Homework Set 2

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1 Problem 1

If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_l \times \mathbb{R}$ and as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$. In each case, it is a familiar topology.

To be precise, we define a line L as a set of points $L = \{x, y \in \mathbb{R}^2 | \ y = mx+b, m, b \in \mathbb{R}\}$. In both cases, the basis elements for the subspace topology L inherits can be defined as $\{B_l \times B \cap L\}$.

As a subspace of $\mathbb{R}_l \times \mathbb{R}$, we have basis elements of the form $[x_1, x_2) \times (y_1, y_2) \cap L$. Since a line is one dimensional, we can parameterize x, y as functions of a dummy t, such that $y(t) = mx(t) + b$, and we are left with a one dimensional set with the lower limit topology, i.e. $\mathbb{R}_l$.

The same holds for L as a subspace of $\mathbb{R}_l \times \mathbb{R}_l$.

2 Problem 2

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

The product topology on $I \times I$ is finer than the the dictionary order topology on $I \times I$, which is both finer and coarser than the subspace topology inherited from $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Proof. Product Topology is finer than the Dictionary Order Topology: The basis for the product topology is sets of the form $((a, b), (c, d))$, $a, b, c, d \in I$. Basis elements of the dictionary order topology are also intervals, with endpoints (p_1, p_2) and (q_1, q_2): $B = \{(a, b), a, b \in I \mid (p_1 < a \leq q_1)$ or $(p_1 = a)$ and $(p_2 < b \leq q_2)\}$.

Clearly, since both elements are drawn from the same set I, we can find a slice B in the dictionary order topology which contains any given element $((a, b),(c, d))$ of the product topology, since elements from the product topology do not contain their endpoints. The product topology is finer than the dictionary order topology.

Subspace topology is equal to the DOT on $I \times I$: This follows from the definition of the subspace topology and the definition of the DOT on $I \times I$. First, the subspace topology:

The subspace topology on $I \times I$ is that topology having as basis the collection of sets $D_I = \{d \cap I | d \in D\}$ where D is the basis of the DOT in $\mathbb{R}^2$. These are intervals $(p, q) \in I$ with the dictionary ordering. The basis of the DOT on $I \times I$ is exactly this collection of sets, as well. The two are equal, and thus comparable. $\square$

3 Problem 3

Let Y be a subspace of X. Prove that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. Let A be closed in Y. Then its complement, $A_C = Y \setminus A$, is open in Y (and thus an element of the topology on Y). Since Y is a subspace of X, A_C is also an element in the topology of X. Topologies are closed under arbitrary unions, so the set $A_{CX} = A_C \cup (Y \setminus X)$ is also in the topology. Thus, the complement of A in X is open, and therefore A is closed in X. $\square$

4 Problem 4

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Proof. Let A be closed in X, and B be closed in Y. Then, $A_C = A \setminus X$ is an element of the topology on X, and $B_C = B \setminus Y$ is an element of the topology on Y. Since the product topology on $X \times Y$ is inherited from the sets X and Y, the set $A_C \times B_C$ is in this product topology. This set is also the complement of $A \times B$ in $X \times Y$. Thus, $A \times B$ is closed in $X \times Y$. $\square$

5 Problem 5

Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology.

Compare this topology with the standard topology on $\mathbb{R}^2$.

Proof. First, recall that a basis element in the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ is an interval of the form $(p, q), p, q \in \mathbb{R} \times \mathbb{R}$.

Next, we note from theorem 15.1 we have that the basis for the product topology on $\mathbb{R}_d \times \mathbb{R}$ is the collection of sets $D = \{b \times c \mid b \in B, c \in C\}$, where B, C are the bases of the discrete and standard topologies on $\mathbb{R}$, respectively.

To the proof. Let $A = (x, (a, b))$ be a basis element of $\mathbb{R}_d \times \mathbb{R}$. Choose $p_1, p_2 \in \mathbb{R} \times \mathbb{R}, p_1 = (a_1, b_1), p_2 = (a_2, b_2)$. Then, let $a_1 = a_2 = x$. Clearly, then, the interval $(p_1, p_2) = (a_1, (b_1, b_2))$, which is equivalent to our chosen A from the product topology. Thus, for any element of the topology on $\mathbb{R}_d \times \mathbb{R}$, we can find an equivalent element in $\mathbb{R} \times \mathbb{R}_{DOT}$. $\mathbb{R}_d \times \mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}_{DOT}$.

Next, let $p = (a, b)$ and $q = (c, d)$. Then, we can choose elements of the basis of $\mathbb{R}_d \times \mathbb{R}$ as $r_i = (p + i\epsilon, (b, d))$. By construction, $r_i \in C$, the basis on $\mathbb{R}_d \times \mathbb{R}$. Let $R = \bigcup_{i \in J} r_i = (p, q)$. Topologies are closed under arbitrary unions, so $R \in C$ as well. But $R = (p, q)$, so $(p, q) \in \mathbb{R}_d \times \mathbb{R}$, and $\mathbb{R} \times \mathbb{R}_{DOT} \subseteq \mathbb{R}_d \times \mathbb{R}$. Therefore, $\mathbb{R} \times \mathbb{R}_{DOT} = \mathbb{R}_d \times \mathbb{R}$. □

Next, we prove that $\mathbb{R}_d \times \mathbb{R}$ is strictly finer than $\mathbb{R}^2_{std}$.

Proof. Let $A = (x, (a, b))$ be a basis element of $\mathbb{R}_d \times \mathbb{R}$. Choose $c < x < d, c, d \in \mathbb{R}$, and let $S = ((c, d), (a, b))$. Clearly, $S \in \mathbb{R}^2_{std}$, by construction, and also $A \subset S$. $\mathbb{R}_d \times \mathbb{R}$ is finer than $\mathbb{R}^2_{std}$.

However, we cannot go the other way. Topologies are only closed under finite intersections, and we cannot make the set A by any finite number of intersections of sets of the form of S, due to the completeness of the reals (equivalently, the nested interval principle).

Thus, $\mathbb{R}_d \times \mathbb{R}$ is strictly finer than $\mathbb{R}^2_{std}$. □