Math 532 Notes

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Contents

1 More on the Product Topology

Sidenote: looks like I missed out some discussion on the metric topology last week when I was out before the exam.

Theorem 1.1. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$ where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each f_α is continuous.*

Proof. Suppose U_β is open in X_β. Then $\pi_\beta^{-1}(U_\beta)$ is a subbasis element for the product topology on $\prod X_\alpha$. So $\pi_\beta : \prod X_\alpha \to X_\beta$ is continuous if $\prod X_\alpha$ has the product topology.

Suppose $f : A \to \prod_{\alpha \in J} X_\alpha$ is continuous. Then $f_\alpha = (\pi_\alpha \cdot f)(a)$ is a composition of continuous functions, so is itself continuous.

Suppose each f_α is continuous for $\alpha \in J$. It suffices to show that the preimage of any subbasis element is open. Such a set has the form $\pi_\beta^{-1}(U_\beta)$ for some U_β open in X_β. So \$f-1(π_β-1(U$_\beta$)). Note: $f_\beta = \pi_{beta} \cdot f$. So $f_\beta^{-1}(S) = f^{-1}(\pi_\beta^{-1}(S)$. Thus, we have that $f_\beta^{-1}(U_\beta) = f^{-1}(\pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$, which is a known continuous function. Thus, the preimage of U_β is open, and f is continuous. $\square$

1.1 Example

Let $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} X_n$ where $X_n = \mathbb{R}$ for $n \in \mathbb{Z}_+$. Define $f : \mathbb{R} \to \mathbb{R}^\omega$ as $f(t) = (t, t, t, \ldots)$. Note: each $f_n(t) = t$ is continuous in the standard topology on $\mathbb{R}$. By the previous theorem, if we give $\mathbb{R}^\omega$ the product topology, then f is continuous. But, f is **not** continuous in the box topology. Consider the set $B = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots$.

We claim that $f^{-1}(B)$ is not open in $\mathbb{R}$. If $f^{-1}(B)$ were open then given $x_0 \in f^{-1}(B), \exists \delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(B)$. Consider $x_0 = 0$. Then if the preimage is open, $\exists \delta > 0$ such that $(-\delta, \delta) \subseteq f^{-1}(B)$. Then, $\forall n$, $f_n((-\delta, \delta)) \subseteq (-1/n, 1/n)$. But $f((-\delta, \delta)) = (-\delta, \delta)$, for all n. This is impossible, and thus a contradiction. $f^{-1}(B)$ is not open.

This demonstrates that the box topology is not a good candidate for working with curves in infinite dimensions. Basically, this motivates the funky definition for the product topology in infinite dimensions.

2 The Metric Topology

Definition 2.1. *A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that*

1. $d(x, y) \geq 0, \forall x, y \in X, d(x, y) = 0 iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$. $(a, b) \subseteq mathbbR$ is equivalent to the ball $B((a + b)/2, (b - a)/2)$.

Definition 2.2. *If d is a metric on a set X, then the collection of all \$$\epsilon$\$-balls $B_d(x, \epsilon)$ for all $x \in X, \epsilon > 0$ is a basis for a topology on X called the metric topology induced by d.*

Here we see that the standard topology on the reals is the same as the metric topology on the reals induced by the standard metric $d(x, y) = |x - y|$. This is also the same as the order topology on the reals.

Given any set X, we can always define a metric as $d(x, y) = 1$ if $x \neq y$ and 0 otherwise. Note: $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$, which for this case would be **only** x. This induces the discrete topology.

Definition 2.3. *If X is a topological space, X is said to be **metrizable** if there exists a metric d on X which induces the topology of X. A {metric space} is a metrizable space X together with a metric d which induces the topology of X.*

Definition 2.4. *$A \subseteq X$, a metric space, is bounded if $\exists M > 0$ such that $d(a_1, a_2) < M, \forall a_1, a_2 \in A$.*

If A is nonempty and bounded, then diam $A = sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

We can always bound a metric and get the same topology.

Definition 2.5. *If X is a metric space with metric d, define $\vec{d} : X \times X \to \mathbb{R}$ as $\vec{d}(x, y) = min\{d(x, y), 1\}$. Then $\vec{d}$ is a metric that induces the same topology as d. $\vec{d}$ is called the **standard bounded metric** corresponding to d.*

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Definition 3.1. *Let $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The **norm** of $\vec{x}$, $||\vec{x}|| = \sqrt{x_1^2 + \ldots + x_n^2}$. The {Euclidean metric} on $\mathbb{R}^n$ is $d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$. The **square metric** is $\rho(\vec{x}, \vec{y}) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.*

Lemma 3.1. *Let d, d' be two metrics on X. Let T, T' be the topologies they induce. Then T' is finer than T iff $\forall x \in X, \forall \epsilon > 0$, \$∃ δ > 0 \$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.*

Proof. Let T' be finer than T. Given a basis element $B_d(x, \epsilon)$ of T, then there exists a basis element $x \in B' \subseteq B_d(x, \epsilon)$. Then, within B' we can find $B_{d'}(x, \delta)$ centered at x, with $\delta > 0$.

Conversely, suppose the $\epsilon - \delta$ holds. Given a basis element B of T containing x. Within this element we can find $B_d(x, \epsilon) \subseteq B$. Then, by assumption, $\exists \delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$. So T' is finer than T. □

Theorem 3.1. *The topologies on $\mathbb{R}^n$ induced by the Euclidean metric and the square metric are the same, and they both give the product topology.*

Proof. By "algebra", $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$. Then, we simply apply the above Lemma to show that we can always subset one into the other, since their elements are always at most a finite different in size.

Next, we show that each basis element $B = (a_1, b_1) \times \ldots \times (a_n, b_n)$ in the product topology is open in both metrics (and the converse). $\square$

The next question we have to ask ourselves is: "How can we put a metric on $\mathbb{R}^\omega$?" Maybe, we can try $d(x, y) = (\sum_{i=1}^{\infty}(x_i - y_i)^2)^{1/2}$, or $\rho(x, y) = \sup\{|x_i - y_i| : n \in \mathbb{N}\}$. But, these won't work. The difficulty is that we want a metric which induces the product topology.

4 The Uniform Metric/Topology

Definition 4.1. *Given any indexed set J, let $\mathbb{R}^J$ be the set of all functions from J to $\mathbb{R}$. (So $\mathbb{R}^\omega$ is all countable sequences of reals). Given $(x_\alpha)_{\alpha \in J}$, $(y_\alpha)_{\alpha \in J} \in \mathbb{R}^J$, define a metric \$$\overline{\rho} as follows : \overline{\rho}(x, y) = sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$, where $\overline{d}$ is the bounded metric. This metric $\overline{\rho}$ is called the **Uniform Metric** on $\mathbb{R}^J$, and the topology it induces is called the **Uniform Topology**.*

Theorem 4.1. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology, and coarser than the box topology. These three are different if J is infinite.*

Proof. Suppose $\prod U_\alpha$ is open in the product topology (even better, assume it's a basis element). Let $\alpha_1, \ldots, \alpha_n$ be the indices so that $U_\alpha \neq \mathbb{R}$. Let $x \in \prod U_\alpha$. Pick, for each i, $(1 \leq i \leq n)$, choose $\epsilon_i > 0$ such that $B_{\overline{d}}(x_{\alpha_i}, \epsilon_i) \subseteq U_{\alpha_i}$. $\square$