

For $1 = 1(1)k$, X_i, $\frac{n_i+1}{n_i}X_i$ and $\sqrt{\frac{n_i+1}{n_i-1}}X_i$ are respectively the maximum likelihood estimator, the uniformly minimum variance unbiased estimator and uniformly minimum risk unbiased estimator of θ_i in component estimation problem. Based on these estimators, following are three natural estimators.

$$\varphi_{N,1}(\mathbf{X}) = \sum_{i=1}^{k} X_i I_{A_i}(\mathbf{X}), \tag{26}$$

$$\varphi_{N,2}(\mathbf{X}) = \sum_{i=1}^{k} \left(\frac{n_i+1}{n_i}\right) X_i I_{A_i}(\mathbf{X}) \tag{27}$$

and

$$\varphi_{N,3}(\mathbf{X}) = \sum_{i=1}^{k} \sqrt{\frac{n_i+1}{n_i-1}} X_i I_{A_i}(\mathbf{X}). \tag{28}$$

7.2 Inadmissibility Results

Now some quantities are to be defined. Let $T_j = \frac{X_j}{X_1}, j = 2(1)k$.

Let $B_1 = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_j < \frac{a_1}{a_j}, j = 2(1)k\}$ and $B_l = \{(t_2, ..., t_k) \in \mathbb{R}_+^{k-1} : t_l > max(\frac{a_1}{a_l}, max_{2\leq j\leq k, j\neq l}\frac{a_j t_j}{a_l}), j = 2(1)k\}, l = 2(1)k$, so that $\{B_1, B_2, ..., B_k\}$ forms a partition of $\mathbb{R}_+^{k-1}$.

Define $\phi(\boldsymbol{t}, \boldsymbol{\theta}) = \sum_{i=1}^{k} \theta_i \sqrt{\frac{E_{\boldsymbol{\theta}}(\frac{1}{X_1}|\mathbf{T}=\mathbf{t})}{E_{\boldsymbol{\theta}}(X_1|\mathbf{T}=\mathbf{t})}} I_{B_i}(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^{k-1}, \boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Define

$$\phi_*(\mathbf{t}) = inf_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} = \begin{cases} \sqrt{\frac{\sum_{j=1}^{k} n_j+1}{\sum_{j=1}^{k} n_j-1}} & if\ \mathbf{t} \in B_1 \\ \sqrt{\frac{\sum_{j=1}^{k} n_j+1}{\sum_{j=1}^{k} n_j-1}} t_l & if\ \mathbf{t} \in B_l, l = 2(1)k. \end{cases}$$

Under the asymmetric loss function (7), the natural estimator $\varphi_{N,1}(\mathbf{X})$ is inadmissible and dominated by

$$\varphi^*_{N,1}(\mathbf{X}) = \sqrt{\frac{\sum_{j=1}^{k} n_j + 1}{\sum_{j=1}^{k} n_j - 1}}\varphi_{N,1}(\mathbf{X}). \tag{29}$$

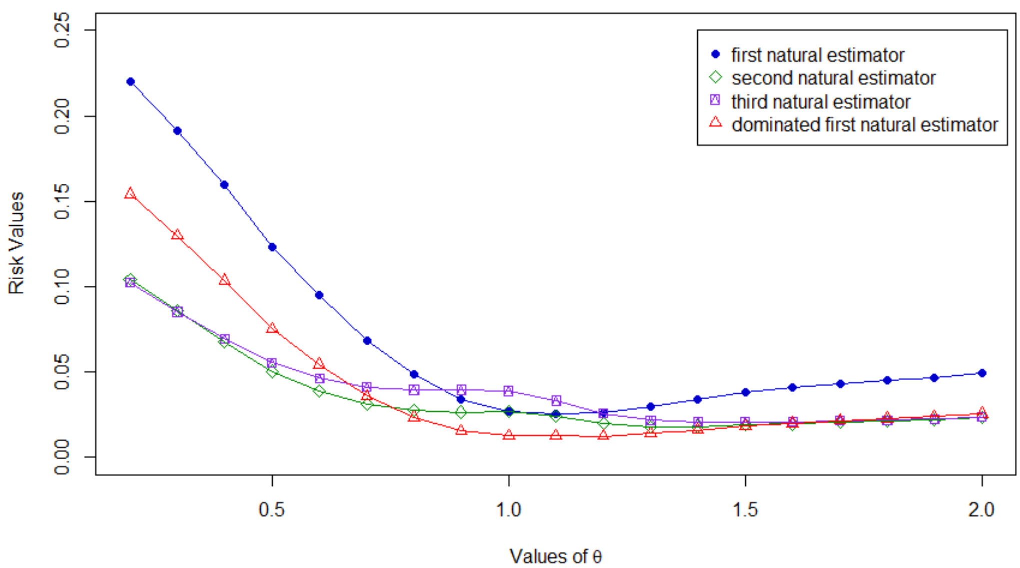
7.3 Generalised Bayes Estimator

With respect to the non-informative prior (18), the natural estimator $\varphi_{N,3}$ is the generalised Bayes estimator of θ_S, under the asymmetric loss function (7).

7.4 Risk Comparison of Various Competing Estimators

For $k = 2$, the risk values of various estimators against various values of $\theta = \frac{\theta_2}{\theta_1}$ are plotted. As the selection rule (10) depends on the sample sizes, the sample sizes are defined before. The sample sizes considered here are $(n_1, n_2) \in \{(3, 6), (9, 7)\}$.

Figure 9: Comparison of Various Estimators under Asymmetric Loss Function for (n_1, n_2)=(3, 6)



From the Figure (9), Figure (10), Figure (11) and Figure (12), the estimator $\varphi^*_{N,2}$ is recommended.

Figure 10: Comparison of Various Estimators under Asymmetric Loss Function for (n_1, n_2)=(6, 3)

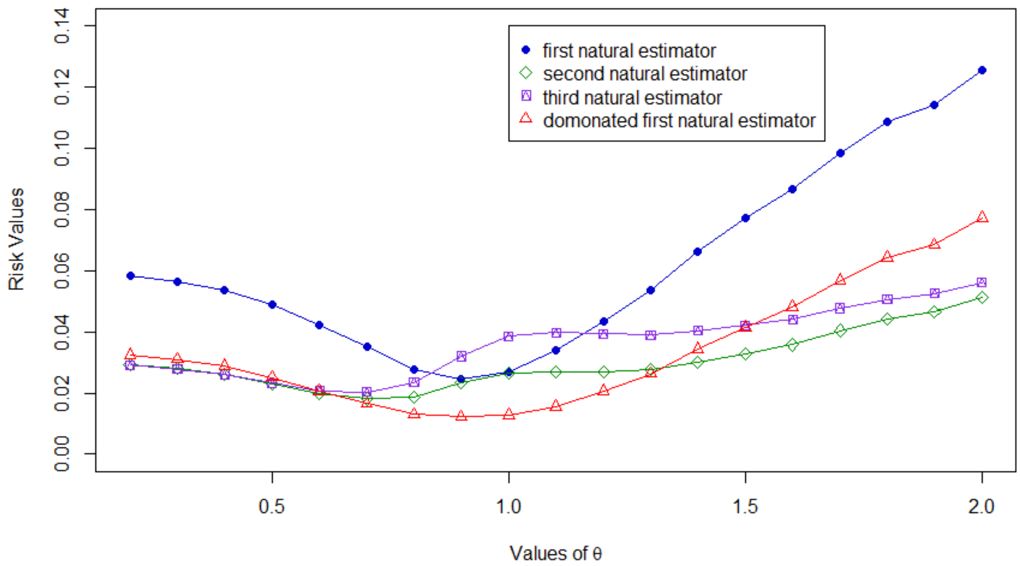


Figure 11: Comparison of Various Estimators under Asymmetric Loss Function for (n_1, n_2)=(7, 9)

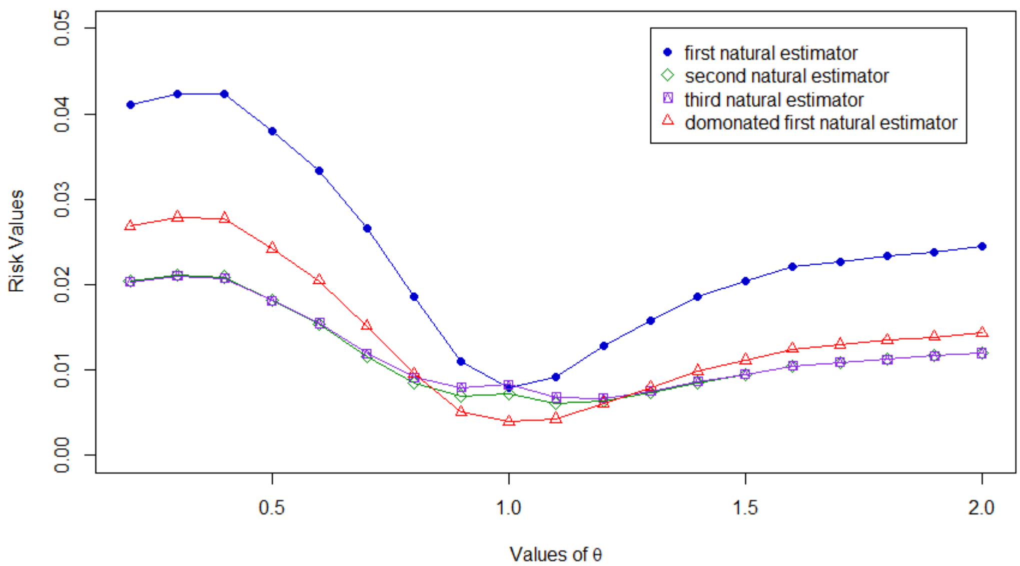
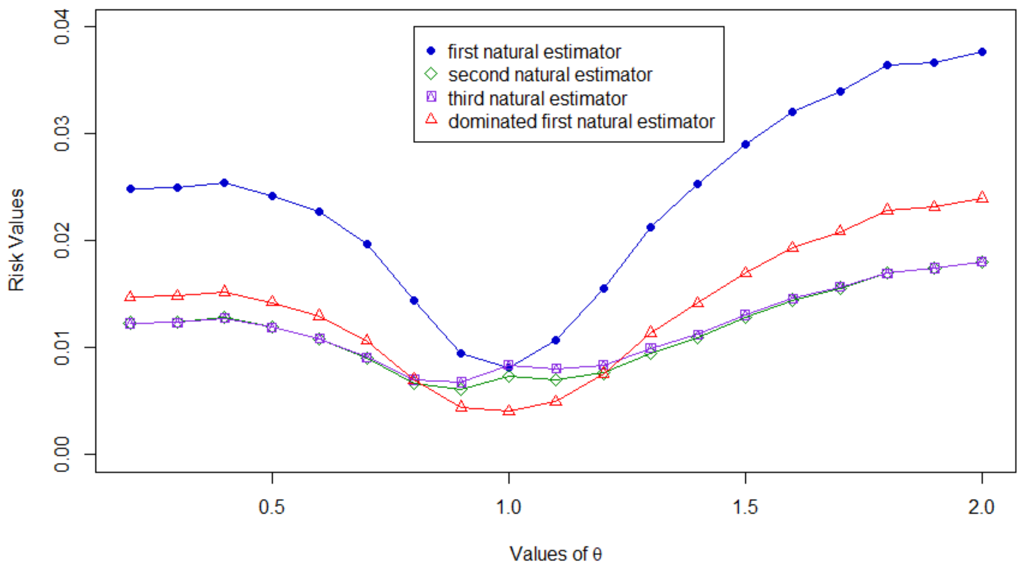


Figure 12: Comparison of Various Estimators under Asymmetric Loss Function for (n_1, n_2)=(9, 7)



8 Data Analysis I

Batch 1 (π_1)	1505	1556	1801	1629	1644	1607	1825	1748
Batch 2 (π_2)	1799	1618	1604	1655	1708	1675	1728	

Table 1: Lives in hours of two batches of electric lamps

The data in Table (1) represent lives (in hours) of two batches of electric lamps (see Gun et al., 2008, page 410). The location of the data is shifted by its minimum value. For the population (π_1), the Kolmogrov-Smirnov (K-S) distance between the shifted data and the fitted uniform (0, 320) distribution is 0.1906 and the corresponding p-value is 0.8844. Similarly for the population (π_2), the Kolmogrov-Smirnov (K-S) distance between the shifted data and the fitted uniform (0, 195) distribution is 0.2213 and the corresponding p-value is 0.8163. So there is no evidence to suspect that the samples are not from uniform distributions.

Suppose the quality of the electric lamps is measured in terms of average lifetime i.e., the population $\pi_1 \equiv U(0, \theta_1)$ is better than $\pi_2 \equiv U(0, \theta_2)$ if $\theta_1 > \theta_2$ and the population $\pi_2 \equiv U(0, \theta_2)$ is better than $\pi_1 \equiv U(0, \theta_1)$ if $\theta_1 \leq \theta_2$. For the goal of selecting the better batch of electric lamps, the minimax selection rule $\boldsymbol{\delta}^{\boldsymbol{a}^*}$ (10) is considered. Since the minimax selection rule $\boldsymbol{\delta}^{\boldsymbol{a}^*}$ (10) depends on sample sizes n_1 and n_2, the following two cases are considered.

8.1 Case I: n_1=6, n_2=4

The first 6 observations from π_1 and first 4 observations from π_2 (excluding 0) are taken. Then a^*=1.0466 and $\mathbf{x}$=(x_1, x_2)=(320, 195). Clearly, $x_1 = 320 > a_* x_2 = 204.0939$. So $\theta_S = \theta_1$.

Now various estimates under scaled-squared error loss function (5) are

$$\varphi_U(\mathbf{x}) = 369.7435$$

$$\varphi_U^*(\mathbf{x}) = 369.7435$$

$$\varphi_{N,1}(\mathbf{x}) = 320.0000$$

$$\varphi_{N.1}^*(\mathbf{x}) = 349.0909$$

$$\varphi_{N,2}(\mathbf{x}) = 373.3333$$

$$\varphi_{N,3}(\mathbf{x}) = 365.7143$$

Various estimates under entropy loss function (6) are

$$\varphi_U(\mathbf{x}) = 378.8993$$

$$\varphi_U^*(\mathbf{x}) = 378.8993$$
$$\varphi_{N,1}(\mathbf{x}) = 320.0000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 355.5556$$
$$\varphi_{N,2}(\mathbf{x}) = 384.0000$$

Various estimates under asymmetric loss function (7) are

$$\varphi_{N,1}(\mathbf{x}) = 320.0000$$
$$\varphi_{N,2}(\mathbf{x}) = 373.3333$$
$$\varphi_{N,3}(\mathbf{x}) = 378.6291$$
$$\varphi_{N.1}^*(\mathbf{x}) = 355.5556$$

8.2 Case II: n_1=4, n_2=6

The first 4 observations from π_1 and first 6 observations from π_2 (excluding 0) are taken. Then a^*=0.9554 and $\mathbf{x}$=(x_1, x_2)=(296, 195). Clearly, $x_1 = 296 > a^* x_2 = 186.3113$. So $\theta_S = \theta_1$.

Now various estimates under scaled-squared error loss function (5) are

$$\varphi_U(\mathbf{x}) = 358.3849$$
$$\varphi_U^*(\mathbf{x}) = 358.3849$$
$$\varphi_{N,1}(\mathbf{x}) = 296.0000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 322.9091$$
$$\varphi_{N,2}(\mathbf{x}) = 370.0000$$
$$\varphi_{N,3}(\mathbf{x}) = 355.2000$$

Various estimates under entropy loss function (6) are

$$\varphi_U(\mathbf{x}) = 375.0443$$
$$\varphi_U^*(\mathbf{x}) = 375.0443$$
$$\varphi_{N,1}(\mathbf{x}) = 296.0000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 328.8889$$
$$\varphi_{N,2}(\mathbf{x}) = 394.6667$$

Various estimates under asymmetric loss function (7) are

$$\varphi_{N,1}(\mathbf{x}) = 296.0000$$
$$\varphi_{N,2}(\mathbf{x}) = 370.0000$$
$$\varphi_{N,3}(\mathbf{x}) = 382.1344$$
$$\varphi_{N.1}^*(\mathbf{x}) = 327.2403$$

9 Data Analysis II

X (π_1)	3.7	2.8	7.1	8.4	6.2	2.7
Y (π_2)	6.4	6.8	9.1	7.4	6.9	6.8

Table 2: Failure times in hundreds of hours of certain type of light bulb manufactured by two different companies

The data in Table (2) represent failure times (in hundreds of hours) of certain type of light bulb manufactured by two different companies (see Rohatgi, 2003, Section 11.6.1, Example 2). The location of the data is shifted by its minimum value. For the population (π_1), the Kolmogrov-Smirnov (K-S) distance between the shifted data and the fitted uniform (0, 5.7) distribution is 0.3246 and the corresponding p-value is 0.4555. Similarly for the population (π_2), the Kolmogrov-Smirnov (K-S) distance between the shifted data and the fitted uniform (0, 2.7) distribution is 0.4815 and the corresponding p-value is 0.1238. So there is no evidence to suspect that the samples are not from uniform distributions.

Suppose the quality of the light bulb is measured in terms of average failure time i.e., the population $\pi_1 \equiv U(0, \theta_1)$ is better than $\pi_2 \equiv U(0, \theta_2)$ if $\theta_1 > \theta_2$ and the population $\pi_2 \equiv U(0, \theta_2)$ is better than $\pi_1 \equiv U(0, \theta_1)$ if $\theta_1 \leq \theta_2$. For the goal of selecting the better batch of electric lamps, the minimax selection rule $\boldsymbol{\delta}^{\boldsymbol{a}^*}$ (10) is considered. Since the minimax selection rule $\boldsymbol{\delta}^{\boldsymbol{a}^*}$ (10) depends on sample sizes n_1 and n_2, the following two cases are considered.

9.1 Case I: n_1=3, n_2=5

The first 3 observations from π_1 and first 5 observations from π_2 (excluding 0) are taken. Then a^*=0.9283 and $\mathbf{x}$=(x_1, x_2)=(4.4, 2.7). Clearly, $x_1 = 4.4 > a^* x_2 = 2.5065$. So $\theta_S = \theta_1$.

Now various estimates under scaled-squared error loss function (5) are

$$\varphi_U(\mathbf{x}) = 5.5956$$

$$\varphi_U^*(\mathbf{x}) = 5.5956$$

$$\varphi_{N,1}(\mathbf{x}) = 4.4000$$

$$\varphi_{N.1}^*(\mathbf{x}) = 4.8889$$

$$\varphi_{N,2}(\mathbf{x}) = 5.8667$$

$$\varphi_{N,3}(\mathbf{x}) = 5.5000$$

Various estimates under entropy loss function (6) are

$$\varphi_U(\mathbf{x}) = 6.0416$$

$$\varphi_U^*(\mathbf{x}) = 6.0416$$
$$\varphi_{N,1}(\mathbf{x}) = 4.4000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 5.0286$$
$$\varphi_{N,2}(\mathbf{x}) = 6.6000$$

Various estimates under asymmetric loss function (7) are

$$\varphi_{N,1}(\mathbf{x}) = 4.4000$$

$$\varphi_{N,2}(\mathbf{x}) = 5.8667$$
$$\varphi_{N,3}(\mathbf{x}) = 6.2225$$
$$\varphi_{N.1}^*(\mathbf{x}) = 4.9891$$

9.2 Case II: n_1=5, n_2=3

The first 5 observations from π_1 and first 3 observations from π_2 (excluding 0) are taken. Then a^*=1.0772 and $\mathbf{x}$=(x_1, x_2)=(5.7, 2.7). Clearly, $x_1 = 5.7 > a_* x_2 = 2.9085$. So $\theta_S = \theta_1$.

Now various estimates under scaled-squared error loss function (5) are

$$\varphi_U(\mathbf{x}) = 6.8006$$

$$\varphi_U^*(\mathbf{x}) = 6.8006$$
$$\varphi_{N,1}(\mathbf{x}) = 5.7000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 6.3333$$
$$\varphi_{N,2}(\mathbf{x}) = 6.8400$$
$$\varphi_{N,3}(\mathbf{x}) = 6.6500$$

Various estimates under entropy loss function (6) are

$$\varphi_U(\mathbf{x}) = 7.0639$$

$$\varphi_U^*(\mathbf{x}) = 7.0639$$
$$\varphi_{N,1}(\mathbf{x}) = 5.7000$$
$$\varphi_{N.1}^*(\mathbf{x}) = 6.5143$$
$$\varphi_{N,2}(\mathbf{x}) = 7.1250$$

Various estimates under asymmetric loss function (7) are

$$\varphi_{N,1}(\mathbf{x}) = 5.7000$$

$$\varphi_{N,2}(\mathbf{x}) = 6.8400$$
$$\varphi_{N,3}(\mathbf{x}) = 6.9810$$
$$\varphi^*_{N.1}(\mathbf{x}) = 6.4632$$

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