EE 779 Advanced Topics in Signal Processing
Assignment 5
Assigned: 08/10/16, Due: 20/10/16
Indian Institute of Technology Bombay

Note

- Submit the written part and print out of simulations, together.

Problems

1. [$*$]When estimating $\mathbf{x}$ using least-squares approach to solve
$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$
determine the bounds on the error
$$\|\tilde{\mathbf{x}} - \mathbf{x}\|_2^2,$$
where we estimate $\tilde{\mathbf{x}}$ as $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{y}$.

2. Show that the minimizer $\hat{\mathbf{x}}$ of
$$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \delta\|\mathbf{x}\|_2^2$$
for $\mathbf{A}$ a $M \times N$ matrix with $M > N$ is
$$\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A} + \delta\mathbf{I}\right)^{-1}\mathbf{A}^T\mathbf{y}.$$
This modification to the standard least-squares problem is a special case of the Tikhonov Regularization. Compare the above solution with the least-squares solution by using SVD representation of $\mathbf{A}$.

3. Suppose $\mathbf{U}$ is an $N \times N$ matrix with orthonormal columns $\mathbf{U}^T\mathbf{U} = \mathbf{I}$. Show that $\|\mathbf{U}\mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$ for every $\mathbf{x} \in \mathbb{R}^N$.

4. [$*$]Use the file `blocks_deconv.mat` from moodle data file. This contains the vectors
   - $\mathbf{x}$: a 512×1 signal
   - $\mathbf{h}$: a 30×1 filter
   - $\mathbf{y}$: a 541×1 vector of observations of $\mathbf{h}$ convolved with $\mathbf{x}$.
   - $\mathbf{yn}$: a noisy observation of $\mathbf{y}$. The noise is iid Guassian with standard deviation 0.01.

   (a) Write a function which takes a vector $\mathbf{h}$ of length L and input signal length N, and returns the $M \times N$ with $M = N + L - 1$ matrix $\mathbf{A}$ such that for any vector $\mathbf{x} \in \mathbb{R}^N$, the product $\mathbf{A}\mathbf{x}$ is the vector of non-zero values of $\mathbf{h}$ convolved with $\mathbf{x}$.

   (b) Use MATLAB's `svd` command to calculate the SVD of $\mathbf{A}$. What are the largest and smallest singular values ? Calculate $\mathbf{A}^{\dagger}\mathbf{y}$ and plot it.

   (c) Apply $\mathbf{A}^{\dagger}$ to the noisy $\mathbf{yn}$. Plot the result. Calculate the mean square error $\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2$ and compare to the measurement error $\|\mathbf{y} - \mathbf{yn}\|_2^2$.

   (d) Approximate $\mathbf{A}$ by truncating the last q terms in the SVD to obtain:
$$\mathbf{A}' = \sum_{k=1}^{p-q} \sigma_k \mathbf{u}_k \mathbf{v}_k^T.$$
   Apply the new pseudo-inverse $\mathbf{A}'^{\dagger}$ to $\mathbf{yn}$ and plot the result. Try different values for q and choose the one which gives the best result. Mention the value you choose for q. Calculate the mean-square reconstruction error.

(e) Form another approximate inverse using the Tikhonov regularization (See Problem 2, above). Try different values of δ and choose the best one. What value of δ gave the best result. Calculate the mean-square reconstruction error.

(f) Summarize your observations and findings by comparing (c), (d), and (e). Include the error for $\|\mathbf{x} - \mathbf{yn'}\|_2^2$ using appropriate part of $\mathbf{yn}$.