Math 532 Notes

Jake Bailey

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Contents

1 Closed Sets and Limit Points *<2020-02-05 Wed 09:04>*

Recall that a set A is closed iff its complement $X \setminus A$ is open.

$int(A)$ [The interior of A] is the union of all open sets contained in A.

$cl(A)$ [The closure of A] is the intersection of all closed sets containing A.

Theorem 1.1. *Let Y be a subspace of X, $A \subseteq Y$. The closure of A in Y equals $cl(A) \cap Y$.*

Definition 1.1. *Two sets A, B **intersect** if their intersection is not empty, i.e. $A \cap B \neq \emptyset$.*

Theorem 1.2. *Let $A \subseteq X$. Then, $x \in cl(A)$ iff every open set containing x intersects A. Also, if the topology of X is given by a basis, $x \in cl(A)$ iff every basis element of B containing x intersects A.*

WEIRD MOMENT: Munkres doesn't mention this, but we worked it through in class. Need to add the requirement/assumption that $x \in X$ in order for this theorem to work.

1.1 Example

$X = \mathbb{R}_{std}, A = (0, 1]$. Then, $cl(A) = [0, 1]$.

$B = \{1/n \mid n \in \mathbb{N}\}$. Then, $cl(B) = \{0\} \cup B$.

$\overline{cl(\mathbb{Q})} = \mathbb{R}$.

$\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Let $Y = (0, 1] \subseteq \mathbb{R}$, $A = (0, 1/2) \subseteq Y$. Closure of A in $\mathbb{R}$ is $[0, 1/2]$. Closure in Y is $Y \cap \overline{A} = (0, 1/2]$.

1.1.1 Sidenote

I noticed this morning, as I was furiously trying to update my config to get it up to speed for notetaking (since I goofed and didn't sync my changes from the desktop last night), that the laptop has code completion in elisp (and I'm suspecting, probably elsewhere). Diff the configs and see what's different, since I'm missing that on both the work and home pcs.

1.2 Limit points

Definition 1.2. *If X is a topological space, $A \subseteq X$, $x \in X$, then x is a limit point of A if every open neighborhood of x intersects A in some point other than x itself.*

Limit points of A may not be in A, but they also could be. It tells us about the structure of A if it contains all of its limit points.

Note: U is an open neighborhood of x iff U is open and $x \in U$.

1.2.1 Examples of limit points

$A = (0, 1]$. Here, 0 is a limit point of A. So is every point in A.

$B = \{1/n \mid n \in \mathbb{N}\}$. 0 is a limit point of B. Points of B are not limit points. Note that the "open neighborhoods" of a point should come from some underlying space, not just the set under consideration. 0 is the only limit point.

Note: x is a limit point of A iff $x \in \overline{A \setminus \{x\}}$.

1.3 The ambitious theorem *<2020-02-05 Wed 09:41>*

Theorem 1.3. *Let $A \subseteq X$, X a topological space. Let A' be the set of limit points of A. Then, $\overline{A} = A \cup A'$.*

Proof. Inclusion in both directions.

As observed earlier, $A \subseteq \overline{A}$. Also, by definition, every limit point of A is in $\overline{A}$, since a limit point has its open neighborhoods intersect A, and the closure points also have open neighborhoods intersect A. Thus, $A \cup A' \subseteq \overline{A}$ is obvious.

Suppose $x \in \overline{A}$. If $x \in A$, then done. Suppose $x \ni A$. $x \in \overline{A}$, so every open neighborhood U of x intersects A. Let $y \in U \cap A$. $y \neq x$, since $y \in A$. So, x is a limit point, and thus $x \in A'$. We've proven both cases, so $\overline{A} \subseteq A \cup A'$. □

If a point is in the closure of a set, it is either *{*in the set itself}, or it its *a limit point* of it. There are no other possibilities.

2 Closed Sets Again, and maybe Hausdorff Spaces? *<2020-02-07 Fri 09:00>*

Definition 2.1. *If X is a topological space, $A \subseteq X$, $x \in X$ then x is a limit point of A if every neighborhood of x intersects A in a point other than x itself.*

I.e. $\forall U$ open with $x \in U$, $(U \setminus \{x\}) \cap A \neq \emptyset$.

Theorem 2.1. *Let $A \subseteq X$, A' a set of limit points of A. Then $\overline{A} = A \cup A'$.*

Corollary 2.1. *A subset A of a topological space is closed iff $\overline{A} = A$. (This holds iff $\overline{A} \subseteq A$ iff \$ A ' $\subseteq$ A\$.*

2.1 Prepping for Hausdorff Spaces

Hausdorff property is important for convergence of sequences.

Consider a one point set in $\mathbb{R}$, say $\{x_0\}$. $\overline{\{x_0\}} = \{x_0\}$.

So one point sets are closed in $\mathbb{R}$, but this is not true in general. This is a convenient property of $\mathbb{R}$.

Another convenient property of $\mathbb{R}$, is that any sequence which converges does so to only one point. In arbitrary topological spaces, sequences may converge to more than one point.

Definition 2.2. *The traditional definition of convergence depends on a metric. A sequence x_n converges to x if, $\forall \epsilon \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that for $n > N$, $|x_n - x| < \epsilon$.*

However, this definition doesn't do us any good if the space we're considering doesn't have a metric.

Definition 2.3. *Topological definition of convergence. In a topological space X we say that the sequence $\backslash x_n \backslash_{n \in N}$ converges to $x \in X$ if for every open neighborhood U of x, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $x_n \in U$.*

2.1.1 Example

Consider the topological space (X, T), $X = \{a, b, c\}$, and $T = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Consider the sequence $\{x_n\}$ where $x_n = b, \forall n$. Does this sequence converge to b? Yes, there is always a neighborhood of b within which x_n lies. It also converges to a, since there is no open neighborhood of a which does not contain b (same for c).

2.2 Introducting Hausdorff Spaces

Definition 2.4. *A topological space X is called a Hausdorff space if, for any two distinct points $x_1, x_2 \in X$, there exist open neighborhoods U_1, U_2, of x_1, x_2 respectively, such that $U_1 \cap U_2 = \emptyset$.*

This is one of the separation axioms? Why is it an axiom? Are we **assuming** this to be true about $\mathbb{R}$? Is this equivalent to the axiom of choice, or completeness? This condition is stronger than the statement that finite point sets are closed (which is also called the T_1 axiom).

Asked in class, turns out these are sort of misnamed as axioms, and really are just definitions. Not on the same level as AoC, the bedrock on which much of modern mathematics lies.

Theorem 2.2. *Let X be a topological space satisfying the T_1 axiom. Let $A \subseteq X$. Then x is a limit point of A iff every neighborhood of x contains infinitely many points of A.*

For a finite set in a T_1 space, there cannot exist any limit points, since A doesn't have infinitely many points. This leads directly to A being closed, though, since it vacuously contains all of its limits points (which don't exist).

Proof. First, if every neighborhood of x contains infinitely many points of A, then every neighborhood of x must contain at least one other point of A than x. x is then a limit point.

Next, we consider x as a limit point of A, with X as above. By contradiction, assume that there exists a neighborhood U, open in X, and $B = U \cap A \setminus \{x\} = \{x_1, x_2, \dots, x_n\}$ is finite. Consider $C = X \setminus B$, which is an open neighborhood of x. Now, $D = U \cap C$ is also open, and contains x. By construction, this set D has empty intersection with $A \setminus \{x\}$. This is a contradiction, since x was assumed to be a limit point. Thus, every open neighborhood of x intersects A at infinitely many points. □

Theorem 2.3. *Let X be a Hausdorff topological space. Then, any sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to at most one point $x \in X$.*

Proof. Let $\{x_n\}$ be a sequence of points in X which converges to x. Let x' be an element of X, $x' \neq x$. Let U and V be disjoint open neighborhoods of x and x', respectively. Then, by definition of convergence, there exists some large, finite integer N such that $x_n \in U$, $\forall n > N$. Since U and V are disjoint, these points cannot be in V, and thus the sequence cannot converge to x'. □

Definition 2.5. *In a Hausdorff space, the unique point to which a convergent sequence converges is called* ***the limit****.*

Theorem 2.4. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is Hausdorff. Subspaces of Hausdorff spaces are Hausdorff.*

This particular theorem will come in handy for problems. Needs a user supplied proof, though.

3 Continuous Functions *<2020-02-07 Fri 09:46>*

Definition 3.1. *Let X, Y be topological spaces. A function $f : X \to Y$ is said to be* ***continuous*** *if for every open $V \subseteq Y$ the preimage $f^{-1}(V)$ is open in X.*