Math 532 Notes

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Contents

1 Compactness <*2020-03-18 Wed 09:15*>

Definition 1.1. *Definition: A collection A of subsets of X is said to cover X, or to be a covering of X, if the union of elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.*

Equivalently, if $X = \bigcup_{a \in A} a$, then A **covers** X

Definition 1.2. *Definition: A space X is said to be compact if every open covering of X contains a finite subcollection that also covers X.*

Example: $\mathbb{R}$ is not compact.

Thinking through this, there's no way to cover $\mathbb{R}$ with a finite number of open sets.

Example: $X = \{0\} \cup \{1/n | n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$

Let A be an open covering of X. Choose $a \in A$ such that $0 \in a = (a, b) \cap X$. By the Archimedean Principle, $\exists N \in \mathbb{Z}_+$ such that $1/n < b$. Then, $1/m \in a, \forall m \geq N$. So a contains all but finitely many elements of X. Choose $a_1, a_2, \ldots$ such that these remaining points are in their own a_i. Then the collection a_i covers X.

Lemma 1.1. *Let Y be a subspace of X. Then Y is compact iff every open covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Suppose Y is compact. Suppose $A = \{a_{\alpha \in J}\}$ is a covering of Y by sets open in X. Let $A' = \{a_\alpha \cap Y\}$. Each \$a$_\alpha \cap Y$ is open in Y. Further, $Y = \bigcup_{\alpha \in J} (a_\alpha \cap Y)$. So, A' is an open covering of Y. So, A' has a finite subcovering, say $\{a_1 \cap Y, \ldots, a_n \cap Y\}$ covers Y, and thus $\{a_1, \ldots, a_n\}$ also covers Y. □

Fun fact: any set with the indiscrete topology, every space is compact (including $\mathbb{R}$).

Finish the other direction of the proof next class.

2 *<2020-03-20 Fri 09:13>* More on Compact Spaces

Theorem 2.1. *Every closed subspace of a compact space is compact.*

Sidenote: an open subspace of a compact space is not necessarily compact.

Proof. Let Y be a closed subspace of the compact space X. Let A be an open covering of Y by sets open in X. Let $B = A \cup \{X \setminus Y\}$. Then B covers X, since A already covers Y. We know X is compact, so B must have a finite subcover (a finite subcollection of B must also cover X). If B_s contains $X \setminus Y$, throw it out, if it doesn't, do nothing. What remains is a finite collection which covers Y. □

Converse is not true: A compact subspace of a compact space does not have to be closed. The spaces must be Hausdorff for this to be the case.

Theorem 2.2. *Every compact subspace of a Hausdorff space is closed.*

Recall that in a Hausdorff space, for any two points in the space, we can find open sets which contain just each point, not the other, and they don't intersect.

Proof. Let Y be a compact subspace of the Hausdorff space X. Claim that $X \setminus Y$ is open. Let $x_0 \in X \setminus Y$. Want to show that there exists U open such that $x_0 \in U \subseteq X \setminus Y$. For each $y \in Y$ choose disjoint neighborhoods, U_y, V_y open in X such that $x_0 \in U_y$ and $y \in V_y$. Then, $\{V_y | y \in Y\}$ is an open covering of Y by sets that are open in X. Y is compact, so, there exists a finite subcovering. I.e. for finitely many V_y, Y is covered. Call these $V_{y1}, V_{y2}, ldots V_{yn}$. Then, $Y \subseteq \bigcup_n V_{yn}$.

Consider $U = \bigcap_n U_{yn}$. V is open and disjoint from U. Let $z \in V$, and then $z \in V_{yi}$ for some i. Then $z \notin U_{yi}$, so \$z $\neg\in U$. So $V \cap U = \emptyset$, so $Y \cap U = \emptyset$, or $U \subseteq X \setminus Y$, and $x_0 \in U$, which is what we set out to show. Thus, Y is closed. $\square$