Regularized Dynamics in the Circular Restricted Three Body Problem

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Abstract

Numerical integration of complex dynamical systems is a nontrivial exercise, and presents numerous difficulties to the aspiring analyst. The analyst must wrestle not only with the difficulties of the physical phenomena arising in the system, but also the numerical difficulties of balancing truncation/roundoff error in the integrator and quantization effects due to floating point represenations in the dynamics functions.

The problem of three bodies under mutual gravitational attraction is one such system. In this paper, we juxtapose the traditional Cowell formulation (directly integrating Newton's equations), and a regularized formulation due to Kustaanheimo and Steifel; investigating their numerical performance and efficiency in treating the circular restricted Earth-Sun-Spacecraft problem. We show the performance of both formulations for multiple spacecraft trajectories in this configuration, and compare/contrast performance for these different scenarios.

1 Introduction

Analytical solutions of the equations of motion for celestial bodies have been sought for a number of specific problems and general cases for centuries. The combined work of the giants in this field are an astrodynamicists jewels, for the solutions we have to various cases of the two, three, and many body problems give significant insight into the behavior of orbits in real situations. We draw from these insights whenever we tackle a new problem, as guiding principles to lead us on to new solutions.

However, analytical solutions are incredibly difficult to come by, in all but the simplest of problems. Thus, we most often have to turn to our computers to look for answers to the more interesting questions in astrodynamics. In this paper, we examine the work of Davide Amato, Guilio Bau, and Claudio Bombardelli in Accurate orbit propagation in the presence of planetary close encounters [1]. The topic of planetary close encounters in numerically determining a satellites trajectory is an interesting problem, one fraught with difficulties as the physics the spacecraft encounters change significantly throughout different portions of its orbit. In using the techniques of regularization in the manner of Kustaanheimo and Steifel [3], we show that the overall computational burden of the task can be reduced, without sacrificing accuracy in the solutions.

2 Dynamics, Singularities and Regularization

2.1 The Cowell Formulation

To set about the task of numerically integrating our system for the restricted three body problem, we first require the equations of motion of said system. The most common method of writing these equations is via Cowell's formulation; this involves simply summing the respective mutual attractions between the N bodies in the system, as shown below in 1.

$$\ddot{\vec{r}}_i = \sum_{j=1}^{N} \frac{Gm_j\left(\vec{r}_j - \vec{r}_i\right)}{r_{ij}^3}, i \neq j \tag{1}$$

Here, $\ddot{\vec{r}}_i$ is the acceleration of the ith body in an inertial frame, m_j is the mass of the jth body, and r_{ij} is the magnitude of the relative distance between the ith and jth bodies. The development of this formulation is discussed in detail in Roy [2].

It is immediately apparent, upon inspection, that these equations contain a few singularities. There are precisely two in this system: one when bodies 1 and 2 collide, and another when bodies 2 and 3 do the same. This creates difficulties in the numerical solution of the system under certain conditions, such as close encounters. An adaptive time-step solver, such as the Runge-Kutta-Fehlberg methods, would be forced into exceedingly small steps during such a close encounter in order to achieve the specified accuracy tolerance, leading to undesirably large numbers of steps and function evaluations to solve the system. This strategy is thus seen, at least at face value, to be less than desirable for some applications.

2.1.1 Hamiltonian of the Sun-Earth-Spacecraft System

We start treatment of the dynamics with the Lagrangian. Ignoring the suns dynamics, as its the center of our chosen coordinate system, we take the kinetic and potential and kinetic energies of our two remaining bodies as:

$$T_{SC} = \frac{1}{2} m_{SC} v_{SC}^2 \tag{2}$$

$$T_E = \frac{1}{2} m_E v_E^2 \tag{3}$$

$$U_{SC} = -\frac{\mu_{Sun}}{\vec{r}_{SC}} - \frac{\mu_E}{(\vec{r}_E - \vec{r}_{SC})} \tag{4}$$

$$U_E = -\frac{\mu_{Sun}}{\vec{r}_E} \tag{5}$$

This leads us to the corresponding Lagrangian:

$$L = \frac{1}{2} m_{SC} v_{SC}^2 + \frac{1}{2} m_E v_E^2 + \frac{\mu_{Sun}}{\vec{r}_{SC}} + \frac{\mu_E}{(\vec{r}_E - \vec{r}_{SC})} + \frac{\mu_{Sun}}{\vec{r}_E} \tag{6}$$

After dropping the final term in the second EOM (an approximation due to the significant differences in magnitude between the accelerations caused on the Earth by the sun and the spacecraft), we arrive at the Hamiltonian of the system via Hamiltons equations:

$$H = \sum \dot{q}_i p_i - L \tag{7}$$

$$H = \frac{1}{2} m_{SC} v_{SC}^2 + \frac{1}{2} m_E v_E^2 - \frac{\mu_{Sun}}{\vec{r}_{SC}} - \frac{\mu_E}{(\vec{r}_E - \vec{r}_{SC})} - \frac{\mu_{Sun}}{\vec{r}_E} \tag{8}$$

Since the Hamiltonian is autonomous and involves no high order functions of the generalized coordinates or momenta, we expect this to be an integral of the motion.

2.2 Regularization of the Equations of Motion

Next, we discuss a method to remove such singularities from the equations of motion, without loss of generality/applicability to the system. There are other methods of doing so, but here we focus on the method originally due to Kustaanheimo, and expounded in the monograph by Stiefel and Scheifele [3]. The full derivation can be found in said monograph. Here, we discuss briefly the salient points of the formulation before presenting it without derivation.

The Kustaanheimo-Stiefel regularized equations of motion are achieved via a two step transformation process. The first is a first order Sundman transformation to create a fictitious time as the independent variable:

$$dt = rds \tag{9}$$

We use this relation to rewrite the equations of motion for the four dependent variables t, x, y, z as functions of the new independent variable, s, which corresponds to the eccentric anomaly. We note here that r is the magnitude of the position vector.

The second step in the regularization process is what is aptly called the K-S tranform. By forming the Levi-Civita matrix with the four dimensional parameter vector $\vec{u}$:

$$L(\vec{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \tag{10}$$

we can transform between the four dimensional parameter space $\vec{u}$ and the three dimensional position space $\vec{r}$ via:

$$\vec{r} = L(\vec{u})\vec{u} \tag{11}$$

Taking this transformation to the equations of motion, and using the relation of the system energy to the parameter vector, we can then write the equations of motion of the system as follows.

$$\vec{u}'' + \frac{h_k}{2}\vec{u} = \frac{\mid \vec{u} \mid^2}{2}\left(-\frac{1}{2}\frac{\partial V}{\partial \vec{u}} + L^T\vec{P}\right) \tag{12}$$

$$h_k' = \left(\frac{\partial V}{\partial \vec{u}}, \vec{u}'\right) - 2\left(\vec{u}', L^T\vec{P}\right) \tag{13}$$

$$t' = (\vec{u}, \vec{u}) \tag{14}$$

Noting here that h_k is the Keplerian orbital energy, V the disturbing potential, $\vec{P}$ the non-potential disturbing force, and that prime notation refers to differentiation with respect to the fictitious time, s, instead of actual time, t. It is worth noting that in this formulation, when the disturbing potential and forces are zero (as in the case of Keplerian motion), this system reduces to a simple harmonic oscillator.

These regularized equations of motion have increased in total order (from order 6 to order 10). However, as we will show, the increase in order is a small price to pay for the increase in numerical efficiency achieved by removing the singularity at the origin.

2.2.1 Transforming Initial Conditions

As a consequence of the extra dimension of $\vec{u}$, there are infinitely many $\vec{u}$ corresponding to any unique $\vec{r}$. As such, we are free to choose any $\vec{u}$ such that equation 11 holds, and the magnitude satisfies the relation:

$$\mid \vec{r} \mid = \mid \vec{u} \mid^2 \tag{15}$$

Once we have a suitable $\vec{u}$, we then transform the initial velocity of the bodies:

$$\vec{u'}(0) = \frac{1}{2 \mid \vec{u}(0) \mid^2} L^T \left(\vec{u}(0)\right) \vec{r'}(0) \tag{16}$$

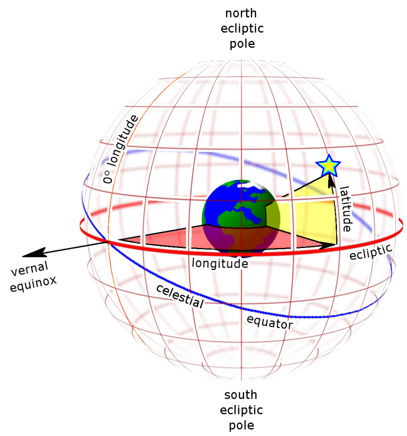
2.3 Switching Primary Bodies

One important point in dealing with the regularized equations of motion 12-14 is that not *all* of the singularities present in a three body system have been removed. We have indeed removed the singularity at the center of the primary body, but our disturbing potential will still have a singularity when the second and third bodies are very close together.

A technique to avoid the same numerical difficulties mentioned with the Cowell formulation in this case is to employ a change of coordinates in the integrating system. At a certain distance threshold between the second and third bodies, we can change the equations of motion to consider the second body as the primary, and the first body as the disturber. Provided the first and second bodies are sufficiently distant, as will be the case for the specific Sun-Earth-Spacecraft scenario we will investigate here, the new equations of motion will no longer be near the singularity in the disturbing potential. The larger consequence of this technique is that the singularity produced by the proximity of the second and third bodies is no longer of importance, since the regularized equations of motion are not singular at the origin.

This technique does require a bit more work, however. Firstly, it is common to specify the "certain distance threshold" in the real coordinates $\vec{r}$ and/or the real time t. However, we are integrating in the parameter space, $\vec{u}$, and fictitious time, s. Thus, some sort of event detection function will be

Figure 1: The Heliocentric-Ecliptic Coordinate System



necessary to determine at what point during the integration the equations of motion should be switched. The techniques developed in [1] will also be employed here for this purpose.

3 Numerical Analysis

In order to investigate the effectiveness of either dynamical formulation in a close encounter situation, we consider a case of the circular, restricted three body problem. Specifically, we'll investigate the Sun-Earth-Spacecraft system, where the infinitesimally small spacecraft will perform a parabolic fly-by of Earth, as might happen in a mission involving a gravity assist to reach the outer planets of the solar system.

To start, we investigate the dynamics in the Cowell formulation, where numerical integration is performed directly on the Cartesian position and velocity vectors of the bodies. The coordinate frame used here, shown in figure 1, is the Heliocentric-Ecliptic: origin at the suns center, primary axis along the vernal equinox, fundamental plane the Earths ecliptic. Additionally, we take the coordinates to be along the three orthogonal axes (rectilinear rather than spherical), and the units of distance to be Earth equatorial radii. This was chosen as a compromise on dynamic range/numerical precision between the Earths orbit and the spacecrafts. Units of time were chosen as seconds, although this was for convenience rather than performance.

In both solutions, the Earth and Sun are considered so massive in comparison to the spacecraft that we neglect its effect on them. Additionally, we also assume that the Earth's orbital eccentricity to be sufficiently low for the orbit to be described by a circle, with angular velocity equal to the mean motion. This reduces the number of equations in the Cowell formulation to 6, with the K-S formulation retaining 10.

3.1 Close Encounters (of the Spacecraft Kind)

For both dynamical formulations, the initial conditions are the same. We take Earth starting at a distance equal to its circular radius along the x axis of the coordinates, with the spacecraft's specific position and velocity relative to the Earth. These can be stated as follows:

$$\vec{r}_{Earth} = r_{Circular}\hat{i} \tag{17}$$

$$\vec{\omega}_{Earth} = \omega_{Circular}\hat{k} \tag{18}$$

$$\vec{r}_{SC} = 6700\hat{i} \tag{19}$$

$$\vec{v}_{SC} = -1\hat{i} + 7\hat{j} \tag{20}$$

where $r_{Circular}$ is equal to roughly 149.6 million kilometers, and $\omega_{Circular}$ is roughly 0.0172 radians/day. The spacecraft position and velocity are specified in kilometers and kilometers/second, respectively. These initial conditions were propagated for a total of 546 days, which was roughly the orbital period of the spacecraft about the sun.

The results of the numerical integration using both schemes are shown in figures 1-4. One difference is immediately apparent: the orbits for the satellite predicted by each formulation are entirely different. The Cowell formulation, shown in figure 2, takes more than a year to exit the Earth's Sphere of Influence, whereas the KS formulation shows the spacecraft exiting the sphere almost immediately, as seen in figure 4.

The Cowell formulation was actually quite difficult to get to converge to a continuous orbit. The time steps necessary for a physically possible orbit were on the order of 30 seconds with the Cowell formulation, which took an overall 4 minutes and 10.8 seconds to complete. The Hamiltonian relative error is shown in figure 3.

For the KS formulation, the story was completely different. The timestep used could be as large as 1 in s (or roughly 1 times the semimajor axis of the Earth). This took a whopping 0.647 seconds to complete. The Hamiltonian relative error is shown in figure 5.

Figure 2: Planar Motion of S/C and Earth in the Cowell formulation

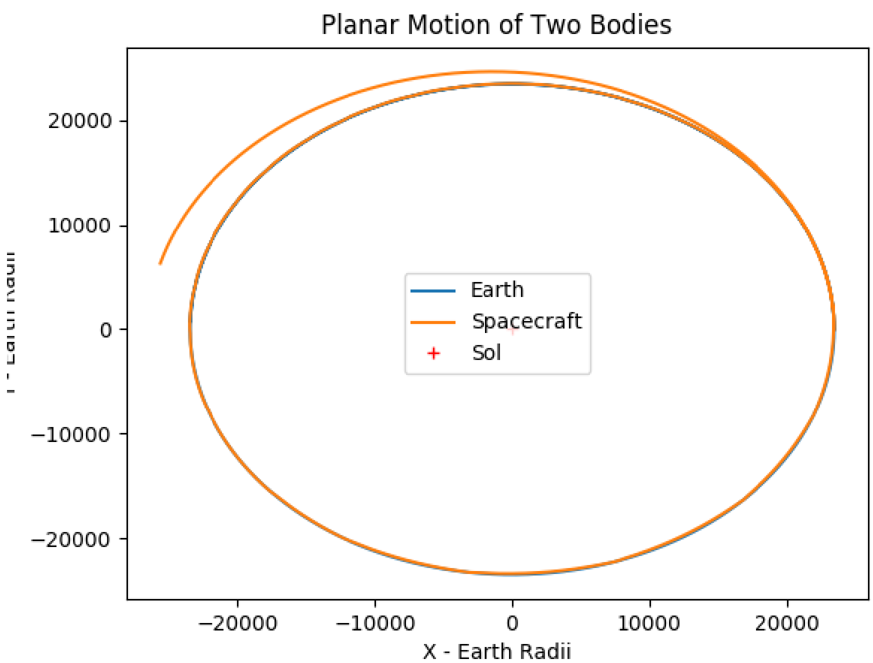


Figure 3: Relative Hamiltonian Error, Cowell

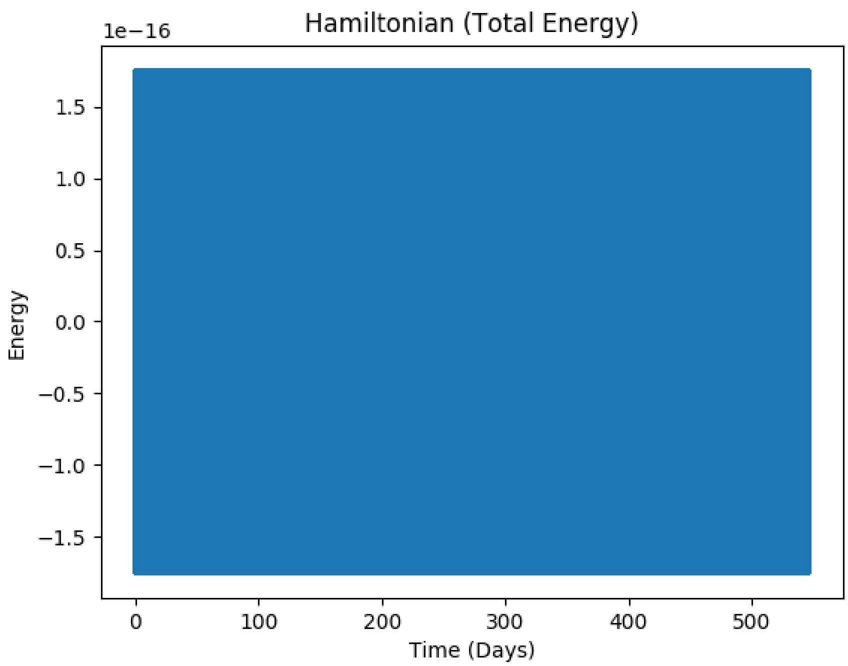


Figure 4: Planar Motion of S/C and Earth in the Kustaanheimo-Steifel formulation

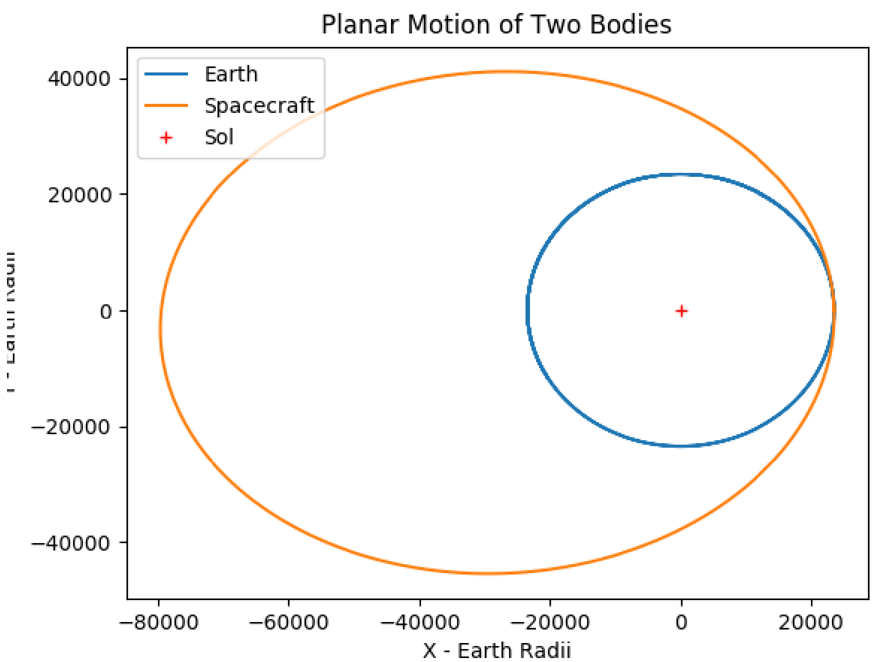
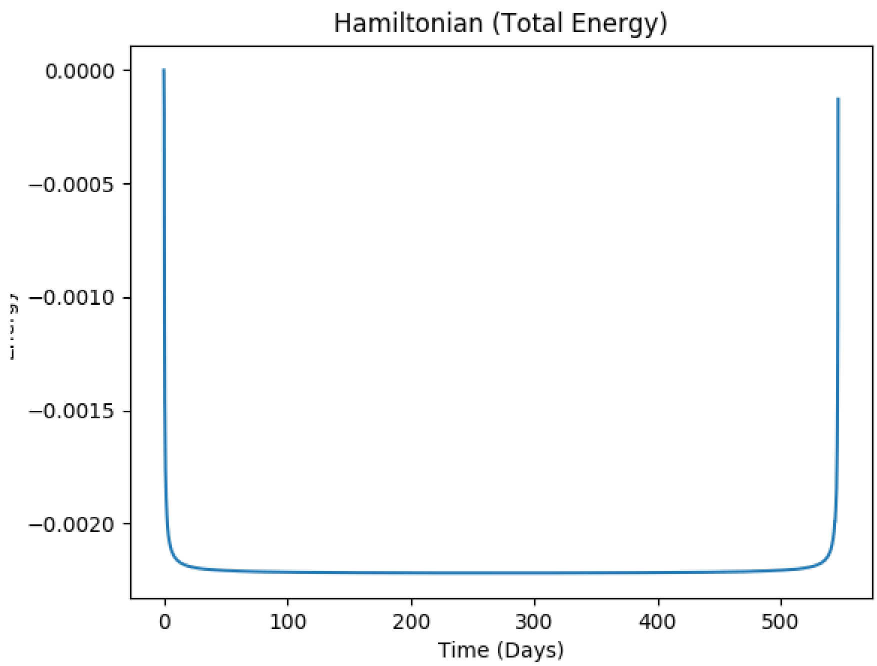


Figure 5: Relative Hamiltonian Error, Kustaanheimo-Steifel



Cowell			Kustaanheimo-Steifel		
dt (s)	RelErr	Run Time	dt (a_earth)	RelErr	Run Time
0.1	8e-5	130.4	0.01	2e-3	61.95
1	8e-4	13.06	1	2e-3	0.647
10	8e-3	0.7	10	8e-3	0.0649

Table 1: Relative error and run time versus time step, Cowell and KS formulations.

Also of note is the interesting form of the relative error for the KS Hamiltonian. While the Cowell Hamiltonian error was oscillatory, the KS Hamiltonian spiked significantly during the close encounters. Since the close encounters (rather, where we switch primaries at the boundary of the Sphere of Influence) are where we expect to see the worst numerical performance due to the singularity in the forcing function, this aligns with expectations. To succinctly summarize the relative accuracies, we show the magnitude of relative errors in the Hamiltonian for both formulations, on various time steps, in table 3.1. The time steps with each formulation were varied to achieve (approximately) the same order of relative error in the Hamiltonian.

4 Investigating the Switching Distance

In order to investigate the performance of the KS formulation further, we consider changing the switching distance for the integrator. In our original formulation, the switching distance was set to

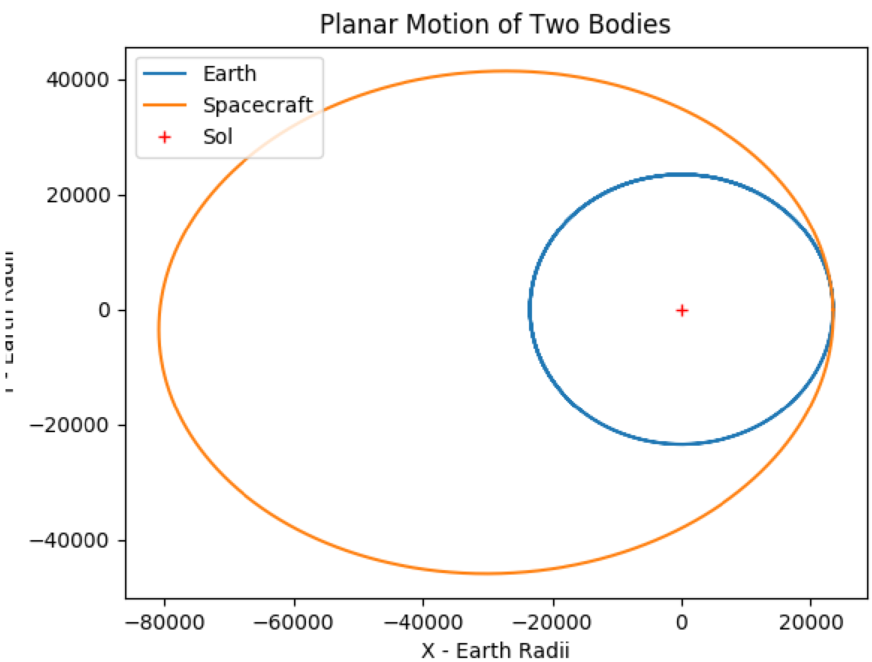
$$\left(\frac{\mu_{Earth}}{\mu_{Sun}}\right)^{\frac{2}{5}} * a_{Earth} \tag{21}$$

To look for improvements, we search for a switching distance which will minimize the spiking seen in the relative error in the Hamiltonian in the original problem. The results are summarized in table 4. We also show the planar path and Hamiltonian relative errors in figures 6 and 7, respectively. Of note here is that the time step, relative and absolute error tolerances to the solver were all left untouched in this variation.

5 Conclusion

Ultimately, we've shown that regularization of the equations of motion can be advantageous in numerical propagation of orbits with close encounters.

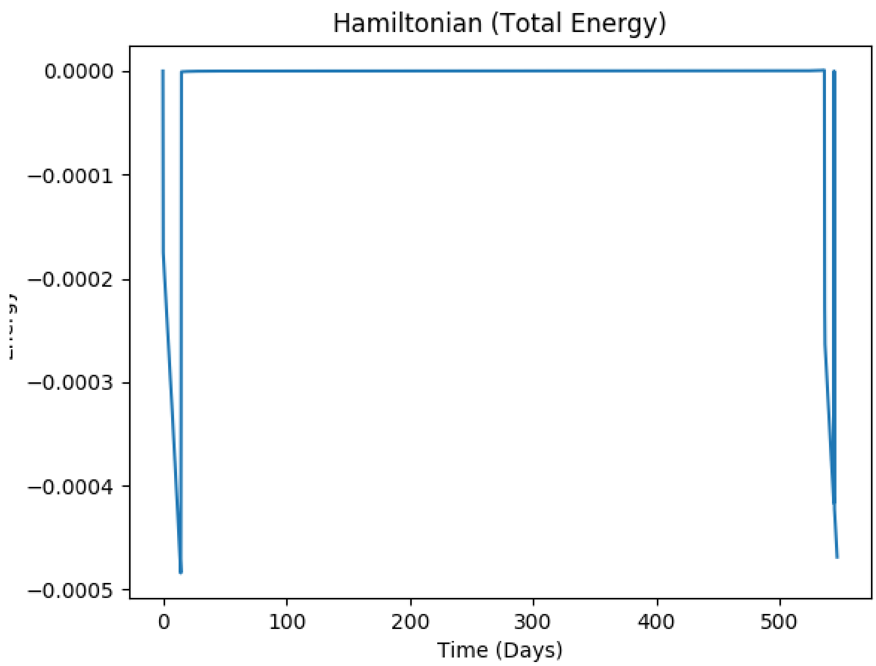
Figure 6: Planar Motion of S/C and Earth



Switching Distance	Relative Error
Original	2.0e-3
10x	5.0e-4
100x	8.0e-4

Table 2: Relative Error in the Hamiltonian for different Primary switching distances.

Figure 7: Relative Hamiltonian Error, 10x Switching Distance



While it does suffer from difficulties near singularities, far away from them it performs admirably, and with far greater efficiency than the Cowell formulation.

While there are difficulties to consider when using a regularized formulation, such as the increased complexity of the equations of motion and the spiking of relative errors during close encounters, these can be mitigated to some extent. In particular, changing the distance from Earth at which we switch the primary bodies, we show that the relative error in the Hamiltonian can be reduced further for a given time step. Especially when considering long time scale problems, or large or highly elliptic orbits, a regularization scheme can significantly increase the numerical efficiency of the analysis, and reduce the computational burden.

References

[1] Davide Amato, Giulio Bau, and Claudio Bombardelli. Accurate orbit propagation in the presence of planetary close encounters. *Monthly Notices of the Royal Astronomical Society*, 470(2):20792099, 2017.

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[3] Eduard L. Stiefel and Gerhard Scheifele. *Linear and regular celestial mechanics: Perturbed Two-body Motion, Numerical Methods, Canonical Theory.* Springer-Verlag, 1971.