Math 532 Notes

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1 Bases again <*2020-01-22 Wed*>

An eloquent summary: Bases must span the underlying space, and intersections of elements must contain other elements (i.e. must have sufficent granularity to contain every element of the set).

Open intervals predate topologies.

1.1 Example

Let B be the collection fo all open intervals on $\mathbb{R}$, together with all sets of the form $(a, b) \setminus K$, where $K = \{1/n \mid n \in \mathbb{N}\}$. This is a basis for a topology on $\mathbb{R}$, called the K-topology, and on $\mathbb{R}$ is also written $\mathbb{R}_K$.

1.2 Back to bas(is)ics

Lemma 1.1. *$\mathbb{R}_l$ and $\mathbb{R}_K$ are both finer than $\mathbb{R}_{std}$, but they are not comparable to each other.*

On $\mathbb{R}^2$ the collection of all open discs is a basis for a topology, as are open balls in $\mathbb{R}^n$. Open rectangles (cubes) are also a basis, for the same topology. Proved by inclusion of open sets.

Question 1.1. *Prove that B_{disc} and B_{rect} generate the same topology.*

Lemma 1.2. *Let (X, T) be a topological space. Suppose C is a collection of open sets such that $\forall U \subseteq X$, and each $x \in U$, $\exists c \in C$, such that $x \in c \subseteq U$. Then C is a basis for T.*

Proof. To be a basis, C a) Must cover X, and b) if $x \in c_1 \cap c_2$, then $\exists c_3$ such that $x \in c_3, c_3 \subseteq c_q \cap c_2$.

claim $\forall x \in X, \exists c \in C$ such that $x \in c$. Let $x \in X$. $X is open in T, so by construction of C, $\exists c \in C$ such that $x \in c \subseteq X$.

Next, claim $\forall c_1, c_2 \in C$ and $\forall x \in c_1 \cap c_2$, $\exists c_3 \in C$ such that $x \in c_3 \subseteq c_1 \cap c_2$. Let $x \in c_1 \cap c_2$, $c_1, c_2 \in C$. C is a collection of open sets, i.e. members of T. T is closed under finite intersections, since it is a topology. Thus $c_1 \cap c_2 \in T$. By construction of C, $\exists c_3 \in C$ such that $x \in c_3 \subseteq c_1 \cap c_2$.

Let $U \in T$. $\forall x \in U$, $\exists c_x \in C$ such that $x \in c_x \subseteq U$. Then, $U = \bigcup_{x \in U} c_x$. T' is closed under unions, so $U \in T'$.

Next, let $U \in T'$. Then $U = \bigcup_{\alpha \in J} c_\alpha$. Each c_α is open (in T), by construction. Thus, $U \in T$. Hence, C is a basis, and it generates T. □

1.3 The Dictionary Order Topology

Definition 1.1. *A relation c on a set X is called an* ***order relation*** *if it has the following properties:*

1. *Comparability: $\forall$ x_1, x_2, either $x_1 c x_2$, or $x_2 c x_1$.*

2. *Non-reflexivity: For no $x \in X$ does xcx hold.*

3. *Transitivity: $\forall x, y, z \in X$, if xcy and ycz, then xcz.*