Homework Set 10

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1 Problem 1

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

There exist two homotopies $H : X \times I \to Y$ between h and h', and $K : Y \times I \to Z$ between k and k'. Consider the map $F : X \times I \to Z$ such that $F(x,t) = K(H(x,t),t)$. Clearly, this is a homotopy between $k \circ h$ and $k' \circ h'$.

2 Problem 2

Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\alpha} * \hat{\beta}$.

$$\hat{\gamma}([f]) = [\overline{\alpha * \beta}] * [f] * [\overline{\beta} * \overline{\alpha}]$$

$$= [\overline{\beta} * \overline{\alpha}] * [f] * [\alpha * \beta]$$

$$= [\overline{\beta}] * [\overline{\alpha}] * [f] * [\alpha] * [\beta]$$

$$= [\overline{\beta}] * \hat{\alpha}([f]) * [\beta]$$

$$= \hat{\beta} \circ \hat{\alpha}([f]).$$

3 Problem 3

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

For every $\alpha, \beta, \hat{\alpha} = \hat{\beta}$ iplies that for every α, β, f, a loop based at $x_0, \alpha(\hat{[f]}) = \beta(\hat{[f]})$. Further, this implies that $[f] \circ [\alpha \circ \beta] = [\alpha \circ \beta] \circ [f]$.

Note that $\alpha \circ \beta$ is an arbitrary loop based at x_0 and passing through x_1. So, if the group of the path homotopy classes of loops based at x_0 is Abelian, then the right hand side expression holds for arbitrary α, β, and f, therefore for every $\alpha, \beta, \hat{\alpha} = \hat{\beta}$.

Vice versa, if for every $\alpha, \beta, \hat{\alpha} = \hat{\beta}$, then we have shown that the group is commutative when at least one of the terms is a path homotopy class of a loop passing through x_1. So, take arbitrary $[f], [g] \in \pi_1(X, x_0)$ and take any path α from x_0 to x_1. Then, $g \circ \alpha \circ \overline{\alpha}$ is a loop based at x_0 passing through x_1. Then, $[f] \circ [g \circ \alpha \circ \overline{\alpha}] = [g \circ \alpha \circ \overline{\alpha}] \circ [f]$.

4 Problem 4

Show that the mapa of Example 3 on page 338 is a covering map. Generalize to the map $p(z) = z^n$.

Example 3: $p : S^1 \to S^1$ given by $p(z) = z^2$ where S^1 is considered as a subspace of the complex plane.

Consider a general map $p(z) = z^n$. If $U = \{z = e^{i\phi} \mid \phi \in (a, b) \subseteq (0, 2\pi)\}$. Then, $p^{-1}(U) = \{z = e^{i\phi} | \ \phi \in (a/n + 2\pi k/n, b/n + 2\pi k/n), k \in \overline{0, n-1}\}$ iis the union of n open intervals in S^1 such that the restriction of p onto each such interval is a homeomorphism of the interval with U. This is also the case if we restrict ϕ to $(-\pi, \pi)$. Overall, ever point of s^1 has such an open neighborhood.

5 Problem 5

Show that if X is path connected, the homomorphism induced by a continuous map is independent of base point, up to isomorphism of the groups involved. More precisely, let $h : X \to Y$ be continuous, with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let α be a path in X from x_0 to x_1, and let $\beta = h \circ \alpha$. Show that

$\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}$.

This equation expresses the fact that the following diagram of maps "commutes."

(See graph in text. That's a bit beyond my own TEX-ing ability, at the moment.)

$$\hat{\beta} \circ (h_{x_0})([f]) = [\overline{\beta}] * (h_{x_0})([f]) * [\beta]$$

$$= [h \circ \overline{\alpha}] * [h \circ f] * [h \circ \alpha]$$
$$= [h \circ (\overline{\alpha} * f * \alpha)]$$
$$= (h_{x_1})[\overline{\alpha} * f * \alpha]$$
$$= (h_{x_1}) \circ \alpha([f])$$