Orbit Determination via Topocentric Angular Observations

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October 16, 2018

Abstract

In this work, a set of topocentric angular observations of a satellite's motion are used to determine the salient parameters of the satellite's orbit. Two different methods of orbit determination are herein examined: the methods of Gauss and Laplace. After discussion of the merits and pitfalls of these methods, we demonstrate the accuracy of the two by computing a best-fit orbit for the Tiangong-1 satellite.

1 Introduction

The determination of patterns of motion for celestial bodies is a surprisingly difficult problem, and one that has been oft studied through the history of celestial mechanics. Although the aim of the method is simple, it has been incredibly fruitful in its products that the rest of science has benefitted from. The struggle of rationalizing Tycho's observational data on the known planets led to Kepler's three laws, which are a fundamental piece of our understanding of the solar system. The determination of the orbit of Ceres as it passed the sun in 1801 led Gauss to develop the method of least squares regression, which has seen considerable use in the last century to fit models to observational data in all branches of science.

2 Theory of Orbit Determination

2.1 Gauss' Method

Gauss developed his method of orbit determination to solve a troubling problem: on January 1st 1801, Giuseppe Piazzi discovered Ceres and was able to track it for 40 days before it was lost in the glare of the sun. As it continued its solar orbit, the problem was to determine the orbital path, and predict the position at which it would again become observable. Gauss is credited with the predictions which allowed another astronomer, Franz Xaver von Zach, to observe the minor planet again on December 31st of the same year.

The following treatment of Gauss' method is the same as described at [1], which was developed in [4].

2.1.1 Observational Quantities

Gauss' method centers on two key quantities: the observer's position vector in the equatorial coordinate system, and unit vectors along the direction of the observation. The latter is a result of the observational technology at the time: ranging methods such as radar were not available at the time, and thus the best observation one could achieve with the technology of the time (telescopes) was simply two angles describing the orientation of the observed object's line of sight vector.

The equatorial position vector of the observer can be found as

$$\begin{aligned} R_n = &\left[\frac{R_e}{\sqrt{(1-(2f-f^2)sin^2\phi)}} + H_n\right] cos\phi_n(cos\theta_n\hat{I} + sin\theta_n\hat{J}) \\ &+ \left[\frac{R_e(1-f)^2}{\sqrt{(1-(2f-f^2)sin^2\phi)}} + H_n\right] sin\phi_n\hat{K} \end{aligned} \tag{1}$$

R_n is the observer's position vector (in Equatorial Coordinate System)
R_e is the equatorial radius of the body (e.g., Earth's Re is 6,378 km)
f is the oblateness (or flattening) of the body (e.g., Earth's f is 0.003353)
ϕ_n is the respective geodetic latitude
ϕ'_n is the respective geocentric latitude
H_n is the respective altitude
θ_n is the respective local sidereal time

The observation unit vector can also be found (in the topocentric coordinate system) via the following [1]:

$$\hat{\rho}_n = cos\delta_n cos\alpha_n\hat{I} + cos\delta_n sin\alpha_n\hat{J} + sin\delta_n\hat{K} \tag{2}$$

2.1.2 Gauss' Algorithm

Once we have in hand at least three line-of-sight observations and the observer's position vector in the equatorial coordinate frame at those times, we can determine the position and velocity vectors of the orbiting object (and thus the classical orbital elements).

We begin with the relevant time intervals:

$$\tau_1 = t_1 - t_2 \tag{3}$$

$$\tau_3 = t_3 - t_2 \tag{4}$$

$$\tau = t_3 - t_1 \tag{5}$$

The next step is to find the common scalar product, D_0:

$$D_0 = \hat{\rho_1} \cdot (\hat{\rho_2} \times \hat{\rho_3}) \tag{6}$$

Followed by the matrix quantities, D_{mn}:

$$D_{11} = R_1 \cdot (\hat{\rho_2} \times \hat{\rho_3}) \quad D_{12} = R_1 \cdot (\hat{\rho_1} \times \hat{\rho_3}) \quad D_{13} = R_1 \cdot (\hat{\rho_1} \times \hat{\rho_2}) \tag{7}$$

$$D_{21} = R_2 \cdot (\hat{\rho_2} \times \hat{\rho_3}) \quad D_{22} = R_2 \cdot (\hat{\rho_1} \times \hat{\rho_3}) \quad D_{23} = R_2 \cdot (\hat{\rho_1} \times \hat{\rho_2}) \tag{8}$$

$$D_{31} = R_3 \cdot (\hat{\rho_2} \times \hat{\rho_3}) \quad D_{32} = R_3 \cdot (\hat{\rho_1} \times \hat{\rho_3}) \quad D_{33} = R_3 \cdot (\hat{\rho_1} \times \hat{\rho_2}) \tag{9}$$

Using the just calculated quantities, we build three coefficients for the scalar position.

$$A = \frac{1}{D_0}\left(-D_{12}\frac{\tau_3}{\tau} + D_{22} - D_{32}\frac{\tau_1}{\tau}\right) \tag{10}$$

$$B = \frac{1}{6D_0}\left(D_{12}\left(\tau_3^2 - \tau^2\right)\frac{\tau_3}{\tau} + D_{32}\left(\tau^2 - \tau_1^2\right)\frac{\tau_1}{\tau}\right) \tag{11}$$

$$E = R_2 \cdot \hat{\rho_2} \tag{12}$$

We will also need the squared magnitude of the second observer position vector

$$R_2^2 = R_2 \cdot R_2 \tag{13}$$

Using the coefficients just built, we build a polynomial in the scalar distance of the observation. Here, μ is the gravitational parameter of the focal body of the orbit.

$$a = -\left(A^2 + 2AE + R_2^2\right) \tag{14}$$

$$b = -2\mu B\left(A+E\right) \tag{15}$$

$$c = -\mu^2 B^2 \tag{16}$$

These quantities are now the coefficients in an 8th order polynomial in the scalar distance of the second observation, r_2.

$$r_2^8 + ar_2^6 + br_2^3 + c = 0 \tag{17}$$

This polynomial can be solved by any suitable root finding routine, such as the Newton-Rhapson method. We note that since this is a radial distance from the focal body of the orbit, the root must be real. In the event there are multiple real roots of the polynomial, other measurements or data must be used to disambiguate the solution.

With the orbital distance of the body fixed for one of the observations, we can now discern the slant range of the object from the observer, ρ_n.

$$\rho_1 = \frac{1}{D_0}\left[\frac{6\left(D_{31}\frac{\tau_1}{\tau_3} + D_{21}\frac{\tau}{\tau_3}\right)r_2^3 + \mu D_{31}\left(\tau^2-\tau_1^2\right)\frac{\tau_1}{\tau_3}}{6r_2^3 + \mu\left(\tau^2-\tau_3^2\right)} - D_{11}\right] \tag{18}$$

$$\rho_2 = A + \frac{\mu B}{r_2^3} \tag{19}$$

$$\rho_3 = \frac{1}{D_0}\left[\frac{6\left(D_{13}\frac{\tau_3}{\tau_1} - D_{23}\frac{\tau}{\tau_1}\right)r_2^3 + \mu D_{13}\left(\tau^2-\tau_3^2\right)\frac{\tau_3}{\tau_1}}{6r_2^3 + \mu\left(\tau^2-\tau_1^2\right)} - D_{33}\right] \tag{20}$$

With the slant ranges in hand, we can now easily calculate the orbital position vectors of the observed body to its focal body, R_n.

$$\vec{r}_n = \vec{R}_n + \rho_n\hat{\rho}_n \tag{21}$$

To find the velocity of the orbiting body, we rely on a series expansion of the orbital motion about the midpoint of the observations.

$$\vec{v}_2 = \frac{1}{f_1 g_3 - f_3 g_1}\left(-f_3\vec{r}_1 + f_1\vec{r}_3\right) \tag{22}$$

Where the expansion terms are:

$$f_1 = 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_1^2 \tag{23}$$

$$f_3 = 1 - \frac{1}{2}\frac{\mu}{r_2^3}\tau_3^2 \tag{24}$$

$$g_1 = \tau_1 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_1^3 \tag{25}$$

$$g_3 = \tau_3 - \frac{1}{6}\frac{\mu}{r_2^3}\tau_3^3 \tag{26}$$

The orbital determination problem is now complete. Since we have assumed a Keplerian orbit, the entirety of the orbit is defined by the six components of the second position vector and its associated velocity, $\vec{r}_2$ and $\vec{v}_2$ [5].

2.2 Laplace's Method

We now turn our attention to the second classical method of orbit determination, that of Simon Pierre Laplace. This treatment will closely follow the one presented in [3].

As before with Gauss' method, we expect the observations to only contain angular quantities, such as right ascension and declination, in the topocentric coordinate system. Begin by expressing the observations as line of sight vectors in the topocentric coordinate system using equations 2. Additionally, we require the position of the observer at the observation epochs, found via equation 1.

2.2.1 The Orbital Position Vector

Noting that the position vector of the orbiting body at each epoch can be written as

$$\vec{r} = \rho_n \hat{\rho}_n + \vec{R}_n \tag{27}$$

with ρ_n the (as yet undetermined) slant range from the observer to the body, we differentiate the position vector twice to arrive at a relation between the body's position and the assumed (Keplerian) form of its dynamics:

$$\hat{\rho}_2 \ddot{\rho}_2 + 2\dot{\hat{\rho}}_2 \dot{\rho}_2 + \left(\ddot{\hat{\rho}}_2 + \frac{\mu}{r^3}\hat{\rho}_2\right)\rho_2 = -\left(\ddot{\vec{R}}_2 + \mu\frac{\vec{R}_2}{r^3}\right) \tag{28}$$

Here, we note that r^3 is simply the magnitude of the observed body's distance from its orbital focus. Also of note is that the above equation is only used for the second observation. The first and third of the set will be used in the numerical differentiation process to determine the derivatives of the line of sight vector, as follows.

$$\dot{\hat{\rho}}(t) = \frac{2t - t_2 - t_3}{(t_1 - t_2)(t_1 - t_3)}\hat{\rho}_1 + \frac{2t - t_1 - t_3}{(t_2 - t_1)(t_2 - t_3)}\hat{\rho}_2 + \frac{2t - t_1 - t_2}{(t_3 - t_1)(t_3 - t_2)}\hat{\rho}_3 \quad (29)$$

$$\ddot{\hat{\rho}}(t) = \frac{2}{(t_1 - t_2)(t_1 - t_3)}\hat{\rho}_1 + \frac{2}{(t_2 - t_1)(t_2 - t_3)}\hat{\rho}_2 + \frac{2}{(t_3 - t_1)(t_3 - t_2)}\hat{\rho}_3 \quad (30)$$

Writing 28 for the central observation, and including the information from 29 and 30, we note that 28 is now a three component equation in four unknowns $\rho, \dot{\rho}, \ddot{\rho}$, and r. Taking the components of this equation and writing the system in matrix form, we can attempt a solution via Cramer's rule (with some additional elimination in the matrix):

$$D = 2\begin{vmatrix} \hat{\rho}_I & \dot{\hat{\rho}}_I & \ddot{\hat{\rho}}_I \\ \hat{\rho}_J & \dot{\hat{\rho}}_J & \ddot{\hat{\rho}}_J \\ \hat{\rho}_K & \dot{\hat{\rho}}_K & \ddot{\hat{\rho}}_K \end{vmatrix} \quad (31)$$

Applying Cramer's rule to equation 28, we can see that

$$D\rho = -\begin{vmatrix} \hat{\rho}_I & 2\dot{\hat{\rho}}_I & \ddot{R}_I + \mu R_I/r^3 \\ \hat{\rho}_J & 2\dot{\hat{\rho}}_J & \ddot{R}_J + \mu R_J/r^3 \\ \hat{\rho}_K & 2\dot{\hat{\rho}}_K & \ddot{R}_K + \mu R_K/r^3 \end{vmatrix} \quad (32)$$

We can further simplify this to

$$\rho = \frac{-2D_1}{D} - \frac{2\mu D_2}{f^3 D}, D \neq 0 \quad (33)$$

With

$$D_1 = -2\begin{vmatrix} \hat{\rho}_I & \dot{\hat{\rho}}_I & \ddot{R}_I \\ \hat{\rho}_J & \dot{\hat{\rho}}_J & \ddot{R}_J \\ \hat{\rho}_K & \dot{\hat{\rho}}_K & \ddot{R}_K \end{vmatrix} \quad (34)$$

$$D_2 = -2\frac{\mu}{r^3}\begin{vmatrix} \hat{\rho}_I & \dot{\hat{\rho}}_I & R_I \\ \hat{\rho}_J & \dot{\hat{\rho}}_J & R_J \\ \hat{\rho}_K & \dot{\hat{\rho}}_K & R_K \end{vmatrix} \quad (35)$$

This yields an expression for the slant range, which is dependent only on the still unknown magnitude of the orbiting body's focal position vector r. Dotting 27 with itself, we find

$$r^2 = \rho^2 + 2\rho\hat{\rho} \cdot \vec{R} + R^2 \tag{36}$$

Once 36 is solved, the resulting magnitude can be substituted to 33 to find the slant range, and the position vector can be determined from 27.

2.2.2 Determining the Velocity Vector

If we return to 28 and again apply Cramer's rule, we find that the velocity can also be expressed as a function of determinants.

$$D\dot{\rho} = -D_3 - \frac{\mu}{r^3} D_4 \tag{37}$$

With

$$D_3 = \begin{vmatrix} \hat{\rho}_I & \ddot{R}_I & \ddot{\hat{\rho}}_I \\ \hat{\rho}_J & \ddot{R}_J & \ddot{\hat{\rho}}_J \\ \hat{\rho}_K & \ddot{R}_K & \ddot{\hat{\rho}}_K \end{vmatrix} \tag{38}$$

$$D_4 = \begin{vmatrix} \hat{\rho}_I & R_I & \ddot{\hat{\rho}}_I \\ \hat{\rho}_J & R_J & \ddot{\hat{\rho}}_J \\ \hat{\rho}_K & R_K & \ddot{\hat{\rho}}_K \end{vmatrix} \tag{39}$$

We can then solve for the time derivative of the slant range as

$$\dot{\rho} = -\frac{D_3}{D} - \frac{\mu}{r^3}\frac{D_4}{D} \tag{40}$$

Finally, with the slant range and slant velocity solved, we can differentiate 27 and obtain the velocity vector:

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{\vec{R}} \tag{41}$$

With the position and velocity vectors defined in three space within a suitable coordinate system, the Keplerian orbit is fully defined. Classical elements of the orbit (semi-major axis, eccentricity, inclination, etc) can be found as in [5].

3 Results

With the theory of orbital determination in place, we turn to the results. The two methods described above were used to transform a set of observations

Julian Date (Days)	Right Ascension (Degrees)	Declination (Degrees)
2458130.5830398300	353.2148120	31.2289572
2458130.5830409615	353.2950670	31.3242459
2458130.5830421210	353.3776091	31.4220063
2458130.5830433145	353.4628954	31.5227576
2458130.5830444675	353.5456008	31.6202080

Table 1: Observational Data of Tiangong-1

	TLE	Gauss	Laplace
Semi-Major Axis	6656.2318 km	6559.578 km	6587.1 km
Eccentricity	0.0017667	0.014807	0.0099
Inclination	42.7537	42.7646	42.7393
Ascending Node	344.4268	345.3355	345.3201
Arg. of Periapsis	147.3056	221.8772	218.1973
True Anomaly	-17.6624	190.2413	193.9572
Epoch (Julian)	2458130.721406	2458130.5830	2458130.5830

Table 2: Orbital Determination Results

made of the chinese satellite Tiangong-1, while it was de-orbiting due to a hardware malfunction.

The observations were taken in the topocentric coordinate system, and reported as 3-tuples of Julian date, right ascension, and declination. The observational data are shown in table 1.

We now present the results of the orbital determination methods. As a baseline, we also show TLE data for the satellite's orbit before it's demise, which was found at [2]. The "official" AFSPC data has been removed from their website, as the satellite is no longer in orbit. The classical elements of the Keplerian orbit are compared between the TLE, Gauss' method, and Laplace's method in table 2.

It is clear that while the general size and shape of the orbit's determined from the observations agree - to some extent - with the published TLE data, the orientation of the orbital plane with respect to the earth does not. This could be explained by the difference in epochs, as the latest TLE data was observed more than 80 Julian days after our observation set was taken, and the satellite was de-orbiting. It could also be explained by the inadequacy of our observations, which spanned a very short arc of the satellite's orbit.

Finally, we show results for the prediction phase. After the initial orbit determination was complete, we propagated the satellites position to two

	Gauss		Laplace	
	60 min	80 d	60 min	80 d
f	-138.564	161.813	78.934	-123.929
rx	775.549	-5346.675	1811.145	752.547
ry	-5111.923	-2430.357	-4916.406	4817.465
rz	-4392.244	-3426.826	-3970.561	4482.459

Table 3: Orbital Propagation Results

points in the future: 60 minutes after the observation epoch, and 80 days after the epoch. This process was repeated for both determination methods. The results of each propagation is shown in table 3.

4 Conclusion

Without a baseline with an epoch much closer to the observations contained here, it's difficult to comment on the accuracy of either orbital determination method with serious confidence. However, we can make a few small notes. It appears that Laplace's method produced a better estimate of the size/shape orbital parameters a, e, and i, at least when compared to the TLE data available.

The propagation results also show considerable variability between the two methods. This is likely an artifact of the considerable disagreement between the method's results for orbital orientation. The magnitude of the difference in predicted position between the two methods is roughly 1200 km at the 60 minute prediction point, but that grows significantly to 12,000 km at the 80 day mark. Given that this is roughly the length of the major axis of the orbit, the two predictions are clearly out of phase, and differences in predicted mean motion have caused divergence in the solution.

If we were to depend on these results for a real world application, such as predicting the next patch of observability for the object, we would be hopelessly lost when turning our telescopes to the sky. Ideally, a significantly larger number of observations would be available, and Gauss' least squares methods could be used to refine the estimate of the orbit.

References

[1] Anon. Gauss' method, Sep 2018.

[2] Anon. Tiangong-1 orbit, Oct 2018.

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[5] Craig A. Kluever. *Space flight dynamics*. Wiley, 2018.