Math 532 Notes

Jake Bailey

April 3, 2020

Contents

1 Compact Subspaces of the Real Line *<2020-03-30 Mon 09:03>*

Theorem 1.1. *The extreme value theorem: Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then $\exists c, d \in X$ such that $f(x) \leq f(x) \leq f(d)$.*

Proof. Assume $f : X \to Y$ be continuous, X compact, Y ordered w/ the order topology. Let $A = f(X)$. The continuous image of a compact space is itself is compact, so A is compact. Claim that A has a largest element M and a smallest element m. I.e. $m, M \in A$, so $\exists c, d \in X$ such that $f(c) = m, f(d) = M$.

If A has no largest element, then we can build an open cover which has no finite subcover: $\{(-\infty, a) \mid a \in A\}$ is an open cover of A. But A is compact, so it has a finite subcovering, say $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$. Let $a = \max\{a_1, \ldots, a_n\}$. This $a \in A$ is not a member of any $(-, a_i)$. This is a contradiction, since these sets were assumed to be a cover. So A has a largest element. The argument for a smallest element is similar. $\square$

Definition 1.1. *Let (X, d) be a metric space. Let $A \subseteq X, A \neq \emptyset$. For $x \in X$ define $d(x, A) = inf\{d(x, y) \mid y \in A\}$ as the distance from x to A.*

Note: for fixed A this is a continuous function of x. Next, we want to find a sort of diameter of containment of the set A. We can do so by finding the supremum of the distance between any two points in the set.

Definition 1.2. *Let (X, d) be a metric space, $A \subseteq X$, A bounded. The diameter of A is: $diamA = sup\{d(x, y) \mid x, y \in A\}$.*

Lemma 1.1. *The Lebesgue Number Lemma: Let A be an open covering of (X, d), a metric space. If X is compact, then $\exists \delta > 0$, called a Lebesgue number, such that every subset of X with diameter less than δ is a subset of some member of A.*

Note: δ is the Lebesgue number of A.

Proof. Let A be an open covering of (X, d), a compact metric space. If $X \in A$, then any positive number will work as a Lebesgue number (and we're done). So, now we assume $X \notin A$.

Choose a finite subcollection of A which will still cover X (since X is compact), say $\{A_1, \ldots, A_n\}$. Each A_i is open, so its complement must be closed. Let $C_i = X \setminus A_i$ be these closed complements.

Define $f : X \to \mathbb{R}$ as $f(x) = 1/n \sum\limits_{i=1}^{n} d(x, C_i)$. Essentially, we're taking the average distance of the closed sets to X. First, we claim that the average distance is not zero. Let $x \in X$, and choose A_i which contains x. Choose $\epsilon > 0$ so that $B_d(x, \epsilon) \subseteq A_i$. Thus, $d(x, C_i) \geq \epsilon$. So \$f(x) $\geq \epsilon$ /n.

Now, we have that f is a continuous map from a compact set to an ordered one, so by the extreme value theorem, it has both a minimum and a maximum. Call the minimum δ, and claim that δ is the Lebesgue number of A.

Let $B \subseteq X$ with diam $B < \delta$. Let $x_o \in B$. Then, $B \subseteq B_d(x_0, \delta)$. Now, $\delta \leq f(x_0) \leq d(x_0, C_m)$ where $d(x_0, C_m)$ is the largest of all the distances from x_0 to each C_i.

So, $B \subseteq B_d(X_0, \delta) \subseteq A_m = X \setminus C_m$. Thus, δ is the Lebesgue number of A. □

Definition 1.3. *A function $f : (X, d_x) \to (Y, d_y)$ between metric spaces is continuous at $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall y \in X, d_x(x_0, y) < \delta \Rightarrow d_y(f(x_0), f(y)) < \epsilon$. This is a pointwise condition, where δ depends on both ϵ and x_0.*

Definition 1.4. *A function $f : (X, d_x) \to (Y, d_y)$ between metric spaces is said to be uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0, \forall x_0, x_1 \in X$ such that $d_x(x_0, x_1) < \delta \Rightarrow d_y(f(x_0), f(x_1)) < \epsilon$.*

Theorem 1.2. *The Uniform Continuity Theorem: If $f : X \to Y$ is a continuous map from a compact metric space to a metric space, then that map is uniformly continuous.*

Proof. Let $\epsilon > 0$, cover y by sets $B(y, \epsilon/2)$ for $y \in Y$. Cover X by $A = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$. Let δ be the Lebesgue number of A. If $x_1, x_2 \in X$, and the $d(x_1, x_2) < \delta$, then $\operatorname{diam}\{x_1, x_2\} < \delta$, and this set is a subset of one of the covering elements in A, i.e. $\{x_1, x_2\} \subseteq f^{-1}(B(y, \epsilon/2))$ for some $y \in Y$. Then, $d_y(f(x_1), f(x_2)) \leq d(f(x_1), y) + d(f(x_2), y) < \epsilon/2 + \epsilon/2 = \epsilon$. □

2 Compact Subsets of the Real Line *<2020-04-01 Wed 08:59>*

Definition 2.1. *$x \in X$ is isolated if $\{x\}$ is open in X.*

Definition 2.2. *A set A is countable if it is finite or countably infinite (i.e. there exists a bijection from the set to the naturals).*

Theorem 2.1. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Step 1: Show that given $U \subseteq X$, open, nonempty, and taking $x \in X$, $\exists V \subseteq U$, nonempty, open, such that $x \notin \overline{V}$. Let $y \in U$, $y \notin x$. If $x \in U$ since x is not isolated, $U \not\{x\}$, so $\exists y \in U, y \neq x$. If $x \not U$, y exists since $U \neq \emptyset$.

Choose W_1, W_2 open such that $x \in W_1$, $y \in W_2$. Then $V = W_2 \cap U$ is the desired open set.

Step 2: Show that a function $f : \mathbb{Z}_+ \to X$ cannot be surjective. Then it follows that X is uncountable. Let $x_n = f(n)$. Apply step 1 with $x_1 = x, U = X_1$ to obtain nonempty open V_1 such that $x_1 \notin \overline{V}_1$. We are going to apply this recursively, choosing $U_{n+1} = V_n$. In general, $V_n \subseteq V_{n-1}, x_n \notin \overline{V}_n$. We have $\overline{V}_1 \overline{V}_2 \ldots$, etc. Thus we have a descending sequence of nonempty closed sets.

Because X is compact, $\{\overline{V}_n \mid n \in \mathbb{N}\}$ is a collection of closed sets with the Finite Intersection Property, and $\exists x \in \cap_n \overline{V}_n$. Then, $\forall n, x \neq x_n$ since $x \in \overline{V}_n$ and $x_n \notin \overline{V}_n$. □

One corollary of this: Every closed interval in $\mathbb{R}$ is uncountable.

3 Limit Point Compactness <*2020-04-01 Wed 09:37*>

Definition 3.1. *A space X is said to be limit point compact if every infinite subset of X has a limit point.*

Theorem 3.1. *Compactness implies limit point compactness.*

Proof. Let X be compact. Suppose $A \subseteq X$ has no limit point (i.e. A is closed, since it contains all (zero) of its limit points). For each $a \in A$, choose open U_a such that $a \in U_a$, and $(U_a \setminus \{a\}) \cap A = \emptyset$. Notice $\{X \setminus A\} \cup \{U_a \mid a \in A\}$ is an open cover of X, so it has a finite subcover (because X is compact). Notice $X \setminus A$ does not intersect A, each U_a contains only one point of A, and there can only be finitely many of them (finite subcover), so A must be finite. Thus, for a subset of X to not have a limit point, it must be finite, and all infinite subsets of X must have a limit point. □

Note that the converse (limit point compactness implies compactness) is not true! Example: Let Y be a two point set in the indiscrete topology. Consider $X = \mathbb{Z}_+ \times Y$. Every nonempty subset of X has a limit point. But, $U_n = \{n\} \times Y$ is an open cover with a finite subcover.

4 More Limit Point Compactness <*2020-04-03 Fri 09:07*>

Definition 4.1. *Let X be a well-ordered. Given $\alpha \in X$, let S_α denote $S_\alpha = \{x \mid x \in X, x < \alpha\}$. This is the section of X by α.*

Lemma 4.1. *There exists a well-ordered set A having largest element Ω so each section S_Ω of A by Ω is uncountable, but evey other section of A is countable.*

Proof. Begin with an uncountable well ordered set B. Let $C = \{1, 2\} \times B$ in the dictionary order topology. Some section of C has to be uncountable. any $1 \times b$ for $b \in B$ is less than any $2 \times b$. A section by $\alpha = 2 \times b$ is uncountable. Let Ω be the least element of X such that the section of C by Ω is uncountable. A is the section of C together with Ω. □

S_Ω is uncountable, well ordered, and every other section is countable, so it is called the minimum uncountable well ordered set. Denote $A = S_\Omega \cup \{\Omega\}$ by $\overline{S}_\Omega$.

Theorem 4.1. *If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.*

Is S_Ω compact? Take S_α for $\alpha \in A$. S_Ω has no largest element, so no finite subcover of this cover. So S_Ω is not compact. S_Ω is limit point compact: Let $A \subseteq S_\Omega$ be infinite. Let $B \subseteq A$ be countably infinite. Let b be an upper bound of B. So $B \subseteq [a_o, b]$ where $a_0 = \min S_\Omega$. Since S_Ω has a least upper bound, $[a_0, b]$ is compact. So B has a limit point in $[a_0, b]$. Then x is also a limit point of A. So S_Ω is limit point compact.

Definition 4.2. *Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence of elements of the topological space X. Let \$$n_1 < n_2 < \ldots < n_i < \ldots$\$ be an increasing sequence of positive integers. Then $(y_i)_{n\in\mathbb{Z}_+}$ defined as $y_i = x_{n_i}$ is a subsequence of (x_n).*

X is sequentially compact if every sequence of points of X has a convergent subsequence.

Theorem 4.2. *Let X be a metrizable space. The following are equivalent:*

1. *X is compact.*
2. *X is limit point compact.*
3. *X is sequentially compact.*

Proof.

1. implies 2): done above!
2. implies 3): Assume X is limit point compact. Let $(x_n)_{n\in\mathbb{Z}_+}$ be a sequence of elements of X.

□