

Chapter 2

Signal and Linear System Analysis

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2.1 Signal Models

2.1.1 Deterministic and Random Signals

- Deterministic Signals, used for this course, can be modeled as completely specified functions of time, e.g.,

$$x(t) = A(t) \cos[2\pi f_0(t)t + \phi(t)]$$

- Note that here we have also made the amplitude, frequency, and phase functions of time
- To be deterministic each of these functions must be completely specified functions of time
- Random Signals, used extensively in Comm Systems II, take on random values with known probability characteristics, e.g.,

$$\mathbf{x}(t) = x(t, \zeta_i)$$

where ζ_i corresponds to an elementary outcome from a sample space in probability theory

- The ζ_i create an ensemble of sample functions $x(t, \zeta_i)$, depending upon the outcome drawn from the sample space

2.1.2 Periodic and Aperiodic Signals

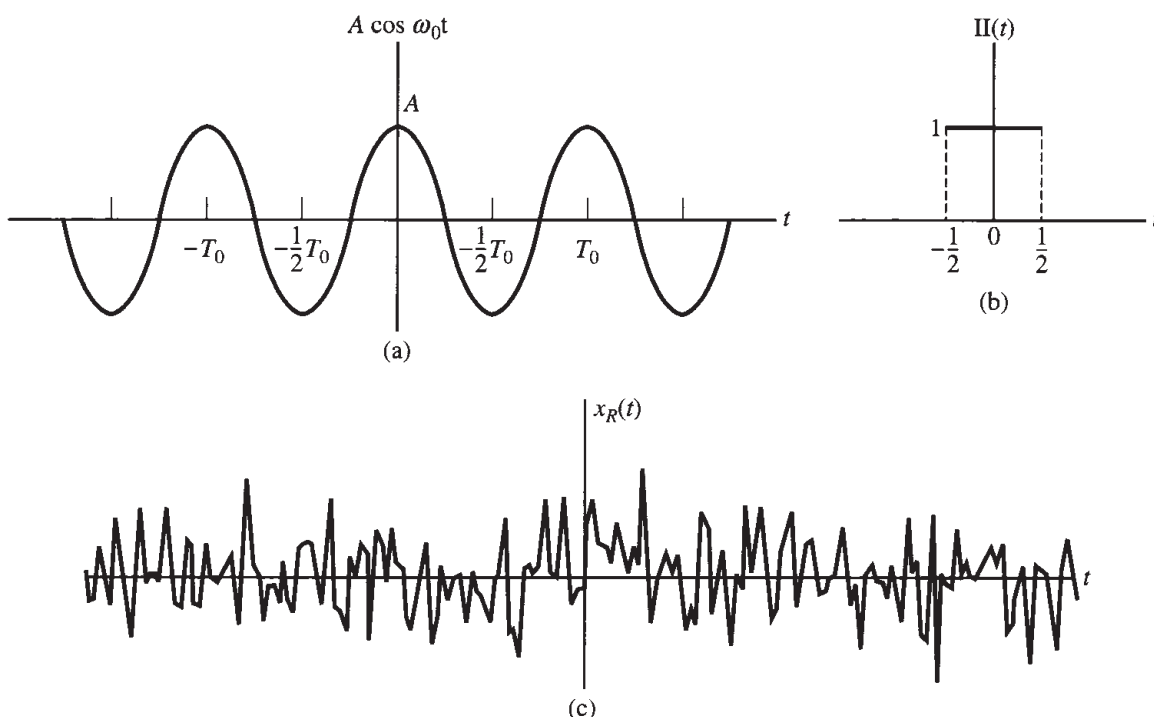
- A deterministic signal is *periodic* if we can write

$$x(t + nT_0) = x(t)$$

for any integer n , with T_0 being the signal fundamental period

- A signal is *aperiodic* otherwise, e.g.,

$$\Pi(t) = \begin{cases} 1, & |t| \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$



(a) periodic signal, (b) aperiodic signal, (c) random signal

2.1.3 Phasor Signals and Spectra

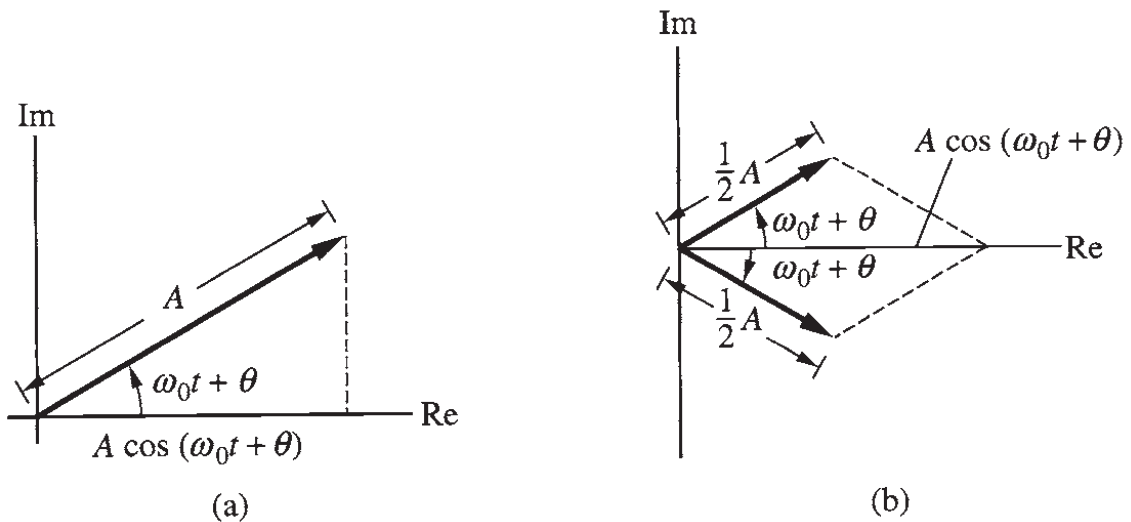
- A complex sinusoid can be viewed as a rotating phasor

$$\tilde{x}(t) = Ae^{j(\omega_0 t + \theta)}, \quad -\infty < t < \infty$$

- This signal has three parameters, amplitude A , frequency ω_0 , and phase θ
- The fixed phasor portion is $Ae^{j\theta}$ while the rotating portion is $e^{j\omega_0 t}$

- This signal is periodic with period $T_0 = 2\pi/\omega_0$
- It is also related to the real sinusoid signal $A \cos(\omega_0 t + \theta)$ via Euler's theorem

$$\begin{aligned}
 x(t) &= \operatorname{Re}\{\tilde{x}(t)\} \\
 &= \operatorname{Re}\{A \cos(\omega_0 t + \theta) + jA \sin(\omega_0 t + \theta)\} \\
 &= A \cos(\omega_0 t + \theta)
 \end{aligned}$$



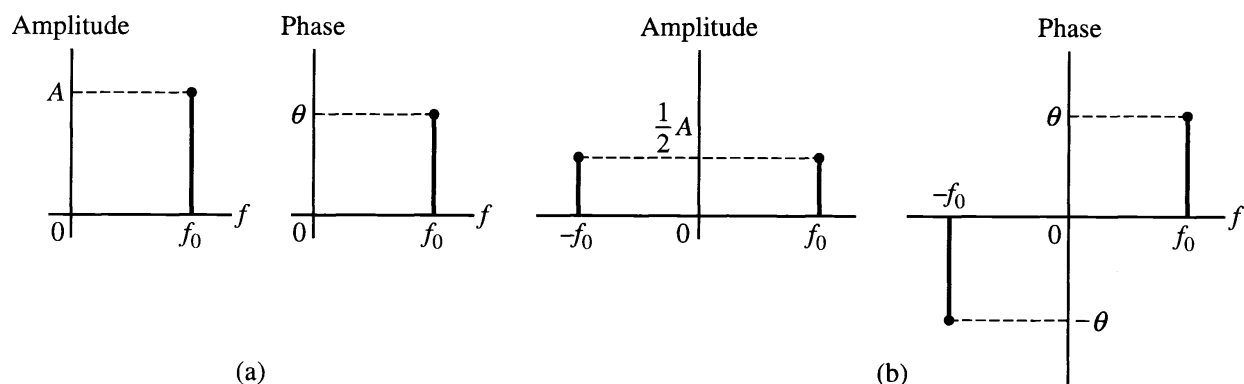
(a) obtain $x(t)$ from $\tilde{x}(t)$, (b) obtain $x(t)$ from $\tilde{x}(t)$ and $\tilde{x}^*(t)$

- We can also turn this around using the inverse Euler formula

$$\begin{aligned}
 x(t) &= A \cos(\omega_0 t + \theta) \\
 &= \frac{1}{2} \tilde{x}(t) + \frac{1}{2} \tilde{x}^*(t) \\
 &= \frac{A e^{j(\omega_0 t + \theta)} + A e^{-j(\omega_0 t + \theta)}}{2}
 \end{aligned}$$

- The frequency spectra of a real sinusoid is the line spectra plotted in terms of the amplitude and phase versus frequency

- The relevant parameters are A and θ for a particular $f_0 = \omega_0/(2\pi)$



- Both the single-sided and double-sided line spectra, shown above, correspond to the real signal $x(t) = A \cos(2\pi f_0 t + \theta)$

Example 2.1: Multiple Sinusoids

- Suppose that

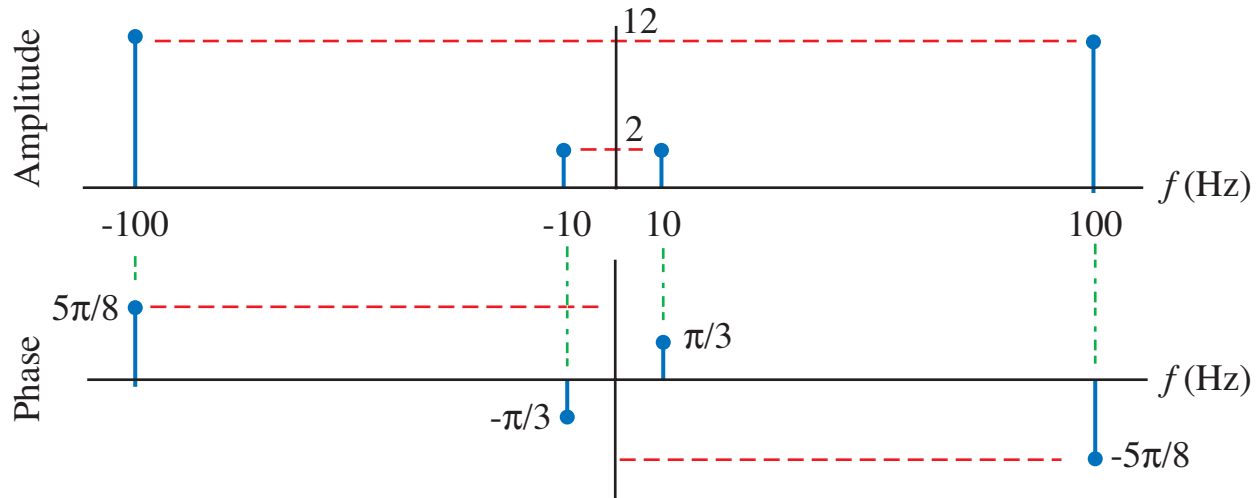
$$x(t) = 4 \cos(2\pi(10)t + \pi/3) + 24 \sin(2\pi(100)t - \pi/8)$$

- Find the two-sided amplitude and phase line spectra of $x(t)$
- First recall that $\cos(\omega_0 t - \pi/2) = \sin(\omega_0 t)$, so

$$x(t) = 4 \cos(2\pi(10)t + \pi/3) + 24 \cos(2\pi(100)t - 5\pi/8)$$

- The complex sinusoid form is directly related to the two-sided line spectra since each real sinusoid is composed of positive and negative frequency complex sinusoids

$$x(t) = 2 \left[e^{j(2\pi(10)t + \pi/3)} + e^{-j(2\pi(10)t + \pi/3)} \right] + 12 \left[e^{j(2\pi(100)t - 5\pi/8)} + e^{-j(2\pi(100)t - 5\pi/8)} \right]$$



Two-sided amplitude and phase line spectra

2.1.4 Singularity Functions

Unit Impulse (Delta) Function

- Singularity functions, such as the delta function and unit step
- The *unit impulse function*, $\delta(t)$ has the operational properties

$$\int_{t_1}^{t_2} \delta(t - t_0) dt = 1, \quad t_1 < t_0 < t_2$$

$$\delta(t - t_0) = 0, \quad t \neq t_0$$

which implies that for $x(t)$ continuous at $t = t_0$, the *sifting property* holds

$$\int_{-\infty}^{\infty} x(t) \delta(t - t_0) dt = x(t_0)$$

- Alternatively the unit impulse can be defined as

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0)$$

- Properties:

1. $\delta(at) = \delta(t)/|a|$
2. $\delta(-t) = \delta(t)$
3. Sifting property special cases

$$\int_{t_1}^{t_2} x(t) \delta(t - t_0) dt = \begin{cases} x(t_0), & t_1 < t_0 < t_2 \\ 0, & \text{otherwise} \\ \text{undefined,} & t_0 = t_1 \text{ or } t_0 = t_2 \end{cases}$$

4. Sampling property

$$x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0)$$

for $x(t)$ continuous at $t = t_0$

5. Derivative property

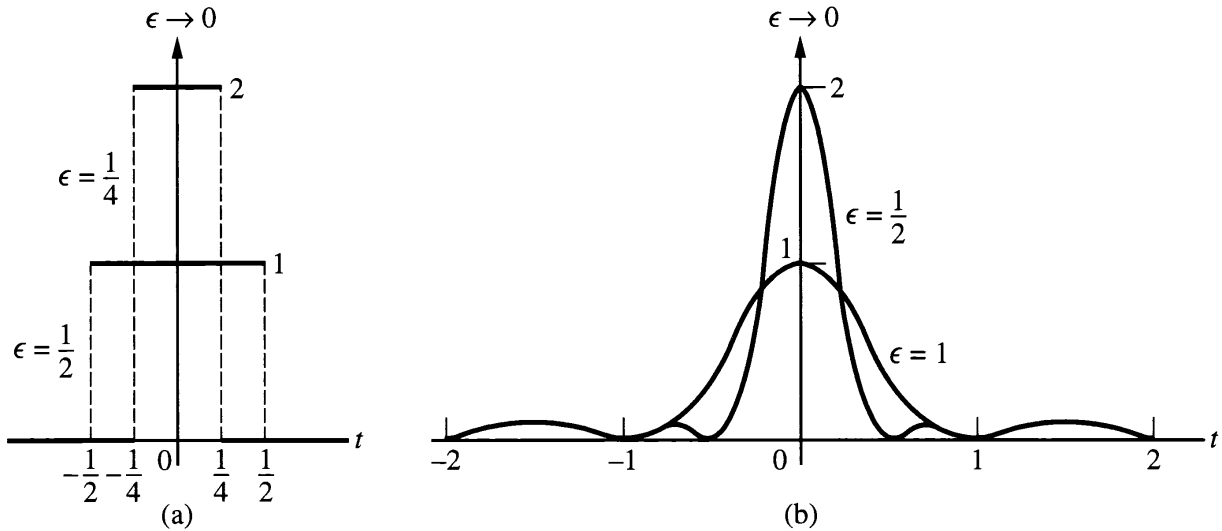
$$\begin{aligned} \int_{t_1}^{t_2} x(t) \delta^{(n)}(t - t_0) dt &= (-1)^n x^{(n)}(t_0) \\ &= (-1)^n \left. \frac{d^n x(t)}{dt^n} \right|_{t=t_0} \end{aligned}$$

Note: Dealing with the derivative of a delta function requires care

- A *test* function for the unit impulse function helps our intuition and also helps in problem solving
- Two functions of interest are

$$\delta_\epsilon(t) = \frac{1}{2\epsilon} \Pi\left(\frac{t}{2\epsilon}\right) = \begin{cases} \frac{1}{2\epsilon}, & |t| \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{1\epsilon}(t) = \epsilon \left(\frac{1}{\pi t} \sin \frac{\pi t}{\epsilon} \right)^2$$



Test functions for the unit impulse $\delta(t)$: (a) $\delta_\epsilon(t)$, (b) $\delta_{1\epsilon}(t)$

- In both of the above test functions letting $\epsilon \rightarrow 0$ results in a function having the properties of a true delta function

Unit Step Function

- The unit step function can be defined in terms of the unit impulse

$$u(t) \equiv \int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \\ \text{undefined,} & t = 0 \end{cases}$$

also

$$\delta(t) = \frac{du(t)}{dt}$$

Example 2.2: Unit Impulse 1st-Derivative

- Consider

$$\int_{-\infty}^{\infty} x(t) \delta'(t) dt$$

- Using the rectangular pulse test function, $\delta_{\epsilon}(t)$, we note that

$$\delta_{\epsilon}(t) = \frac{1}{2\epsilon} \Pi\left(\frac{t}{2\epsilon}\right) \stackrel{\text{also}}{=} \frac{1}{2\epsilon} [u(t + \epsilon) - u(t - \epsilon)]$$

and

$$\frac{d\delta_{\epsilon}(t)}{dt} = \frac{1}{2\epsilon} [\delta(t + \epsilon) - \delta(t - \epsilon)]$$

- Placing the above in the integrand with $x(t)$ we obtain, with the aid of the sifting property, that

$$\begin{aligned} \int_{-\infty}^{\infty} x(t) \delta'(t) dt &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} [x(t + \epsilon) - x(t - \epsilon)] \\ &= \lim_{\epsilon \rightarrow 0} \frac{-[x(t - \epsilon) - x(t + \epsilon)]}{2\epsilon} \\ &= -x'(0) \end{aligned}$$

2.2 Signal Classifications

- From circuits and systems we know that a real voltage or current waveform, $e(t)$ or $i(t)$ respectively, measured with respect to a real resistance R , the instantaneous power is

$$P(t) = e(t)i(t) = i^2(t)R \text{ W}$$

- On a per-ohm basis, we obtain

$$p(t) = P(t)/R = i^2(t) \text{ W/ohm}$$

- The average energy and power can be obtained by integrating over the interval $|t| \leq T$ with $T \rightarrow \infty$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T i^2(t) dt \text{ Joules/ohm}$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T i^2(t) dt \text{ W/ohm}$$

- In system engineering we take the above energy and power definitions, and extend them to an arbitrary signal $x(t)$, possibly complex, and define the normalized energy (e.g. 1 ohm system) as

$$E \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

- Signal Classes:

1. $x(t)$ is an *energy signal* if and only if $0 < E < \infty$ so that $P = 0$
2. $x(t)$ is a *power signal* if and only if $0 < P < \infty$ which implies that $E \rightarrow \infty$

Example 2.3: Real Exponential

- Consider $x(t) = Ae^{-\alpha t}u(t)$ where α is real
- For $\alpha > 0$ the energy is given by

$$\begin{aligned} E &= \int_0^{\infty} (Ae^{-\alpha t})^2 dt = \left. \frac{A^2 e^{-2\alpha t}}{-2\alpha} \right|_0^{\infty} \\ &= \frac{A^2}{2\alpha} \end{aligned}$$

- For $\alpha = 0$ we just have $x(t) = Au(t)$ and $E \rightarrow \infty$
- For $\alpha < 0$ we also have $E \rightarrow \infty$
- In summary, we conclude that $x(t)$ is an energy signal for $\alpha > 0$
- For $\alpha > 0$ the power is given by

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \frac{A^2}{2\alpha} (1 - e^{-\alpha T}) = 0$$

- For $\alpha = 0$ we have

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \cdot A^2 T = \frac{A^2}{2}$$

- For $\alpha < 0$ we have $P \rightarrow \infty$
 - In summary, we conclude that $x(t)$ is a power signal for $\alpha = 0$
-

Example 2.4: Real Sinusoid

- Consider $x(t) = A \cos(\omega_0 t + \theta)$, $-\infty < t < \infty$
- The signal energy is infinite since upon squaring, and integrating over one cycle, $T_0 = 2\pi/\omega_0$, we obtain

$$\begin{aligned}
 E &= \lim_{N \rightarrow \infty} \int_{-NT_0/2}^{NT_0/2} A^2 \cos^2(\omega_0 t + \theta) dt \\
 &= \lim_{N \rightarrow \infty} N \int_{-T_0/2}^{T_0/2} A^2 \cos^2(\omega_0 t + \theta) dt \\
 &= \lim_{N \rightarrow \infty} N \frac{A^2}{2} \int_{-T_0/2}^{T_0/2} [1 + \cos(2\omega_0 t + 2\theta)] dt \\
 &= \lim_{N \rightarrow \infty} N \frac{A^2}{2} \cdot T_0 \rightarrow \infty
 \end{aligned}$$

- The signal average power is finite since the above integral is normalized by $1/(NT_0)$, i.e.,

$$P = \lim_{N \rightarrow \infty} \frac{1}{NT_0} \cdot N \frac{A^2}{2} \cdot T_0 = \frac{A^2}{2}$$

2.3 Generalized Fourier Series

The goal of generalized Fourier series is to obtain a representation of a signal in terms of points in a *signal space* or abstract vector space. The coordinate vectors in this case are orthonormal functions. The complex exponential Fourier series is a special case.

- Let \vec{A} be a vector in a three dimensional vector space
- Let \vec{a}_1, \vec{a}_2 , and \vec{a}_3 be *linearly independent* vectors in the same three dimensional space, then

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = \mathbf{0} \text{ (zero vector)}$$

only if the constants $c_1 = c_2 = c_3 = 0$

- The vectors \vec{a}_1, \vec{a}_2 , and \vec{a}_3 also *span* the three dimensional space, that is for any vector \vec{A} there exists a set of constants c_1, c_2 , and c_3 such that

$$\vec{A} = c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3$$

- The set $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ forms a *basis* for the three dimensional space
- Now let $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ form an *orthogonal* basis, which implies that

$$\vec{a}_i \cdot \vec{a}_j = (\vec{a}_i, \vec{a}_j) = \langle \vec{a}_i, \vec{a}_j \rangle = 0, \quad i \neq j$$

which says the basis vectors are *mutually orthogonal*

- From analytic geometry (and linear algebra), we know that we can find a representation for \vec{A} as

$$\vec{A} = \frac{(\vec{a}_1 \cdot \vec{A})}{|\vec{a}_1|^2} + \frac{(\vec{a}_2 \cdot \vec{A})}{|\vec{a}_2|^2} + \frac{(\vec{a}_3 \cdot \vec{A})}{|\vec{a}_3|^2}$$

which implies that

$$\vec{A} = \sum_{i=1}^3 c_i \vec{a}_i$$

where

$$c_i = \frac{\vec{a}_i \cdot \vec{A}}{|\vec{a}_i|^2}, \quad i = 1, 2, 3$$

is the component of \vec{A} in the \vec{a}_i direction

- We now extend the above concepts to a set of orthogonal functions $\{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\}$ defined on $t_0 \leq t \leq t_0 + T$, where the dot product (inner product) associated with the ϕ_n 's is

$$\begin{aligned} (\phi_m(t), \phi_n(t)) &= \int_{t_0}^{t_0+T} \phi_m(t) \phi_n^*(t) dt \\ &= c_n \delta_{mn} = \begin{cases} c_n, & n = m \\ 0, & n \neq m \end{cases} \end{aligned}$$

- The ϕ_n 's are thus orthogonal on the interval $[t_0, t_0 + T]$
- Moving forward, let $x(t)$ be an arbitrary function on $[t_0, t_0 + T]$, and consider approximating $x(t)$ with a linear combination of ϕ_n 's, i.e.,

$$x(t) \simeq x_a(t) = \sum_{n=1}^N X_n \phi_n(t), \quad t_0 \leq t \leq t_0 + T,$$

where a denotes approximation

- A measure of the approximation error is the integral squared error (ISE) defined as

$$\epsilon_N = \int_T |x(t) - x_a(t)|^2 dt,$$

where \int_T denotes integration over any T long interval

- To find the X_n 's giving the minimum ϵ_N we expand the above integral into three parts (see homework problems)

$$\begin{aligned} \epsilon_N = & \int_T |x(t)|^2 dt - \sum_{n=1}^N \frac{1}{c_n} \left| \int_T x(t) \phi_n^*(t) dt \right|^2 \\ & + \sum_{n=1}^N c_n \left| X_n - \frac{1}{c_n} \int_T x(t) \phi_n^*(t) dt \right|^2 \end{aligned}$$

- Note that the first two terms are independent of the X_n 's and the last term is nonnegative (missing steps are in text homework problem 2.14)
- We conclude that ϵ_N is minimized for each n if each element of the last term is made zero by setting

$$X_n = \frac{1}{c_n} \int_T x(t) \phi_n^*(t) dt \quad \text{Fourier Coefficient}$$

- This also results in

$$(\epsilon_N)_{\min} = \int_T |x(t)|^2 dt - \sum_{n=1}^N c_n |X_n|^2$$

- **Definition:** The set of ϕ_n 's is *complete* if

$$\lim_{N \rightarrow \infty} (\epsilon_N)_{\min} = 0$$

for $\int_T |x(t)|^2 dt < \infty$

- Even if though the ISE is zero when using a complete set of orthonormal functions, there may be isolated points where $x(t) - x_a(t) \neq 0$

- Summary

$$x(t) = \text{l.i.m.} \sum_{n=1}^{\infty} X_n \phi_n(t)$$

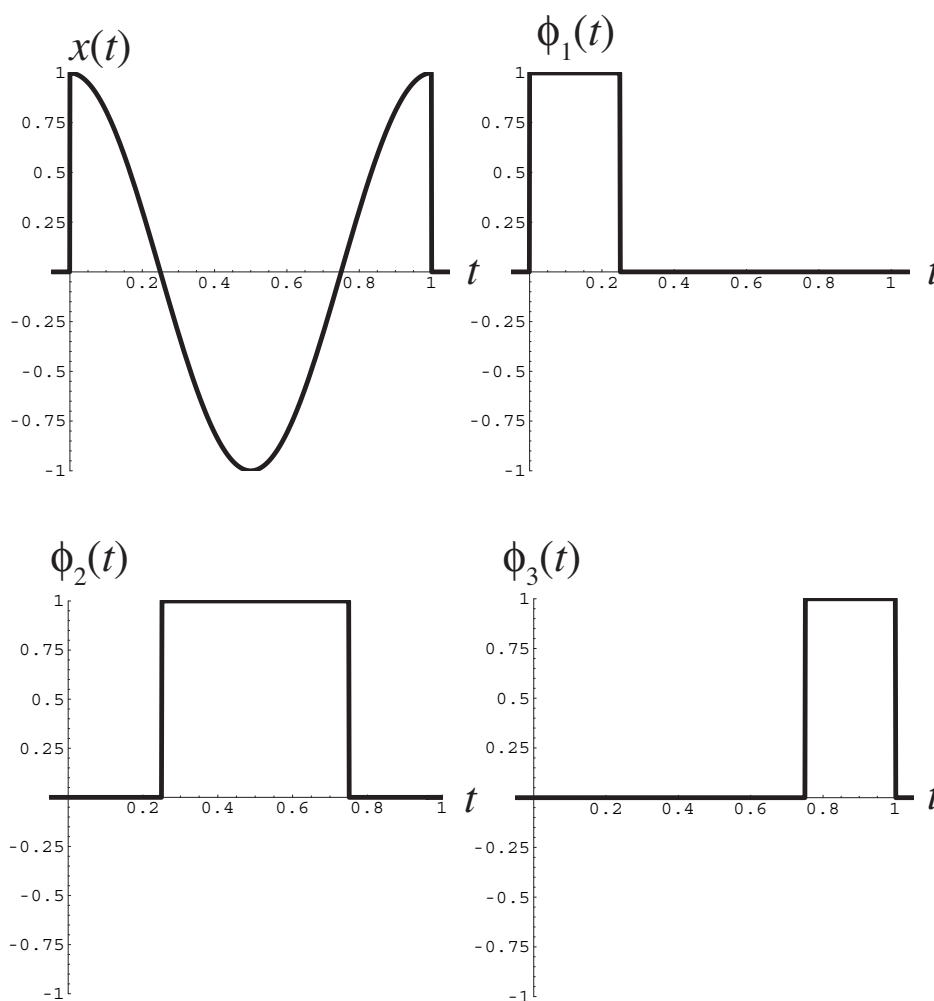
$$X_n = \frac{1}{c_n} \int_T x(t) \phi_n^*(t) dt$$

- The notation l.i.m. stands for *limit in the mean*, which is a mathematical term referring to the fact that ISE is the convergence criteria
- Parseval's theorem: A consequence of completeness is

$$\int_T |x(t)|^2 dt = \sum_{n=1}^{\infty} c_n |X_n|^2$$

Example 2.5: A Three Term Expansion

- Approximate the signal $x(t) = \cos 2\pi t$ on the interval $[0, 1]$ using the following basis functions



Signal $x(t)$ and basis functions $\phi_i(t)$, $i = 1, 2, 3$

- To begin with it should be clear that $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ are mutually orthogonal since the integrand associated with the inner product, $\phi_i(t) \cdot \phi_j^*(t) = 0$, for $i \neq j$, $i, j = 1, 2, 3$

- Before finding the X_n 's we need to find the c_n 's

$$c_1 = \int_T |\phi_1(t)|^2 dt = \int_0^{1/4} |1|^2 dt = 1/4$$

$$c_2 = \int_T |\phi_2(t)|^2 dt = 1/2$$

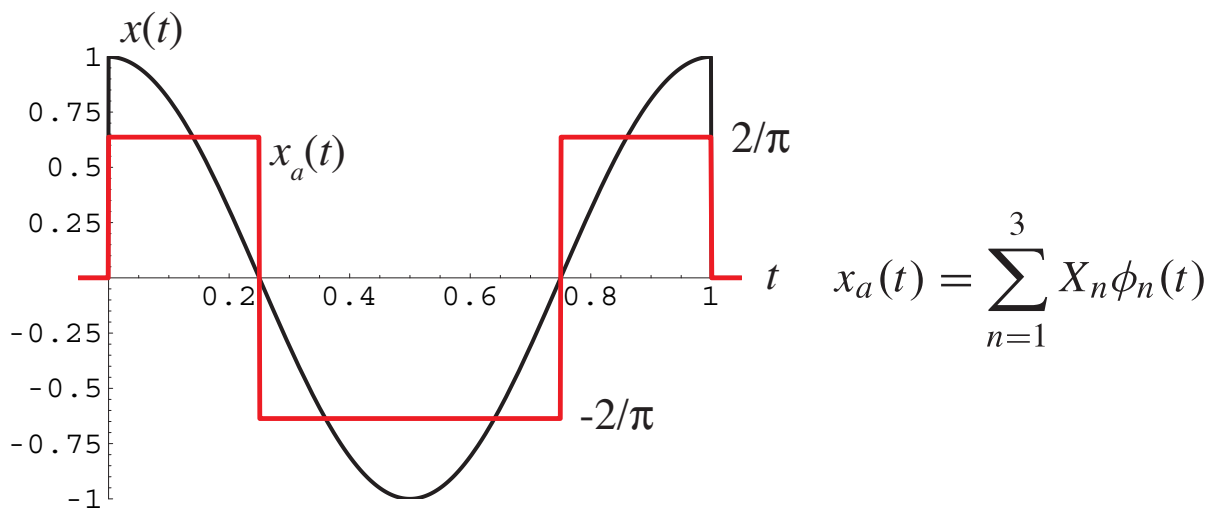
$$c_3 = \int_T |\phi_3(t)|^2 dt = 1/4$$

- Now we can compute the X_n 's:

$$\begin{aligned} X_1 &= 4 \int_T x(t) \phi_1^*(t) dt \\ &= 4 \int_0^{1/4} \cos(2\pi t) dt = \frac{2}{\pi} \sin(2\pi t) \Big|_0^{1/4} = \frac{2}{\pi} \end{aligned}$$

$$X_2 = 2 \int_{1/4}^{3/4} \cos(2\pi t) dt = \frac{1}{\pi} \sin(2\pi t) \Big|_{1/4}^{3/4} = \frac{-2}{\pi}$$

$$X_3 = 4 \int_{3/4}^1 \cos(2\pi t) dt = \frac{2}{\pi} \sin(2\pi t) \Big|_{3/4}^1 = \frac{2}{\pi}$$



Functional approximation

- The integral-squared error, ϵ_N , can be computed as follows:

$$\begin{aligned}
 \epsilon_N &= \int_T \left| x(t) - \sum_{n=1}^3 X_n \phi_n(t) \right|^2 dt \\
 &= \int_T |x(t)|^2 dt - \sum_{n=1}^3 c_n |X_n|^2 \\
 &= \frac{1}{2} - \frac{1}{4} \left| \frac{2}{\pi} \right|^2 - \frac{1}{2} \left| \frac{2}{\pi} \right|^2 - \frac{1}{4} \left| \frac{2}{\pi} \right|^2 \\
 &= \frac{1}{2} - \left| \frac{2}{\pi} \right|^2 = 0.0947
 \end{aligned}$$

2.4 Fourier Series

When we choose a particular set of basis functions we arrive at the more familiar Fourier series.

2.4.1 Complex Exponential Fourier Series

- A set of ϕ_n 's that is complete is

$$\phi_n(t) = e^{jn\omega_0 t}, \quad n = 0, \pm 1, \pm 2, \dots$$

over the interval $(t_0, t_0 + T_0)$ where $\omega_0 = 2\pi/T_0$ is the period of the expansion interval

proof of orthogonality

$$\begin{aligned}
 (\phi_m(t), \phi_n(t)) &= \int_{t_0}^{t_0+T_0} e^{jm\frac{2\pi t}{T_0}} e^{-jn\frac{2\pi t}{T_0}} dt = \int_{t_0}^{t_0+T_0} e^{j\frac{2\pi}{T_0}(m-n)t} dt \\
 &= \begin{cases} \int_{t_0}^{t_0+T_0} dt, & m = n \\ \int_{t_0}^{t_0+T_0} [\cos[2\pi(m-n)t/T_0] \\ + j \sin[2\pi(m-n)t/T_0]] dt, & m \neq n \end{cases} \\
 &= \begin{cases} T_0, & m = n \\ 0, & m \neq n \end{cases}
 \end{aligned}$$

We also conclude that $c_n = T_0$

- Complex exponential Fourier series summary:

$$\begin{aligned}
 x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \quad t_0 \leq t \leq t_0 + T_0 \\
 \text{where } X_n &= \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t}
 \end{aligned}$$

- The Fourier series expansion is unique

Example 2.6: $x(t) = \cos^2 \omega_0 t$

- If we expand $x(t)$ into complex exponentials we can immediately determine the Fourier coefficients

$$\begin{aligned}
 x(t) &= \frac{1}{2} + \frac{1}{2} \cos 2\omega_0 t \\
 &= \frac{1}{2} + \frac{1}{4} e^{j2\omega_0 t} + \frac{1}{4} e^{-j2\omega_0 t}
 \end{aligned}$$

- The above implies that

$$X_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{1}{4}, & n = \pm 2 \\ 0, & \text{otherwise} \end{cases}$$

Time Average Operator

- The time average of signal $v(t)$ is defined as

$$\langle v(t) \rangle \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T v(t) dt$$

- Note that

$$\langle av_1(t) + bv_2(t) \rangle = a\langle v_1(t) \rangle + b\langle v_2(t) \rangle,$$

where a and b are arbitrary constants

- If $v(t)$ is periodic, with period T_0 , then

$$\langle v(t) \rangle = \frac{1}{T_0} \int_{T_0} v(t) dt$$

- The Fourier coefficients can be viewed in terms of the time average operator
- Let $v(t) = x(t)e^{-jn\omega_0 t}$ using $e^{-j\theta} = \cos \theta - j \sin \theta$, we find that

$$\begin{aligned} X_n &= \langle v(t) \rangle = \langle x(t)e^{-jn\omega_0 t} \rangle \\ &= \langle x(t) \cos n\omega_0 t \rangle - j \langle x(t) \sin n\omega_0 t \rangle \end{aligned}$$

2.4.2 Symmetry Properties of the Fourier Coefficients

- For $x(t)$ real, the following coefficient symmetry properties hold:

1. $X_n^* = X_{-n}$
2. $|X_n| = |X_{-n}|$
3. $\angle X_n = -\angle X_{-n}$

proof

$$\begin{aligned} X_n^* &= \left(\frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt \right)^* \\ &= \frac{1}{T_0} \int_{T_0} x(t) e^{-j(-n)\omega_0 t} dt = X_{-n} \end{aligned}$$

since $x^*(t) = x(t)$

- Waveform symmetry conditions produce special results too

1. If $x(-t) = x(t)$ (even function), then

$$X_n = \operatorname{Re}\{X_n\}, \text{ i.e., } \operatorname{Im}\{X_n\} = 0$$

2. If $x(-t) = -x(t)$ (odd function), then

$$X_n = \operatorname{Im}\{X_n\}, \text{ i.e., } \operatorname{Re}\{X_n\} = 0$$

3. If $x(t \pm T_0/2) = -x(t)$ (odd half-wave symmetry), then

$$X_n = 0 \text{ for } n \text{ even}$$

Example 2.7: Odd Half-wave Symmetry Proof

- Consider

$$X_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) e^{-jn\omega_0 t} dt + \frac{1}{T_0} \int_{t_0+T_0/2}^{t_0+T_0} x(t') e^{-jn\omega_0 t'} dt'$$

- In the second integral we change variables by letting $t = t' - T_0/2$

$$\begin{aligned} X_n &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) e^{-jn\omega_0 t} dt \\ &\quad + \frac{1}{T_0} \int_{t_0}^{t_0+T_0/2} \underbrace{x(t - T_0/2)}_{-x(t)} e^{-jn\omega_0(t+T_0/2)} dt \\ &= \left(1 - e^{-jn\omega_0 T_0/2}\right) \frac{1}{T_0} \int_{t_0}^{t_0+T_0/2} x(t) e^{-jn\omega_0 t} dt \end{aligned}$$

but $n\omega_0(T_0/2) = n(2\pi/T_0)(T_0/2) = n\pi$, thus

$$1 - e^{-jn\pi} = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

- We thus see that the even indexed Fourier coefficients are indeed zero under odd half-wave symmetry

2.4.3 Trigonometric Form

- The complex exponential Fourier series can be arranged as follows

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \\ &= X_0 + \sum_{n=1}^{\infty} [X_n e^{jn\omega_0 t} + X_{-n} e^{-jn\omega_0 t}] \end{aligned}$$

- For real $x(t)$, we may know that $|X_{-n}| = |X_n|$ and $\angle X_{-n} = -\angle X_n$, so

$$\begin{aligned} x(t) &= X_0 + \sum_{n=1}^{\infty} [|X_n| e^{j[n\omega_0 t + \angle X_n]} + |X_n| e^{-j[n\omega_0 t + \angle X_n]}] \\ &= X_0 + 2 \sum_{n=1}^{\infty} |X_n| \cos[n\omega_0 t + \angle X_n] \end{aligned}$$

since $\cos(x) = (e^{jx} + e^{-jx})/2$

- From the trig identity $\cos(u + v) = \cos u \cos v - \sin u \sin v$, it follows that

$$x(t) = X_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

where

$$\begin{aligned} A_n &= 2\langle x(t) \cos(n\omega_0 t) \rangle \\ B_n &= 2\langle x(t) \sin(n\omega_0 t) \rangle \end{aligned}$$

2.4.4 Parseval's Theorem

- Fourier series analysis are generally used for periodic signals, i.e., $x(t) = x(t + nT_0)$ for any integer n
- With this in mind, Parseval's theorem becomes

$$\begin{aligned}
 P &= \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \\
 &= X_0^2 + 2 \sum_{n=1}^{\infty} |X_n|^2 \quad (\text{W})
 \end{aligned}$$

Note: A 1 ohm system is assumed

2.4.5 Line Spectra

- Line spectra was briefly reviewed earlier for simple signals
- For any periodic signal having Fourier series representation we can obtain both single-sided and double-sided line spectra
- The double-sided magnitude and phase line spectra is most easily obtained from the complex exponential Fourier series, while the single-sided magnitude and phase line spectra can be obtained from the trigonometric form

$$\begin{array}{l} \text{Double-sided} \\ \text{mag. and phase} \end{array} \iff \sum_{n=-\infty}^{\infty} X_n e^{j2\pi(nf_0)t}$$

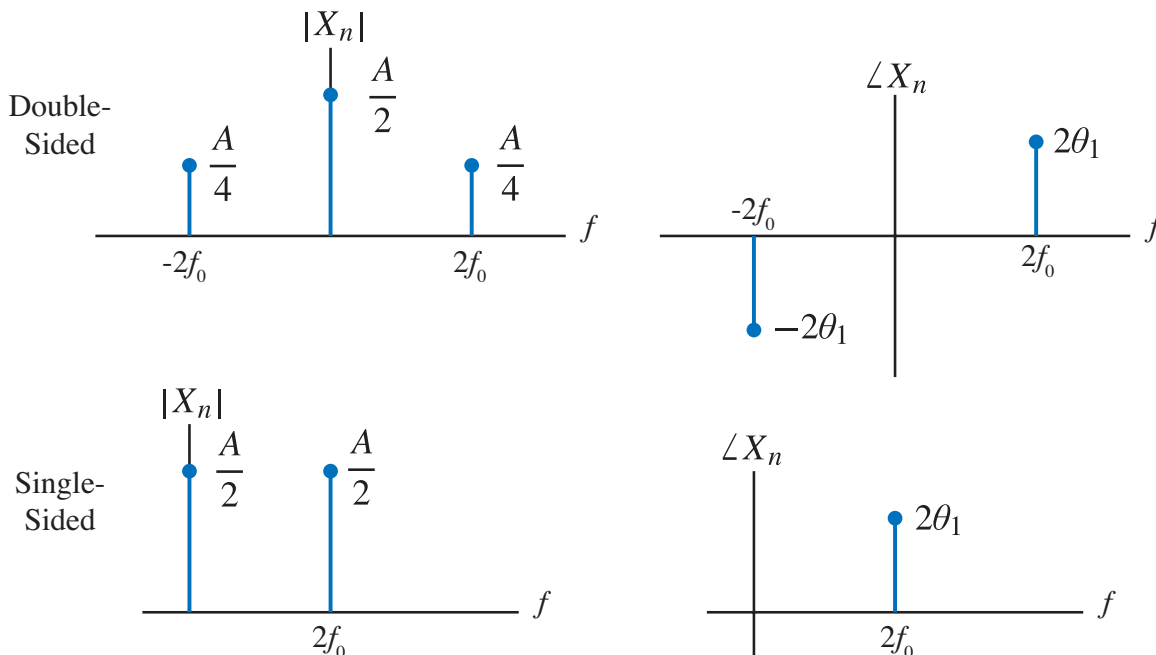
$$\begin{array}{l} \text{Single-sided} \\ \text{mag. and phase} \end{array} \iff X_0 + 2 \sum_{n=1}^{\infty} |X_n| \cos[2\pi(nf_0)t + \angle X_n]$$

- For the double-sided simply plot as lines $|X_n|$ and $\angle X_n$ versus nf_0 for $n = 0, \pm 1, \pm 2, \dots$
- For the single-sided plot $|X_0|$ and $\angle X_0$ as a special case for $n = 0$ at $nf_0 = 0$ and then plot $2|X_n|$ and $\angle X_0$ versus nf_0 for $n = 1, 2, \dots$

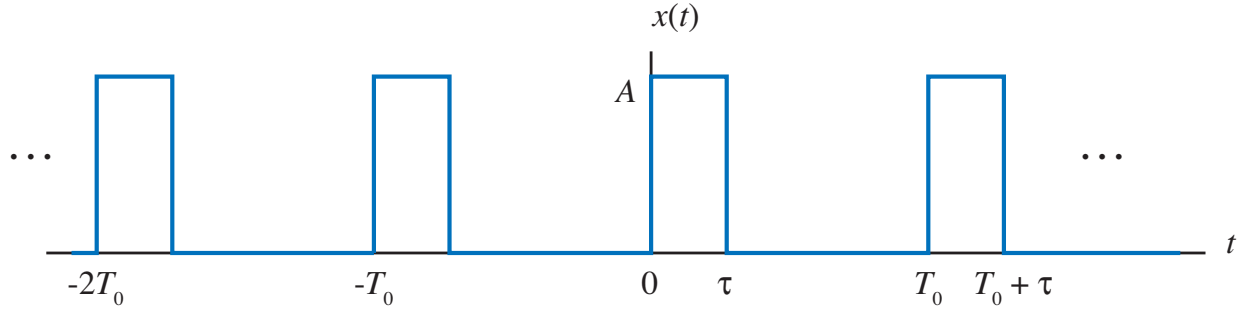
Example 2.8: Cosine Squared

- Consider

$$\begin{aligned} x(t) &= A \cos^2(2\pi f_0 t + \theta) = \frac{A}{2} + \frac{A}{2} \cos[2\pi(2f_0)t + 2\theta_1] \\ &= \frac{A}{2} + \frac{A}{4} e^{j2\theta_1} e^{j2\pi(2f_0)t} + \frac{A}{4} e^{-j2\theta_1} e^{-j2\pi(2f_0)t} \end{aligned}$$



Example 2.9: Pulse Train



Periodic pulse train

- The pulse train signal is mathematically described by

$$x(t) = \sum_{n=-\infty}^{\infty} A \Pi \left(\frac{t - nT_0 - \tau/2}{\tau} \right)$$

- The Fourier coefficients are

$$\begin{aligned} X_n &= \frac{1}{T_0} \int_0^{\tau} A e^{-j2\pi(nf_0)t} dt = \frac{A}{T_0} \cdot \frac{e^{-j2\pi(nf_0)t}}{-j2\pi(nf_0)} \Big|_0^{\tau} \\ &= \frac{A}{T_0} \cdot \frac{1 - e^{-j2\pi(nf_0)\tau}}{j2\pi(nf_0)} \\ &= \frac{A\tau}{T_0} \cdot \frac{e^{j\pi(nf_0)\tau} - e^{-j\pi(nf_0)\tau}}{(2j)\pi(nf_0)\tau} \cdot e^{-j\pi(nf_0)\tau} \\ &= \frac{A\tau}{T_0} \cdot \frac{\sin[\pi(nf_0)\tau]}{[\pi(nf_0)\tau]} \cdot e^{-j\pi(nf_0)\tau} \end{aligned}$$

- To simplify further we define

$$\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

- Finally,

$$X_n = \frac{A\tau}{T_0} \text{sinc}(nf_0\tau) e^{-j\pi(nf_0)\tau}, \quad n = 0, \pm 1, \pm 2, \dots$$

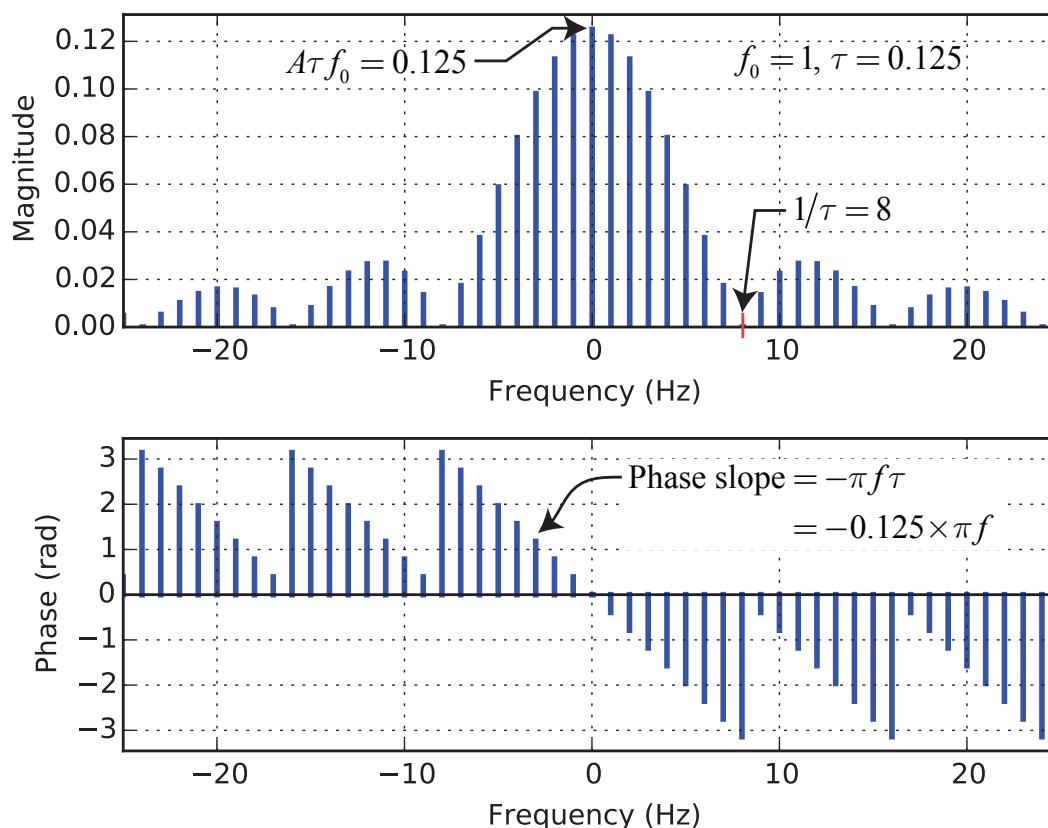
- To plot the line spectra we need to find $|X_n|$ and $\angle X_n$

$$|X_n| = \frac{A\tau}{T_0} |\text{sinc}(nf_0\tau)|$$

$$\angle X_n = \begin{cases} -\pi(nf_0)\tau, & \text{sinc}(nf_0\tau) > 0 \\ -\pi(nf_0)\tau + \pi, & nf_0 > 0 \text{ and } \text{sinc}(nf_0\tau) < 0 \\ -\pi(nf_0)\tau - \pi, & nf_0 < 0 \text{ and } \text{sinc}(nf_0\tau) < 0 \end{cases}$$

- Pulse train double-sided line spectra for $\tau = 0.125$ using Python, specifically use `ssd.line_spectra()`
- As a specific example enter the following at the IPython command prompt

```
n = arange(0,25+1) # Get 0 through 25 harmonics
tau = 0.125; f0 = 1; A = 1;
Xn = A*tau*f0*sinc(n*f0*tau)*exp(-1j*pi*n*f0*tau)
figure(figsize=(6,2))
f = n # Assume a fundamental frequency of 1 Hz so f = n
ssd.line_spectra(f,Xn,mode='mag',fsize=(6,2))
xlim([-25,25]);
figure(figsize=(6,2))
ssd.line_spectra(f,Xn,mode='phase',fsize=(6,2))
xlim([-25,25]);
```



2.4.6 Numerical Calculation of X_n

- Here we consider a purely numerical calculation of the X_k coefficients from a single period waveform description of $x(t)$
- In particular, we will use the numpy *fast Fourier transform* (FFT) function to carry out the numerical integration
- By definition

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-j2\pi k f_0 t} dt, \quad k = 0, \pm 1, \pm 2, \dots$$

- A simple rectangular integration approximation to the above integral is

$$X_k \simeq \frac{1}{T_0} \sum_{n=0}^{N-1} x(nT) e^{-jk2\pi(nf_0)T_0/N} \cdot \frac{T_0}{N}, \quad k = 0, \pm 1, \pm 2, \dots$$

where N is the number of points used to partition the time interval $[0, T_0]$ and $T = T_0/N$ is the time step

- Using the fact that $2\pi f_0 T_0 = 2\pi$, we can write that

$$X_k \simeq \frac{1}{N} \sum_{n=0}^{N-1} x(nT) e^{-\frac{j2\pi kn}{N}}, \quad k = 0, \pm 1, \pm 2, \dots$$

- Note that the above must be evaluated for each Fourier coefficient of interest
- Also note that the accuracy of the X_k values depends on the value of N
 - For k small and $x(t)$ smooth in the sense that the harmonics rolloff quickly, N on the order of 100 may be adequate
 - For k moderate, say 5–50, N will have to become increasingly larger to maintain precision in the numerical integral

Calculation Using the FFT

- The FFT is a powerful digital signal processing (DSP) function, which is a computationally efficient version of the *discrete Fourier transform* (DFT)

- For the purposes of the problem at hand, suffice it to say that the FFT is just an efficient algorithm for computing

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \dots, N-1$$

- If we let $x[n] = x(nT)$, then it should be clear that

$$X_k \simeq \frac{1}{N}X[k], \quad k = 0, 1, \dots, \frac{N}{2}$$

- To obtain X_k for $k < 0$ note that

$$\begin{aligned} X_{-k} &\simeq \frac{1}{N}X[-k] = \frac{1}{N} \sum_{n=0}^{N-1} x(nT)e^{\frac{-j2\pi(-k)n}{N}} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(nT)e^{\frac{-j2\pi(N-k)n}{N}} = X[N-k] \end{aligned}$$

since $e^{-j2\pi Nn/N} = e^{-j2\pi n} = 1$

- In summary

$$X_k \simeq \begin{cases} X[k]/N, & 0 \leq k \leq N/2 \\ X[N-k]/N, & -N/2 \leq k < 0 \end{cases}$$

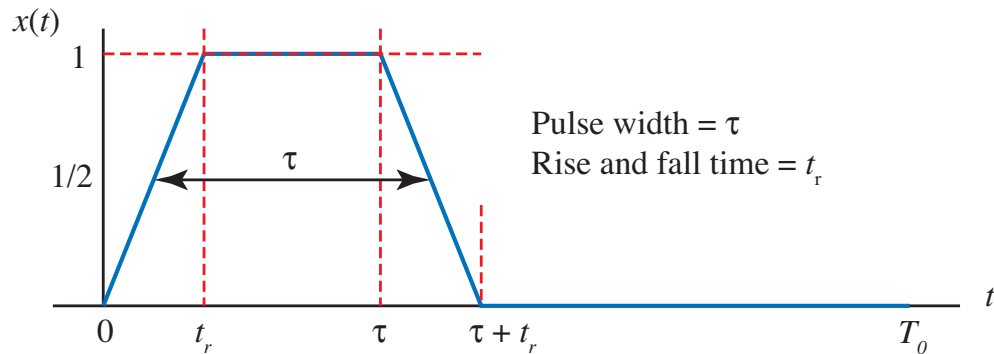
- To use the Python function `fft.fft()` to obtain the X_k we simply let

$$X = \text{fft.fft}(x)$$

where $x = \{x(t) : t = 0, T_0/N, 2T_0/N, \dots, T_0(N-1)/N\}$

- Unlike in MATLAB $X[0]$ is really found in $X[0]$

Example 2.10: Finite Rise/Fall-Time PulseTrain



Pulse train with finite rise and fall time edges

- Shown above is one period of a finite rise and fall time pulse train
- We will numerically compute the Fourier series coefficients of this signal using the FFT
- The Python function `trap_pulse` was written to generate one period of the signal using N samples

```
def trap_pulse(N,tau,tr):
    """
    xp = trap_pulse(N,tau,tr)

    Mark Wickert, January 2015
    """
    n = arange(0,N)
    t = n/N
    xp = zeros(len(t))
    # Assume tr and tf are equal
    T1 = tau + tr
    # Create one period of the trapezoidal pulse waveform
    for k in n:
        if t[k] <= tr:
            xp[k] = t[k]/tr
        elif (t[k] > tr and t[k] <= tau):
```

```

        xp[k] = 1
    elif (t[k] > tau and t[k] < T1):
        xp[k] = -t[k]/tr + 1 + tau/tr;
    else:
        xp[k] = 0
    return xp, t

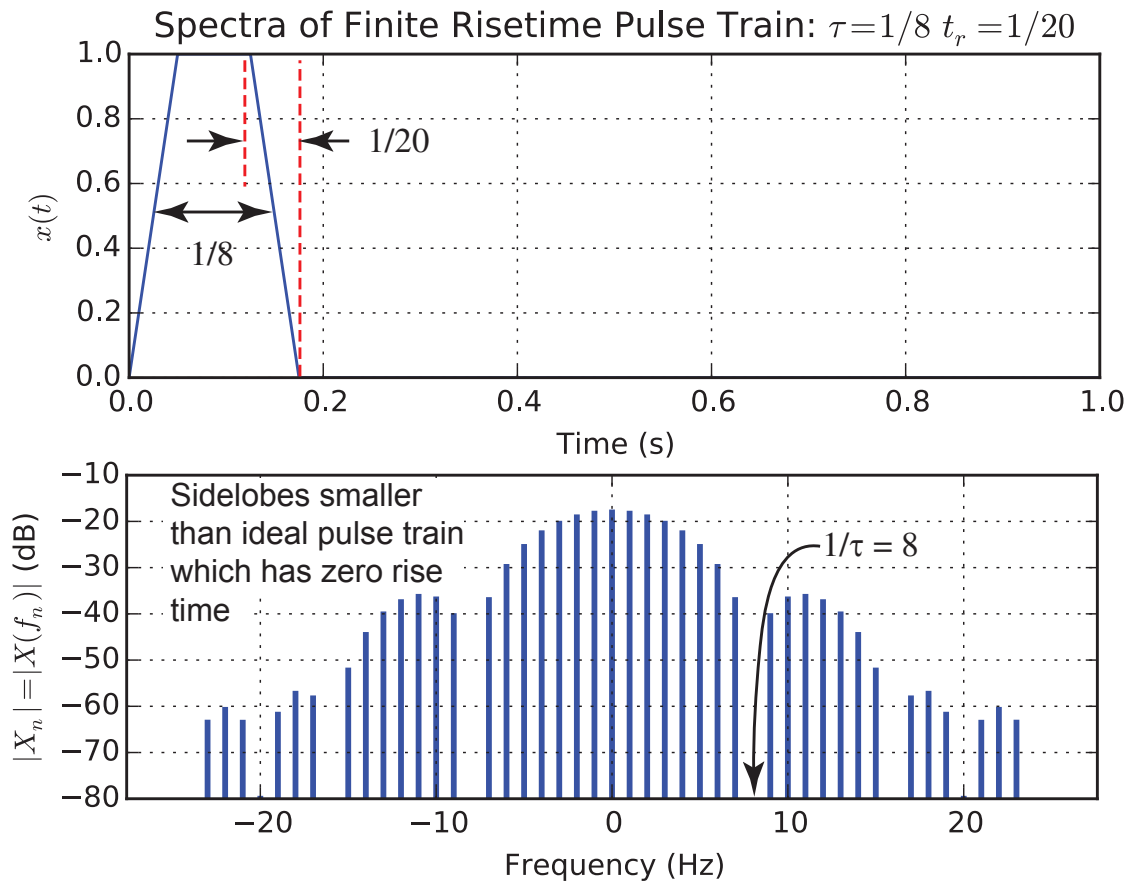
```

- We now plot the double-sided line spectra for $\tau = 1/8$ and two values of rise-time t_r

```

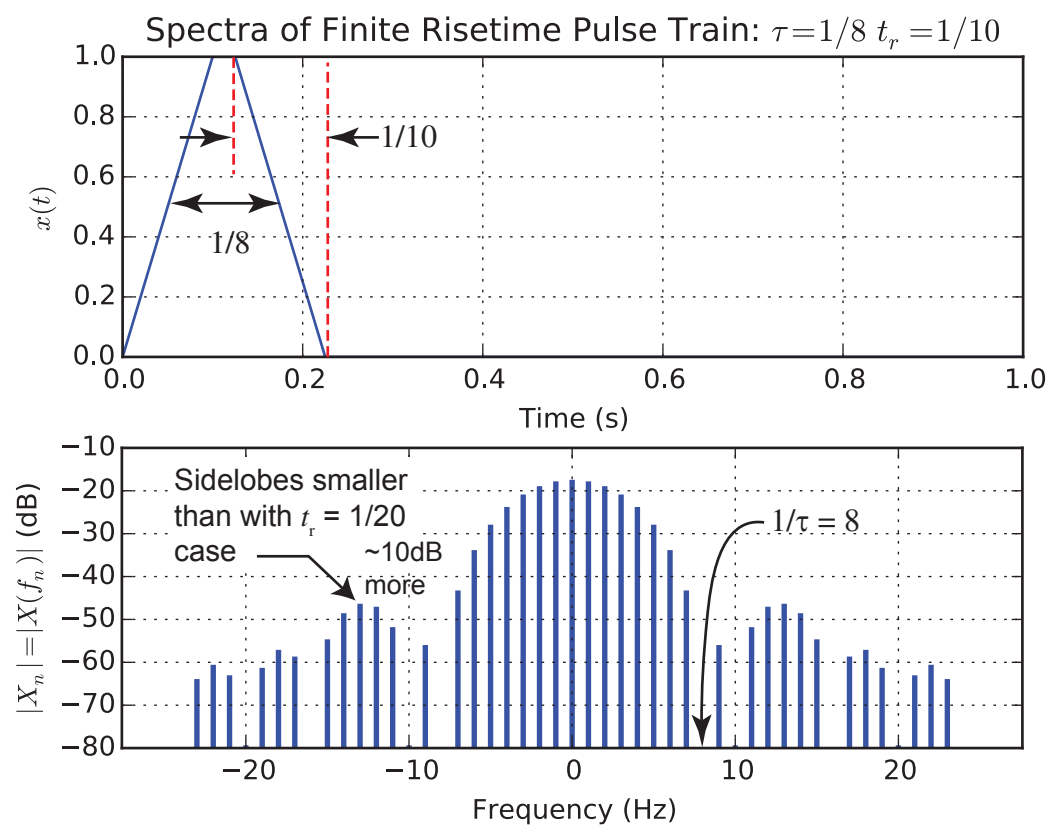
# tau = 1/8, tr = 1/20
N = 1024
xp,t = trap_pulse(N,1/8,1/20)
Xp = fft.fft(xp)
figure(figsize=(6,2))
plot(t,xp)
grid()
title(r'Spectra of Finite Risetime Pulse Train: $\tau = 1/8$ $t_r = 1/20$')
ylabel(r'$x(t)$')
xlabel('Time (s)')
f = arange(0,N/2)
ssd.line_spectra(f[0:25],Xp[0:25]/N,'magdB',floor_dB=-80,fsize=(6,2))
ylabel(r'$|X_n| = |X(f_n)|$ (dB)');
## tau = 1/8, tr = 1/10
xp,t = trap_pulse(N,1/8,1/10)
Xp = fft.fft(xp)
figure(figsize=(6,2))
plot(t,xp)
grid()
title(r'Spectra of Finite Risetime Pulse Train: $\tau = 1/8$ $t_r = 1/10$')
ylabel(r'$x(t)$')
xlabel('Time (s)')
ssd.line_spectra(f[0:25],Xp[0:25]/N,'magdB',floor_dB=-80,fsize=(6,2))
ylabel(r'$|X_n| = |X(f_n)|$ (dB)');

```



Signal $x(t)$ and line spectrum for $\tau = 1/8$ and $t_r = 1/20$

- The spectral roll-off rate is faster with the trapezoid
- Clock edges are needed in digital electronics, but by slowing the edge speed down the clock harmonic generation can be reduced
- The second plot is a more extreme example



Signal $x(t)$ and line spectrum for $\tau = 1/8$ and $t_r = 1/10$

2.4.7 Other Fourier Series Properties

- Given $x(t)$ has Fourier series (FS) coefficients X_n , if

$$y(t) = A + Bx(t)$$

it follows that

$$Y_n = \begin{cases} A + BX_0, & n = 0 \\ BX_n, & n \neq 0 \end{cases}$$

proof:

$$\begin{aligned} Y_n &= \langle y(t)e^{-j2\pi(nf_0)t} \rangle \\ &= A\langle e^{-j2\pi(nf_0)t} \rangle + B\langle x(t)e^{-j2\pi(nf_0)t} \rangle \\ &= A \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} + BX_n \end{aligned}$$

QED

- Likewise if

$$y(t) = x(t - t_0)$$

it follows that

$$Y_n = X_n e^{-j2\pi(nf_0)t_0}$$

proof:

$$Y_n = \langle x(t - t_0)e^{-j2\pi(nf_0)t} \rangle$$

Let $\lambda = t - t_0$ which implies also that $t = \lambda + t_0$, so

$$\begin{aligned} Y_n &= \langle x(\lambda)e^{-j2\pi(nf_0)(\lambda+t_0)} \rangle \\ &= \langle x(\lambda)e^{-j2\pi(nf_0)\lambda} \rangle e^{-j2\pi(nf_0)t_0} \\ &= X_n e^{-j2\pi(nf_0)t_0} \end{aligned}$$

QED

2.5 Fourier Transform

- The Fourier series provides a frequency domain representation of a periodic signal via the Fourier coefficients and line spectrum
- The next step is to consider the frequency domain representation of aperiodic signals using the Fourier transform
- Ultimately we will be able to include periodic signals within the framework of the Fourier transform, using the concept of *transform in the limit*
- The text establishes the Fourier transform by considering a limiting case of the expression for the Fourier series coefficient X_n as $T_0 \rightarrow \infty$
- The Fourier transform (FT) and inverse Fourier transform (IFT) is defined as

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (\text{FT}) \\ x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (\text{IFT}) \end{aligned}$$

- Sufficient conditions for the existence of the Fourier transform are
 1. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
 2. Discontinuities in $x(t)$ be finite
 3. An alternate sufficient condition is that $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$, which implies that $x(t)$ is an energy signal

2.5.1 Amplitude and Phase Spectra

- FT properties are very similar to those obtained for the Fourier coefficients of periodic signals
- The FT, $X(f) = \mathcal{F}\{x(t)\}$, is a complex function of f

$$\begin{aligned} X(f) &= |X(f)|e^{j\theta(f)} = |X(f)|e^{j\angle X(f)} \\ &= \text{Re}\{X(f)\} + j\text{Im}\{X(f)\} \end{aligned}$$

- The magnitude $|X(f)|$ is referred to as the *amplitude* spectrum
- The angle $\angle X(f)$ is referred to as the *phase* spectrum
- Note that

$$\begin{aligned} \text{Re}\{X(f)\} &= \int_{-\infty}^{\infty} x(t) \cos 2\pi f t \, dt \\ \text{Im}\{X(f)\} &= \int_{-\infty}^{\infty} x(t) \sin 2\pi f t \, dt \end{aligned}$$

2.5.2 Symmetry Properties

- If $x(t)$ is real it follows that

$$\begin{aligned} X(-f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi(-f)t} \, dt \\ &= \left(\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt \right)^* = X^*(f) \end{aligned}$$

thus

$$\begin{aligned} |X(-f)| &= |X(f)| \quad (\text{even in frequency}) \\ \angle X(-f) &= -\angle X(f) \quad (\text{odd in frequency}) \end{aligned}$$

- Additionally,
 1. For $x(-t) = x(t)$ (even function), $\text{Im}\{X(f)\} = 0$
 2. For $x(-t) = -x(t)$ (odd function), $\text{Re}\{X(f)\} = 0$

2.5.3 Energy Spectral Density

- From the definition of signal energy,

$$\begin{aligned}
 E &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
 &= \int_{-\infty}^{\infty} x^*(t) \left[\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \right] dt \\
 &= \int_{-\infty}^{\infty} X(f) \left[\int_{-\infty}^{\infty} x^*(t) e^{j2\pi ft} dt \right] df
 \end{aligned}$$

but

$$\int_{-\infty}^{\infty} x^*(t) e^{j2\pi ft} dt = \left(\int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \right)^* = X^*(f)$$

- Finally,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

which is known as *Rayleigh's Energy Theorem*

- Are the units consistent?
 - Suppose $x(t)$ has units of volts
 - $|X(f)|^2$ has units of (volts-sec)²

– In a 1 ohm system $|X(f)|^2$ has units of Watts-sec/Hz = Joules/Hz

- The *energy spectral density* is defined as

$$G(f) \triangleq |X(f)|^2 \text{ Joules/Hz}$$

- It then follows that

$$E = \int_{-\infty}^{\infty} G(f) df$$

Example 2.11: Rectangular Pulse

- Consider

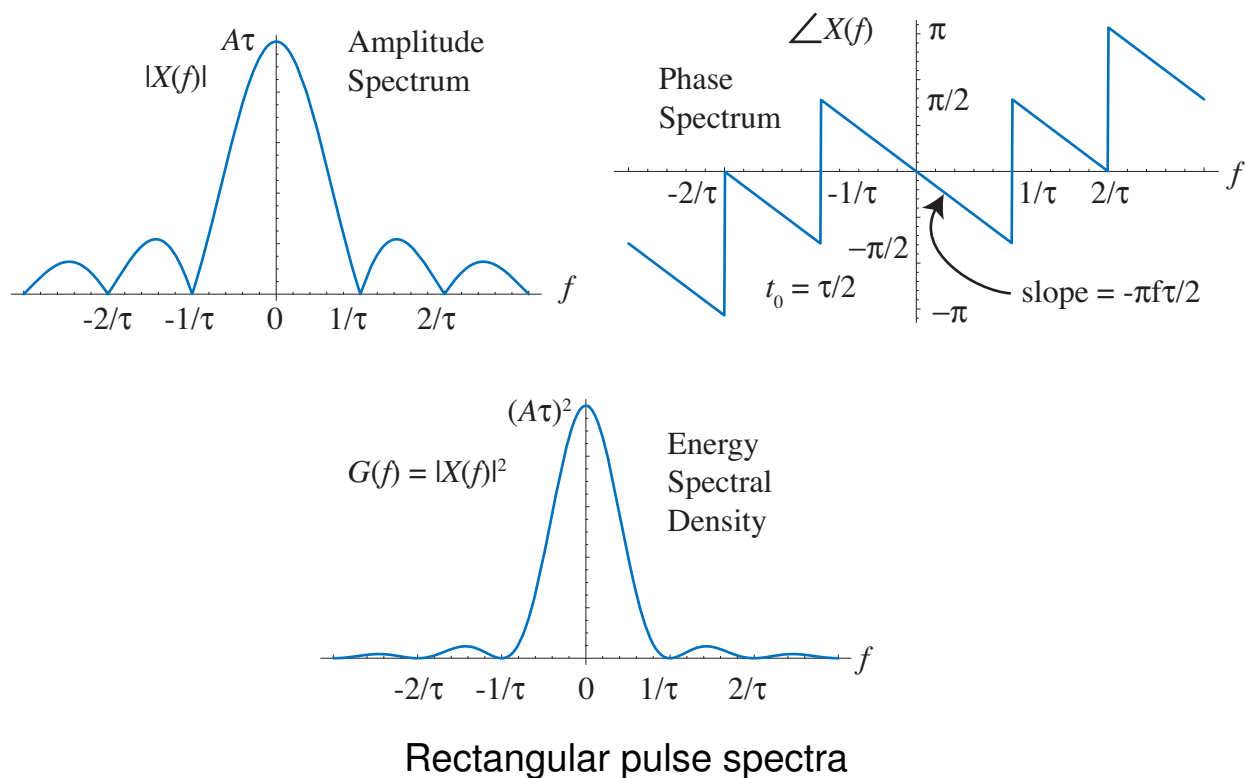
$$x(t) = A\Pi\left(\frac{t - t_0}{\tau}\right)$$

- FT is

$$\begin{aligned} X(f) &= A \int_{t_0 - \tau/2}^{t_0 + \tau/2} e^{-j2\pi ft} dt \\ &= A \cdot \frac{e^{-j2\pi ft}}{-j2\pi f} \Big|_{t_0 - \tau/2}^{t_0 + \tau/2} \\ &= A\tau \cdot \left[\frac{e^{j\pi f\tau} - e^{-j\pi f\tau}}{(j2)\pi f\tau} \right] \cdot e^{-j2\pi ft_0} \\ &= A\tau \text{sinc}(f\tau) e^{-j2\pi ft_0} \end{aligned}$$

$$A\Pi\left(\frac{t - t_0}{\tau}\right) \xleftrightarrow{\mathcal{F}} A\tau \text{sinc}(f\tau) e^{-j2\pi ft_0}$$

- Plot $|X(f)|$, $\angle X(f)$, and $G(f)$



2.5.4 Transform Theorems

- Be familiar with the FT theorems found in the table of Appendix G.6 of the text

Superposition Theorem

$$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{\mathcal{F}} a_1X_1(f) + a_2X_2(f)$$

proof:

Time Delay Theorem

$$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(f)e^{-j2\pi f t_0}$$

proof:

Frequency Translation Theorem

- In communications systems the frequency translation and modulation theorems are particularly important

$$x(t)e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} X(f - f_0)$$

proof: Note that

$$\int_{-\infty}^{\infty} x(t)e^{j2\pi f_0 t} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t} dt$$

so

$$\mathcal{F}\{x(t)e^{j2\pi f_0 t}\} = X(f - f_0)$$

QED

Modulation Theorem

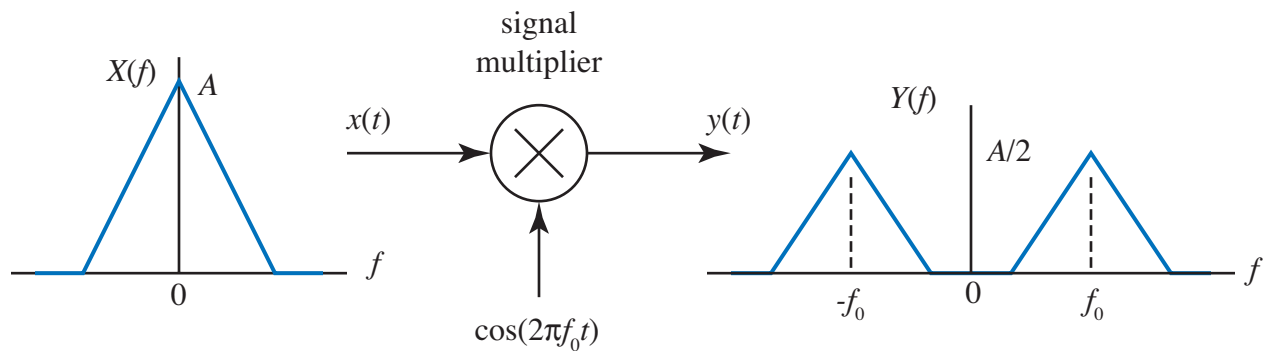
- The modulation theorem is an extension of the frequency translation theorem

$$x(t) \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$$

proof: Begin by expanding

$$\cos(2\pi f_0 t) = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}$$

Then apply the frequency translation theorem to each term separately



A simple *modulator*

Duality Theorem

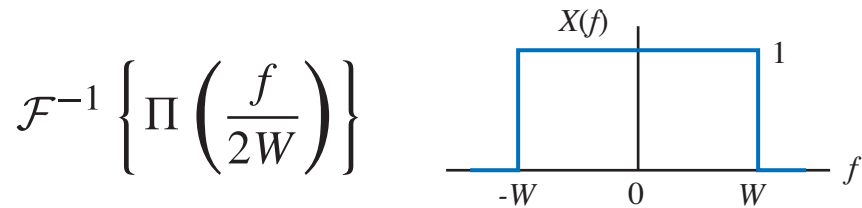
- Note that

$$\mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} X(t)e^{j2\pi(-f)t} dt$$

which implies that

$$X(t) \xleftrightarrow{\mathcal{F}} x(-f)$$

Example 2.12: Rectangular Spectrum



- Using duality on the above we have

$$X(t) = \Pi \left(\frac{t}{2W} \right) \xleftrightarrow{\mathcal{F}} 2W \operatorname{sinc}(2Wf) = x(-f)$$

- Since $\operatorname{sinc}(\cdot)$ is an even function ($\operatorname{sinc}(x) = \operatorname{sinc}(-x)$), it follows that

$$2W \operatorname{sinc}(2Wt) \xleftrightarrow{\mathcal{F}} \Pi \left(\frac{f}{2W} \right)$$

Differentiation Theorem

- The general result is

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f)$$

proof: For $n = 1$ we start with the integration by parts formula, $\int u dv = uv \Big| - \int v du$, and apply it to

$$\begin{aligned} \mathcal{F} \left\{ \frac{dx}{dt} \right\} &= \int_{-\infty}^{\infty} \frac{dx}{dt} e^{-j2\pi ft} dt \\ &= \underbrace{x(t)e^{-j2\pi ft} \Big|_{-\infty}^{\infty}}_0 + j2\pi f \underbrace{\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt}_{X(f)} \end{aligned}$$

alternate — From Leibnitz's rule for differentiation of integrals,

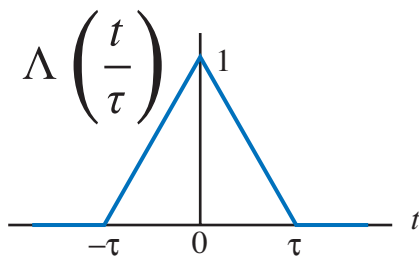
$$\frac{d}{dt} \int_{-\infty}^{\infty} F(f, t) df = \int_{-\infty}^{\infty} \frac{\partial F(f, t)}{\partial t} df$$

so

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{d}{dt} \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \\ &= \int_{-\infty}^{\infty} X(f) \frac{\partial e^{j2\pi ft}}{\partial t} df \\ &= \int_{-\infty}^{\infty} j2\pi f X(f) e^{j2\pi ft} df \end{aligned}$$

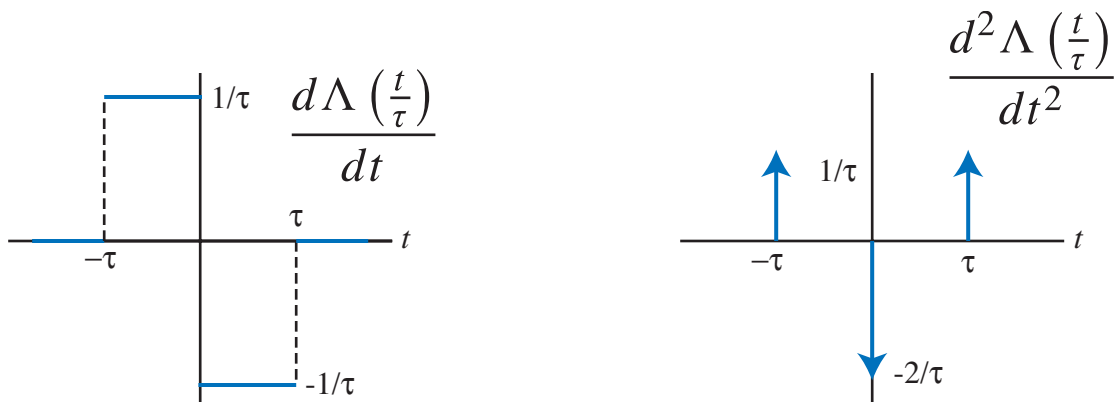
$$\Rightarrow dx/dt \xleftrightarrow{\mathcal{F}} j2\pi f X(f) \quad \text{QED}$$

Example 2.13: FT of Triangle Pulse



$$\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{|t|}{\tau}, & |t| < \tau \\ 0, & \text{otherwise} \end{cases}$$

- Note that



- Using the differentiation theorem for $n = 2$ we have that

$$\begin{aligned}
 \mathcal{F} \left\{ \Lambda \left(\frac{t}{\tau} \right) \right\} &= \frac{1}{(j2\pi f)^2} \mathcal{F} \left\{ \frac{1}{\tau} \delta(t + \tau) - \frac{2}{\tau} \delta(t) + \frac{1}{\tau} \delta(t - \tau) \right\} \\
 &= \frac{\frac{1}{\tau} e^{j2\pi f \tau} - \frac{2}{\tau} + \frac{1}{\tau} e^{-j2\pi f \tau}}{(j2\pi f)^2} \\
 &= \frac{2 \cos(2\pi f \tau) - 2}{-\tau (2\pi f)^2} \\
 &= \tau \frac{4 \sin^2(\pi f \tau)}{4(\pi f \tau)^2} = \tau \text{sinc}^2(f \tau)
 \end{aligned}$$

$$\Lambda \left(\frac{t}{\tau} \right) \xleftrightarrow{\mathcal{F}} \tau \text{sinc}^2(f \tau)$$

Convolution and Convolution Theorem

- Before discussing the convolution theorem we need to review convolution
- The convolution of two signals $x_1(t)$ and $x_2(t)$ is defined as

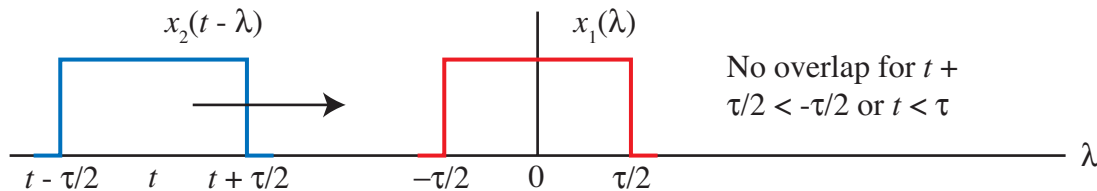
$$\begin{aligned}
 x(t) &= x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) x_2(t - \lambda) d\lambda \\
 &= x_2(t) * x_1(t) = \int_{-\infty}^{\infty} x_2(\lambda) x_1(t - \lambda) d\lambda
 \end{aligned}$$

- A special convolution case is $\delta(t - t_0)$

$$\begin{aligned}
 \delta(t - t_0) * x(t) &= \int_{-\infty}^{\infty} \delta(\lambda - t_0) x(t - \lambda) d\lambda \\
 &= x(t - \lambda) \Big|_{\lambda=t_0} = x(t - t_0)
 \end{aligned}$$

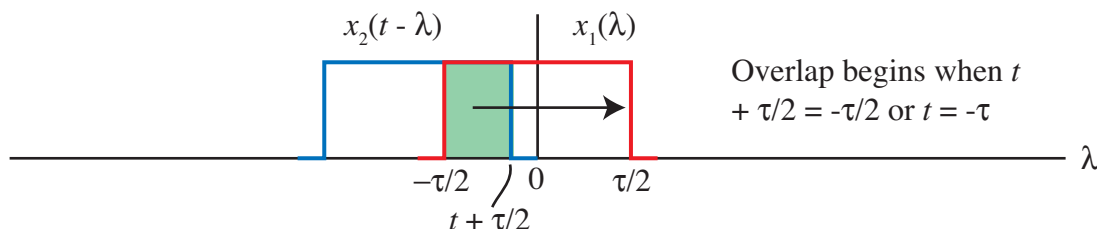
Example 2.14: Rectangular Pulse Convolution

- Let $x_1(t) = x_2(t) = \Pi(t/\tau)$
- To evaluate the convolution integral we need to consider the integrand by sketching of $x_1(\lambda)$ and $x_2(t - \lambda)$ on the λ axis for different values of t
- For this example four cases are needed for t to cover the entire time axis $t \in (-\infty, \infty)$
- **Case 1:** When $t < -\tau$ we have no overlap so the integrand is zero and $x(t)$ is zero



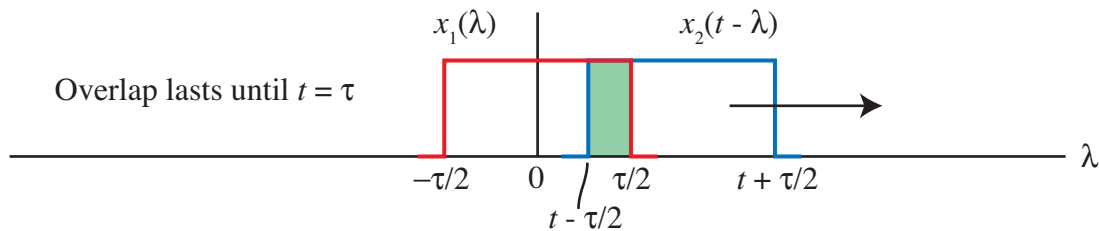
- **Case 2:** When $-\tau < t < 0$ we have overlap and

$$\begin{aligned}
 x(t) &= \int_{-\infty}^{\infty} x_1(\lambda) x_2(t - \lambda) d\lambda \\
 &= \int_{-\tau/2}^{t + \tau/2} d\lambda = \lambda \Big|_{-\tau/2}^{t + \tau/2} \\
 &= t + \tau/2 + \tau/2 = \tau + t
 \end{aligned}$$



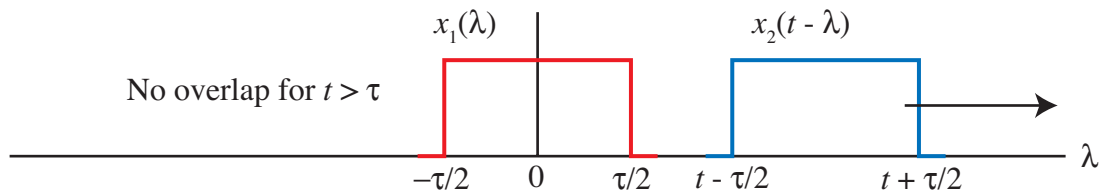
- **Case 3:** For $0 < t < \tau$ the leading edge of $x_2(t - \lambda)$ is to the right of $x_1(\lambda)$, but the trailing edge of the pulse is still overlapped

$$x(t) = \int_{t-\tau/2}^{\tau/2} d\lambda = \tau/2 - t + \tau/2 = \tau - t$$



- **Case 4:** For $t > \tau$ we have no overlap, and like case 1, the result is

$$x(t) = 0$$



- Collecting the results

$$x(t) = \begin{cases} 0, & t < -\tau \\ \tau + t, & -\tau \leq t < 0 \\ \tau - t, & 0 \leq t < \tau \\ 0, & t \geq \tau \end{cases}$$

$$= \begin{cases} \tau - |t|, & |t| \leq \tau \\ 0, & \text{otherwise} \end{cases}$$

- Final summary,

$$\Pi\left(\frac{t}{\tau}\right) * \Pi\left(\frac{t}{\tau}\right) = \tau \Lambda\left(\frac{t}{\tau}\right)$$

- **Convolution Theorem:** We now consider $x_1(t) * x_2(t)$ in terms of the FT

$$\begin{aligned} & \int_{-\infty}^{\infty} x_1(\tau) x_2(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} X_2(f) e^{j2\pi f(t-\tau)} df \right] d\tau \\ &= \int_{-\infty}^{\infty} X_2(f) \left[\int_{-\infty}^{\infty} x_1(\tau) e^{-j2\pi f\tau} d\tau \right] e^{j2\pi ft} df \\ &= \int_{-\infty}^{\infty} X_1(f) X_2(f) e^{j2\pi ft} df \end{aligned}$$

which implies that

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f) X_2(f)$$

Example 2.15: Revisit $\Pi(t/\tau) * \Pi(t/\tau)$

- Knowing that $\Pi(t/\tau) * \Pi(t/\tau) = \tau \Lambda(t/\tau)$ in the time domain, we can follow-up in the frequency domain by writing

$$\mathcal{F}\{\Pi(t/\tau)\} \cdot \mathcal{F}\{\Pi(t/\tau)\} = (\tau \text{sinc}(f\tau))^2$$

- We have also established the transform pair

$$\tau \Lambda\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau^2 \text{sinc}^2(f\tau) = \tau (\tau \text{sinc}^2(f\tau))$$

or

$$\Lambda\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau \operatorname{sinc}^2(f\tau)$$

Multiplication Theorem

- Having already established the convolution theorem, it follows from the duality theorem or direct evaluation, that

$$x_1(t) \cdot x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f) * X_2(f)$$

2.5.5 Fourier Transforms in the Limit

- Thus far we have considered two classes of signals
 1. Periodic power signals which are described by line spectra
 2. Non-periodic (aperiodic) energy signals which are described by continuous spectra via the FT
- We would like to have a unifying approach to spectral analysis
- To do so we must allow impulses in the frequency domain by using limiting operations on conventional FT pairs, known as *Fourier transforms-in-the-limit*
 - Note: The corresponding time functions have infinite energy, which implies that the concept of energy spectral density will not apply for these signals (we will introduce the concept of power spectral density for these signals)

Example 2.16: A Constant Signal

- Let $x(t) = A$ for $-\infty < t < \infty$

- We can write

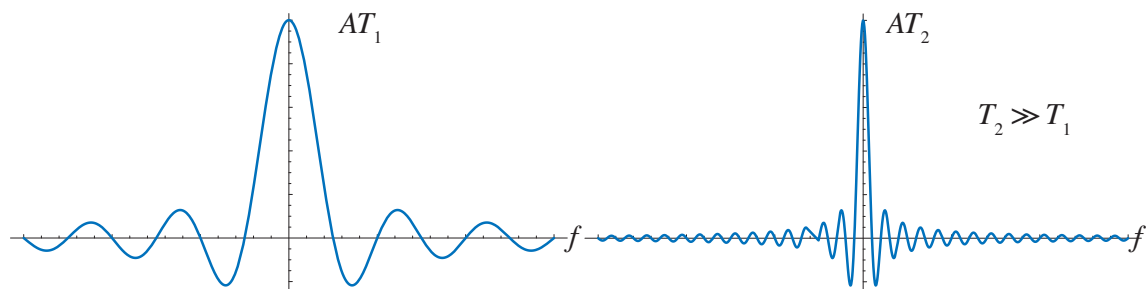
$$x(t) = \lim_{T \rightarrow \infty} A\Pi(t/T)$$

- Note that

$$\mathcal{F}\{A\Pi(t/T)\} = AT \operatorname{sinc}(fT)$$

- Using the transform-in-the-limit approach we write

$$\mathcal{F}\{x(t)\} = \lim_{T \rightarrow \infty} AT \operatorname{sinc}(fT)$$



Increasing T in $AT \operatorname{sinc}(fT)$

- Note that since $x(t)$ has no time variation it seems reasonable that the spectral content ought to be confined to $f = 0$
- Also note that it can be shown that

$$\int_{-\infty}^{\infty} AT \operatorname{sinc}(fT) df = A, \quad \forall T$$

- Thus we have established that

$$A \xleftrightarrow{\mathcal{F}} A\delta(f)$$

- As a further check

$$\mathcal{F}^{-1}\{A\delta(f)\} = \int_{-\infty}^{\infty} A\delta(f)e^{j2\pi f t} df = Ae^{j2\pi f t}\Big|_{f=0} = A$$

- As a result of the above example, we can obtain several more FT-in-the-limit pairs

$$\begin{aligned} Ae^{j2\pi f_0 t} &\xleftrightarrow{\mathcal{F}} A\delta(f - f_0) \\ A\cos(2\pi f_0 t + \theta) &\xleftrightarrow{\mathcal{F}} \frac{A}{2}[e^{j\theta}\delta(f - f_0) + e^{-j\theta}\delta(f + f_0)] \\ A\delta(t - t_0) &\xleftrightarrow{\mathcal{F}} Ae^{-j2\pi f t_0} \end{aligned}$$

- **Reciprocal Spreading Property:** Compare

$$A\delta(t) \xleftrightarrow{\mathcal{F}} A \quad \text{and} \quad A \xleftrightarrow{\mathcal{F}} A\delta(f)$$

A constant signal of infinite duration has *zero* spectral width, while an impulse in time has *zero* duration and infinite spectral width

2.5.6 Fourier Transforms of Periodic Signals

- For an arbitrary periodic signal with Fourier series

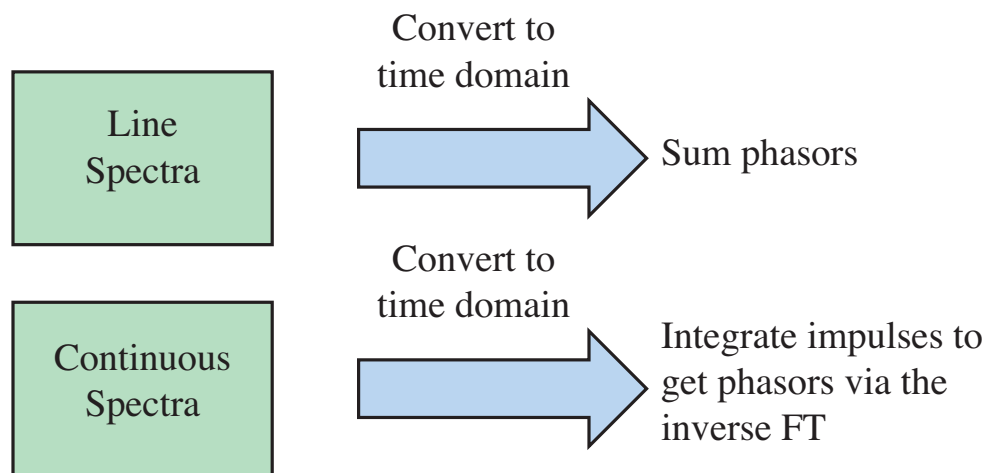
$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t}$$

we can write

$$\begin{aligned} X(f) &= \mathcal{F} \left[\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n f_0 t} \right] \\ &= \sum_{n=-\infty}^{\infty} X_n \mathcal{F} \{ e^{j2\pi n f_0 t} \} \\ &= \sum_{n=-\infty}^{\infty} X_n \delta(f - n f_0) \end{aligned}$$

using superposition and $\mathcal{F}\{Ae^{j2\pi f_0 t}\} = A\delta(f - f_0)$

- What is the difference between *line spectra* and *continuous spectra*? none!
- Mathematically,



- The Fourier series coefficients need to be known before the FT spectra can be obtained
- A technique that obtains the FT directly will be discussed shortly (page 2–58))

Example 2.17: Ideal Sampling Waveform

- When we discuss sampling theory it will be useful to have the FT of the periodic impulse train signal

$$y_s(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_s)$$

where T_s is the *sample spacing* or period

- Since this signal is periodic, it must have a Fourier series representation too
- In particular

$$Y_n = \frac{1}{T_s} \int_{T_s} \delta(t) e^{-j2\pi(nf_s)t} dt = \frac{1}{T_s} = f_s, \text{ any } n$$

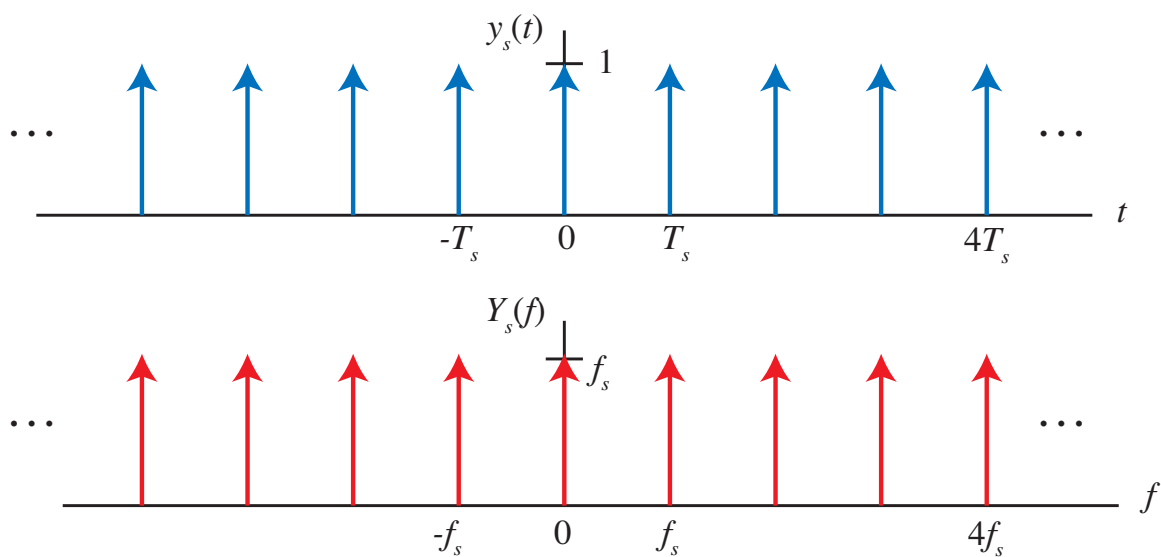
where f_s is the sampling rate in Hz

- The FT of $y_s(t)$ is given by

$$Y_s(f) = f_s \sum_{n=-\infty}^{\infty} \mathcal{F}\{e^{j2\pi n f_0 t}\} = f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)$$

- Summary,

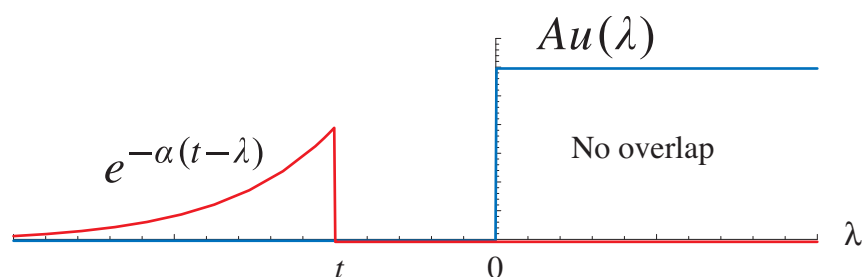
$$\sum_{m=-\infty}^{\infty} \delta(t - mT_s) \xleftrightarrow{\mathcal{F}} f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)$$



An impulse train in time is an impulse train in frequency

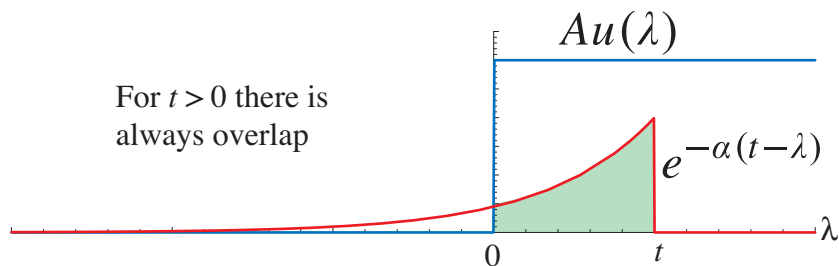
Example 2.18: Convolve Step and Exponential

- Find $y(t) = Au(t) * e^{-\alpha t}u(t)$, $\alpha > 0$
- For $t \leq 0$ there is no overlap so $Y(t) = 0$



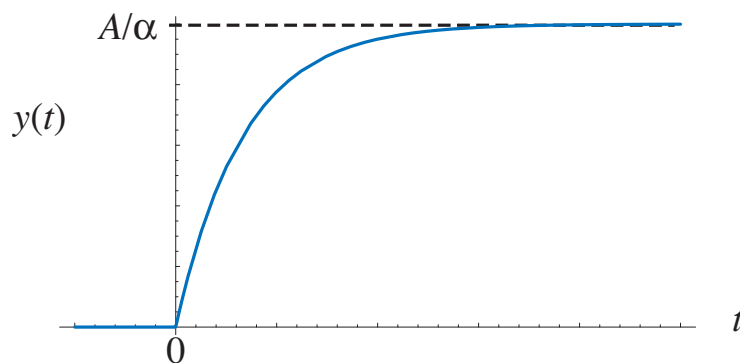
- For $t > 0$ there is always overlap

$$\begin{aligned}
 y(t) &= \int_0^t A \cdot e^{-\alpha(t-\lambda)} d\lambda \\
 &= Ae^{-\alpha t} \cdot \frac{e^{\alpha\lambda}}{\alpha} \Big|_0^t \\
 &= Ae^{-\alpha t} \cdot \frac{e^{\alpha t} - 1}{\alpha}
 \end{aligned}$$



- Summary,

$$y(t) = \frac{A}{\alpha} [1 - e^{-\alpha t}] u(t)$$



Direct Approach for the FT of a Periodic Signal

- The FT of a periodic signal can be found directly by expanding $x(t)$ as follows

$$x(t) = \left[\sum_{m=-\infty}^{\infty} \delta(t - mT_s) \right] * p(t) = \sum_{m=-\infty}^{\infty} p(t - mT_s)$$

where $p(t)$ represents one period of $x(t)$, having period T_s

- From the convolution theorem

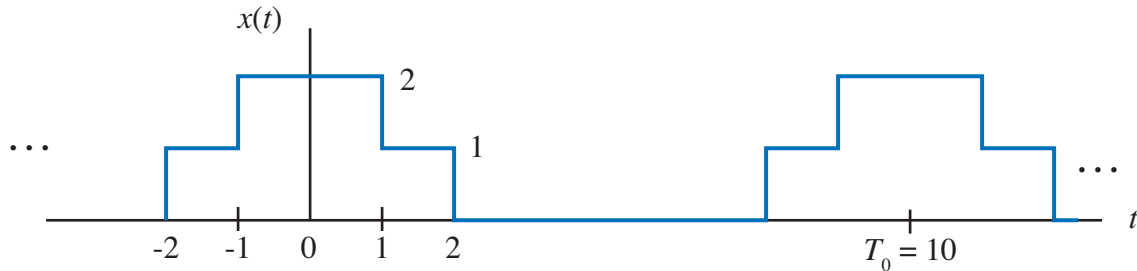
$$\begin{aligned} X(f) &= \mathcal{F} \left\{ \sum_{m=-\infty}^{\infty} \delta(t - mT_s) \right\} \cdot P(f) \\ &= f_s P(f) \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \\ &= f_s \sum_{n=-\infty}^{\infty} P(nf_s) \delta(f - nf_s) \end{aligned}$$

where $P(f) = \mathcal{F}\{p(t)\}$

- The FT transform pair just established is

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) \xleftrightarrow{\mathcal{F}} \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s)$$

Example 2.19: $p(t) = \Pi(t/2) + \Pi(t/4)$, $T_0 = 10$



Stacked pulses periodic signal

- We begin by finding $P(f)$ using $\mathcal{F}\{\Pi(t/\tau)\} = \tau \text{sinc}(f\tau)$

$$P(f) = 2\text{sinc}(2f) + 4\text{sinc}(4f)$$

- Plugging into the FT pair derived above with $nf_s = n/10$,

$$X(f) = \frac{1}{10} \sum_{n=-\infty}^{\infty} \left[2\text{sinc}\left(\frac{n}{5}\right) + 4\text{sinc}\left(\frac{2n}{5}\right) \right] \delta\left(f - \frac{n}{10}\right)$$

2.5.7 Poisson Sum Formula

- The *Poisson sum formula* from mathematics can be derived using the FT pair

$$e^{-j2\pi(nf_s)t} \xleftrightarrow{\mathcal{F}} \delta(f - nf_s)$$

by writing

$$\mathcal{F}^{-1} \left\{ \sum_{n=-\infty}^{\infty} f_s P(nf_s) \delta(f - nf_s) \right\} = f_s \sum_{n=-\infty}^{\infty} P(nf_s) e^{j2\pi(nf_s)t}$$

- From the earlier developed FT of periodic signals pair, we know that the left side of the above is also equal to

$$\sum_{m=-\infty}^{\infty} p(t - mT_s) \stackrel{\text{also}}{=} f_s \sum_{n=-\infty}^{\infty} P(nf_s) e^{j2\pi(nf_s)t}$$

- We can finally relate this back to the Fourier series coefficients, i.e.,

$$X_n = f_s P(nf_s)$$

2.6 Power Spectral Density and Correlation

- For energy signals we have the energy spectral density, $G(f)$, defined such that

$$E = \int_{-\infty}^{\infty} G(f) df$$

- For power signals we can define the *power spectral density* (PSD), $S(f)$ of $x(t)$ such that

$$P = \int_{-\infty}^{\infty} S(f) df = \langle |x(t)|^2 \rangle$$

- Note: $S(f)$ is real, even and nonnegative
- If $x(t)$ is periodic $S(f)$ will consist of impulses at the harmonic locations
- For $x(t) = A \cos(\omega_0 t + \theta)$, intuitively,

$$S(f) = \frac{1}{4} A^2 \delta(f - f_0) + \frac{1}{4} A^2 \delta(f + f_0)$$

since $\int S(f) df = A^2/2$ as expected (power on a per ohm basis)

- To derive a general formula for the PSD we first need to consider the *autocorrelation function*

2.6.1 The Time Average Autocorrelation Function

- Let $\phi(\tau)$ be the autocorrelation function of an energy signal

$$\begin{aligned}\phi(\tau) &= \mathcal{F}^{-1}\{G(f)\} \\ &= \mathcal{F}^{-1}\{X(f)X^*(f)\} \\ &= \mathcal{F}^{-1}\{X(f)\} * \mathcal{F}^{-1}\{X^*(f)\}\end{aligned}$$

but $x(-t) \xleftrightarrow{\mathcal{F}} X^*(f)$ for $x(t)$ real, so

$$\phi(\tau) = x(t) * x(-t) = \int_{-\infty}^{\infty} x(t)x(t + \tau) d\tau$$

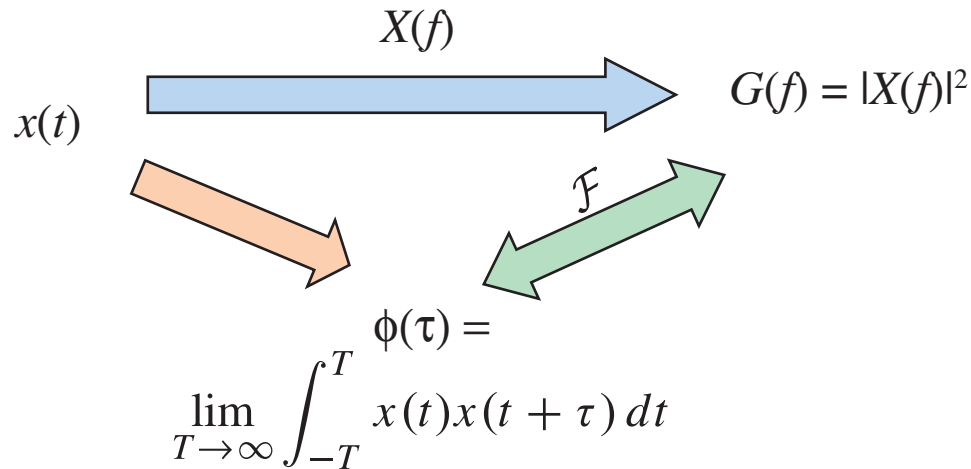
or

$$\phi(\tau) = \lim_{T \rightarrow \infty} \int_{-T}^T x(t)x(t + \tau) d\tau$$

- Observe that

$$G(f) = \mathcal{F}\{\phi(\tau)\}$$

- The autocorrelation function (ACF) gives a measure of the similarity of a signal at time t to that at time $t + \tau$; the coherence between the signal and the delayed signal



Energy spectral density and signal relationships

2.6.2 Power Signal Case

For power signals we define the autocorrelation function as

$$\begin{aligned} R_x(\tau) &= \langle x(t)x(t + \tau) \rangle \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t + \tau) dt \\ &\stackrel{\text{if periodic}}{=} \frac{1}{T_0} \int_{T_0} x(t)x(t + \tau) dt \end{aligned}$$

- Note that

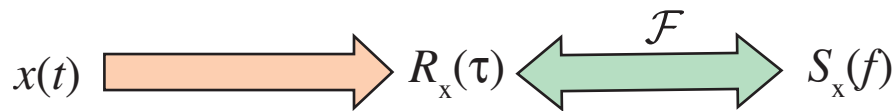
$$R_x(0) = \langle |x(t)|^2 \rangle = \int_{-\infty}^{\infty} S_x(f) df$$

and since for energy signals $\phi(\tau) \xleftrightarrow{\mathcal{F}} G(f)$, a reasonable assumption is that

$$R_x(\tau) \xleftrightarrow{\mathcal{F}} S_x(f)$$

- A formal statement of this is the *Wiener-Kinchine theorem* (a proof is given in text Chapter 7)

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$



Power spectral density (PSD) and signal relationships

2.6.3 Properties of $R(\tau)$

- The following properties hold for the autocorrelation function
 1. $R(0) = \langle |x(t)|^2 \rangle \geq |R(\tau)|$ for all values of τ
 2. $R(-\tau) = \langle x(t)x(t - \tau) \rangle = R(\tau) \Rightarrow$ an even function
 3. $\lim_{|\tau| \rightarrow \infty} R(\tau) = \langle x(t) \rangle^2$ if $x(t)$ is not periodic
 4. If $x(t)$ is periodic, with period T_0 , then $R(\tau) = R(\tau + T_0)$
 5. $\mathcal{F}\{R(\tau)\} = S(f) \geq 0$ for all values of f
- The power spectrum and autocorrelation function are frequently used for systems analysis with random signals
- For the case of random signal, $x(t)$, the Fourier transform $X(f)$ is also a random signal, but now in the frequency domain
- The autocorrelation and PSD of $x(t)$, under typical assumptions, are both deterministic functions; this turns out to be vital in problem solving in Comm II and beyond

Example 2.20: Single Sinusoid

- Consider the signal $x(t) = A \cos(2\pi f_0 t + \theta)$, for all t

$$\begin{aligned}
 R_x(\tau) &= \frac{1}{T_0} \int_0^{T_0} A^2 \cos(2\pi f_0 t + \theta) \cos(2\pi(t + \tau) + \theta) dt \\
 &= \frac{A^2}{2T_0} \int_{T_0} [\cos(2\pi f_0 \tau) + \cos(2\pi(2f_0)t + 2\pi f_0 \tau + 2\theta)] dt \\
 &= \frac{A^2}{2} \cos(2\pi f_0 \tau)
 \end{aligned}$$

- Note that

$$\mathcal{F}\{R_x(\tau)\} = S_x(f) = \frac{A^2}{4} [\delta(f - f_0) + \delta(f + f_0)]$$

More Autocorrelation Function Properties

- Suppose that $x(t)$ has autocorrelation function $R_x(\tau)$
- Let $y(t) = A + x(t)$, $A = \text{constant}$

$$\begin{aligned}
 R_y(\tau) &= \langle [A + x(t)][A + x(t + \tau)] \rangle \\
 &= \langle A^2 \rangle + \langle Ax(t + \tau) \rangle + \langle Ax(t) \rangle + \langle x(t)x(t + \tau) \rangle \\
 &= \underbrace{A^2 + 2A\langle x(t) \rangle}_{\text{const. terms}} + R_x(\tau)
 \end{aligned}$$

- Let $z(t) = x(t - t_0)$

$$\begin{aligned}
 R_z(\tau) &= \langle z(t)z(t + \tau) \rangle = \langle x(t - t_0)x(t - t_0 + \tau) \rangle \\
 &= \langle x(\lambda)x(\lambda + \tau) \rangle, \quad \text{with } \lambda = t - t_0 \\
 &= R_x(\tau)
 \end{aligned}$$

- The last result shows us that the autocorrelation function is *blind* to time offsets

Example 2.21: Sum of Two Sinusoids

- Consider the sum of two sinusoids

$$y(t) = x_1(t) + x_2(t)$$

where $x_1(t) = A_1 \cos(2\pi f_1 t + \theta_1)$ and $x_2(t) = A_2 \cos(2\pi f_2 t + \theta_2)$ and we assume that $f_1 \neq f_2$

- Using the definition

$$\begin{aligned} R_y(\tau) &= \langle [x_1(t) + x_2(t)][x_1(t + \tau) + x_2(t + \tau)] \rangle \\ &= \langle x_1(t)x_1(t + \tau) \rangle + \langle x_2(t)x_2(t + \tau) \rangle \\ &\quad + \langle x_1(t)x_2(t + \tau) \rangle + \langle x_2(t)x_1(t + \tau) \rangle \end{aligned}$$

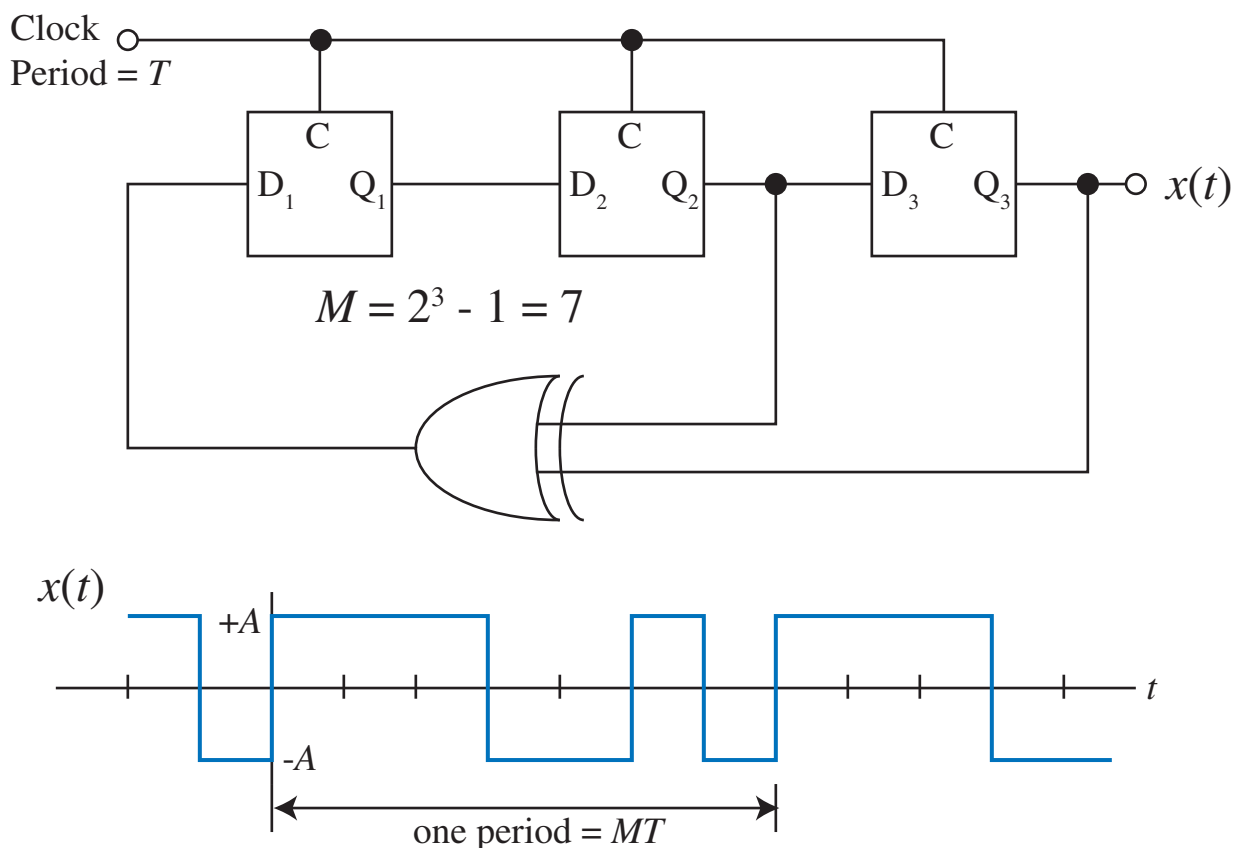
- The last two terms are zero since $\langle \cos((\omega_1 \pm \omega_2)t) \rangle = 0$ when $f_1 \neq f_2$ (why?), hence

$$\begin{aligned} R_y(\tau) &= R_{x_1}(\tau) + R_{x_2}(\tau), \text{ for } f_1 \neq f_2 \\ &= \frac{A_1^2}{2} \cos(2\pi f_1 \tau) + \frac{A_2^2}{2} \cos(2\pi f_2 \tau) \end{aligned}$$

Example 2.22: PN Sequences

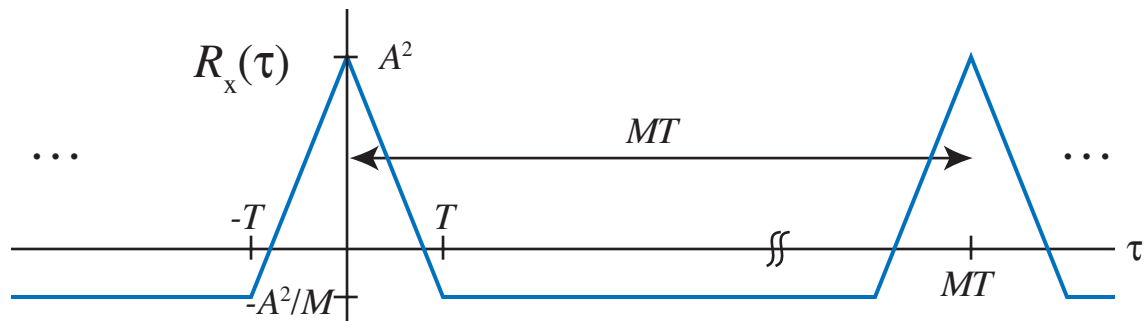
- In the testing and evaluation of digital communication systems a source of *known* digital data (i.e., ‘1’s and ‘0’s) is required (see also text Chapter 10 p. 524–527)

- A *maximal length* sequence generator or *pseudo-noise* (PN) code is often used for this purpose
- Practical implementation of a PN code generator can be accomplished using an N -stage shift register with appropriate exclusive-or feedback connections
- The sequence length or period of the resulting PN code is $M = 2^N - 1$ bits long



Three stage PN (m -sequence) generator using logic circuits

- PN sequences have quite a number of properties, one being that the time average autocorrelation function is of the form shown below



PN sequence autocorrelation function

- The calculation of the power spectral density will be left as a homework problem (a specific example is text Example 2.20)

– Hint: To find $S_x(f) = \mathcal{F}\{R_x(\tau)\}$ we use

$$\sum_n p(t - nT_s) \xleftrightarrow{\mathcal{F}} f_s \sum_n P(nf_s) \delta(f - nf_s)$$

where $T_s = MT$

- One period of $R_x(\tau)$ is a triangle pulse with a level shift
- Suppose the logic levels are switched from $\pm A$ to positive levels of say v_1 to v_2
 - Using the additional autocorrelation function properties this can be done
 - You need to know that a PN sequence contains one more ‘1’ than ‘0’
- Python code for generating PN sequences from 2 to 12 stages plus 16, is found `ssd.py`:

```
def PN_gen(N_bits,m=5):
    """
    Maximal length sequence signal generator.
```

Generates a sequence 0/1 bits of N_bits duration. The bits themselves are obtained from an m-sequence of length m. Available m-sequence (PN generators) include m = 2,3,...,12, & 16.

Parameters

N_bits : the number of bits to generate

m : the number of shift registers. 2,3, ..., 12, & 16

Returns

PN : ndarray of the generator output over N_bits

Notes

The sequence is periodic having period $2^m - 1$ ($2^m - 1$).

Examples

```
>>> # A 15 bit period signal nover 50 bits
>>> PN = PN_gen(50,4)
"""
c = m_seq(m)
Q = len(c)
max_periods = int(np.ceil(N_bits/float(Q)))
PN = np.zeros(max_periods*Q)
for k in range(max_periods):
    PN[k*Q:(k+1)*Q] = c
PN = np.resize(PN, (1,N_bits))
return PN.flatten()
```

```
def m_seq(m):
```

"""

Generate an m-sequence ndarray using an all-ones initialization.

Available m-sequence (PN generators) include m = 2,3,...,12, & 16.

Parameters

m : the number of shift registers. 2,3, ..., 12, & 16

Returns

c : ndarray of one period of the m-sequence

Notes

The sequence period is $2^m - 1$ ($2^m - 1$).

Examples

```
>>> c = m_seq(5)
"""
# Load shift register with all ones to start
sr = np.ones(m)
# M-sequence length is:
Q = 2**m - 1
c = np.zeros(Q)
```

```

if m == 2:
    taps = np.array([1, 1, 1])
elif m == 3:
    taps = np.array([1, 0, 1, 1])
elif m == 4:
    taps = np.array([1, 0, 0, 1, 1])
elif m == 5:
    taps = np.array([1, 0, 0, 1, 0, 1])
elif m == 6:
    taps = np.array([1, 0, 0, 0, 0, 1, 1])
elif m == 7:
    taps = np.array([1, 0, 0, 0, 1, 0, 0, 1])
elif m == 8:
    taps = np.array([1, 0, 0, 0, 1, 1, 1, 0, 1])
elif m == 9:
    taps = np.array([1, 0, 0, 0, 0, 1, 0, 0, 0, 1])
elif m == 10:
    taps = np.array([1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1])
elif m == 11:
    taps = np.array([1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1])
elif m == 12:
    taps = np.array([1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1])
elif m == 16:
    taps = np.array([1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1])
else:
    print 'Invalid length specified'
for n in range(Q):
    tap_xor = 0
    c[n] = sr[-1]
    for k in range(1,m):
        if taps[k] == 1:
            tap_xor = np.bitwise_xor(tap_xor,np.bitwise_xor(int(sr[-1]),
                int(sr[m-1-k])))
    sr[1:] = sr[:-1]
    sr[0] = tap_xor
return c

```

$R(\tau)$, $S(f)$, and Fourier Series

- For a periodic power signal, $x(t)$, we can write

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi(nf_0)t}$$

- There is an interesting linkage between the Fourier series representation of a signal, the power spectrum, and then back to the autocorrelation function

- Using the orthogonality properties of the Fourier series expansion we can write

$$\begin{aligned}
 R(\tau) &= \left\langle \left(\sum_{n=-\infty}^{\infty} X_n e^{j2\pi(nf_0)t} \right) \left(\sum_{m=-\infty}^{\infty} X_m e^{j2\pi(mf_0)(t+\tau)} \right)^* \right\rangle \\
 &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_n X_m^* \underbrace{\langle e^{j2\pi(nf_0)t} e^{-j2\pi(mf_0)(t+\tau)} \rangle}_{n \neq m \text{ terms are zero, why?}} \\
 &= \sum_{n=-\infty}^{\infty} |X_n|^2 \langle e^{j2\pi(nf_0)t} e^{-j2\pi(nf_0)(t+\tau)} \rangle \\
 &= \sum_{n=-\infty}^{\infty} |X_n|^2 e^{j2\pi(nf_0)\tau}
 \end{aligned}$$

- The power spectral density can be obtained by Fourier transforming both sides of the above

$$S(f) = \sum_{n=-\infty}^{\infty} |X_n|^2 \delta(f - nf_0)$$

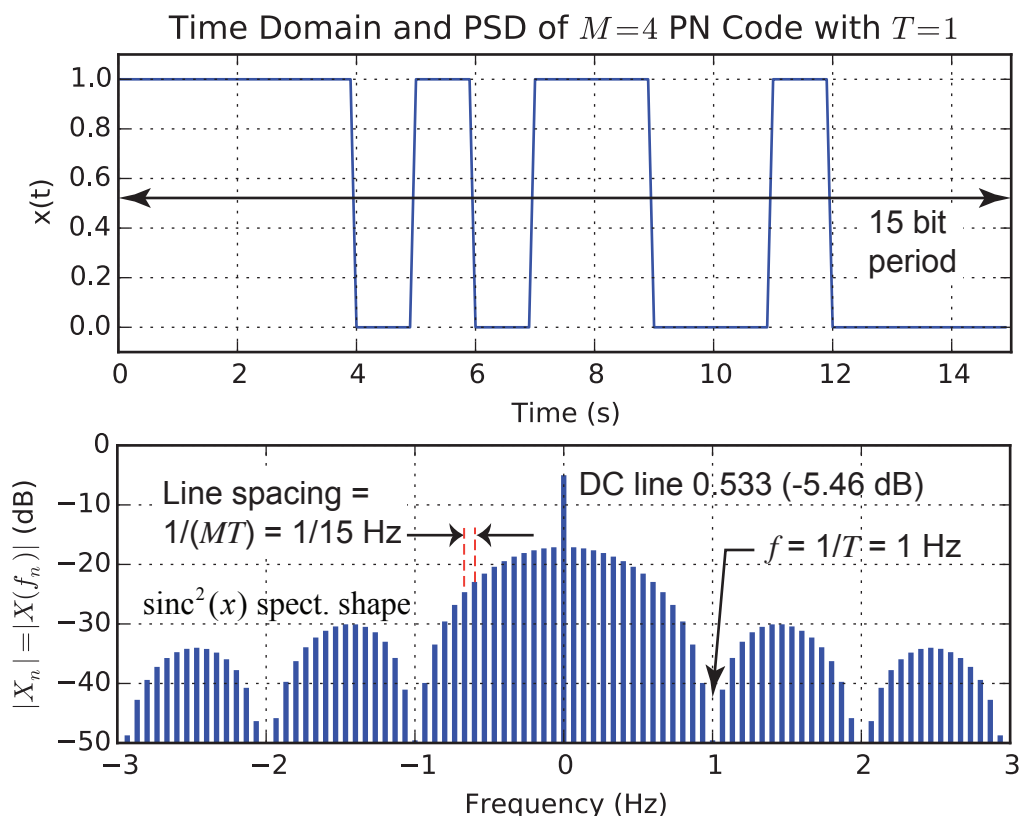
Example 2.23: PN Sequence Analysis and Simulation

- In this example I consider an $N = 4$ or $M = 15$ generator from two points of view:
 1. First using the FFT to calculate the Fourier coefficients X_n using one period of the waveform (in discrete-time I use 10 samples per bit)
 2. Second, I just generate a waveform of 10,000 bits (again at 10 samples per bit) and use standard signal processing

tools to estimate the time average autocorrelation function and the PSD

- **Numerical Fourier Series Analysis:** Generate one period of the waveform using `ssd.m_seq(4)` and then upsample and interpolate to create a waveform of 10 samples per bit
- The waveform amplitude levels are $[0, 1]$ so there is a large DC component visible in the spectrum (why?); eight ones and seven zeros makes the average value $X_0 = 8/15 = 0.533$
- The function `ssd.m_seq()` returns an array of zeros and ones of length 15, i.e., $\text{len}(x_{\text{PN4}}) = 15$
- To create a waveform I upsample the signal by 10 and then filter using a finite impulse response of exactly 10 ones
- To plot the line spectrum I use the `ssd.line_spectra()` function used earlier

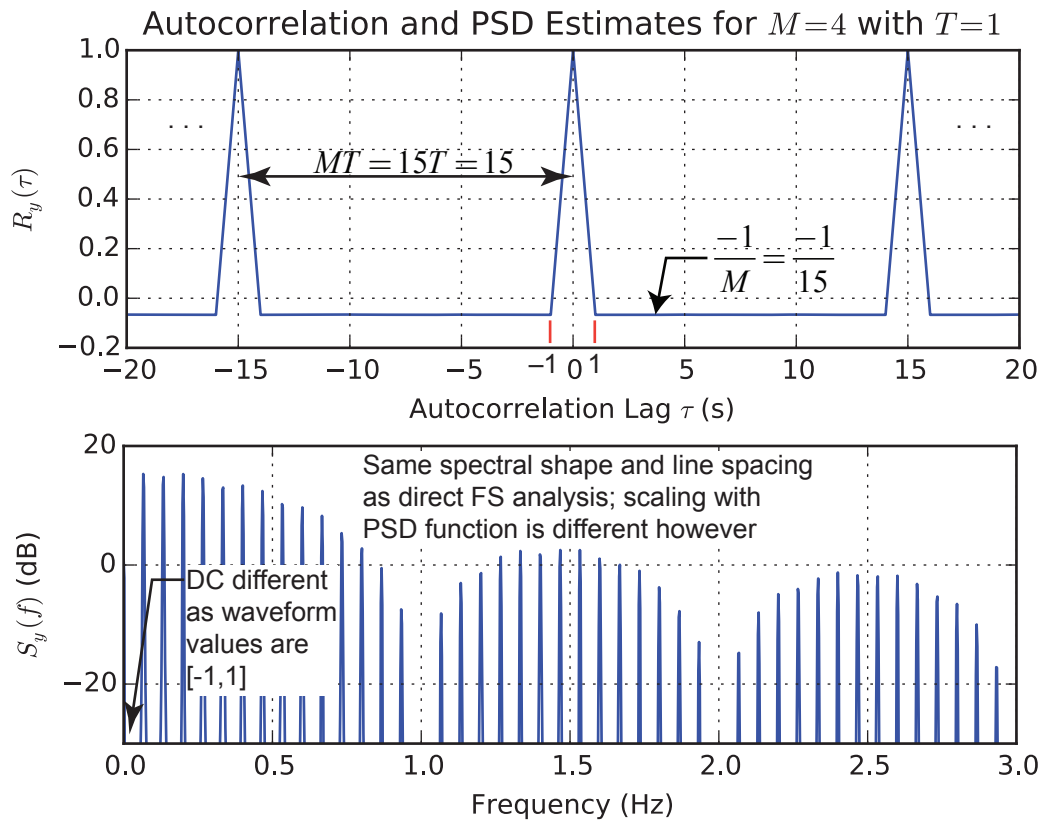
```
x_PN4 = ssd.m_seq(4)
x = signal.lfilter(ones(10), 1, ssd.upsample(x_PN4, 10))
t = arange(0, len(x))/10
figure(figsize=(6, 2))
plot(t, x);
title(r'Time Domain and PSD of $M=4$ PN Code with $T = 1$')
xlabel(r'Time (s)')
ylabel(r'$x(t)$')
axis([0, 15, -.1, 1.1]);
grid()
X = fft.fft(x, 10*15) # 10 samples/bit so 150 samples/period
f = arange(0, 7*10)/15 # harmonics spaced by 1/(15*T) = 1/15
ssd.line_spectra(f[0:45], abs(X[0:45])/150, 'magdB', floor_dB=-50, fsize=(6, 2))
xlim([-3, 3])
ylabel(r'$|X_n| = |X(f_n)|$ (dB)');
```



Fourier series based spectral analysis of PN code

```
import digitalcom as dc
y_PN4_bits = ssd.PN_gen(10000,4)
# Convert to waveform level shifted to +/-1 amplitude
y = 2*signal.lfilter(ones(10),1,ssd.upsample(y_PN4_bits,10))-1
# Find the time averaged autocorrelation function normalized
# to have a peak amplitude of 1
Ry,tau = dc.xcorr(y,y,200)
# We know Ry is real
Ry = Ry.real

tau_s = tau/10
figure(figsize=(6,2))
plot(tau_s,Ry)
title(r'Autocorrelation and PSD Estimates for $M=4$ with $T = 1$')
xlabel(r'Autocorrelation Lag $\tau$ (s)')
ylabel(r'$R_y(\tau)$')
grid();
figure(figsize=(6,2))
psd(y,2*12,10)
xlabel(r'Frequency (Hz)')
ylabel(r'$S_y(f)$ (dB)')
xlim([0,3]);
ylim([-30,20]);
```

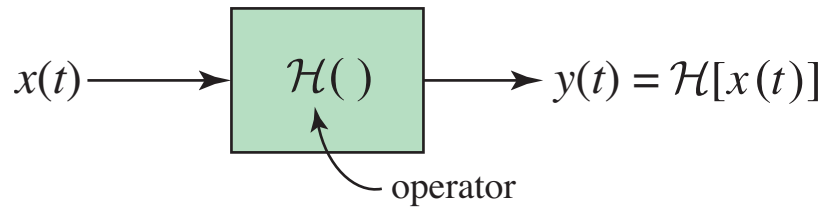



Using simulation to estimate $R_y(\tau)$ and $S_y(f)$ of a PN code

- In generating an estimate of the autocorrelation function I use the FFT to find the time averaged autocorrelation function in the frequency domain
- In generating the spectral estimate, the Python function `psd()` (from `matplotlib`) function uses *Welch's method of averaged periodograms*¹
- Here the PSD estimate uses a 4096 point FFT and assumed sampling rate of 10 Hz; the spectral resolution is $10/4096 = 0.002441$ Hz

¹<http://en.wikipedia.org/wiki/Periodogram>

2.7 Linear Time Invariant (LTI) Systems



Linear system block diagram

Definition

- *Linearity* (superposition) holds, that is for input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, α_1 and α_2 constants,

$$\begin{aligned} y(t) &= \mathcal{H}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\ &= \alpha_1 \mathcal{H}[x_1(t)] + \alpha_2 \mathcal{H}[x_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

- A system is *time invariant* (fixed) if for $y(t) = \mathcal{H}[x(t)]$, a delayed input gives a correspondingly delayed output, i.e.,

$$y(t - t_0) = \mathcal{H}[x(t - t_0)]$$

Impulse Response and Superposition Integral

- The *impulse response* of an LTI system is denoted

$$h(t) \triangleq \mathcal{H}[\delta(t)]$$

assuming the system is initially at rest

- Suppose we can write $x(t)$ as

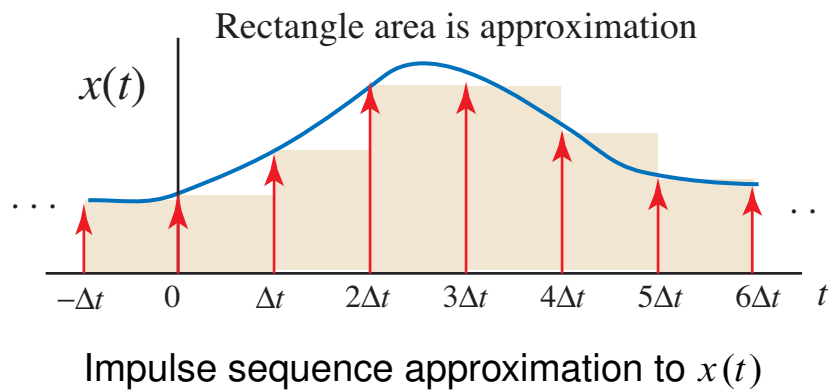
$$x(t) = \sum_{n=1}^N \alpha_n \delta(t - t_n)$$

- For an LTI system with impulse response $h(\cdot)$

$$y(t) = \sum_{n=1}^N \alpha_n h(t - t_n)$$

- To develop the superposition integral we write

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x(\lambda) \delta(t - \lambda) d\lambda \\ &\simeq \lim_{N \rightarrow \infty} \sum_{n=-N}^N x(n\Delta t) \delta(t - n\Delta t) \Delta t, \text{ for } \Delta t \ll 1 \end{aligned}$$



- If we apply \mathcal{H} to both sides and let $\Delta t \rightarrow 0$ such that $n\Delta t \rightarrow \lambda$ we have

$$\begin{aligned} y(t) &\simeq \lim_{N \rightarrow \infty} \sum_{n=-N}^N x(n\Delta t) h(t - n\Delta t) \Delta t \\ &= \int_{-\infty}^{\infty} x(\lambda) h(t - \lambda) d\lambda = x(t) * h(t) \\ &\stackrel{\text{or}}{=} \int_{-\infty}^{\infty} x(t - \sigma) h(\sigma) d\sigma = h(t) * x(t) \end{aligned}$$

2.7.1 Stability

- In signals and systems the concept of *bounded-input bounded-output* (BIBO) stability is introduced
- Satisfying this definition requires that every bounded-input ($|x(t)| < \infty$) produces a bounded output ($|y(t)| < \infty$)
- For LTI systems a fundamental theorem states that a system is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- Further implications of this will be discussed later

2.7.2 Transfer Function

- The frequency domain result corresponding to the convolution expression $y(t) = x(t) * h(t)$ is

$$Y(f) = X(f)H(f)$$

where $H(f)$ is known as the *transfer function* or *frequency response* of the system having impulse response $h(t)$

- It immediately follows that

$$h(t) \xleftrightarrow{\mathcal{F}} H(f)$$

and

$$y(t) = \mathcal{F}^{-1}\{X(f)H(f)\} = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft} df$$

2.7.3 Causality

- A system is *causal* if the present output relies only on past and present inputs, that is the output does not anticipate the input
- The fact that for LTI systems $y(t) = x(t) * h(t)$ implies that for a causal system we must have

$$h(t) = 0, \quad t < 0$$

- Having $h(t)$ nonzero for $t < 0$ would incorporate future values of the input to form the present value of the output
- Systems that are causal have limitations on their frequency response, in particular the *Paley–Wiener* theorem states that for $\int_{-\infty}^{\infty} |h(t)|^2 dt < \infty$, $H(f)$ for a causal system must satisfy

$$\int_{-\infty}^{\infty} \frac{|\ln |H(f)||}{1 + f^2} df < \infty$$

- In simple terms this means:
 1. We cannot have $|H(f)| = 0$ over a finite band of frequencies (isolated points ok)
 2. The roll-off rate of $|H(f)|$ cannot be too great, e.g., $e^{-k_1|f|}$ and $e^{-k_2|f|^2}$ are not allowed, but polynomial forms such as $\sqrt{1/(1 + (f/f_c)^{2N})}$, N an integer, are acceptable
 3. Practical filters such as Butterworth, Chebyshev, and elliptical filters can come close to ideal requirements

2.7.4 Properties of $H(f)$

- For $h(t)$ real it follows that

$$|H(-f)| = |H(f)| \text{ and } \angle H(-f) = -\angle H(f)$$

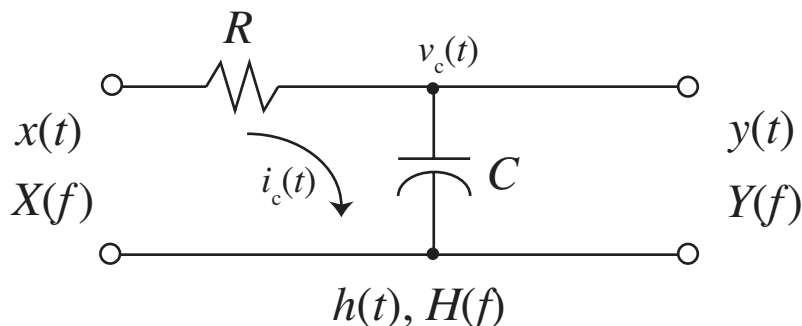
why?

- Input/output relationships for spectral densities are

$$G_y(f) = |Y(f)|^2 = |X(f)H(f)|^2 = |H(f)|^2 G_x(f)$$

$$S_y(f) = |H(f)|^2 S_x(f) \quad \text{proof in text chap. 6}$$

Example 2.24: RC Lowpass Filter



RC lowpass filter schematic

- To find $H(f)$ we may solve the circuit using AC steady-state analysis

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

so

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + jf/f_3}, \quad \text{where } f_3 = 1/(2\pi RC)$$

- From the circuit differential equation

$$x(t) = i_c(t)R + y(t)$$

but

$$i_c(t) = C \frac{dv_c(t)}{dt} = C \frac{y(t)}{dt}$$

thus

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

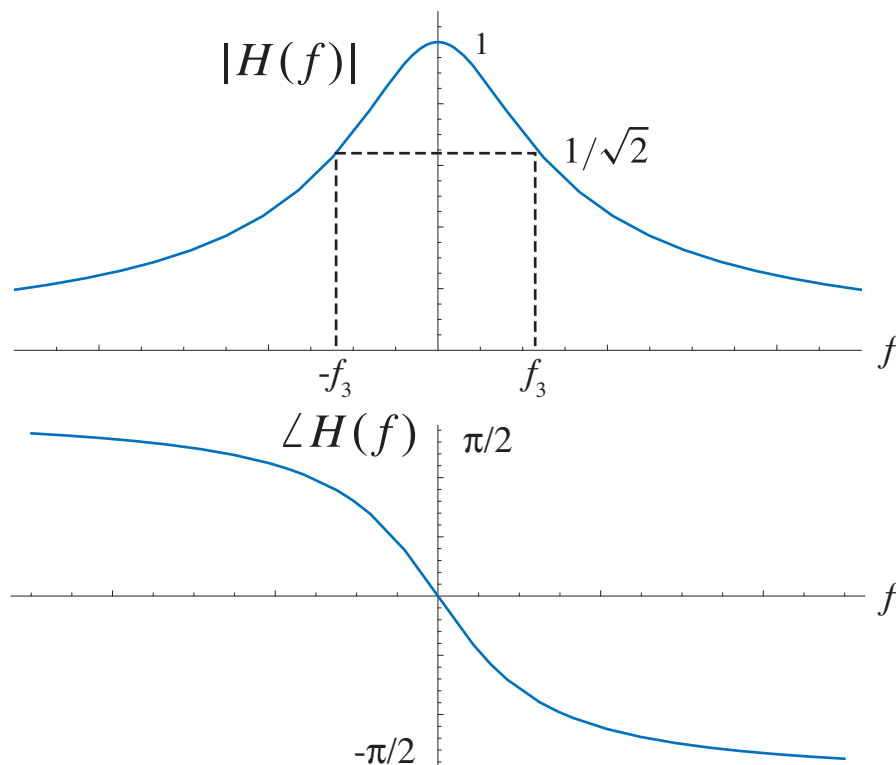
- FT both sides using $dx/dt \xleftrightarrow{\mathcal{F}} j2\pi f X(f)$

$$j2\pi f RC Y(f) + Y(f) = X(f)$$

so again

$$\begin{aligned} H(f) &= \frac{Y(f)}{X(f)} = \frac{1}{1 + jf/f_3} \\ &= \frac{1}{\sqrt{1 + (f/f_3)^2}} e^{-j \tan^{-1}(f/f_3)} \end{aligned}$$

- The Laplace transform could also be used here, and perhaps is preferred; we just need to substitute $s \rightarrow j\omega \rightarrow j2\pi f$



RC lowpass frequency response

- Find the system response to

$$x(t) = A\Pi\left(\frac{(t - T/2)}{T}\right)$$

- Finding $Y(f)$ is easy since

$$Y(f) = X(f)H(f) = AT\text{sinc}(fT) \left[\frac{1}{1 + jf/f_s} \right] e^{-j\pi ft}$$

- To find $y(t)$ we can IFT the above, use Laplace transforms, or convolve directly
- From the FT tables we know that

$$h(t) = \frac{1}{RC} e^{-t/(RC)} u(t)$$

- In Example 2.18 we showed that

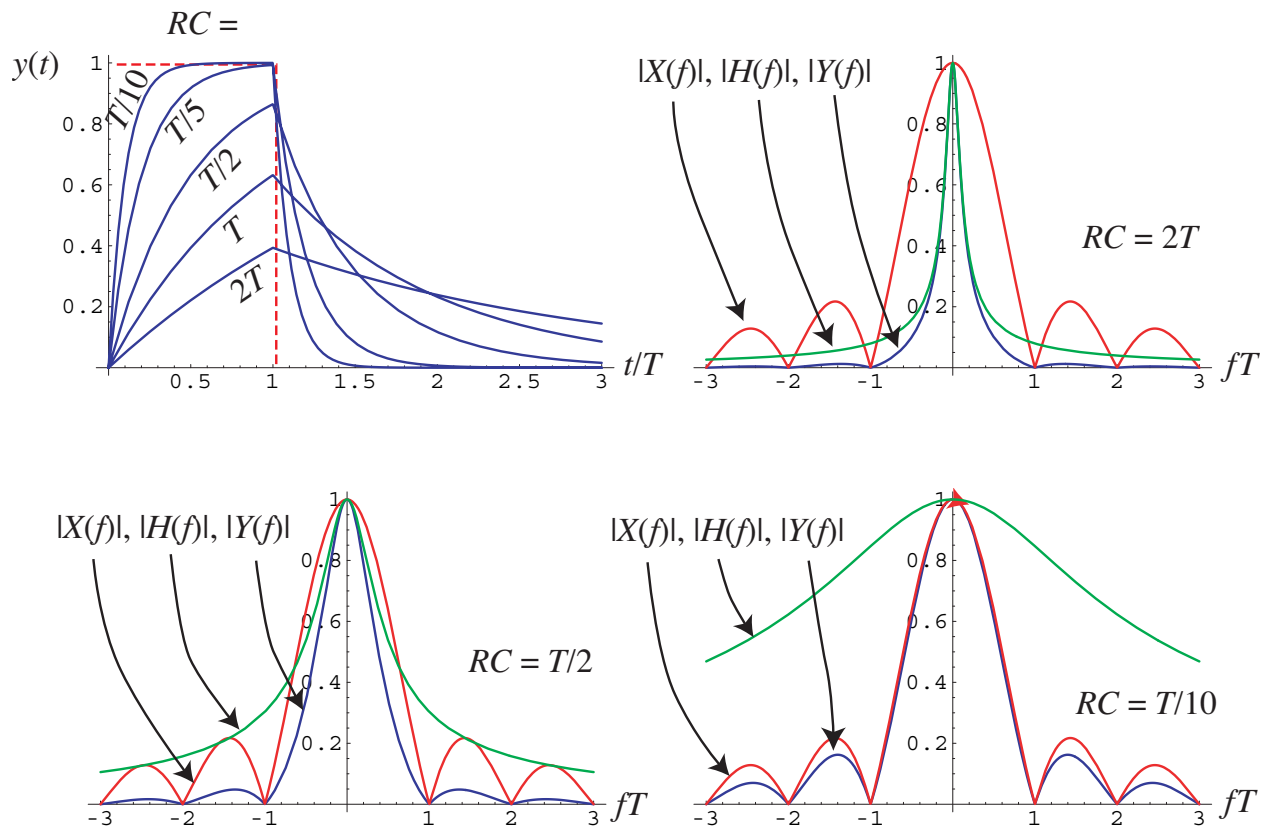
$$Au(t) * e^{-\alpha t}u(t) = \frac{A}{\alpha}[1 - e^{-\alpha t}]u(t)$$

- Note that

$$A\Pi\left(\frac{t - T/2}{T}\right) = A[u(t) - u(t - T)]$$

and here $\alpha = 1/(RC)$, so

$$y(t) = \frac{A}{RC}RC[1 - e^{-t/(RC)}]u(t) - \frac{A}{RC}RC[1 - e^{-(t-T)/(RC)}]u(t - T)$$



Pulse time response and frequency spectra with $A = 1$

2.7.5 Input/Output with Spectral Densities

- We know that for an LTI system with frequency response $H(f)$ and input Fourier transform $X(f)$, the output Fourier transform is given by $Y(f) = H(f)X(f)$
- It is easy to show that in terms of energy spectral density

$$G_y(f) = |H(f)|^2 G_x(f)$$

where $G_x(f) = |X(f)|^2$ and $G_y(f) = |Y(f)|^2$

- For the case of power signals a similar relationship holds with the power spectral density (proof found in Chapter text 7, i.e., Comm II)

$$S_y(f) = |H(f)|^2 S_x(f)$$

2.7.6 Response to Periodic Inputs

- When the input is periodic we can write

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi(nf_0)t}$$

which implies that

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - nf_0)$$

- It then follows that

$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(nf_0) \delta(f - nf_0)$$

and

$$\begin{aligned}
 y(t) &= \sum_{n=-\infty}^{\infty} X_n H(n f_0) e^{j 2 \pi (n f_0) t} \\
 &= \sum_{n=-\infty}^{\infty} |X_n| |H(n f_0)| e^{j [2 \pi (n f_0) t + \angle X_n + \angle H(n f_0)]}
 \end{aligned}$$

- This is a steady-state response calculation, since the analysis assumes that the periodic signal was applied to the system at $t = -\infty$

2.7.7 Distortionless Transmission

- In the time domain a distortionless system is such that for any input $x(t)$,

$$y(t) = H_0 x(t - t_0)$$

where H_0 and t_0 are constants

- In the frequency domain this implies a frequency response of the form

$$H(f) = H_0 e^{-j 2 \pi f t_0},$$

that is the amplitude response is constant and the phase shift is linear with frequency

- Distortion types:
 1. Amplitude response is not constant over a frequency band (interval) of interest \leftrightarrow *amplitude distortion*
 2. Phase response is not linear over a frequency band of interest \leftrightarrow *phase distortion*

3. The system is non-linear, e.g., $y(t) = k_0 + k_1x(t) + k_2x^2(t) \leftrightarrow \text{nonlinear distortion}$

2.7.8 Group and Phase Delay

- The phase distortion of a linear system can be characterized using *group delay*, $T_g(f)$,

$$T_g(f) = -\frac{1}{2\pi} \frac{d\theta(f)}{df}$$

where $\theta(f)$ is the phase response of an LTI system

- Note that for a distortionless system $\theta(f) = -2\pi f t_0$, so

$$T_g(f) = -\frac{1}{2\pi} \frac{d}{df} - 2\pi f t_0 = t_0 \text{ s,}$$

clearly a constant group delay

- $T_g(f)$ is the delay that two or more frequency components undergo in passing through an LTI system
 - If say $T_g(f_1) \neq T_g(f_2)$ and both of these frequencies are in a band of interest, then we know that delay distortion exists
 - Having two different frequency components arrive at the system output at different times produces signal dispersion
- An LTI system passing a single frequency component, $x(t) = A \cos(2\pi f_1 t)$, always appears distortionless since at a single

frequency the output is just

$$\begin{aligned} y(t) &= A|H(f_1)| \cos [2\pi f_1 t + \theta(f_1)] \\ &= A|H_1(f)| \cos \left[2\pi f_1 \left(t - \frac{-\theta(f_1)}{2\pi f_1} \right) \right] \end{aligned}$$

which is equivalent to a delay known as the *phase delay*

$$T_p(f) = -\frac{\theta(f)}{2\pi f}$$

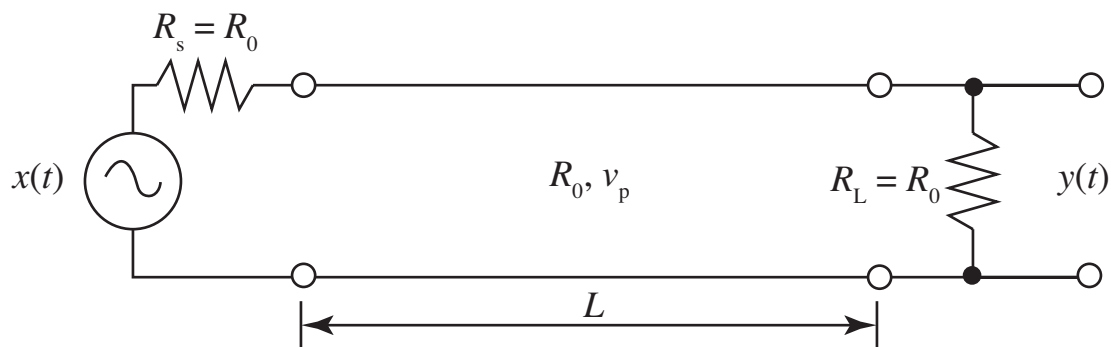
- The system output now is

$$y(t) = A|H(f_1)| \cos [2\pi f_1 (t - T_p(f_1))]$$

- Note that for a distortionless system

$$T_p(f) = -\frac{1}{2\pi f}(-2\pi f t_0) = t_0$$

Example 2.25: Terminated Lossless Transmission Line

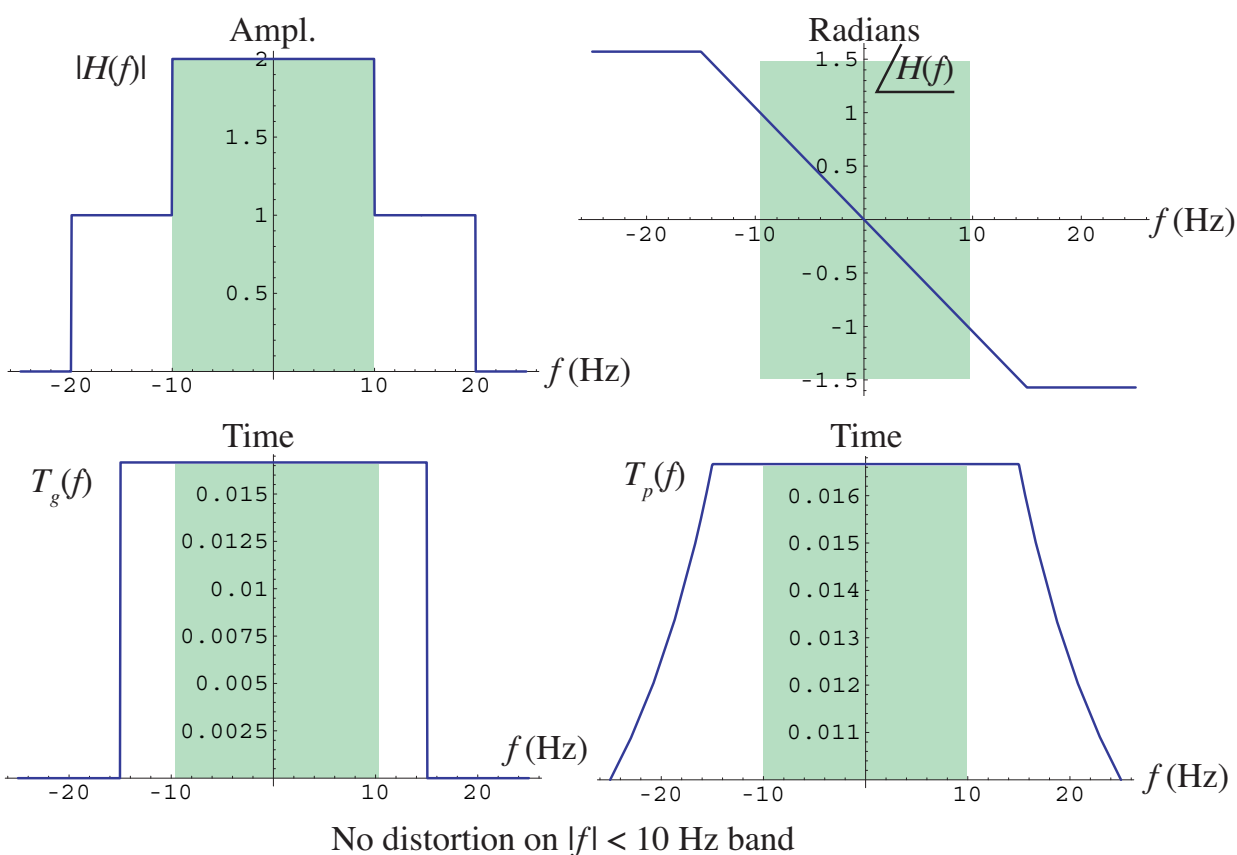


$$y(t) = \frac{1}{2}x\left(t - \frac{L}{v_p}\right)$$

Lossless transmission line

- We conclude that $H_0 = 1/2$ and $t_0 = L/v_p$
- Note that a real transmission line does have losses that introduces dispersion on a wideband signal

Example 2.26: A Fictitious System



Amplitude, phase, group delay, phase delay

- The system in this example is artificial, but the definitions can be observed just the same
- For signals with spectral content limited to $|f| < 10$ Hz there is no distortion, amplitude or phase/group delay

- For $10 < |f| < 15$ amplitude distortion is present
 - For $|f| > 15$ both amplitude and phase distortion are present
 - What about the interval $10 < |f| < 15$?
-

2.7.9 Nonlinear Distortion

- In the time domain a nonlinear system may be written as

$$y(t) = \sum_{n=0}^{\infty} a_n x^n(t)$$

- Specifically consider

$$y(t) = a_1 x(t) + a_2 x^2(t)$$

- Let

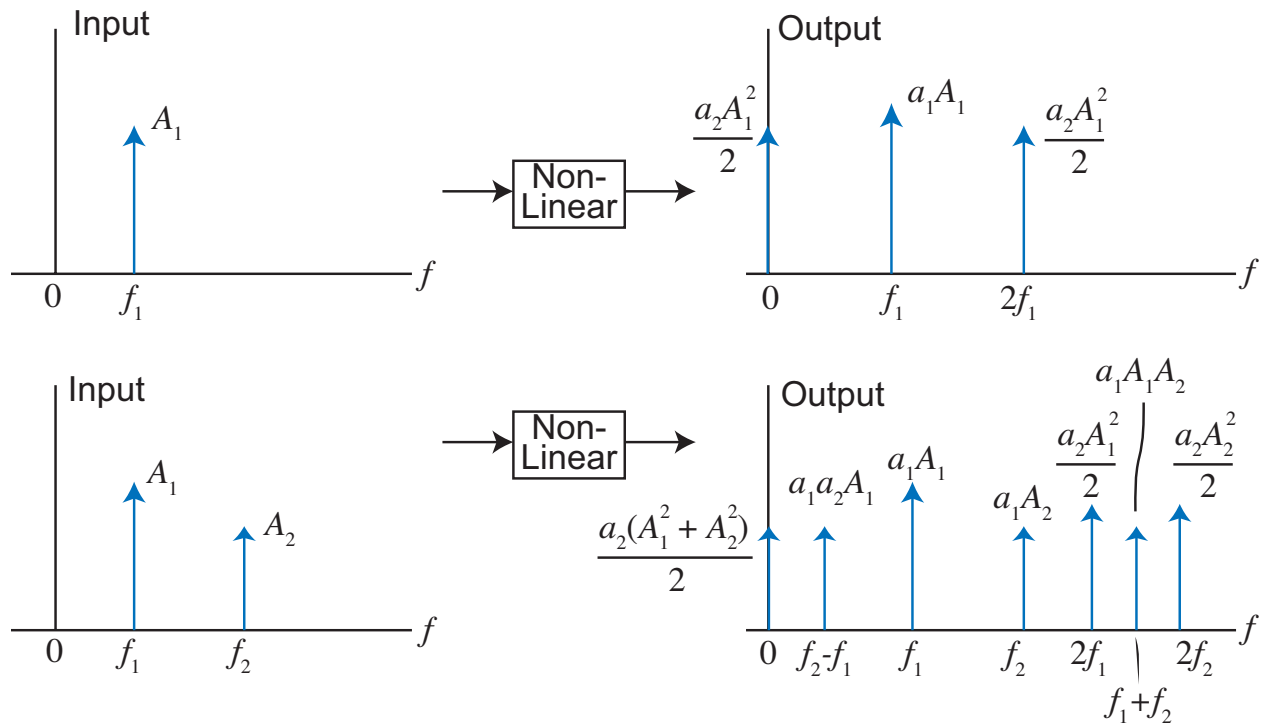
$$x(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$

- Expanding the output we have

$$\begin{aligned} y(t) &= a_1 [A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)] \\ &\quad + a_2 [A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)]^2 \\ &= a_1 [A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)] \\ &\quad + \left\{ \frac{a_2}{2} (A_1^2 + A_2^2) + \frac{a_2}{2} [A_1^2 \cos(2\omega_1 t) + A_2^2 \cos(2\omega_2 t)] \right\} \\ &\quad + a_2 A_1 A_2 \{ \cos[(\omega_1 + \omega_2)t] + \cos[(\omega_1 - \omega_2)t] \} \end{aligned}$$

- The third line is the desired output

- The fourth line is termed *harmonic distortion*
- The fifth line is termed *intermodulation distortion*



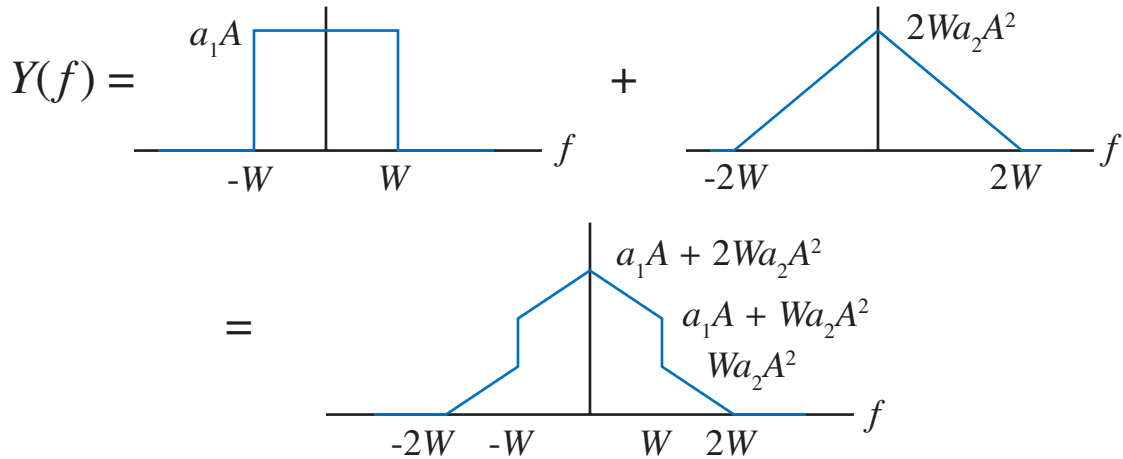
One and two tones in $y(t) = a_1 x(t) + a_2 x^2(t)$ device

- In general if $y(t) = a_1 x(t) + a_2 x^2(t)$ the multiplication theorem implies that

$$Y(f) = a_1 X(f) + a_2 X(f) * X(f)$$

- In particular if $X(f) = A \Pi(f/(2W))$

$$Y(f) = a_1 A \Pi\left(\frac{f}{2W}\right) + a_2 2W A^2 \Lambda\left(\frac{f}{2W}\right)$$

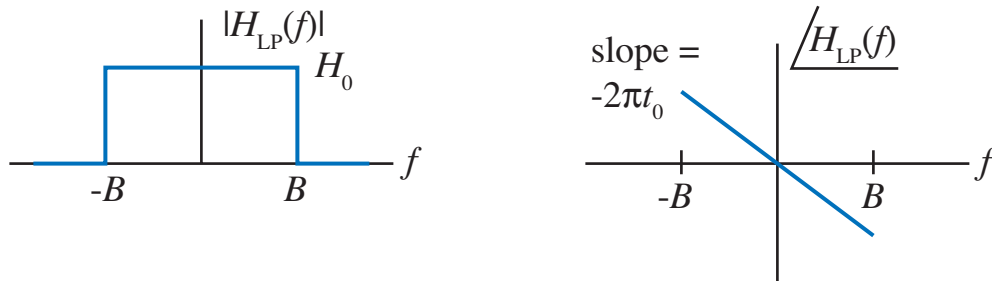


Continuous spectrum in $y(t) = a_1x(t) + a_2x^2(t)$ device

2.7.10 Ideal Filters

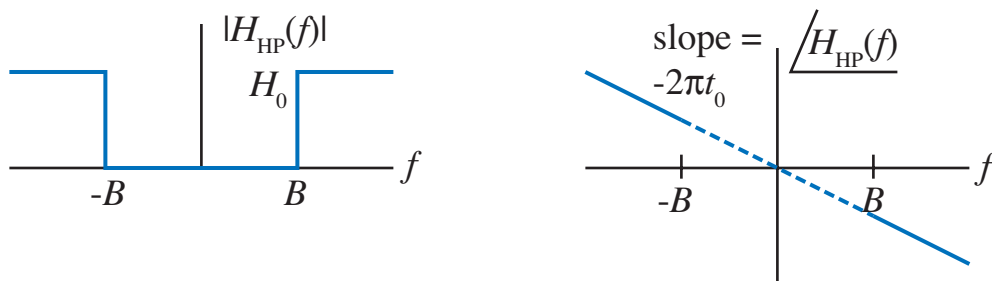
1. Lowpass of bandwidth B

$$H_{\text{LP}}(f) = H_0 \Pi\left(\frac{f}{2B}\right) e^{-j2\pi f t_0}$$



2. Highpass with cutoff B

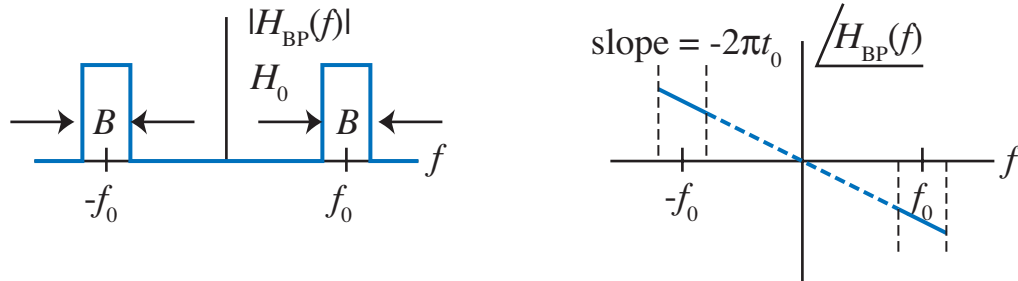
$$H_{\text{HP}}(f) = H_0 [1 - \Pi(f/(2B))] e^{-j2\pi f t_0}$$



3. Bandpass of bandwidth B

$$H_{BP}(f) = [H_l(f - f_0) + H_l(f + f_0)]e^{-j2\pi f t_0}$$

where $H_l(f) = H_0\Pi(f/B)$



- The impulse response of the lowpass filter is

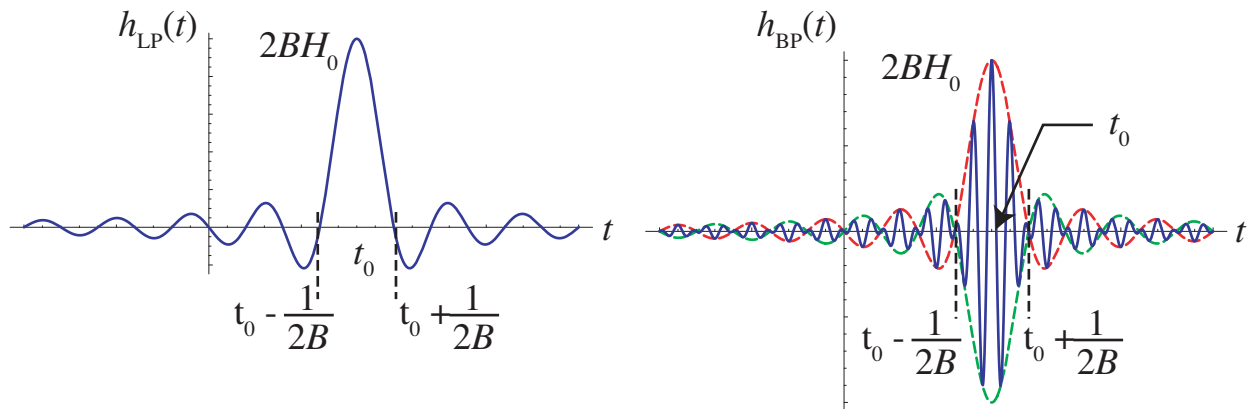
$$\begin{aligned} h_{LP}(t) &= \mathcal{F}^{-1}\{H_0\Pi(f/(2B))e^{-j2\pi f t_0}\} \\ &= 2BH_0\text{sinc}[2B(t - t_0)] \end{aligned}$$

- Ideal filters are not realizable, but simplify calculations and give useful performance upper bound results

- Note that $h_{LP}(t) \neq 0$ for $t < 0$, thus the filter is noncausal and unrealizable

- From the modulation theorem it also follows that

$$\begin{aligned} h_{BP}(t) &= 2h_l(t - t_0) \cos[2\pi f_0(t - t_0)] \\ &= 2BH_0\text{sinc}[B(t - t_0)] \cos[2\pi f_0(t - t_0)] \end{aligned}$$



Ideal lowpass and bandpass impulse responses

2.7.11 Realizable Filters

- We can approximate ideal filters with realizable filters such as Butterworth, Chebyshev, and Bessel, to name a few
- We will only consider the lowpass case since via frequency transformations we can obtain the others

Butterworth

- A Butterworth filter has a *maximally flat* (flat in the sense of derivatives of the amplitude response at dc being zero) pass-band
- In the s -domain ($s = \sigma + j\omega$) the transfer function of a lowpass design is

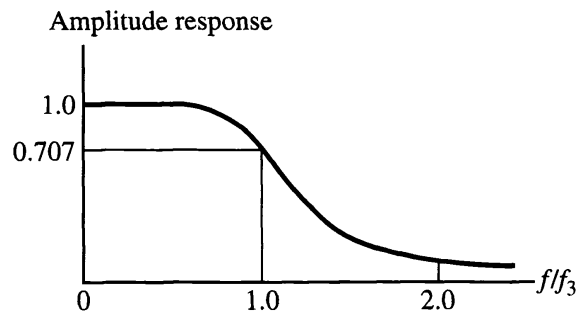
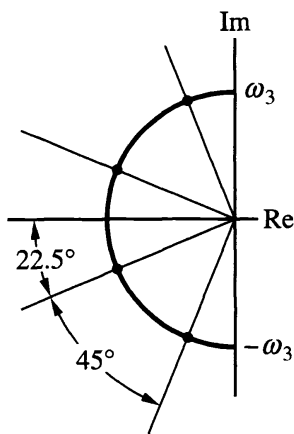
$$H_{\text{BU}}(s) = \frac{\omega_c^n}{(s - s_1)(s - s_2) \cdots (s - s_n)}$$

where

$$s_k = \omega_c \exp \left[\pi \left(\frac{1}{2} + \frac{2k-1}{2n} \right) j \right], \quad k = 1, 2, \dots, n$$

- Note that the poles are located on a semi-circle of radius $\omega_c = 2\pi f_c$, where f_c is the 3dB cutoff frequency of the filter
- The amplitude response of a Butterworth filter is simply

$$|H_{BU}(f)| = \frac{1}{\sqrt{1 + (f/f_c)^{2n}}}$$


 Butterworth $n = 4$ lowpass filter

Chebyshev

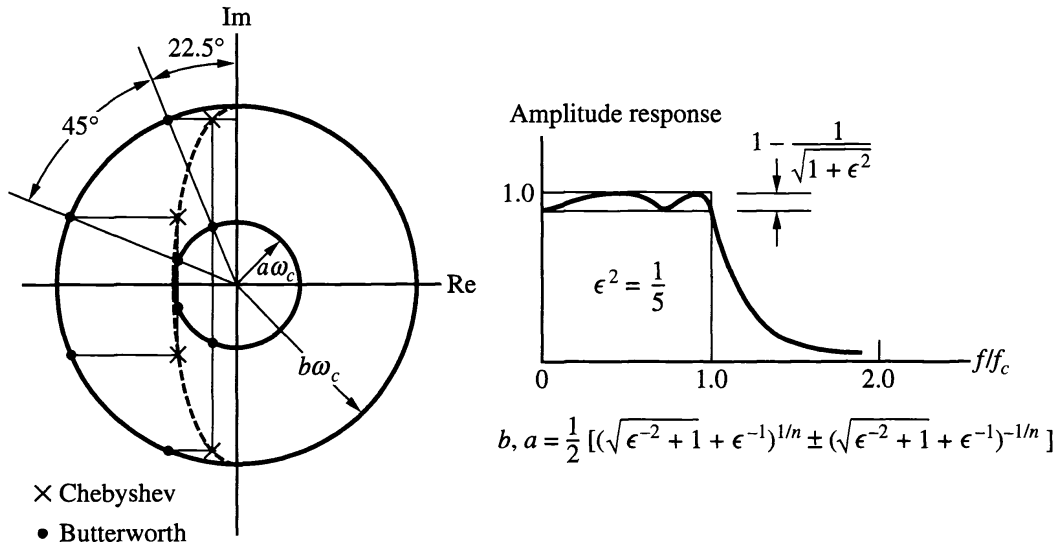
- A Chebyshev type I filter (ripple in the passband), is designed to maintain the maximum allowable attenuation in the passband yet have maximum stopband attenuation
- The amplitude response is given by

$$|H_C(f)| = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2(f/f_c)}}$$

where

$$C_n(f) = \begin{cases} \cos(n \cos^{-1}(f/f_c)), & 0 \leq |f| \leq f_c \\ \cosh(n \cosh^{-1}(f/f_c)), & |f| > f_c \end{cases}$$

- The poles are located on an ellipse as shown below



Bessel

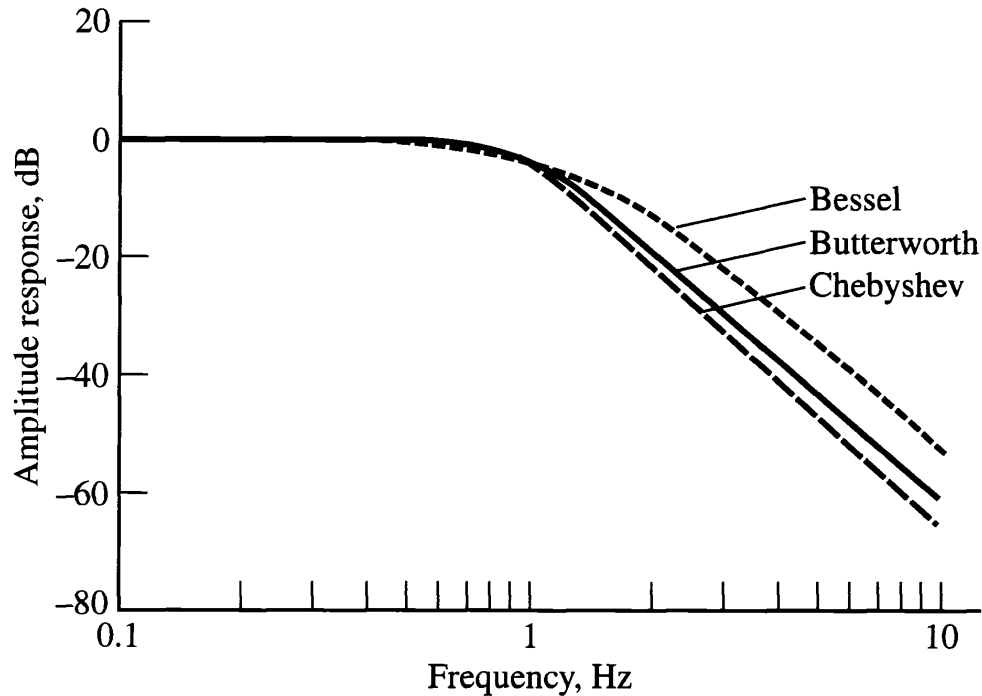
- A Bessel filter is designed to maintain linear phase in the pass-band at the expense of the amplitude response

$$H_{BE}(f) = \frac{K_n}{B_n(f)}$$

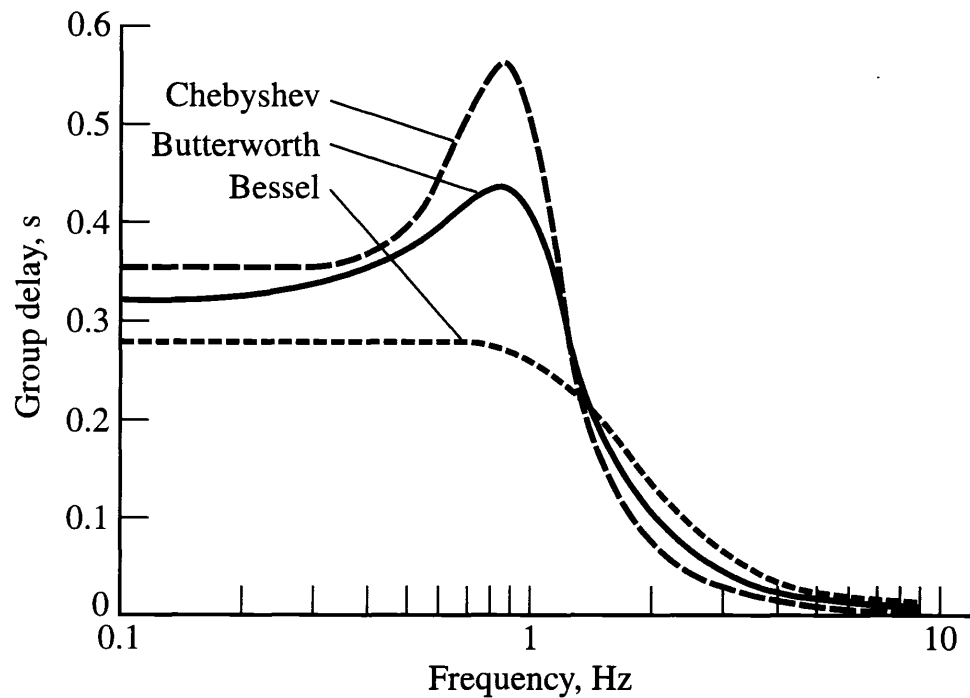
where $B_n(f)$ is a Bessel polynomial of order n (see text) and K_n is chosen so that the filter gain is unity at $f = 0$

Amplitude Rolloff and Group Delay Comparison

- Compare Butterworth, 0.1 dB ripple Chebyshev, and Bessel



$n = 3$ Amplitude response



$n = 3$ Group delay

- Despite DSP being pervasive in communication systems, there is still a great need for analog filter design and implementation technologies
- The table below describes some of the well known construction techniques

Filter Construction Techniques

Construction Type	Description of Elements or Filter	Center Frequency Range	Fre- (typical)	Unloaded Q	Filter Application	Appli-
<i>LC</i> (passive)	lumped elements	DC–300 MHz or higher in integrated form		100	Audio, video, IF and RF	
Active	R , C , op-amps	DC–500 kHz or higher using WB op-amps		200	Audio and low RF	
Crystal	quartz crystal	1kHz – 100 MHz		100,000	IF	
Ceramic	ceramic disks with electrodes	10kHz – 10.7 MHz		1,000	IF	
Surface acoustic waves (SAW)	interdigitated fingers on a Piezo-electric substrate	10-800 MHz,		variable	IF and RF	
Transmission line	quarterwave stubs, open and short ckt	UHF and microwave		1,000	RF	
Cavity	machined and plated metal	Microwave		10,000	RF	

Example 2.27: Use Python to Characterize Standard Filters

- You can use the filter design capability of Python with Scipy or MATLAB with the signal processing to study lowpass, bandpass, bandstop, and highpass filters
- Here I will consider two functions written in Python, one for digital filters and one for analog filters, to allow plotting of gain in dB, phase in radians, and group delay in samples or seconds

```

def freqz_resp(b,a=[1],mode = 'dB',fs=1.0,Npts = 1024,fsize=(6,4)):
    """
    A method for displaying digital filter frequency response magnitude,
    phase, and group delay. A plot is produced using matplotlib

    freq_resp(self,mode = 'dB',Npts = 1024)

    A method for displaying the filter frequency response magnitude,
    phase, and group delay. A plot is produced using matplotlib

    freqs_resp(b,a=[1],Dmin=1,Dmax=5,mode = 'dB',Npts = 1024,fsize=(6,4))

        b = ndarray of numerator coefficients
        a = ndarray of denominator coefficients
        Dmin = start frequency as 10**Dmin
        Dmax = stop frequency as 10**Dmax
        mode = display mode: 'dB' magnitude, 'phase' in radians, or
              'groupdelay_s' in samples and 'groupdelay_t' in sec,
              all versus frequency in Hz
        Npts = number of points to plot; default is 1024
        fsize = figure size; default is (6,4) inches

    Mark Wickert, January 2015
    """
    f = np.arange(0,Npts)/(2.0*Npts)
    w,H = signal.freqz(b,a,2*np.pi*f)
    plt.figure(figsize=fsize)
    if mode.lower() == 'db':
        plt.plot(f*fs,20*np.log10(np.abs(H)))
        plt.xlabel('Frequency (Hz)')
        plt.ylabel('Gain (dB)')
        plt.title('Frequency Response - Magnitude')

    elif mode.lower() == 'phase':
        plt.plot(f*fs,np.angle(H))
        plt.xlabel('Frequency (Hz)')
        plt.ylabel('Phase (rad)')
        plt.title('Frequency Response - Phase')

    elif (mode.lower() == 'groupdelay_s') or (mode.lower() == 'groupdelay_t'):
        """
        Notes
        -----

        Since this calculation involves finding the derivative of the
        phase response, care must be taken at phase wrapping points

```


and when the phase jumps by $\pm\pi$, which occurs when the amplitude response changes sign. Since the amplitude response is zero when the sign changes, the jumps do not alter the group delay results.

```

"""
theta = np.unwrap(np.angle(H))
# Since theta for an FIR filter is likely to have many pi phase
# jumps too, we unwrap a second time 2*theta and divide by 2
theta2 = np.unwrap(2*theta)/2.
theta_dif = np.diff(theta2)
f_dif = np.diff(f)
Tg = -np.diff(theta2)/np.diff(w)
max_Tg = np.max(Tg)
#print(max_Tg)
if mode.lower() == 'groupdelay_t':
    max_Tg /= fs
    plt.plot(f[:-1]*fs,Tg/fs)
    plt.ylim([0,1.2*max_Tg])
else:
    plt.plot(f[:-1]*fs,Tg)
    plt.ylim([0,1.2*max_Tg])
plt.xlabel('Frequency (Hz)')
if mode.lower() == 'groupdelay_t':
    plt.ylabel('Group Delay (s)')
else:
    plt.ylabel('Group Delay (samples)')
plt.title('Frequency Response - Group Delay')
else:
    s1 = 'Error, mode must be "dB", "phase, '
    s2 = '"groupdelay_s", or "groupdelay_t"'
    print(s1 + s2)

def freqs_resp(b,a=[1],Dmin=1,Dmax=5,mode = 'dB',Npts = 1024,fs=(6,4)):
    """
    A method for displaying analog filter frequency response magnitude,
    phase, and group delay. A plot is produced using matplotlib

    freqs_resp(b,a=[1],Dmin=1,Dmax=5,mode='dB',Npts=1024,fs=(6,4))

    b = ndarray of numerator coefficients
    a = ndarray of denominator coefficients
    Dmin = start frequency as 10**Dmin
    Dmax = stop frequency as 10**Dmax
    mode = display mode: 'dB' magnitude, 'phase' in radians, or
           'groupdelay', all versus log frequency in Hz
    Npts = number of points to plot; default is 1024

```

```

fsize = figure size; defult is (6,4) inches

Mark Wickert, January 2015
"""
f = np.logspace(Dmin,Dmax,Npts)
w,H = signal.freqs(b,a,2*np.pi*f)
plt.figure(figsize=fsize)
if mode.lower() == 'db':
    plt.semilogx(f,20*np.log10(np.abs(H)))
    plt.xlabel('Frequency (Hz)')
    plt.ylabel('Gain (dB)')
    plt.title('Frequency Response - Magnitude')

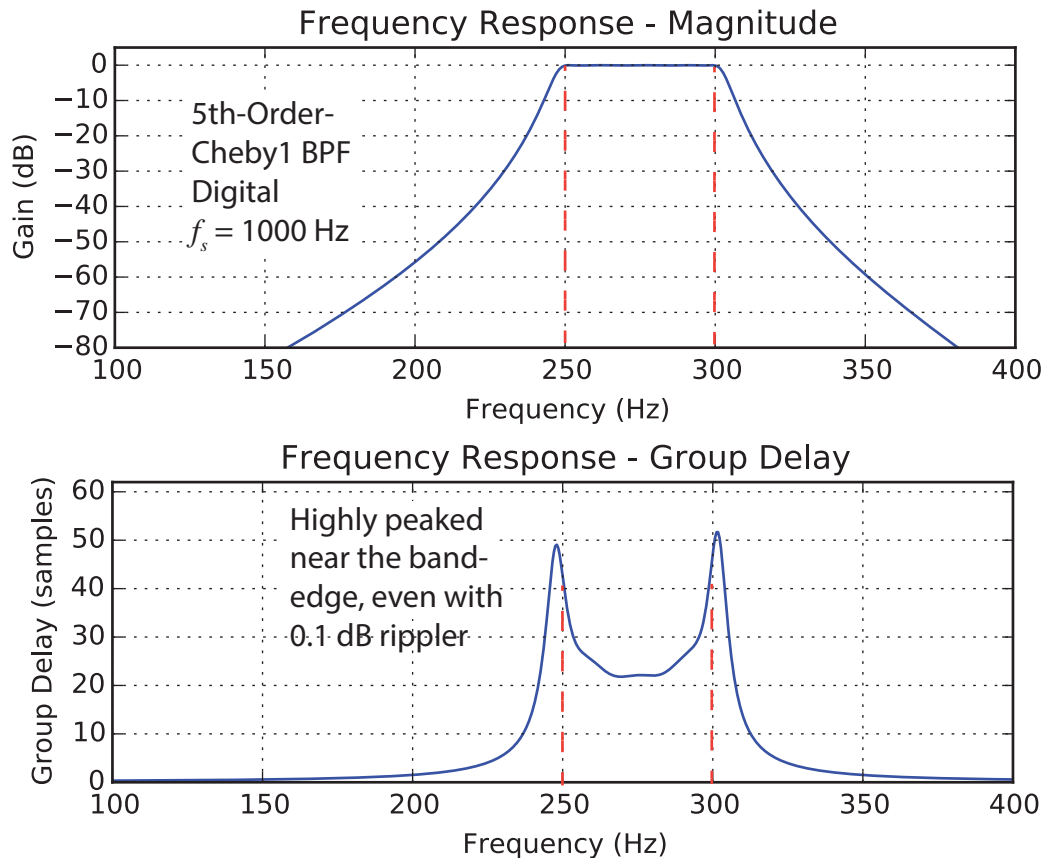
elif mode.lower() == 'phase':
    plt.semilogx(f,np.angle(H))
    plt.xlabel('Frequency (Hz)')
    plt.ylabel('Phase (rad)')
    plt.title('Frequency Response - Phase')

elif mode.lower() == 'groupdelay':
    """
    Notes
    -----

    See freqz_resp() for calculation details.
    """
    theta = np.unwrap(np.angle(H))
    # Since theta for an FIR filter is likely to have many pi phase
    # jumps too, we unwrap a second time 2*theta and divide by 2
    theta2 = np.unwrap(2*theta)/2.
    theta_dif = np.diff(theta2)
    f_dif = np.diff(f)
    Tg = -np.diff(theta2)/np.diff(w)
    max_Tg = np.max(Tg)
    #print(max_Tg)
    plt.semilogx(f[:-1],Tg)
    plt.ylim([0,1.2*max_Tg])
    plt.xlabel('Frequency (Hz)')
    plt.ylabel('Group Delay (s)')
    plt.title('Frequency Response - Group Delay')
else:
    print('Error, mode must be "dB" or "phase or "groupdelay"')

```

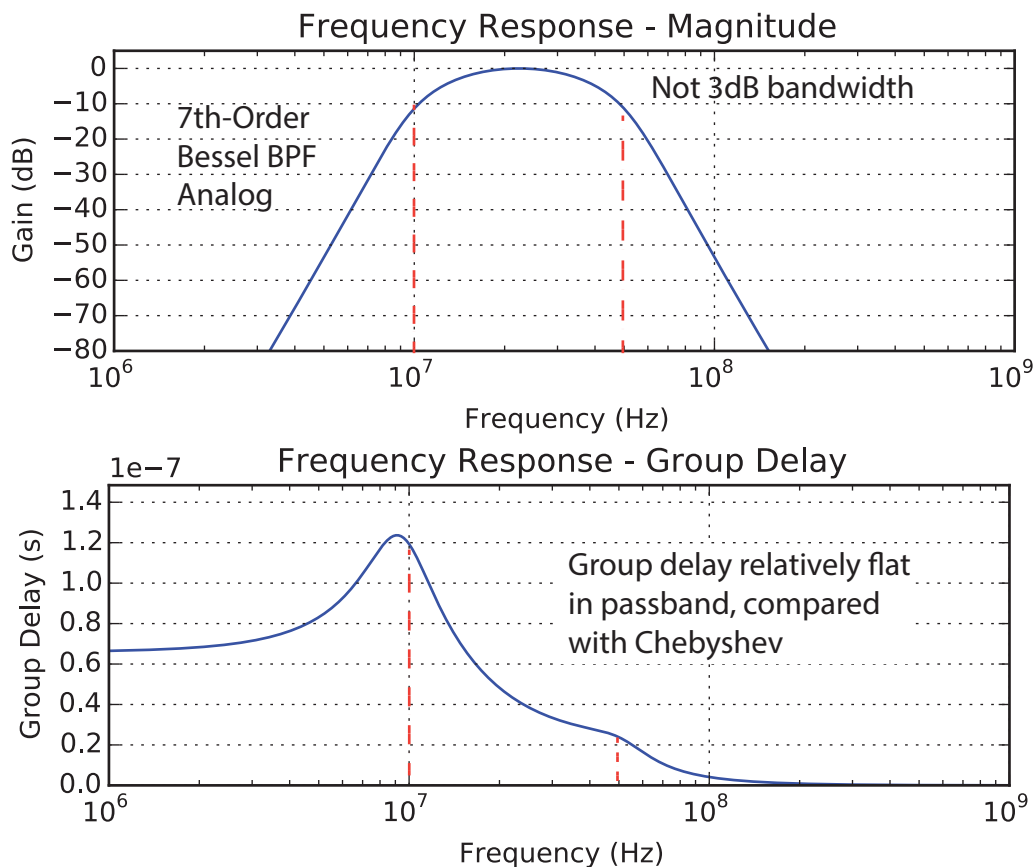
- **Case 1:** A 5th-order Chebyshev type 1 digital bandpass filter, having 1 dB ripple and passband of [250, 300] Hz relative to a sampling rate of $f_s = 1000$ Hz



Digital bandpass with $f_s = 1000$ Hz (Chebyshev)

- The Chebyshev in both analog and digital forms still has a large peak in the group delay, even with a small ripple (here 0.1 dB)

- **Case 2:** A 7th-order Bessel analog bandpass filter, having pass-band of [10, 50] MHz



Analog bandpass (Bessel)

- The Bessel filter has a much lower and smoother group delay, but the magnitude response is rather sloppy
- The filter passband is far from being flat and the roll-off is gradual considering the filter order is seven

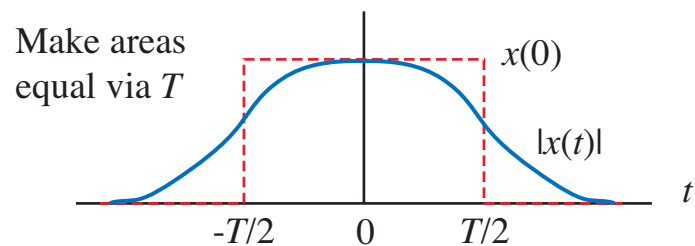
2.7.12 Pulse Resolution, Risetime, and Bandwidth

Problem: Given a non-bandlimited signal, what is a reasonable estimate of the signals transmission bandwidth?

We would like to obtain a relationship to the signals time duration

- Step 1: We first consider a time domain relationship by seeking a constant T such that

$$Tx(0) = \int_{-\infty}^{\infty} |x(t)| dt$$



- Note that

$$\int_{-\infty}^{\infty} x(t) dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \Big|_{f=0} = X(0)$$

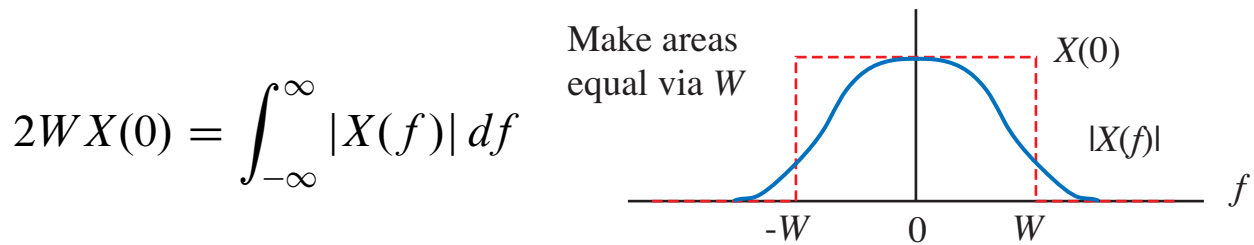
and

$$\int_{-\infty}^{\infty} |x(t)| dt \geq \int_{-\infty}^{\infty} x(t) dt$$

which implies

$$Tx(0) \geq X(0)$$

- Step 2: Find a constant W such that



- Note that

$$\int_{-\infty}^{\infty} X(f) df = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \Big|_{t=0} = x(0)$$

and

$$\int_{-\infty}^{\infty} |X(f)| df \geq \int_{-\infty}^{\infty} X(f) df$$

which implies that

$$2WX(0) \geq x(0)$$

- Combining the results of Step 1 and Step 2, we have

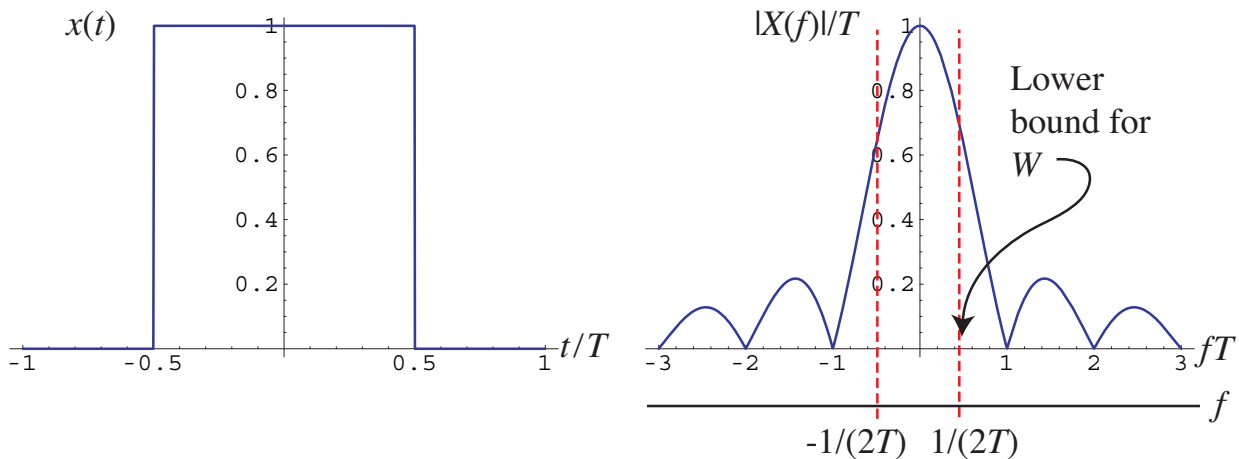
$$2WX(0) \geq x(0) \geq \frac{1}{T}X(0)$$

or

$$2W \geq \frac{1}{T} \quad \text{or} \quad W \geq \frac{1}{2T}$$

Example 2.28: Rectangle Pulse

- Consider the pulse $x(t) = \Pi(t/T)$
- We know that $X(f) = T \text{sinc}(fT)$



Pulse width versus Bandwidth, is $W \geq 1/(2T)$

- We see that for the case of the sinc() function the bandwidth, W , is clearly greater than the simple bound predicts

Risetime

- There is also a relationship between the risetime of a pulse-like signal and bandwidth
- Definition: The risetime, T_R , is the time required for the leading edge of a pulse to go from 10% to 90% of its final value
- Given the impulse response $h(t)$ for an LTI system, the step response is just

$$\begin{aligned}
 y_s(t) &= \int_{-\infty}^{\infty} h(\lambda) u(t - \lambda) d\lambda \\
 &= \int_{-\infty}^t h(\lambda) d\lambda \stackrel{\text{if causal}}{=} \int_0^t h(\lambda) d\lambda
 \end{aligned}$$

Example 2.29: Risetime of RC Lowpass

- The RC lowpass filter has impulse response

$$h(t) = \frac{1}{RC} e^{-t/(RC)} u(t)$$

- The step response is

$$y_s(t) = [1 - e^{-t/(RC)}] u(t)$$

- The risetime can be obtained by setting $y_s(t_1) = 0.1$ and $y_s(t_2) = 0.9$

$$\begin{aligned} 0.1 &= [1 - e^{-t_1/(RC)}] \Rightarrow \ln(0.9) = \frac{-t_1}{RC} \\ 0.9 &= [1 - e^{-t_2/(RC)}] \Rightarrow \ln(0.1) = \frac{-t_2}{RC} \end{aligned}$$

- The difference $t_2 - t_1$ is the risetime

$$T_R = t_2 - t_1 = RC \ln(0.9/0.1) \simeq 2.2RC = \frac{0.35}{f_3}$$

where f_3 is the RC lowpass 3dB frequency

Example 2.30: Risetime of Ideal Lowpass

- The risetime of an ideal lowpass filter is of interest since it is used in modeling and also to see what an ideal filter does to a step input
- The impulse response is

$$h(t) = \mathcal{F}^{-1} \left\{ \Pi \left(\frac{f}{2B} \right) \right\} = 2B \text{sinc}[2Bt]$$

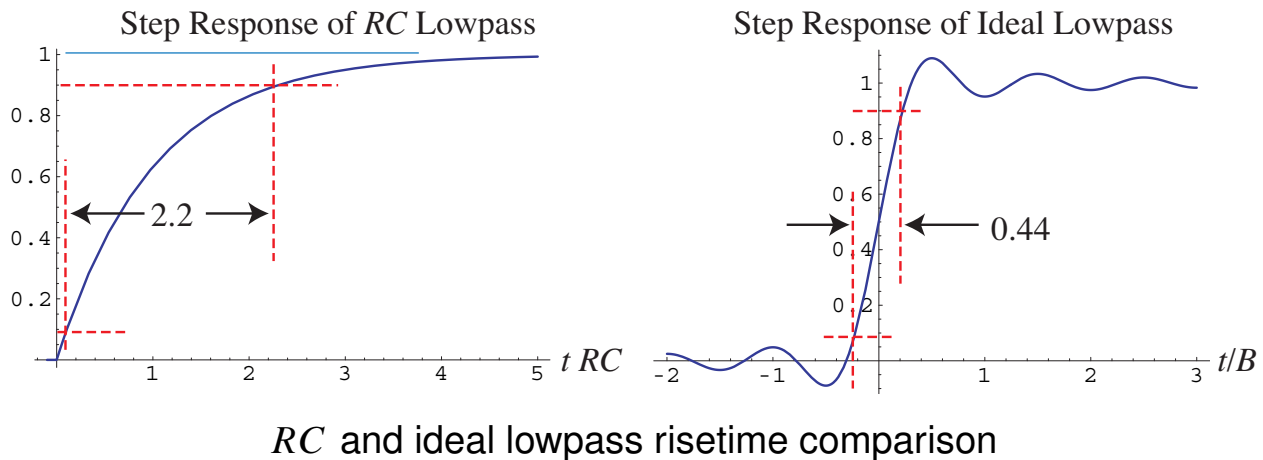
- The step response then is

$$\begin{aligned}
 y_s &= \int_{-\infty}^t 2B \operatorname{sinc}[2B\lambda] d\lambda \\
 &= \frac{1}{\pi} \int_{-\infty}^{2\pi Bt} \frac{\sin u}{u} du \\
 &= \frac{1}{2} + \frac{1}{\pi} \operatorname{Si}[2\pi Bt]
 \end{aligned}$$

where $\operatorname{Si}(\cdot)$ is a special function known as the *sine integral*

- We can numerically find the risetime to be

$$T_R \simeq \frac{0.44}{B}$$



2.8 Sampling Theory

Integrate with Chapter 3 material.

2.9 The Hilbert Transform

Integrate with Chapter 3 material.

2.10 The Discrete Fourier Transform and FFT

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