Homework Set 3

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1 Problem 1

Let A, B, and A_α denote subsets of a topological space X. Determine which of the following equations hold; if an equality fails, give an example, and determine whether one of the inclusions $\supset$ or $\subset$ holds.

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$
2. $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$
3. $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$

1.1 $\overline{A \cap B} = \overline{A} \cap \overline{B}$

This equality holds. The intersection of two closed sets equals a closed set, since the complement of an intersection is the union of the complements, and the union of open sets is open in a topological space. Since the two sides have the same interiors and are both closed, they are equal.

1.2 $\overline{\cap A_\alpha} = \cap \overline{A_\alpha}$

This equality also holds. Argument is the same as above, since topologies are closed under arbitrary unions of open sets (thus the arbitrary intersection of closed sets is closed).

1.3 $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$

This does not hold. $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$. Consider, as example, $A = (0, 2)$ and $B = (1, 2)$. Then $\overline{A \setminus B} = [0, 1]$, but $\overline{A} \setminus \overline{B} = [0, 1)$.

2 Problem 2

Show that every order topology is Hausdorff.

Definition 2.1. *A topological space is said to be **Hausdorff** if, for any two distinct points $x_1, x_2 \in X$, there exist open neighborhoods U_1, U_2 ofx_1, x_2 respectively, such that $U_1 \cap U_2 = \emptyset$.*

Proof. Let X be a topological space with the order topology, and let $x, y \in X$, distinct, such that \$ x < y \$. Then, we have two cases:

Case 1: $\exists z$ such that $x < z < y$. Then, $U = (-\infty, z)$ is open and contains x. Similarly, $V = (z, \infty)$ is open and contains y. But, by construction, $U \cap V = \emptyset$.

Case 2: Such a z does not exist (i.e. y is the "next point" in the order after x). Then, We modify U and V as $U = (-\infty, y) = (-\infty, x]$, and $V = (x, \infty) = [y, \infty)$. Clearly, we still have that $U \cap V = \emptyset$.

Since our x, y are assumed distinct and arbitrary, this shows that X must be Hausdorff. □

3 Problem 3

Show that X is Hausdorff iff the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Proof. Let $X \times X$ have the product topology, and let Δ be closed in $X \times X$. Choose (x, y) in $X \times X$, such that $x \neq y$. $(x, y) \in X \times X \setminus \Delta$. Then, there exist open neighborhoods U and V of x and y, respectively, such that $(U \times V) \cap \Delta = \emptyset$, since $X \times X \setminus \Delta$ is open, due to Δ's closedness. Consequently, $U \cap V = \emptyset$, for if there was an x such that $x \in U \cap V$, $(x, x) \in (U \times V) \subseteq \Delta$, a contradiction. X is Hausdorff.

Next, let X be Hausdorff. Take x and y as above, such that $(x, y) \in X \times X \setminus \Delta$. Since X is Hausdorff, we have that $\exists U, V \ni U \cap V = \emptyset$, U, V open. Since the product topology is closed under unions, we can build up all of $X \times X \setminus \Delta$ via these neighborhoods, for all distinct $x, y \in X$. Thus, $X \times X \setminus \Delta$ is open, and its complement, Δ, is therefore closed. □

4 Problem 4

In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

The sequence converges (in the topological sense, i.e. for every open neighborhood U of x, $\exists N \in \mathbb{N}$ such that $\forall n > N, x_n \in U$) to every point of $\mathbb{R}$. This is because the open neighborhoods must all be infinite (and, in fact, of infinite measure), since they contain all but finitely many points of $\mathbb{R}$. Thus, for any point in $\mathbb{R}$, we can find a suitable N such that x_N is in its neighborhood.

5 Problem 5

Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis of Exercise 8 in Section 13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

In the lower limit topology, $\overline{A} = [0, \sqrt{2})$ and $\overline{B} = [\sqrt{2}, 3)$.

In the topology defined in exercise 8 of section 13, both are the same.