Math 532 Notes

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The order topology vs subspace, continued

From examples, we know that the product and subspace topologies "commute": the subspace topology on a product space is the same as the product topology on a subspace.

The order topology, however, is a different story. It does not commute with other topology combinations. Starting to see these as operations on collections of sets.

Idea: Let X be an ordered set $Y \subseteq X$. The order on X restricted to Y makes Y an ordered set. However, the resulting order topology on Y may **not** be the same as the topology that Y inherits as a subspace of X.

1.1 Example 1

$Y = [0, 1) \cup \{2\} \subseteq \mathbb{R}_{std}$.

$\{2\} = Y \cap (1,3)$ is open in the subspace topology.
But $\{2\}$ is not open in the order topology on Y.

1.2 Example 2

Let $I = [0,1]$. Dictionary order on $I \times I$ is the restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. But, the dictionary order topology is not the same as the subspace topology on $I \times I$. For example, consider the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$. This set is open in the subspace topology, but **not** in the order topology on $I \times I$. This is because the subspace topology can create the closed interval via intersection with the endpoints of I, but the order topology does not contain half open intervals except for the endpoints ("corners of the box").

1.2.1 Key note

In the dictionary order topology, half open intervals are only allowed when they contain the **maximum** or **minimum** points of the underlying set. That doesn't mean "the line on which the max x/y/z coordinate lies", that mean the singular extrema (points).

1.3 Convexity and Commutativity of Order and Subspace

Definition 1.1. *Given an ordered set X, $Y \subseteq X$ is said to be convex if $\forall a, b \in Y$, we have that $(a,b) \subseteq Y$.*

Theorem 1.1. *Let X be an ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.*

Proof. Let Y be a convex subset of Y. Consider $(a, \infty) \subseteq X$. Then $(a, \infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\}$ is an open ray of Y. Consider the case where $a \notin Y$. Then either a is a lower bound of Y, or a is an upper bound of Y, since Y is convex.

More succinctly, if a is lower bound, then $a \cap Y = Y$, and if a is an upper bound, $a \cap Y = \emptyset$. Similarly, $(-\infty, a) \cap Y = \emptyset$ (a is a lower bound), or $(-\infty, a) \cap Y = Y$ (a is an upper bound).

We know that open rays form a subbasis for a topology. In this case, they form a subbasis for the subspace topology on Y. Each of these is open in the order topology, so the order topology contains the subspace topology (because the subbasis elements are open in the OT).

To prove the converse, note that any open ray of Y is the intersection of an open ray of X and the set Y. Since the open rays of Y are a subbasis for the order topology on Y (see previous paragraph), this topology is contained in the subspace topology.

The missing part of this proof is for the case of rays where $a \in Y$. If you intersect an open ray with Y, no matter what, you get an open set of Y. My last question is, do the union of those two raysets give you back Y if $a \in Y$? No, but you can choose all a in Y, because a subbasis does not have to be a partition (open sets can overlap). $\square$

2 Closed sets and limit points

Limit points are my old nemesis from analysis.

A set $A \subseteq X$ is closed if $X \setminus A$ is open.

2.1 Example 1

$[a, b] \subseteq \mathbb{R}$. $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$.

2.2 Example 2

\ (x,y)\ |\ x≥ 0, y≥ 0\

2.3 Finite Complement Topology

On $\mathbb{R}$, the finite complement topology takes as basis elements all sets U such that $\mathbb{R} \setminus U$ is finite (as in finite number of points). Use DeMorgan's laws to prove this is a topology. Think about whether this is a topology, or its complement.