# Math 532 Notes

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## Contents



## 1 Continuous Functions <*2020-02-10 Mon 09:06*>

**Definition 1.1.** *Let $X, Y$ be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for every open $V \subseteq Y$ the preimage $f^{-1}(V)$ is open in $X$.*

Note: If topology is given by a basis, it suffices to show above condition for the basis. Same goes for a subbasis.

The above definition of continuity is equivalent to the $\epsilon - \delta$ definition of continuity in $\mathbb{R}^n$.

### 1.1 Examples

The identity function $id : \mathbb{R} \to \mathbb{R}_l$ is not continuous, but $id : \mathbb{R}_l \to \mathbb{R}$ is, due to the inclusion of open sets of $\mathbb{R}$ in $\mathbb{R}_l$, but not the other way around.

## 2 Exam Notes

Expect 5-6 problems on day one, 1 extra for grad students on day two. Still get a chance to rework one of the problems the second day.

A couple problems will be on definitions/theorems, and the rest on problems like the homework.

Probably a good idea to work through some definition/theorem flashcarding or the like this week, brush up.

## 3 More on continuous functions

**Theorem 3.1.** *Let $X$ and $Y$ be topological spaces. The following are equivalent:*

- $f : X \to Y$ *is continuous*
- $\forall A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$
- $\forall B \subseteq Y$, $B$ *closed,* $f^{-1}(B)$ *is closed*
- $\forall x \in X$, *and for every neighborhood $V$ of $f(x)$, $\exists U$ (neighborhood of $x$) such that $f(U) \subseteq V$.*

For proof, we'll show 1 -> 2 -> 3 -> 1 = 4

*Proof.* 1 implies 2: Assume $f$ is continuous, and $A \subseteq X$. Let $x \in \overline{A}$. Claim that $f(x) \in f(\overline{A})$. Recall that a point is in the closure of a set if every neighborhood of the point intersects the set. Let $V$ be a neighborhood of $f(x)$. Then $x \in f^{-1}(V)$. But $x \in \overline{A}$, so every neighborhood of $x$ intersects $A$ at some point. Let $y \in f^{-1}(V) \cap A$. Then, $f(y) \in V \cap f(A)$. So $V \cap f(A) \neq \emptyset$, and thus $f(x) \in \overline{f(A)}$.

2 implies 3: Assume that $f(\overline{A}) \subseteq \overline{f(A)}$. Let $B \subseteq Y$ be closed. Claim that $A = f^{-1}(B)$ is closed in $X$, i.e. claim $\overline{A} = A$. By set theory we have $f(A) = f(f^{-1}(B)) \subseteq B$. Let $x \in \overline{A}$. Then $f(x) \in f(overline A)$. By the above, $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$. So $x \in f^1(B) = A$. Thus, we've shown that $\overline{A} \subseteq A$, and we know that $A \subseteq \overline{A}$, thus $A = \overline{A}$, and $A$ is closed.

3 implies 1: Assume that $\forall B \subseteq Y$ closed, $A = f^{-1}(B)$ is also closed. Let $U \subseteq Y$ be open. Then, claim that $f^{-1}(U)$ is open in $X$. Let $V = Y - U$ be the complement of $U$. $V$ is closed, so by assumption, $T = f^{-1}(V)$ is also closed. But, $T = f^{-1}(Y) - f^{-1}(U) = X - f^{-1}(U)$. Thus, $C = f^{-1}(U)$ is open, since its complement is closed. $f$ is continuous.

4 equals 1: Both ways

1 implies 4: Let $f$ be continuous. Let $x \in X, V_{f(x)} \subseteq Y$. Claim that $\exists U_x$ such that $f(U_x) \subseteq V$. To see this, take $U_x = f^{-1}(V_{f(x)})$, which is an open neighborhood of $f(x)$. Then, $f(U) \subseteq V$.

4 implies 1: Assume that $\forall x \in X$ and $\forall V_{f(x)} \subseteq Y$, $\exists U_x$ such that $f(U) \subseteq V$. Claim that $f$ is continuous, i.e. that $f^{-1}(V)$ is open, for all open $V$. Let $V \subseteq Y$ be open. Let $f(x) \in V$. Let $V_x$ be an open neighborhood of $f(x)$ such that $V_x \subseteq V$. By our assumption, $\exists U_x \subseteq X$ open for each $V_x$, and $f(U_x) \subseteq V_x$. Then, let $U = \bigcup U_x$. This $U$ is still open, and it is the pre-image of $V$ under $f$. $f$ is continuous. $\square$

# 4 Homeomorphisms *<2020-02-12 Wed 09:13>*

**Definition 4.1.** *Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be bijective (1-1 and onto). Then $f$ is called a homeomorphism if and only if both $f$ and the inverse function $f^{-1} : Y \to X$ are continuous.*

NOTE: In homeworks and exam, don't need to prove that a function is continuous.

Equivalently, a bijection $f : X \to Y$ is a homeomorphism if $f(U)$ is open iff $U$ is open.

Note: A homeomorphism gives a bijection between both the spaces $X$ and $Y$ and their topologies.

**Definition 4.2.** *A property of a space expressed solely in terms of the topology on the space is called a **topological property**.*

**Definition 4.3.** *If $f : X \to Y$ is an injective continuous map, and we consider $Z = f(X)$ as a subspace of $Y$, then if $f' : X \to X$ is defined as the restriction of $f$'s range to $f(X) = Z$, then $f'$ is a homeomorphism, and we call $f$ a **topological embedding**.*

## 4.1 Examples

$f : \mathbb{R} \to \mathbb{R}, f(x) = 3x + 1$ is a homeomorphism. $g(x) = \frac{1}{3} * (x - 1)$ is its inverse.

A bijective function can be continuous and not be a homeomorphism. Consider $S^1$ (the unit circle) in the subspace topology. Then, consider $F : [0, 1) \to S^1, F(t) = (cos 2\pi t, sin 2\pi t)$. Notice $[0, 1/4)$ is open in $[0, 1)$, but $F([0, 1/4))$ is not open in $S^1$.

$F : (-1, 1) \to \mathbb{R}, F(x) = \frac{x}{1-x^2}$. Then, $G(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$. This bijection is order preserving, so it's a homeomorphism.

### 4.2 Back to homeomorphism theorems

**Theorem 4.1.** *(Rules for constructing continuous functions):*

1. *[The constant function] If $f : X \to Y$ maps all of $X$ to a single point $y_0 \in Y$, then $f$ is continuous.*
2. *[Inclusion] If $A$ is a subspace of $X$ the inclusion $j : A \to X$ is continuous.*
3. *[Composites] If $f : X \to Y$, $g : Y \to Z$ are continuous, then $g \circ f$ is continuous.*
4. *[Restricting Domain] If $f : X \to Y$ is continuous, and $A$ a subspace of $X$, then $f|A : A \to Y$ is continuous.*
5. *[Restricting or Expanding the Codomain] of a continuous function gives a continuous function.*
6. *[Local Formulation of Continuity] The map $f : X \to Y$ is continuous if $X$ can be written as a union of sets $U_\alpha$ such that $f|U_\alpha$ is continuous for each $\alpha$.*

*Proof.* (Just of part 6): Let $U \subseteq Y$ be open. Then, $f^{-1}(U) = \bigcup f|U_\alpha^{-1}(U)$, and each $(f|U_\alpha(U))$ is open. □

**Lemma 4.1.** *(The Pasting Lemma) Let $X = A \cup B$, where $A$ and $B$ are closed in $X$. Let $f : A \to Y$, and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$, the combination of the two, is continuous.*

*Proof.* Let $C \subseteq Y$ be closed. Then, $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. $f^{-1}(C)$ is closed in $A$, same for $g$ and $B$. But, $A$ and $B$ are both closed in $X$, so the preimages of $C$ are also closed in $X$ (what theorem is this?). So, $f^{-1}(C) \cup g^{-1}(C)$ is closed in $X$, and $h$ is continuous. □

# 5 More on Maps *<2020-02-14 Fri 08:58>*

**Three weeks until spring break. Starts 3/9/2020.**

**Theorem 5.1.** *(Maps into Products) Let $f : A \to X \times Y$ be given by $ f(a) = (f_1(a), f_2(a))$. Then $f$ is continuous iff the coordinate functions $f_1 : A \to X, f_2 : A \to Y$ are continuous.*

*Proof.* First observe that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous. E.g. $U \subseteq X$ is open, then $\pi_1^{-1}(U) = U \times Y$ which is open. Note that for each $a \in A$, $f_1(a) = \pi_1(f(a)), f_2(a) = \pi_2(f(a))$. So if $f$ is continuous, then $f_1$ and $f_2$ are compositions of continuous functions, so they're continuous.

Now, in the opposite direction, suppose that $f_1$ and $f_2$ are continuous. Consider $U \times V$ open in $X \times Y$. The preimage is $f^{-1}(U \times V) = \{a \in A \mid (f_1(a), f_2(a)) \in U \times V\}$. But $f_1^{-1}(U) = \{a \in A \mid f_1(a) \in U\}$, and $f_2^{-1}(U) = \{a \in A \mid f_2(a) \in V\}$. Thus, $f^{-1}(U \times V) = \{a \in A \mid (f_1(a) \in U)\text{and}(f_2(a) \in V)\} = f_1^{-1}(U) \cap f_2^{-1}(V)$, which is the intersection of two open sets (by assumption). Thus, the preimage under $f$ is open, and therefore $f$ is continuous. ☐

# 6 The Product Topology in Detail *<2020-02-14 Fri 09:11>*

Section 19 in Munkres.

Consider the Cartesian product $X \times \ldots \times X_n$, $n$ finite, and $X_1 \times X_2 \times \ldots$, where each $X_i$ is a topological space. Two possible topologies:

1. Basis is all sets of the form $U_1 \times \ldots \times U_n$ in finite case, or $U_1 \times U_2 \times \ldots$ in infinite case where each $U_i$ is open in $X_i$. (This is called a "Box Topology").

2. Take as subbasis all sets of the form $\pi_i^{-1}(U)$ where $U$ is open in $X_i$. This looks like the product of complete spaces with $U$ in place of the i-th component, i.e. $X_1 \times X_2 \times \ldots U \times X_{i+1} \ldots$. This gives the product topology.

Note that in the case of a finite product, these two definitions are equivalent. They only get weird and different when we go to infinite products. A basis element in the product topology is a finite intersection of subbasis elements generated as in 2).

E.G $\pi_{i_1}^{-1}(U_1) \cap \pi_{i_2}^{-1}(U_2) \cap \ldots \cap \pi_{i_k}^{-1}(U_k) = X_1 \times \ldots \times U_1 \times X_{i_1+1} \times \ldots \times U_2 \times X_{i_2+1} \ldots$.

So $\vec{x} \in B$ iff $\pi_1(x) \in U_i$ for $i = 1, \ldots k$. (So no restriction on other components of $\vec{x}$. Note: These constructions give same topology for finite products, but differ for infinite products.

Notation: If $J$ is an arbitrary index set, we say a function $x : J \to X$ is a J-tuple. We often write this function as $(x_\alpha)_{\alpha \in J}$. If $\{A_\alpha\}_{\alpha \in J}$ is an indexed family of sets, then the cartesian product of this indexed family denoted as $\Pi_{\alpha \in J} A_\alpha$ is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of $X$ such that $x_\alpha \in A_\alpha$, $\forall \alpha \in J$. (So its just $x : J \to \bigcup A_\alpha$ such that $x(\alpha) \in A_\alpha, \forall \alpha$).

**Definition 6.1.** *$\pi_\beta : \Pi_{\alpha \in J} X_\alpha \to X_\beta$ is called the projection mapping with index $\beta$: $\pi_\beta(\vec{x}) = \vec{x}(\beta) = x_\beta$.*

**Definition 6.2.** *Let $S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta open\ in X_\beta\}$. Let $SB = \bigcup_{\beta \in J} S_\beta$. The topology generated by this subbasis is called the product topology. $\Pi_{\beta \in J} X_\beta$ with this topology is called a product space.*

A typical basis element is of the form $B = \pi_{\beta_1}^{-1}(U_1) \cap \ldots \cap \pi_{\beta_n}^{-1}(U_n)$ where $\beta_1, \beta_2, \ldots \beta_n \in J$, and each $U_i$ is open in $X_{\beta_i}$. We will often abbreviate this as $B = \Pi_{\beta \in J} B_\beta$ where $B_\beta = X_\beta$ if $\beta \neq \beta_1, \ldots \beta_n, B_\beta = U_i$ if $\beta = \beta_i$.

Is the box topology finer than the product topology, or the other way around? Being open in the product topology means being open in the box topology, but not the other way around. This means the box topology is generally finer than the product topology.

**Theorem 6.1.** *Suppose the topology on each space $X_\alpha$ is given by a basis $B_\alpha$. The collection of sets of the form $|Pi_{\alpha \in J} b_\alpha$, where $b_\alpha \in B_\alpha$ will serve as a basis for the box topology. The collection of all sets of the same form, where $b_\alpha \in B_\alpha$ for finitely many $\alpha$, and $B_\alpha = X_\alpha$ for the remaining indices, is a basis for the product topology.*