Math 532 Notes

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Contents

1 Lecture 1 <*2020-01-15 Wed*>

Basic introductions to the course. Office hours M 2-3 (Math 220), TW 2-3 (Math 219), and by appointment. Book is Topology by Munkres.

1.1 Topological Spaces

What are open sets in $\mathbb{R}$? We know they can be intervals, like $(1, 2)$ or (a, b). $\mathbb{R}$ itself is another example, as well as its complement, $\emptyset$.

Question 1.1. *What questions does Topology answer. Which does it ask?*

Question 1.2. *What is the origin of the name "Point-Set Topology"?*

The set of all open intervals defines the "standard topology" on $\mathbb{R}$.

We'll also be concerned with homeomorphisms (Donuts + Coffee mugs).

Much of "Point-Set Topology" is motivated by necessity from real analysis.

Metrics induce a topology. A reasonable question in topology is "how close is this space to being metrizable?"

Summary 1.1. *We'll cover fundamental Point-Set Topology as well as som algebraic topology in this course. We begin with open sets in $\mathbb{R}$.*

1.2 Class Outline

- Basics of Topology
- Waht is a topology
- Continuous functions
- Building new topologies
  - Product, Subspace, Metric, Quotient T's
- Connectedness
- Compactness
- Separation Axioms
- Metrizability
- Tychonoff's Theorem
- Topics in Algebraic Topology
  - Homotopy
  - Fundamental Groups
  - Boraz-Luanne?

1.3 Back to Topology

Definition 1.1. *Given a set X, a topology on X is a collection T of subsets of X with the following properties:*

1. $\emptyset, X \in T$

2. *If $X_\alpha \in T, \forall \alpha \in J$, then $\bigcup_{\alpha \in J} X_\alpha \in T$. (J arbitrary indices).*

3. *If J is finite, then $\cap_{\alpha \in J} X_\alpha \in T$.*

Topologies are collections of sets that

1. Contain the empty set and the whole space
2. Are closed under arbitrary unions
3. Are closed under finite intersections

Question 1.3. *Where/how does an infinite intersection break a topology? Why are finite intersections the only kind allowed?*

Summary 1.2. *A topology is a collection of sets with three key properties*

1.4 Topologies on $\mathbb{R}$

1. The standard topology, T_{std}.
2. The lower limit topology, $\mathbb{R}_l$
   (a) Basis of elements as $[a, b)$
3. Any set X has two topologies
   (a) The discrete topology, $T = 2^X$
   (b) The indiscrete topology, $T = \{\emptyset, X\}$

Question 1.4. *Are there sets for which the discrete and indiscrete topologies are equivalent?*

1.5 Example

Take $X = \{1, 2, 3\}$. Then, the following are topologies on X: $T = \{\{1\}, \{1, 2\}, X, \emptyset\}$, $T = \{\{2\}, \{1, 2\}, \{2, 3\}, X, \emptyset\}$. The following is not: $T = \{\{1\}, \{3\}, X, \emptyset\}$.

Summary 1.3. *Topologies can be formed on all sets, even ones we wouldn't categorize with others, e.g. $\mathbb{R}$ vs $\mathbb{N}$, etc.*

2 More on topological spaces <*2020-01-17 Fri*>

Definition 2.1. *Suppose that T, T' are two topologies on X. If $T \subseteq T'$, we say that T' is finer than T, and that T is coarser than T'. We say that T and T' are comparable if either is contained in the other.*

Definition 2.2. *If X is a set, a basis for a topology on X is a collection B of subsets such that*

1. *$\forall x \in X, \exists b \in B$ such that $x \in b$.*

2. *If $b_1, b_2 \in B$, then $\exists b_3 \in B$ such that $b_3 \subseteq b_1 \cap b_2$.*

Question 2.1. *Is the standard topology in $\mathbb{R}$ comparable with the lower limit topology?*

Definition 2.3. *A set X along with a topology is called a {topological space}.*

Definition 2.4. *If (X, T) is a topological space, than a set $Y \subseteq X$ is open iff $Y \in T$. $Z \subseteq X$ is closed iff $Z \setminus X$ is open. (Open means a set is a member of the topology).*

2.1 Example

$(R, 2^{R})$.

All sets in $\mathbb{R}$ are open, and they're also closed.

$(\mathbb{R}, \{\emptyset, \mathbb{R}\})$.

Open sets are only $\emptyset, \mathbb{R}$. These are also the only closed sets.

$(\mathbb{R}, T_{std})$

$(0,5)$ is open. $\mathbb{R} \setminus (0,5)$ is closed.

Recall that T_{std} is generated by the collection of open intervals (a, b) in $\mathbb{R}$.

Question 2.2. *Do bases have to be minimal and/or orthonormal? Does that question even make sense?*

Question 2.3. *Can we characterize topologies where open sets are also closed?*

2.2 Bases

The topology generated by a basis B is defined as: U is open in X if $\forall x \in U, \exists b \in B$ such that $x \in b \subseteq U$.

To check is a set U is open,

1. Pick $x \in U$
2. Find $b \in B$ such that $x \in b \subseteq U$.

2.2.1 Example

Is $U = (0, 5) \cup (15, 20)$ open in $\mathbb{R}_{std}$? Yes.

Proof. Let $x \in U$. Then $x \in (0, 5)$ or $x \in (15, 20)$. Case 1: $x \in (0, 5)$. $(0, 5) \in B$, and $(0, 5) \subseteq (0, 5)$. Case 2 is equivalent. □

Checking for openness is a *pointwise* operation.

2.3 Generating Topologies from Bases

Theorem 2.1. *Given a set X and a basis B, the collection of sets T generated by B as described above is a topology.*

#+BEGIN$_{\text{proof}}$
We look to show 3 properties:

1. $\emptyset, X \in T$
2. T is closed under unions
3. T is closed under finite intersections

1. $\emptyset \in T$ is vacuously true. For $X \in T$, Choose $U = X$. Then, $\exists b \in B$ such that $x \in b$. Satisfies clause 1.
2. Let $U_\alpha \in T$. For any $x \in \cup U_\alpha$, $\exists b \in B$ such that $x \in b, b \subseteq U_\alpha$, since $\exists \alpha'$ such that $x \in \cup U_{\alpha'}$, and $U_{\alpha'}$ is open.
3. For finite intersections: induction. Prove T is closed under pairwise intersections, then assume n intersections, prove $n + 1$. Similar to unions, since $x \in U = U_1 \cap U_2$ implies $x \in U_1$ and $x \in U_2$, so $\exists b \in B$ such that $x \in n$, and $b \subseteq U_1$ and $b \subseteq U_2$.

Lemma 2.1. *Let X be a set. Let B be a basis for a topology on X. Then T equals the collection of all unions of elements of B.*

Proof by subsetting both ways.

Lemma 2.2. *Let (X, T) be a topological space. Suppose C is a collection of open sets of X such that $\forall U \subseteq X$, U open, and $\forall x \in U$, $\exists c \in C$ such that $x \in c \subseteq U$. Then C is a basis for a topology on X.*