Homework Set 8

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1 Problem 1

Let X be limit point compact.

a) If $f : X \to Y$ is continuous, does it follow that $f(X)$ is limit point compact?

No. If X is compact, the image is compact, so we need an example which is limit point compact, but not compact. We take X as defined in Example 1 of section 28, i.e. $X = \mathbb{Z}_+ \times Y$, where Y is a two point set with the indiscrete topology. Then, the projection map in the first coordinate to $\mathbb{Z}_+$ is continuous, but maps the limit point compact space X to the non-limi point compact space $\mathbb{Z}_+$.

b) If A is a closed subset of X, does it follow that A is limit point compact?

Yes. An infinite subset of A has a limit point in X which is a limit point of A as well.

c) If X is a subspace of the Hausdorff space Z, does it follow that X is closed in Z?

We comment that it is not generally true that the product of two limit point copact spaces is itself limit point compact, even if the Hausdorff condition is assumed. But the examples are fairly sophisticated.

No. For a counterexample, we take the space from example 2: $S_\Omega \subseteq \overline{S}_\Omega$. S is not compact, but $\overline{S}_\Omega$ is both Hausdorff and compact, and thus limit point compact. Closedness would require that S_Ω also be compact, by theorem 26.3.

2 Problem 2

A space X is said to be *countably compact* if every countable open covering of X contains a finite subcollection that covers X. Show that for a T_1 space

X, countable compactness is equivalent to limit point compactness. [*Hint:* If no finite subcollection of U_n covers X, choose $x_n \notin U_1 \cup \ldots \cup U_n$ for each n.]

Proof. Let X be a limit point compact T_1 space and $\{U_n\}$ be a countable open covering of X such that there is no finite subcovering of X. Let $V_n = U_1 \cup \ldots \cup U_n$. Note that for every n, V_n does not cover X, but for every $x \in X$ there is a minimal n_x such that $x \in V_{n_x}$. Let $x_0 \in X$. For each $n \geq 1$, let $x_n \in X \setminus V_{n_{x_{n-1}}}$. This defines an infinite subset of X that must have a limit point a. But then the neighborhood V_{n_a} of a contains only a finite number of elements in the sequence, and for each of them that is not a we can find a neighborhood of a that does not contain it (since X is T_1). The finite intersection of all these neighborhoods with V_{n_a} is a neighborhood of a that does not contain any point of the sequence which is not a. This contradicts the fact that a is a limit point of the sequence, and thus X must be countably compact.

Next, suppose X is countably compact and Y is an infinite subset of X. There is a countably infinite subset $Z \subseteq Y$ and every limit point of Z is a limit point of Y, as well. If no point in Z is a limit point, then every point in Z has a neighborhood that does not contain any other points, and the countable collection of such neighborhoods covers Z. Since each set in the collection covers one point of Z and Z is infinite, there is no finite subcollection which also covers Z. Therefore, Z is not closed (as a closed subset of a countably compact space must also be countably compact). Thus, if Z is closed, then it must have a limit point, and therefore be limit point compact. $\square$

3 Problem 3

Let $\{X_\alpha\}$ be an indexed family of nonempty spaces.

a) Show that if $\prod X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

The projection map is continuous. Therefore, we may use the result of the next exercise (exercise 3, page 186) to argue that all X_α are compact. A compact subspace of the product containing an open set has all but finitely many projectionsequal to the whole corresponding space, since the projection is continuous, these spaces must be compact.

b) Prove the converse, assuming the Tychonoff theorem, that an arbitrary product of compact spaces is compact.

Assuming the Tychonoff theorem, all we need to prove is that the product of two locally compact spaces is locally compact. For any product of two locally compact spaces $X \times Y$, we can simply find the corresponding compact subsets and take their products. These product spaces are guaranteed to be compact, by theorem 26.7.

4 Problem 4

Prove that the one-point compactification of $\mathbb{R}$ is homeomorphic with the circle S^1.

Proof. The circle without a single point is homeomorphic to the real line. Next, we use two facts: first, that the one-point compactification of $\mathbb{R}$ is unique up to homeomorphism; second, that the compactification of the punctured circle is the whole circle. Thus, the one-point compactification of $\mathbb{R}$ is homeomorphic to the circle. □

5 Problem 5

Let C be the Cantor set.

a) Show that C is totally disconnected.

Proof. For any two points x, y, $x \neq y$, there exists n such that they cannot lie in the same closed interval A_n. Therefore, there is a point z beteen them which is not in C, and the set can be separated by the rays $(-\infty, z)$ and (z, ∞). □

b) Show that C is compact.

Proof. C is a closed subspace of a compact space, $[0, 1]$. Thus, by theorem 26.2, it is compact. □