Homework Set 7

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1 Problem 1

Let $f : X \to Y$; let Y be compact Hausdorff. Then, f is continuous if and only if the graph of f, $G_f = \{x \times f(x) \mid x \in X\}$ is closed in $X \times Y$. [*Hint:* If G_f is closed and V is a neighborhood of $f(x_0)$ then the intersection of G_f and $X \times (Y \setminus V)$ is closed.] (You may assume the results of Exercise 7 on page 171).

Proof. Two directions, first: If f is continuous, then the graph of f, G_f, is closed in $X \times Y$. Let f be continuous. Since f is continuous and Y compact, f carries closed sets to closed sets. Thus, the set $Y = \{f(x) \mid x \in X\}$ is closed only if X is closed, which it must be, and the graph itself is closed in the product topology.

Next: If the graph of f, G_f, is closed in $X \times Y$, then f is continuous. We take the "hint" here, and note that since G_f is closed, $A = G_f \cap X \times (Y \setminus V)$ is also closed, where V is a neighborhood of $f(x_0)$. We also have, by exercise 7, that the projection $\pi_1 : X \times Y \to X$ is a closed map.

A is the complement of V and is closed, thus V is open. Further, since the projection map is a closed map, we know that the projection of A is also closed. Finally, since the projection of A is closed, its complement (a neighborhood in X of x_0) is open. The preimage under f of an open set is open, and thus f is continuous. $\square$

2 Problem 2

Let (X, d) be a metric space, and let $A \subseteq X$ be nonempty.

a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Proof. If $d(x, A) = 0$, then either x is in A or x is a limit point of A. $x \in \overline{A}$.

If $x \in \overline{A}$, then either x is in A, and thus $d(x, A) = 0$, or x is a limit point of A, and similarly, $d(x, A) = 0$. □

b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

Proof. This is simply the definition of the distance to a set. $d(x, A) = \inf_{a \in A} d(x, a)$. □

c) Define the ϵ-neighborhood of A in X to be the set $U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}$. Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

Proof. Assume (X, d) a metric space, and $A \subseteq X$ nonempty. Consider the set $B = \bigcup_{a \in A} B_d(a, \epsilon)$. Choose $x \in B$. Then, by construction, $d(x, a) < \epsilon$ for some $a \in A$. Clearly, $x \in U(A, \epsilon)$, and $B \subseteq U(A, \epsilon)$.

Next, consider $y \in U(A, \epsilon)$. By defintion, $d(y, A) = \inf_{a \in A} d(y, a)$, so if we require $d(y, A) < \epsilon$, $\exists a \in A$ such that $d(y, a) < \epsilon$. Then, $y \in B_d(a, \epsilon)$, and $y \in B$. $U(A, \epsilon) \subseteq B$, and $U(A, \epsilon) = B$. □

d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

Proof. □

e) Show that the result of (d) need not hold if A is closed but not compact.

3 Problem 3

Recall that $\mathbb{R}_k$ denotes $\mathbb{R}$ in the \$k\$-topology.

a) show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_k$.

Take the collection of sets $(1/n, 2)$ for all $n \in \mathbb{N}$, in addition to $\mathbb{R} \setminus K$. This collection covers the interval $[0, 1]$, but contains no finite subcollection which contains 0. $[0, 1]$ is not compact.

b) Show that $\mathbb{R}_k$ is connected. [*Hint:* $(-\infty, 0)$ and $(0, \infty)$ inherit their usual topologies as subspaces of $\mathbb{R}_k$.]

Proof. Similar to the proof that $\mathbb{R}$ in the standard topology is connected, we cannot find a separation by open sets that covers the entire space, since 0 is a member of the space. I.e. $(-\infty, 0)$ and $(0, \infty)$ does not cover the space, since 0 is not included in the separation.

Let $A = (-\infty, a)$ and $B = (b, \infty)$. If $a < b$, the sets A and B are disjoint, but do not cover $\mathbb{R}_k$. If $a > b$, the sets cover, but are not disjoint. If $a = b$, the sets are disjoint, but do not cover, as they do not contain the point a. □

c) Show that $\mathbb{R}_k$ is not path connected.

Proof. Here we offer a simple counterexample. Choose points 0 and 1. If there exists a path between these two, then the path f is a continuous function from a compact connected space, and hence the the image must also be compact and connected. This is a contradiction with part a. $\square$

4 Problem 4

Show that a connected metric space having more than one point is uncountable.

Proof. Let there be a bijection from X to $\mathbb{Z}_+$, constructed from the metric d as $x_0 \in X$, $f : X \to \mathbb{Z}_+$, $f(y) = d(x_0, y)$. Then, we can separate the image of this bijection into two open sets, those with odd and even distances from the point x_0. If f is a bijection, then these two disjoint open sets cover $\mathbb{Z}_+$, and thus form a separation on X, a contradiction. $\square$

5 Problem 5

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X. Show that if each set A_n has empty interior in X, then the union $\bigcup A_n$ has empty interior in X. [*Hint:* Imitate the proof of theorem 27.7]. This is a special case of the *Baire category theorem.*

Proof. Let X and A_n be as above, and let each A_n have empty interior. The interior of a set is the union of all open sets contained within the given set. Thus, each A_n contains no open sets.

Let U_0 be a nonempty open set of X. It is enough to show that any such set cannot be contained in $A = \bigcup A_n$, i.e. $\exists x \in U_0$ such that $x \notin A$. By assumption, A_1 has empty interior, so U_0 is not contained in A_1, and there exists $y_1 \in U_0$ such that $y_1 \notin A_1$.

Next, choose a nonempty open set $U_1 \subset U_0$, such that $y_1 \notin \overline{U}_1$. We know that such a set exists because U_0 is open and X is Hausdorff.

We can continue this process, i.e. given U_{n-1} choose a point $y_n \in U_{n-1}$, $y_n \notin A_n$ and a nonempty open set $U_n \subset U_{n-1}$, $y_n \notin \overline{U}_{n-1}$. Because X is compact, this nested collection of open sets will have nonempty intersection, i.e. $\bigcap U_n \neq \emptyset$. Finally, we choose $x \in \bigcap \overline{U}_n$. We have that $x \in U_0$ but not in any A_n, and thus not in A. A has empty interior. $\square$