Homework Set 1

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1 Problem 1

Let T_1 and T_2 be topologies on a set X. Show that $T_1 \cap T_2$ is also a topology on X. Give an example to show that $T_1 \cup T_2$ need not be a topology on X.

Recall that by definition, T_1 and T_2 both, as topologies, satisfy the following properties:

- Both include $\emptyset$ and X
- Both are closed under arbitrary unions
- Both are closed under finite intersections

Proof. Consider $T_3 = T_1 \cap T_2$. It is immediate that T_3 contains both $\emptyset$ and X, since T_1 and T_2 do.

Next, we consider unions of members of T_3. By definition, for a set $U_\alpha \in T_3$, U_α must also be in T_1 and T_2. Since both constituents are closed under unions, we have that for any $U_3 = \bigcup\limits_{\alpha \in J} U_\alpha$, U_3 is also in T_3 by closedness of T_1 and T_2. Hence, T_3 is closed under arbitrary unions.

Finally, we consider finite intersections. Again, by definition of T_3, we have that for any set $U \in T_3$, $(U \in T_1)$ and $(U \in T_2)$. Thus, for a set $V = \bigcap\limits_{\alpha \in J} U_\alpha$, J finite, $(V \in T_1)$ and $(V \in T_2)$. Hence, $V \in T_3$, and T_3 is closed under finite intersections.

We have shown that T_3 is a collection of sets on X which satisfies the three properties of a topology. Hence, T_3 is a topology on X. $\square$

For an example of how $T_1 \cup T_2$ need not be a topology, consider $\mathbb{R}_{std} \cup \mathbb{R}_l$. We choose the sets $(1, 2)$ and $[3, 4)$, which are in $\mathbb{R}_{std}$ and $\mathbb{R}_l$, respectively. However, their union $(1, 2) \cup [3, 4)$ is not in T_3, since it is the union of the collections of sets, not the sets themselves. Thus, T_3 isn't closed under unions, and cannot be a topology.

2 Problem 2

Show that the topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Proof. First, we recall the definition of $\mathbb{R}_K$: the topology generated by the basis of sets $B = \{(a, b) - K \mid a, b \in \mathbb{R}\}$, where $K = \{\frac{1}{n} | \ n \in \mathbb{N}\}$.

Given an element of the basis of $\mathbb{R}_l$, say $b_l = [a, b)$. Then, choose $x = a$. Clearly, there is no open set in the topology generated by B_K that both contains x and is a subset of $[a, b)$.

Similarly, take $b_K = (-1, 1) - K$, and let $x = 0$. Then, there is no element of B_l that both contains x and lies within b_K, since there is a point $y = 1/n, n \in \mathbb{N}$ that lies arbitrarily close to x which is not in b_K.

Thus, $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable. □

3 Problem 3

3.1 Part a

Apply lemma 13.2 to show that the countable collection $B = \{(a, b) | \ a < b, \ a, b \in \mathbb{Q}\}$ is a basis that generates $\mathbb{R}_{std}$.

Lemma 13.2: Let X be a topological space. Suppose that C is a collection of open sets, such that $\forall U \in X$ and $\forall x \in U$, $\exists c \in C$ such that $x \in C \subseteq U$. Then C is a basis for the topology in X.

Proof. Define $B_{\mathbb{Q}} = \{(a, b) | \ a < b, a, b \in \mathbb{Q}\}$.

Let U be in the topology generated by $B_{\mathbb{Q}}$. Then, U is some combination of finite unions and arbitrary intersections of elements of $B_{\mathbb{Q}}$, which are all also open sets in $\mathbb{R}_{std}$. Thus, $U \in \mathbb{R}_{std}$.

Conversely, let U be in T_{std}, and take $x \in U$. Then, $\exists (a, b) \in \mathbb{R}$ such that $x \in (a, b) \subseteq U$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find c, d such that $a < c < x < d < b$, and thus $x \in (c, d) \subseteq U$. Thus, by lemma 13.2, our collection $B_{\mathbb{Q}}$ is a basis for the standard topology in $\mathbb{R}$. □

3.2 Part b

Show that the collection $C = \{[a, b) | \ a < b, \ a, b \in \mathbb{Q}\}$ is a basis that generates a topology different from $\mathbb{R}_l$.

Proof. Clearly, C is a basis for a topology in $\mathbb{R}$, again by lemma 13.2. However, it cannot generate $[i, j), i, j \notin \mathbb{Q}$. $[i, j) \notin C$, since C contains only intervals with rational endpoints. Further, we cannot construct $[i, j)$ as

$[a, b) \cap [c, d)$, since $[a, b) \cap [c, d) = [\max(a, c), \min(b, d))$, and $a, b, c, d \in \mathbb{Q}$. Finally, we note that we also cannot construct $[i, j)$ as a union of intervals $[a_i, b_i), a_i, b_i \in \mathbb{Q}$, for the same reason as stated above. Since a topology must be closed under arbitrary unions and finite intersections, and we cannot construct $[i, j), i, j \notin \mathbb{Q}$ from C, the topology generated by C cannot be the lower limit topology on $\mathbb{R}$. $\square$

4 Problem 4

Show that if A is a basis for a topology on a set X, then the topology generated by A equals the intersection of all topologies on X that contain A. Prove the same if A is a subbasis.

Lemma 13.1: The topology generated by A is the collection of all unions of all elements of A.

Proof. Let the intersection of all topologies containing A be $T_x = \bigcap_{\alpha \in J} T_\alpha$. Since they contain A and are topologies, by property 2 of a topology, any T_α must be closed under arbitrary unions. Thus, $T_A = \bigcup_{\alpha \in J} U_\alpha$, the topology generated by A (by lemma 13.1), is in any of the above T_α. Therefore, $T_A \subseteq T_x$.

Conversely, T_x is a topology, and closed under unions. Since $T_A \subseteq T_x$, then there is no set $U \in T_x$ for which $U \notin T_A$. Assume there is. By our construction of T_x, this requires U to be in every T_α on X. However, T_A is a topology on X which contains A, and thus must be one of the T_α. Therefore, $U \in T_A$, a contradiction. Hence, $\forall U \in T_x, U \in T_A$, and $T_x \subseteq T_A$. Therefore, $T_x = T_A$. $\square$

Next, we prove that the topology generated by a subbasis A equals the intersection of all topologies on X that contain A.

Definition: A *subbasis* S for a topology on X is a collection of subsets of X whose union equals X. The *topology generated by* S is defined to be the collection T of all unions of finite intersections of elements of S.

Proof. Any topology that contains A must also contain the topology generated by A, since any topology must be closed under arbitrary unions and finite intersections (properties 2 and 3 of topologies). Thus, the intersection of all topologies on X containing A must also include the topology generated by A. That the two are equal then follows from the subsetting arguments above. $\square$