Homework Set 5

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1 Problem 1

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

2 Problem 2

Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_{1\text{x}1} + b_1, a_{2\text{x}2} + b + 2, \ldots). \tag{1}$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ and itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Proof. Let $\mathbb{R}^\omega$ have the product topology, and h be as defined aboved. If we know the sequences of a_i and b_i, we can define a function $h^{-1}((x_1, x_2, \ldots)) = ((x_1 - b_1)/a_1, (x_2 - b_2)/a_2, \ldots)$. Since all a_i are assumed to be greater than zero, and thus nonzero, h^{-1} is obviously continuous. Since h is similarly continuous, we have that it is a bijection with a continuous inverse between $\mathbb{R}^\omega$ and itself, i.e. a homeomorphism. $\square$

If we instead give $\mathbb{R}^\omega$ the box topology, the result is unchaged.

3 Problem 3

Prove that if (X, d) is a metric space and X has the topology induced by d, then $d : X \times X \to [0, \infty)$ is continuous, where $X \times X$ has the product topology.

Proof. Let X be a metric space, with d a metric, and let $X \times X$ have the product topology. Call the interval $A = [0, \infty)$, and let $U \subseteq A$ be an open subset. Then, $d^{-1}(U)$ is of the form $\bigcup\limits_{x \in X} \{x\} \times V_x$, where $V_x \subseteq X$ is a an open subset of X whose points are less than $a \in U$ (and greater than 0) in distance from the point x. The union of each of these V_x is obviously X, as is the union of the $\{x\}$. $X \times X$ is open in the product topology. We have considered all of the open sets in A, as the point $d = 0$ cannot be in an open set (I've assumed the standard topology in $\mathbb{R}$). d is continuous. $\square$

4 Problem 4

Show that $\mathbb{R}_l$ and the ordered square satisfy the first countability axiom. (This does not, of course, imply that they are metrizable).

Proof. Let $x \in \mathbb{R}_l$. Then, we construct a countable collection of neighborhoods of x as $V_n = [x - 1/n, x + 1/n)$, $\forall n \in \mathbb{N}$. It is clear from the axiom of completeness (eq. the nested interval principle, the continuum hypothesis, etc) that for any neighborhood U of x we choose, we can always find an $n \in \mathbb{N}$ such that $\{V_i\}_{i \geq n}$ are all contained within U. An almost identical argument holds for the ordered square. $\square$

5 Problem 5

Show that the axiom of choice is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq \emptyset$, the Cartesian product $\prod\limits_{\alpha \in J} A_\alpha$ is not empty.

Proof. First, we define the axiom of choice as the statement that "For every indexed family $\text{\$}\backslash\mathrm{A}_\alpha\backslash_{\alpha \in \mathrm{J}}$ of nonempty sets, there exists an indexed family $(x_\alpha)_{\alpha \in J}$ of elements such that $x_\alpha \in A_\alpha$ for every $\alpha \in J$."

Next, we let J be an index, such that $J \neq \emptyset$. Consider the Cartesian product $\prod\limits_{\alpha \in J} A_\alpha$. By the axiom of choice above, we have that there exists a set of elements $(x_\alpha)_{\alpha \in J}$, such that $x_\alpha \in A_\alpha$ for each α. Thus, the Cartesian product cannot be empty. $\square$

Note: The definition I use above for the axiom of choice can be found at `https://en.wikipedia.org/wiki/Axiom_of_choice`. It was significantly easier to use here than the version presented in Munkres' text.