Aaron J. Rosengren

Project 2 - Team

Fall 2018

AME 457/557 – Orbital Mechanics and Space Flight

Due Thursday, November 8, 2017 at 17:00 PM (MST)

*This is a team project — **collaboration with other teams is encouraged**; however, you must do you own work and are not allowed to copy answers or algorithms. It is also permissible to use existing resources (computer programs/websites/books/notes) to check answers or help facilitate the solutions. **All collaboration and use of other resources must be properly acknowledged and described. Those who fail to adhere to this will receive a zero on the technical component of the project.***

Project Description . You are to develop software for orbit propagation and orbital lifetime estimation under the disturbing influences of Earth oblateness and atmospheric drag perturbations. The objective is to create an algorithm (e.g., MATLAB script files) that will convert an initial Cartesian state $(\boldsymbol{r}_0, \boldsymbol{v}_0)$ in the Earth-centered inertial (ECI) frame at epoch t_0 into the initial orbital elements $(a, e, i, \Omega, \omega, M)_0$, and then to propagate this state ahead in time until the satellite's demise (predicted to occur whenever the perigee altitude or instantaneous radius is below the Earth's surface). Orbit propagation is according to either the averaged theory for oblateness and drag perturbations or a numerical integration of the (non-averaged) Newtonian equations of motion governing perturbed Keplerian motion. In any case, you are to describe the effects of each perturbing force and to predict the lifetime of a low-altitude Earth-orbiting satellite (e.g., the Tiangong 1 Chinese Space Station).

Input . The inputs to the software are:

1. The position and velocity vectors $(\boldsymbol{r}, \boldsymbol{v})$ of the Tiangong 1 Chinese Space Station in km, km/s, respectively, as determined from the middle epoch of the observations provided to your group (you are to use the solutions obtained by the TA for this purposes, to avoid using an incorrect starting orbit).
2. The ballistic coefficient $BC = m/C_d/A$ of the Tiangong 1 Chinese Space Station as determined from the TLE provided (use the empirical formula $BC = 1/12.741621/B^*$ kg/m^2, where B^* can be read from the two-line element set).

Output . The outputs to the software are:

1. The predicted position and velocity vectors, and orbital elements, at Earth reentry.
2. The estimated lifetime of the satellite in days.

Structure . There is no specified structure, but your codes should be logical and all sub-functions should be called from a "main" program that also feeds in your specific inputs.

Final Presentation . Present your results in the PowerPoint-style presentation format. Your slides should be similar to a a contractor's presentation to a customer (e.g., SpaceX) that would serve as a "User's Guide" for the orbit propagation and lifetime estimation software. Your presentation should be self-contained, so that the reader can understand it without additional information not contained in the report. The slides should be concise, thorough, clear, well organized, have good grammar, and provide sufficient technical background. Your slides will be graded on these criteria:

- organization and presentation (**20%**),
- technical merit (does your code produce the correct answer?!) (**40%**),
- clarity, style, and grammar (**20%**), and
- completeness (**20%**).

Technical Aspects (Update) . A simplified model for the disturbing acceleration caused by atmospheric drag is given by

$$\boldsymbol{a}_{\text{drag}} = -\frac{1}{2}\left(\frac{A}{m}\right) C_d \rho v^2 \hat{\boldsymbol{v}}, \tag{1}$$

where A is the satellite cross-sectional area along the velocity vector $\hat{\boldsymbol{v}}$, m is the satellite mass, C_d is the drag coefficient, ρ is the local atmospheric density, and v is the inertial orbital speed. Note that we are neglecting the oblateness and rotation of the atmosphere in this analysis. The quantity $m/(C_d A)$ is called the ballistic coefficient, BC. There are a number of models that describe the variation of atmospheric properties with altitude. We will employ here the `USSA76`, which assumes that the atmospheric density varies exponentially with altitude through

$$\rho(h) = \rho_0 \exp\left[-\frac{h - h_0}{H}\right], \tag{2}$$

where h is the satellite's altitude, h_0 and ρ_0 are a reference height and density, respectively, and H is the scale height (fractional change in density with height). For a given orbit radius r, the satellite's altitude can be found by subtracting the Earth's radius R; i.e., $h = r - R$, and the corresponding atmospheric density should then be obtained from the provided MATLAB function `atmosphere.m`. In inertial Cartesian coordinates, the drag acceleration can be stated as:

$$\boldsymbol{a}_{J_2} = -\frac{1}{2}\left(\frac{A}{m}\right) C_d \rho v \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}.$$

The disturbing acceleration caused by the Earth's oblateness is given by

$$\boldsymbol{a}_{J_2} = -\frac{3\mu J_2 R^2}{2r^4}\left[\left(1 - \frac{5z^2}{r^2}\right)\hat{\boldsymbol{r}} + \frac{2z}{r}\hat{\boldsymbol{z}}\right], \tag{3}$$

where $R = 6378.137$ km is the Earth's equatorial radius, $\mu = 3.98600442(10^5)$ km^3/s^2 is the gravitational parameter, and $J_2 = 0.0010826267$ is the oblateness gravity-field coefficient. In ECI coordinates, we have

$$\boldsymbol{a}_{J_2} = -\frac{3\mu J_2 R^2}{2r^4} \begin{bmatrix} \left(1 - \frac{5z^2}{r^2}\right) \frac{x}{r} \\ \left(1 - \frac{5z^2}{r^2}\right) \frac{y}{r} \\ \left(3 - \frac{5z^2}{r^2}\right) \frac{z}{r} \end{bmatrix}.$$

The Newtonian equations of motion for the perturbed two-body problem can then be stated as

$$\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{a}_{\text{drag}} + \boldsymbol{a}_{J_2}, \tag{4}$$

which is equivalent to three simultaneous second-order, nonlinear, differential equations. For the purpose of numerical integration, using, say, `ode45.m` in MATLAB, they must be reduced to six first-order ODEs. This can be accomplished by introducing a vector containing six auxiliary variables $\boldsymbol{y} = \begin{bmatrix} y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \end{bmatrix}^T$, such that

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}. \tag{5}$$

This implies that

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \\ \dot{y}_6 \end{bmatrix} = \begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ a_x(\boldsymbol{y}) \\ a_y(\boldsymbol{y}) \\ a_z(\boldsymbol{y}) \end{bmatrix}. \tag{6}$$

An alternate formulation can be given through the Lagrange planetary or Gauss variational equations, expressed as six first-order differential equations in the elements. To understand the main characteristics of motion, these equations can be averaged over the satellites orbital period, providing a particularly effective approximation of the dynamics.

For the oblateness perturbations, the average rates of change of a, e, and i are zero, while the angular variables have rates given by

$$\dot{\bar{a}} = 0, \tag{7}$$
$$\dot{\bar{e}} = 0, \tag{8}$$
$$\dot{\bar{i}} = 0, \tag{9}$$
$$\dot{\bar{\Omega}} = -\frac{3J_2R^2n}{2p^2}\cos i, \tag{10}$$
$$\dot{\bar{\omega}} = \frac{3J_2R^2n}{4p^2}(5\cos^2 i - 1), \tag{11}$$
$$\dot{\overline{M_0}} = \frac{3J_2R^2n}{4p^2}\sqrt{1-e^2}(3\cos^2 i - 1), \tag{12}$$

where $n = \sqrt{\mu/a^3}$ is the satellite's mean motion and $p = a(1-e^2)$ is the semi-parameter. This asymmetry in the potential thus results primarily into a precession of the orbital plane about the polar axis (nodal precession) and a steady motion of the major axis in the moving orbital plane (apsidal precession), the rates of which are dependent on the constant (averaged) values of (a, e, i). It's important to note that when treating multiple perturbations simultaneously, the orbit parameters (a, e, i) can change during the evolution, and hence the rates $(\dot{\bar{\Omega}}, dot\bar{\omega}$ become time varying.

Under our simplified model, atmospheric drag causes no perturbations on the orbital inclination or ascending node (i.e., $\mathrm{d}i/\mathrm{d}t = \mathrm{d}\Omega/\mathrm{d}t = 0$). Since we are only interested here in the form of the trajectory, we ignore the equation for the mean anomaly, and recall that the (non-averaged) Gauss equations for a, e, and ω are governed by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left(\frac{A}{m}\right)\frac{C_d}{n^2a}\rho v^3 \tag{13}$$
$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\left(\frac{A}{m}\right)C_d\rho(e+\cos f)v \tag{14}$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\left(\frac{A}{m}\right)\frac{C_d}{e}\rho\sin f v \tag{15}$$

In order to average these equations, we note that Eq. 2 for the atmospheric density can be rewritten as a function of the eccentric anomaly as

$$\rho = \rho_a \exp\left(\nu\cos E\right), \tag{16}$$

where $\rho_a = \rho(a-R)$ (i.e., the density at height $a - R$) and $\nu = ae/H$. Noting that

$$v = na\sqrt{\frac{1+e\cos E}{1-e\cos E}}, \quad \cos f = \frac{\cos E - e}{1-e\cos E}, \quad \sin f = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E},$$

we can rewrite the Gauss equations as a function of eccentric anomaly:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d n a^2 \rho_a \exp\left(\nu \cos E\right) \left(\frac{1 + e\cos E}{1 - e\cos E}\right)^{2/3} \tag{17}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d n a(1-e^2) \rho_a \exp\left(\nu \cos E\right) \frac{\cos E}{1 - e\cos E}\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \tag{18}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\left(\frac{A}{m}\right) C_d n a \frac{\sqrt{1-e^2}}{e} \rho_a \exp\left(\nu \cos E\right) \frac{\sin E}{1 - e\cos E}\sqrt{\frac{1 + e\cos E}{1 - e\cos E}}. \tag{19}$$

Noting the differential relationship between the mean anomaly and eccentricity anomaly, $\mathrm{d}M = (1 - e\cos E)\mathrm{d}E$, we can change the time average to an average over E. Doing so for the argument of perigee rate yields

$$\dot{\bar{\omega}} = -\left(\frac{A}{m}\right) C_d n a \frac{\sqrt{1-e^2}}{e} \rho_a \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\nu \cos E\right) \sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \sin E \,\mathrm{d}E, \tag{20}$$

$$= 0. \tag{21}$$

The averaged equations of motion for a and e can be stated as:

$$\dot{\bar{a}} = -\left(\frac{A}{m}\right) C_d n a^2 \rho_a \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\nu \cos E\right) \sqrt{\frac{\left(1 + e\cos E\right)^3}{1 - e\cos E}} \,\mathrm{d}E, \tag{22}$$

$$\dot{\bar{e}} = -\left(\frac{A}{m}\right) C_d n a(1-e^2) \rho_a \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\nu \cos E\right) \cos E\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \,\mathrm{d}E. \tag{23}$$

These integrals can be evaluated to a high degree of accuracy when a/H is large, for all values of e, by expanding in ascending powers of $\sin^2 E$, a process which is suggested by the method of steepest descents. For our purposes, however, considering small eccentricities, we can expand all terms in the integrands but the exponential factor in ascending powers of e. The Taylor series expansions at $e = 0$ for the integrand terms are:

$$\sqrt{\frac{\left(1 + e\cos E\right)^3}{1 - e\cos E}} = 1 + 2e\cos E + \frac{3}{4}e^2(1 + \cos 2E) + \mathcal{O}(e^3), \tag{24}$$

$$\cos E\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} = \cos E + \frac{1}{2}e(1 + \cos 2E) + \frac{1}{8}e^2(3\cos E + \cos 3E) + \mathcal{O}(e^3). \tag{25}$$

Thus, Eqs. 22 and 23 can be rewritten to $\mathcal{O}(e^3)$ as

$$\dot{\bar{a}} = -\left(\frac{A}{m}\right) C_d n a^2 \rho_a \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\nu \cos E\right) \left[1 + 2e\cos E + \frac{3}{4}e^2(1 + \cos 2E)\right] \mathrm{d}E,$$

$$\dot{\bar{e}} = -\left(\frac{A}{m}\right) C_d n a(1-e^2) \rho_a \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\nu \cos E\right) \left[\cos E + \frac{1}{2}e(1 + \cos 2E) + \frac{1}{8}e^2(3\cos E + \cos 3E)\right] \mathrm{d}E,$$

which can be integrated to give

$$\dot{\bar{a}} = -\left(\frac{A}{m}\right) C_d n a^2 \rho_a \left\{I_0(\nu) + 2eI_1(\nu) + \frac{3}{4}e^2\left[I_0(\nu) + I_2(\nu)\right]\right\},$$

$$\dot{\bar{e}} = -\left(\frac{A}{m}\right) C_d n a(1-e^2) \rho_a \left\{I_1(\nu) + \frac{1}{2}e\left[I_0(\nu) + I_2(\nu)\right] + \frac{1}{8}e^2\left[3I_1(\nu) + I_3(\nu)\right]\right\},$$

where $I_k(\nu)$ are the modified Bessel functions of the first kind; defined by:

$$I_k(\nu) = \frac{1}{2\pi}\int_0^{2\pi} \exp(\nu\cos E)\cos kE\,\mathrm{d}E. \tag{26}$$

In MATLAB, e.g., $I_1(\nu) =$ `besseli(1,nu)` for the specific value of ν. Note that the scale height H in ν is a function of the satellite's instantaneous altitude, which we lose track of in averaging; thus, for our purposes here, we'll take its value at the mean perigee altitude $a(1-e)-R$.

In summary, the following averaged equations of motion for oblateness and drag perturbations must be integrated:

$$\dot{\bar{a}} = -\left(\frac{A}{m}\right) C_d n a^2 \rho_a \left\{ I_0(\nu) + 2eI_1(\nu) + \frac{3}{4}e^2\left[I_0(\nu)+I_2(\nu)\right]\right\}, \tag{27}$$

$$\dot{\bar{e}} = -\left(\frac{A}{m}\right) C_d n a(1-e^2)\rho_a \left\{ I_1(\nu) + \frac{1}{2}e\left[I_0(\nu)+I_2(\nu)\right] + \frac{1}{8}e^2\left[3I_1(\nu)+I_3(\nu)\right]\right\}, \tag{28}$$

$$\dot{\bar{\Omega}} = -\frac{3J_2R^2n}{2p^2}\cos i, \tag{29}$$

$$\dot{\bar{\omega}} = \frac{3J_2R^2n}{4p^2}(5\cos^2 i - 1). \tag{30}$$

To numerically integrate this set of four first-order ODEs, introduce a vector containing four auxiliary variables $\boldsymbol{y} = \begin{bmatrix} y_1, & y_2, & y_3, & y_4 \end{bmatrix}^T$, such that

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} a \\ e \\ \Omega \\ \omega \end{bmatrix}. \tag{31}$$

This implies that

$$\dot{\boldsymbol{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} \dot{\bar{a}}(y_1,y_2) \\ \dot{\bar{e}}(y_1,y_2) \\ \dot{\bar{\Omega}}(y_1,y_2) \\ \dot{\bar{\omega}}(y_1,y_2) \end{bmatrix}. \tag{32}$$

Note that $A, m, C_d, \mu, J_2, R, i$ are all constant parameters in the above equations, and that $n = \sqrt{\mu/a^3}$, $p = a(1-e^2)$, $\rho_a = \rho(a-R)$, and ν (being a function of H, evaluated at altitude $a(1-e)-R$ are functions of y_1 and y_2.

Graduate Students . Graduate students must use both propagation methods and comment on the differences observed. Use the same initial state in your osculating and mean propagators.

Extra Credit (10 points) . Undergraduate teams who complete the "Graduate Student" task can receive up to 10 points of extra credit.

Important Note . You must submit your codes and slides in a single zip file (only one submission per group). All codes must be portable, must contain a main function that provides the inputs and calls the various subroutines. If they do not run properly on our machines, your technical scores will be docked 20% (that is, you are responsible for checking dependencies, paths, etc., to ensure portability). Finally, your slides must be converted into a single PDF file (PowerPoint or other formats will not be accepted). Your code does not need to be appended to the PDF.