Homework Set 4

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1 Problem 1

Show that X is Hausdorff if and only if the digaonal $\Delta = \{\ x \times x \mid x \in X\}$ is closed in $X \times X$.

This problem was solved in Homework 3. Per Dr. Aubrey's instructions, we're skipping it this time around.

2 Problem 2

Show that the T_1 axiom is equivalent to the condition that for each pair of points in X, each has a neighborhood not containing the other.

Proof. First, let the T_1 axiom hold, such that for our given topological space X, finite point sets are closed. Then, let $A = \{x_1, x_2\}$. A is closed by T_1, and thus $U = X \setminus A$ is open. We then consider the set $B = U \cup \{x_1\}$. B is also open, since its complement, $X \setminus B = \{x_2\}$ is finite, and therefore closed. We can similarly construct a set $C = U \cup \{x_2\}$, which is again open. Thus, we have constructed neighborhoods of x_1, x_2 which do not contain the other point.

Conversely, let $\exists U, V \subseteq X$, $x_1 \in U$, $x_2 \in V$, $x_2 \notin U$, $x_1 \notin V$, and U, V open. We can combine sets as $C = ((-\infty, x_2) \cup U \cup (x_2, \infty)) \cap X$, and $D = ((-\infty, x_1) \cup V \cup (x_1, \infty)) \cap X$. Both of these are open, since unions of open sets are open. Their complements, C^c, D^c, are thus closed. They are also finite (single) point sets, effectively $C^c = \{x_2\}$ and $D^c = \{x_1\}$. Since X is a topological space, finitely many unions of similarly constructed closed sets will also be closed. We can therefore construct any set of finitely many points which will be closed. Thus, T_1 holds. $\square$

3 Problem 3

If $A \subseteq X$, we define the **boundary** of A by the equation
$\text{Bd } A = \overline{A} \cap \overline{(X \setminus A)}$.
a) Show that Int A and Bd A are disjoint and that $\overline{A} = \text{Int} A \cup \text{Bd} A$.

Proof. First, we note that Int A is the union of all open sets contained within A, and that Int $A \subseteq \overline{A}$. If A is open, then Int $A = A$, $\overline{(X \setminus A)} = (X \setminus A)$, and $(X \setminus A) \cap A = \emptyset$. If A is closed, then $A = \overline{A}$, and $(X \setminus A)$ is open. Further, since Int $A \subset A \subseteq \overline{A}$, Int $A \cap$ Bd $A = \emptyset$. Thus, Int A and Bd A are disjoint. □

b) Show that Bd $A = \emptyset$ iff A is both open and closed.

Proof. If A is closed, then $A = \overline{A}$. If A is open, then $X \setminus A$ is closed, and also equal to its closure. Then, Bd $A = \overline{A} \cap \overline{(X \setminus A)} = A \cap (X \setminus A) = \emptyset$.

If Bd $A = \emptyset$, then $\overline{A} \cap \overline{X \setminus A} = \emptyset$. This implies $A = \overline{A}$, and $(X \setminus A) = \overline{(X \setminus A)}$. Thus, A is both closed and open. □

c) Show that U is open iff Bd $U = \overline{U} \setminus U$.

Proof. If U is open, $X \setminus U$ is closed. Bd $U = \overline{U} \cap \overline{(X \setminus U)} = \overline{U} \cap (X \setminus U) = (\overline{U} \cap X) \setminus (\overline{U} \cap U) = \overline{U} \setminus ((U \cup U') \cap U) = \overline{U} \setminus U$.

Next, let Bd $U = \overline{U} \setminus U$. But, Bd $U = \overline{U} \cap \overline{(X \setminus U)}$. We can rewrite the first equation as Bd $U = (\overline{U} \cap X) \setminus (\overline{U} \cap U)$, which is equivalent to $\overline{U} \cap (X \setminus U)$, as shown in the preceding paragraph. Then, we have that $\overline{U} \cap \overline{(X \setminus U)} = \overline{U} \cap (X \setminus U)$, such that $\overline{(X \setminus U)} = (X \setminus U)$, and thus $(X \setminus U)$ is closed. U is open. □

d) If U is open, is it true that $U = \text{Int } \overline{U}$? Justify your answer.

By definition, Int U is the union of all open sets contained in U. If U is open, then Int $U = U$. We know from theorem 17.6 of Munkres that $\overline{U} = U \cup U'$, where U' is the set of all limit points of U. We also know that adding the limit points to the set creates a closed set, not an open one, so we conclude that U is the "biggest" open set contained within $\overline{U}$.

4 Problem 4

Prove that for functions $F : \mathbb{R} \to \mathbb{R}$ the $\epsilon - \delta$ definition of continuity implies the open set definition.

Proof. Assume f is continuous. Thus, by the "$\delta - \epsilon$" definition of continuity, we have that $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Let $U \subseteq \mathbb{R}$ be open. Then, for any $y \in U$, $\exists x$ such that $f(x) = y$, and there are neighborhoods $y \in U_y$ and $x \in V_x$. Additionally, we can choose to let $U = \bigcup U_y$ for our chosen y, and since the U_y are open (in the standard topology, assumed here), U is also open. Finally, we note that all of the V_x are open in the standard topology, so their union $V = \bigcup V_x$ is as well. But, by construction, $V = f^{-1}(U)$, so the preimage of an open set is open. $\square$

5 Problem 5

Let a, b, and c be real numbers with $a \leq b \leq c$, and $a < c$. Let X denote the set $[a, c] \cup \{b'\}$, where $[a,c] denotes a closed interval in the real line and b' is a point not in $[a, c]$. Let F be the family of subsets of X consisting of all open subsets of $[a, c]$ together with all subsets of the form $(U \setminus \{b\}) \cup \{b'\}$, where U is an open subset of $[a, c]$ **which contains** b. (Emphasis mine).

a) Show that F is a basis for a topology on X.

Proof. For notation's sake, we split F into two categories of subsets: U the open subsets of the interval $[a, c]$, and W, the collection of sets of the form $(U \setminus \{b\}) \cup \{b'\}$.

We consider the two pieces of the definition of basis separately. Trivially, we note that elements of F cover X, for if $x \in [a, c]$, $\exists U \in F$ such that $x \in U$, and if $x = b'$, $\exists W \in F$ such that $x \in W$.

Next, we look to intersections of elements of F. Let $x \in X, x \in f_1 \cap f_2$. Either $x \in [a, c]$, or $x = b'$. If $x \in [a, c]$, $f_1, f_2 \subseteq [a, c]$. Thus, $\exists f_3 \subseteq [a, c]$ such that $x \in f_3 \subseteq f_1 \cap f_2$.

Finally, consider when $x = b'$. Then, $f_1, f_2 \in W$. We see that $\exists f_3 \in W$ such that $b' \in f_3 \subseteq f_1 \cap f_2$, and we're done. $\square$

b) Show that the map which interchanges b and b' and is the identity elsewhere is a homeomorphism.

Proof. Clearly, the map $f : X \to X$ described above is a bijection. Further, we also see that $f = f^{-1}$. It is then enough to show that f is continuous. Let $U \subseteq X$ be open. Then, we have four cases: $\{b, b'\} \cap U = \{\emptyset, \{b\}, \{b'\}, \{b, b'\}\}$.

Case 1: $f^{-1}(U) = U$. Done.

Case 2: $f^{-1}(U) = (U \setminus \{b\}) \cup \{b'\} \in W$, which is open.

Case 3: $f^{-1}(U) = (U \setminus \{b'\}) \cup \{b\} \subseteq [a, c]$, which is also open.

Case 4: $f^{-1}(U) = U$. Done. $\square$

c) Show that this topology on X is not Hausdorff

Proof. Consider the two points b and b'. By construction of X, any neighborhood of b' must also contain some neighborhood of b, no matter how small the neighborhood (see my emphasis in the question text above). X is not Hausdorff. $\square$

d) Show that if $f : X \to \mathbb{R}$ is continuous, then $f(b) = f(b')$.

Proof. We'll assume $\mathbb{R}$ to have the standard topology here. Using the topological definition of continuity, we have that $\forall U \subseteq \mathbb{R}$, U open, $V = f^{-1}(U)$ is also open.

Any open set containing b' also includes $(b-\epsilon, b) \cup (b, b+\epsilon)$, by construction of X. Assume $f(b) \neq f(b')$. Then $\exists U, U'$, $f(b) \in U, f(b') \in U'$, and $U \cap U' = \emptyset$, since $\mathbb{R}$ is Hausdorff. But, $f^{-1}(U) \cap f^{-1}(U') \neq \emptyset$, so $\exists x$ such that $f(x) \in U$ and $f(x) \in U'$. Since $U \cap U' = \emptyset$, this implies that x maps to two distinct points in $\mathbb{R}$, a violation of the function rule. This is a contradiction, so $f(b) = f(b')$. $\square$