Exam 2

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1 Problem 1

Let Y_1 and Y_2 be compact subspaces of a topological space X.

a) Prove that the union $Y_1 \cup Y_2$ is compact if both Y_1 and Y_2 are compact. Give an example to show that the converse is false.

Proof. Let Y_1 and Y_2 be compact. By definition, any covering of each of these sets has a finite subcover. Let A_1 and A_2 be the corresponding finite subcovers for arbitrary covers of each Y. Then, $A = A_1 \cup A_2$ covers $Y_1 \cup Y_2$. Since the union of two finite sets is itself finite, A is a finite cover of $Y_1 \cup Y_2$, and thus the union is compact. $\square$

b) Does the result of part (a) together with the induction principle imply that a *finite* union of compact subspaces is compact? Explain why, or find a counterexample.

Yes, because the union of finitely many finite sets is still itself finite.

c) Does the result of part (a) together with the induction principle imply that a *countable* union of compact subspaces is compact? Explain why, or find a counter example.

No, because a countable union of finite sets could itself no longer be finite. A counterexample would be a collection of compact subspaces of $\mathbb{R}$ of the form $[a, a+1], a \in \mathbb{Z}$. The countable union of these sets forms $\mathbb{R}$, which is known to not be compact.

2 Problem 2

Let $X_3 = \{a, b, c\}$ be a topological space containing exactly three points, and let $X_4 = \{a, b, c, d\}$ be a topological space containing exactly four points.

a) Prove that if there is no open singleton in X_3, then X_3 is connected.

Proof. By assumption, no subset of X_3 of the form $\{x\}$ is open. Thus, our only options for open sets are the empty set, X_3 itself, and sets of the form $\{x_1, x_2\}$. Clearly, no combination of two *distinct* sets of this form and/or the emptyset and/or X_3 can contain all the points of X_3, so X_3 must be connected. □

b) Is the following true? If X_3 is connected, then there is no open singleton in X_3.

If X_3 is connected, there exists no possible separation (two distinct open sets, whose intersection is the empty set) in X_3. The only way to construct such a separation is with singletons, due to the three pointed nature of the set.

c) Is the following true? If there is no open singleton in X_4, then X_4 is connected.

No. $\{a, b\}$ and $\{c, d\}$ are a separation in X_4 that does not require singletons for construction. X_4 is not connected.

3 Problem 3

Let $X = \mathbb{R}^\omega$ with the product topology. Let $A \subseteq X$ be the subset $A = [0,1] \times [0,2] \times [0,3] \times \dots$ endowed with the subspace topology. Explain each answer below.

a) Is X metrizable?

Yes. See theorem 20.5, page 125 of Munkres.

b) Is X compact?

No. None of its constituents ($\mathbb{R}$) is compact, so the product cannot be either.

c) Is X sequentially compact? If not, construct a sequence in X that has no convergent subsequence.

No, since it is metrizable but not compact. The sequence $\{0, \dots x_i, \dots 0 \dots\}_{i \in \mathbb{Z}_+}$ for a given nonzero point x seems to fit the bill.

d) Is X connected?

Yes, since its constituents are all connected. A path function on each $\mathbb{R}$ can be found that is continuous, and the product of such continuous functions is continuous (theorem 19.6), so actually it's path connected.

e) Is A metrizable?

Yes. The Euclidean metric applies equally well to subspaces of $\mathbb{R}$ as it does to all of $\mathbb{R}$.

f) A is compact by the Tychonoff theorem. Is A sequentially compact? If not, construct a sequence in X that has no convergent subsequence.

A is sequentially compact, since it is compact and metrizable, by theorem 28.2.

g) Is A connected?

Yes, by a similar argument to the above for all of X. That each constituent is a single connected interval is the main point.

4 Problem 4

Let X be $\mathbb{R}$ with the countable complement topology.

a) Find the closure and the interior of $Y = (0, \infty) \subseteq X$.

The closure is the smallest closed set which contains Y. Since $[0, \infty)$ is not countable, it cannot be the complement of an open set (and thus not closed). Since it is uncountable, the next largest closed set (as long as we're subscribing to the continuum hypothesis, anyway) is $\mathbb{R}$. $\overline{Y} = \mathbb{R}$.

Next, the interior. The is the largest open set which is contained within Y. Since any open set in the countable complement topology must either be the emptyset or $\mathbb{R}$ minus some countable set, we conclude that the largest open set which is completely contained within Y is the empty set. $\text{int } Y = \emptyset$.

b) Consider the map $f : X \to X$ defined by $f(x) = cos(x)$. Is f continuous on X with the countable complement topology? (*Hint:* It may be helpful to know that the countable union of countable sets is countable.)

Indeed, f is continuous, albeit vacuously. The range of f is actually $[-1, 1]$, whose complement is obviously not countable, and the complement of any subset of it is also not countable. Thus, no subset of the range of f is open. Therefore, vacuously, for every open set $A \subseteq [-1, 1]$, $f^{-1}(A)$ is also open. (Actually, $\emptyset$ is open, but its preimage, also the empty set, is open).

c) Is X compact? (*Hint:* Consider the cover $\{U_k\}_{k \in \mathbb{Z}_+}$ where $U_k = (\mathbb{R} \setminus \mathbb{Z}_+) \cup \{k\}$.)

No. A finite subcover must contain finitely many *finite sets*. No set which is open in this topology (aside from the empty set, which by definition covers nothing) is finite.

d) Is X connected?

Yes. We require two *distinct* sets (i.e $A \cap B = \emptyset$) whose union covers the space for a separation.

If we consider the complements, $(A \cap B)^c = \mathbb{R} = A^c \cup B^c$ by DeMorgan's laws, we see that this would require that we be able to form $\mathbb{R}$ from the union of two countable sets (complements must be countable for A and B to be open). The union of countably many (2 is countable!) countable sets is itself countable, while $\mathbb{R}$ is uncountable.

5 Problem 5

Let X be a Hausdorff space, let C be a compact subset of X, and let a be a point of X which is not in C. Prove that there are disjoint open sets U and V with $C \subseteq U$ and $a \in V$.

This is Lemma 26.4 of Munkres, on page 166. Proof is found on the preceding page, 165, as part of the proof of theorem 26.3. Below I present an excerpt from that proof that deals just with this lemma.

Proof. Let $X,C,$ and a be as above. That is, a is in $X \setminus C$. For each point $c \in C$, let us choose disjoint neighborhoods U_c and V_c of the points a and c, respectively. We know these exist since X is Hausdorff.

The collection $\{V_c \mid c \in C\}$ is a covering of C by sets open in X. Since C is compact, we know that finitely many of the V_c cover C. Thus, we have that the open set $V = V_{c_1} \cup \ldots \cup V_{c_n}$ contains C, and it is disjoint from the open set $U = U_{c_1} \cap \ldots \cap U_{c_n}$ formed by taking the intersection of the corresponding neighborhoods of a. For if z is a point of V, then $z \in V_{c_i}$ for some i, and hence $z \notin U_{c_i}$ by construction, and then $z \notin U$.

Thus, $U \cap V = \emptyset$, $a \in U$, and $Y \subseteq V$. $\square$