Homework Set 6

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1 Problem 1

Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point $x \in X$ such that $f(x) = x$. The point x is called a fixed point of f. What happens if X is $[0, 1)$ or $(0, 1]$?

Proof. In all three cases considered, X is connected. By construction, $f(0) \geq 0$ and $f(1) \leq 1$. Next, we construct a new function $g(x) = f(x) - x$. This function is also continuous, and we have that $g(0) > 0$ and $g(1) < 0$. By the intermediate value theorem, we now have that $\exists x \in (0, 1)$ such that $g(x) = 0$, or $f(x) = x$. □

If we remove the endpoints from X, we are not guaranteed to find a construction g which enables this proof.

2 Problem 2

Let X be an ordered set in the order topology. Show that if X is connected, then X is a linear continuum.

Proof. A linear continuum is an ordered set with two additional properties: the least upper bound property, and that for any $x, y \in X, x < y$, $\exists z \in X$ such that $x < z < y$.

It is trivial that the second property holds for X, for if it didn't, X would contain a separation.

To show the least upper bound property, we consider another set A. If A is non-empty and has an upper bound u but no least upper bound, consider the sets $A_1 = \{x \in X \mid \exists y(x) \in A : x \leq y(x)\}$ and $A_2 = X \setminus A_1$. As $A \subset A_1$, it is non-empty. As u cannot be in A (as an upper bound for A that is in A is a maximum which is also a least upper bound), $u > y$ for all $y \in A$, so $u \in A_2$, so A_2 is non-empty as well. Both of these sets are disjoint, and cover X by construction.

A_1 is open, as for all $x \in A_1$, $x \in (-\infty, y(x)) \subset A_1$. If $x = y(x) \in A$, we know that some $x' \in A$ must exist with $x' > x$, as otherwise x would be a maximum of A, which we have assumed not to exist. Then $x \in (-\infty, x') \subset A_1$ as well.

A_2 is open, because if $x \in A_2$, x is an upper bound for A, but it cannot be the smallest such (as A has been assumed to have no smallest upper bound), so there is a smaller upper bound z for A and clearly $x \in (z, \infty) \subset A_2$.

X having a cover by two open sets contradicts its connectedness, and so we conclude that X must have a least upper bound.

By both portions, we have shown that X satisfies the properties of a linear continuum. X is a linear continuum. □

3 Problem 3

a) Is a product of path-connected spaces necessarily path connected? b) If $A \subseteq X$ and A is path connected, is $\overline{A}$ necessarily path connected? c) If $f : X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected? d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$ is $\bigcup A_\alpha$ necessarily path connected?

a)

Proof. Let x and y be two points in the product. We can construct a curve $z(t)$ between them as $t \in [0, 1]$, and each $z_i(t)$ is a path connecting x_i and y_i within X_i. Since each of the X_i is path connected, these maps z_i are continuous. Then, by theorem 19.6, the overall map $z(t)$ is also continuous. □

b) No. A counterexample is the "Topologist's sine curve", the set of points $T = \{(x, sin(1/x)) | x \in (0, 1]\} \cup \{(0, 0)\}$.

c) Yes, the composition of two continuous functions is itself continuous, and continuous maps preserve connectedness.

d)

Proof. Take any two points. If they are in the same set in the collection, there is a path between them. If they are not in the same set in the collection then there is a path connecting the first point to a common point of all sets in the collection and another one connecting the common point to the second point, the joint path is still continuous and is a path connecting the point. □

4 Problem 4

Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Proof. Let S be a compact subspace of the metric space X. Pick a point $x \in S$. Now, we consider the cover $\{B(x, n) \mid n \in \mathbb{N}\}$, where $B(x, n)$ denotes the open ball centered at x of radius n. We then simply note that we will have a finite subcover, since S is compact. Choosing k as the largest index to the cover, such that $B(x, k)$ is the largest ball in the cover, we see that S will be contained in $B(x, k)$, and thus it is bounded. It must also be closed: otherwise, $B(x, k-1) \cup (B(x, k) \setminus S$ would be a cover of S, and it would not be compact. □

As an example of a metric space in which not every closed bounded subspace is compact, take $\mathbb{R}$ with the discrete metric $d(x, y) = 1$ iff $x \neq y$.

5 Problem 5

Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets).

Proof. Let A be closed in X. Then A, by theorem 26.2, A is compact. f is continuous, so $f(A)$ must be compact, since A is compact. Then, by theorem 26.3, $f(A)$ is closed. Since A was arbitrary, this holds for any closed set in X. f is a closed map. □

6 Problem 6

Assume that $\mathbb{R}$ is uncountable. Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus A$ is path connected. [Hint: How many lines are there passing through a given point of $\mathbb{R}^2$?]

Proof. For any point there is an uncountable number of lines passing through it which do not intersect A. For any two points there is a pair of lines that do intersect each other but do not intersect the set A. So, both points are connected to the point of intersection of the lines, and, therefore, are connected. □