Orbit Determination via Topocentric Angular Observations

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Abstract

In this work, a set of topocentric angular observations of a satellite's motion are used to determine the salient parameters of the satellie's orbit. Two different methods of orbit determination are herein examined: the methods of Gauss and Laplace. After discussion of the merits and pitfalls of these methods, we demonstrate the accuracy of the two by computing a best-fit orbit for the Tiangon-1 satellite.

1 Introduction

The determination of patterns of motion for celestial bodies is a surprisingly difficult problem, and one that has been oft studied through the history of celestial mechanics. Although the aim of the method is simple, it has been incredibly fruitful in its products that the rest of science has benefitted from. The struggle of rationalizing Tycho's observational data on the known planets led to Kepler's three laws, which are a fundamental piece of our understanding of the solar system. The determination of the orbit of Ceres as it passed the sun in 1801 led Gauss to develop the method of least squares regression, which has seen considerable use in the last century to fit models to observational data in all branches of science.

2 Theory of Orbit Determination

2.1 Gauss' Method

Gauss developed his method of orbit determination to solve a troubling problem: on January 1st 1801, Giuseppe Piazzi discovered Ceres and was able to track it for 40 days before it was lost in the glare of the sun. As it continued its solar orbit, the problem was to determine the orbital path, and predict the position at which it would again become observable. Gauss is credited with the predictions which allowed another astronomer, Franz Xaver von Zach, to observe the minor planet again on December 31st of the same year.

2.1.1 Observational Quantities

Gauss' method centers on two key quantities: the observer's position vector in the equatorial coordinate system, and unit vectors along the direction of the observation. The latter is a result of the observational technology at the time: ranging methods such as radar were not available at the time, and thus the best observation one could achieve with the technology of the time (telescopes) was simply two angles describing the orientation of the observed object's line of sight vector.

The equatorial position vector of the observer can be found as [1]

$$R_n = \left[\frac{R_e}{\sqrt{(1-(2f-f^2)sin^2\phi)}} + H_n\right] cos\phi_n(cos\theta_n\hat{I} + sin\theta_n\hat{J}) + \left[\frac{R_e(1-f)^2}{\sqrt{(1-(2f-f^2)sin^2\phi)}} + H_n\right] sin\phi_n\hat{K} \tag{1}$$

R_n is the observer's position vector (in Equatorial Coordinate System)
R_e is the equatorial radius of the body (e.g., Earth's Re is 6,378 km)
f is the oblateness (or flattening) of the body (e.g., Earth's f is 0.003353)
ϕ_n is the respective geodetic latitude
ϕ'_n is the respective geocentric latitude
H_n is the respective altitude
θ_n is the respective local sidereal time

The observation unit vector can also be found (in the topocentric coordinate system) via the following [1]:
$\hat{\rho}_n = cos\delta_n cos\alpha_n\hat{I} + cos\delta_n sin\alpha_n\hat{J} + sin\delta_n\hat{K}$

2.1.2 Gauss' Algorithm

Once we have in hand at least three line-of-sight observations and the observer's position vector in the equatorial coordinate frame at those times, we can determine the position and velocity vectors of the orbiting object (and thus the classical orbital elements).

We begin with the relevant time intervals:

$$\tau_1 = t_1 - t_2 \tag{2}$$

$$\tau_3 = t_3 - t_2 \tag{3}$$

$$\tau = t_3 - t_1 \tag{4}$$

The next step is to find the common scalar product, D_0:

$$D_0 = \hat{\rho}_1 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \tag{5}$$

Followed by the matrix quantities, D_{mn}:

$$D_{11} = R_1 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \quad D_{12} = R_1 \cdot (\hat{\rho}_1 \times \hat{\rho}_3) \quad D_{13} = R_1 \cdot (\hat{\rho}_1 \times \hat{\rho}_2) \tag{6}$$

$$D_{21} = R_2 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \quad D_{22} = R_2 \cdot (\hat{\rho}_1 \times \hat{\rho}_3) \quad D_{23} = R_2 \cdot (\hat{\rho}_1 \times \hat{\rho}_2) \tag{7}$$

$$D_{31} = R_3 \cdot (\hat{\rho}_2 \times \hat{\rho}_3) \quad D_{32} = R_3 \cdot (\hat{\rho}_1 \times \hat{\rho}_3) \quad D_{33} = R_3 \cdot (\hat{\rho}_1 \times \hat{\rho}_2) \tag{8}$$

Using the just calculated quantities, we build three coefficients for the scalar position.

$$A = \frac{1}{D_0} \left(-D_{12} \frac{\tau_3}{\tau} + D_{22} - D_{32} \frac{\tau_1}{\tau} \right) \tag{9}$$

$$B = \frac{1}{6D_0} \left(D_{12} \left(\tau_3^2 - \tau^2 \right) \frac{\tau_3}{\tau} + D_{32} \left(\tau^2 - \tau_1^2 \right) \frac{\tau_1}{\tau} \right) \tag{10}$$

$$E = R_2 \cdot \hat{\rho}_2 \tag{11}$$

We will also need the squared magnitude of the second observer position vector

$$R_2^2 = R_2 \cdot R_2 \tag{12}$$

Using the coefficients just built, we build a polynomial in the scalar distance of the observation. Here, μ is the gravitational parameter of the focal body of the orbit.

$$a = -\left(A^2 + 2AE + R_2^2 \right) \tag{13}$$

$$b = -2\mu B \left(A + E \right) \tag{14}$$

$$c = -\mu^2 B^2 \tag{15}$$

These quantities are now the coefficients in an 8th order polynomial in the scalar distance of the second observation, r_2.

$$r_2^8 + ar_2^6 + br_2^3 + c = 0 \tag{16}$$

This polynomial can be solved by any suitable root finding routine, such as the Newton-Rhapson method. We note that since this is a radial distance from the focal body of the orbit, the root must be real. In the event there are multiple real roots of the polynomial, other measurements or data must be used to disambiguate the solution.

With the orbital distance of the body fixed for one of the observations, we can now discern the slant range of the object from the observer, ρ_n.

$$\rho_1 = \frac{1}{D_0}\left[\frac{6\left(D_{31}\frac{\tau_1}{\tau_3} + D_{21}\frac{\tau}{\tau_3}\right)r_2^3 + \mu D_{31}\left(\tau^2 - \tau_1^2\right)\frac{\tau_1}{\tau_3}}{6r_2^3 + \mu\left(\tau^2 - \tau_3^2\right)} - D_{11}\right] \tag{17}$$

$$\rho_2 = A + \frac{\mu B}{r_2^3} \tag{18}$$

$$\rho_3 = \frac{1}{D_0}\left[\frac{6\left(D_{13}\frac{\tau_3}{\tau_1} - D_{23}\frac{\tau}{\tau_1}\right)r_2^3 + \mu D_{13}\left(\tau^2 - \tau_3^2\right)\frac{\tau_3}{\tau_1}}{6r_2^3 + \mu\left(\tau^2 - \tau_1^2\right)} - D_{33}\right] \tag{19}$$

With the slant ranges in hand, we can now easily calculate the orbital position vectors of the observed body to its focal body, R_n.

$$\vec{r}_n = \vec{R}_n + \rho_n\hat{\rho}_n \tag{20}$$

To find the velocity of the orbiting body, we rely on a series expansion of the orbital motion about the midpoint of the observations.

$$\vec{v}_2 = \frac{1}{f_1 g_3 - f_3 g_1}\left(-f_3\vec{r}_1 + f_1\vec{r}_3\right) \tag{21}$$

Where the expansion terms are:

$$f_1 = 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2 \tag{22}$$

$$f_3 = 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2 \tag{23}$$

$$g_1 = \tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3 \tag{24}$$

$$g_3 = \tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3 \tag{25}$$

The orbital determination problem is now complete. Since we have assumed a Keplerian orbit, the entirety of the orbit is defined by the six components of the second position vector and its associated velocity, $\vec{r}_2$ and $\vec{v}_2$ [2].

2.2 Laplace's Method

We now turn our attention to the second classical method of orbit determination, that of Laplace.

References

[1] Anon. Gauss' method, Sep 2018.

[2] Craig A. Kluever. *Space flight dynamics*. Wiley, 2018.