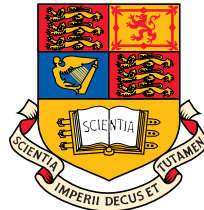# Spectral Estimation and Adaptive Signal Processing
Lecture 1 - Background

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Outline

Background on:

- Linear algebra

- Estimation theory

- Gaussianity

- Bias-variance dilemma

- Sequential and block estimators

- Special matrices, matrix decompositions

- Phase space, attractor dynamics

Is there such a thing as a nonlinear signal?

Linear System

System which obeys superposition and scaling principles:

$$f(ax + by) = a\ f(x) + b\ f(y)$$

Nonlinear System
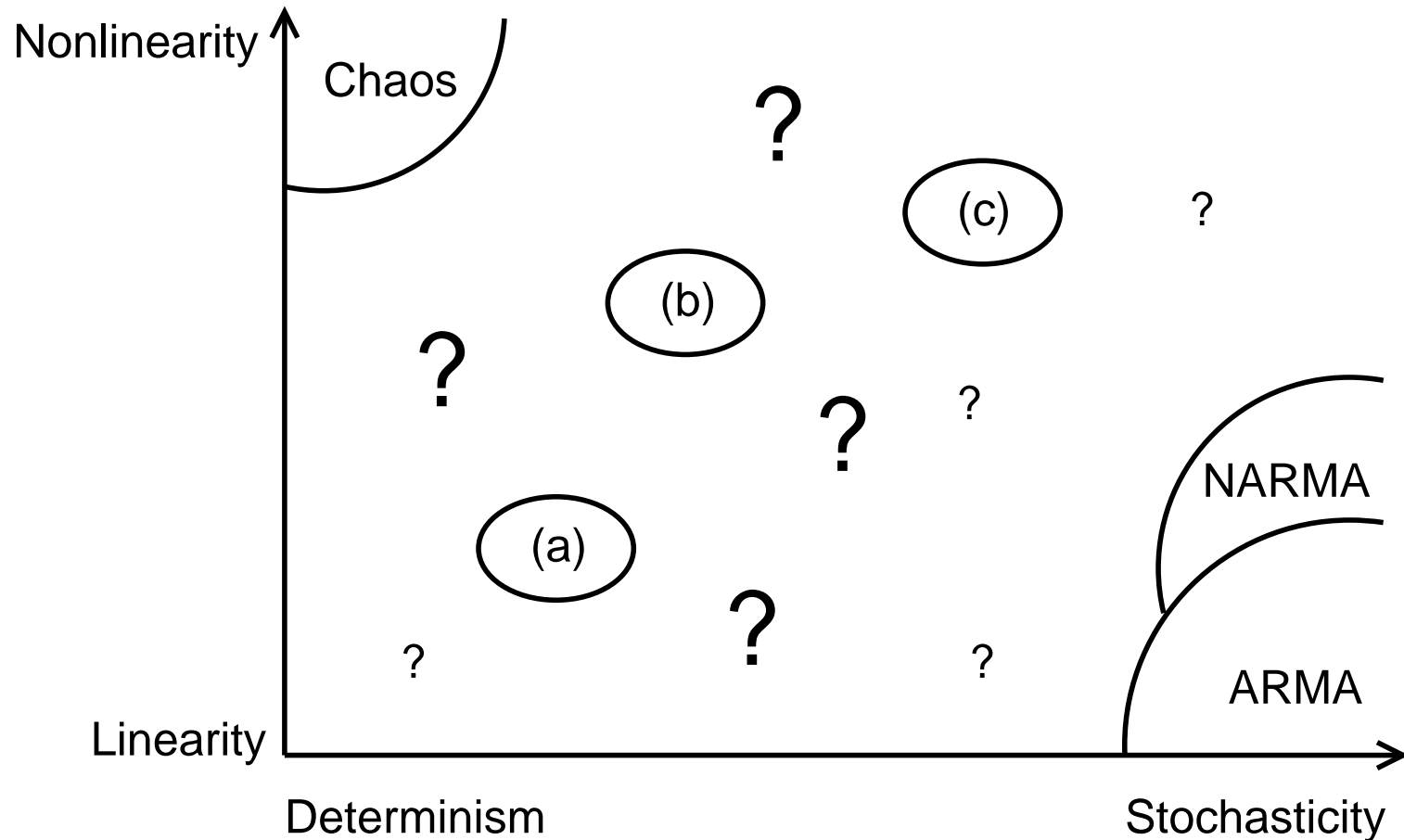
System which does *not* obey these principles

Linear Signal

Signal generated by a linear system driven by white (Gaussian) noise

Nonlinear Signal

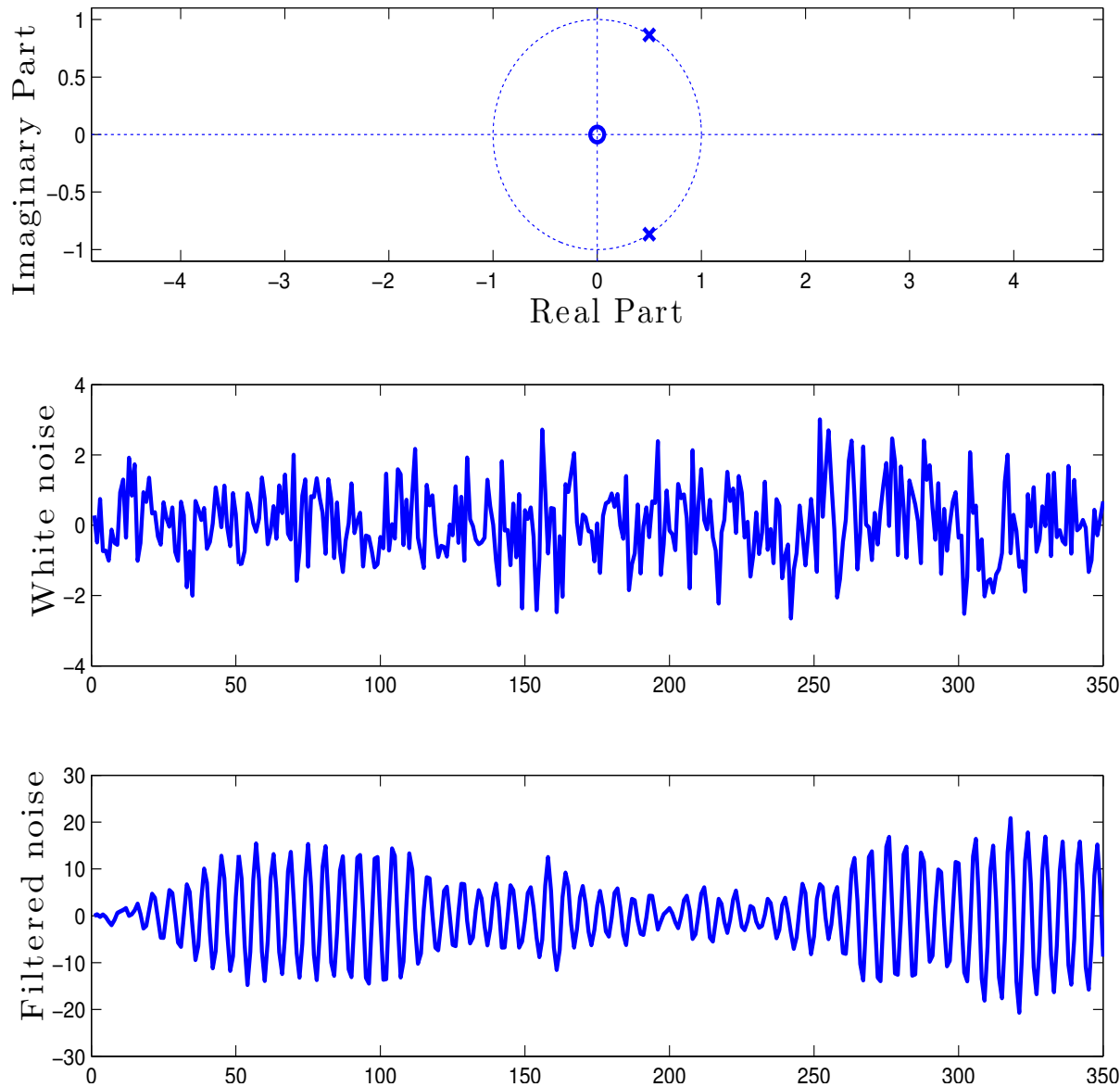**Any (!) signal which is not linear**

Real world signals – 'Nonlinearity' and 'Stochasticity'

a) Periodic oscillations b) Small nonlinearity c) Route to chaos

d) Route to chaos e) small noise f) HMM and others

Example 1.1. So - how about a real-world sinewave?

Spectral estimation or adaptive filtering to estimate it?

Matlab code:

```
z1=0;
p1=[0.5+0.866i,0.5-0.866i];
[num1,den1]=zp2tf(z1,p1,1);
zplane(num1,den1);
s=randn(1,1000);
s1=filter(num1,den1,s);
figure;
subplot(311),plot(s),
subplot(313),plot(s1),
subplot(312),;
zplane(num1,den1)
```

The AR model of a sinewave

```
x(k)=a1*x(k-1)+a2*x(k-2)+w(k)
a1=-1, a2=0.98, w~N(0,1)
```

System nonlinearity detection

$$\boxed{\textbf{Parametric System NL Testing}}$$

- develop parametric nonlinear model
- drive unknown system with known input
- fit model parameters to match system output

$$\boxed{\textbf{Nonparametric System NL Testing}}$$

- drive unknown system with different stimuli
- test the superposition and scaling principles
  1) apply short and long stimulus $\rightarrow R_{\mathrm{s}}$, R_{l}
  $\rightsquigarrow$ can R_{l} be predicted from time shifted R_{s}
  2) apply small and large amplitude stimulus $\rightarrow R_{\mathrm{sa}}$, R_{la}
  $\rightsquigarrow$ can R_{la} be predicted from superposition of R_{sa}

Example 1.2. Analysis in the phase space

How do we differentiate between signals, second order statistics is not enough
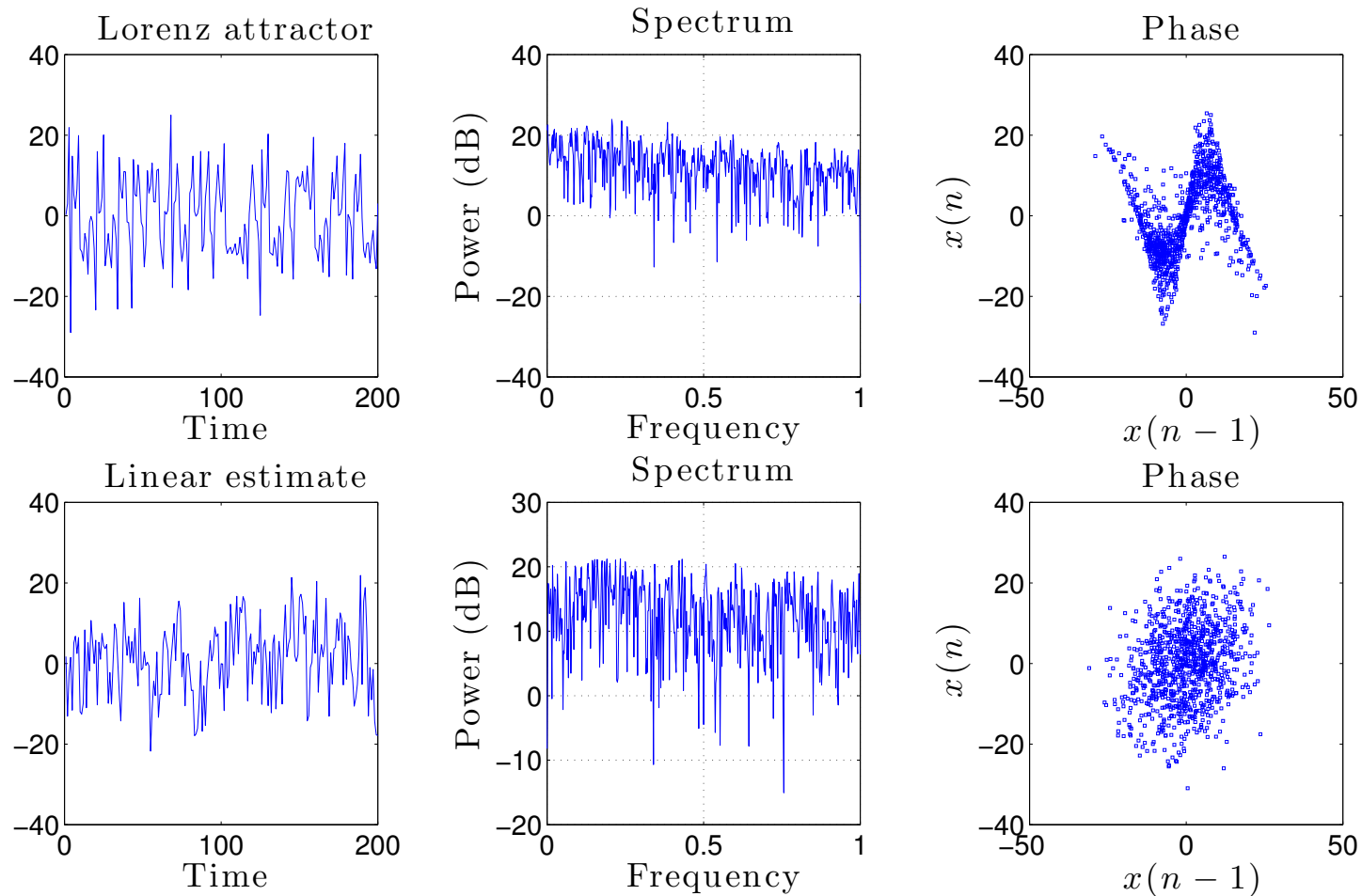
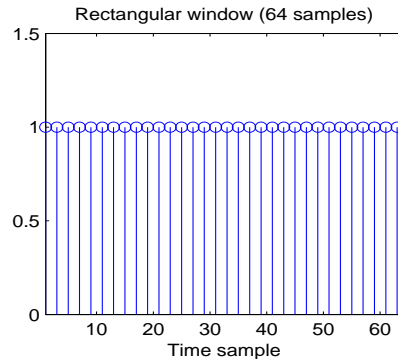The chaotic Lorenz signal and its linear estimate have the same spectral properties.

The Lorenz signal has an attractor in phase space (higher order properties).

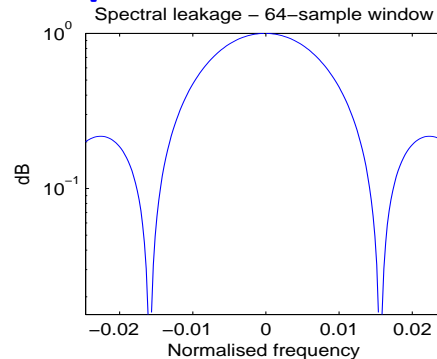# Practical issues in correlation and spectrum estimation: Data length and data windows
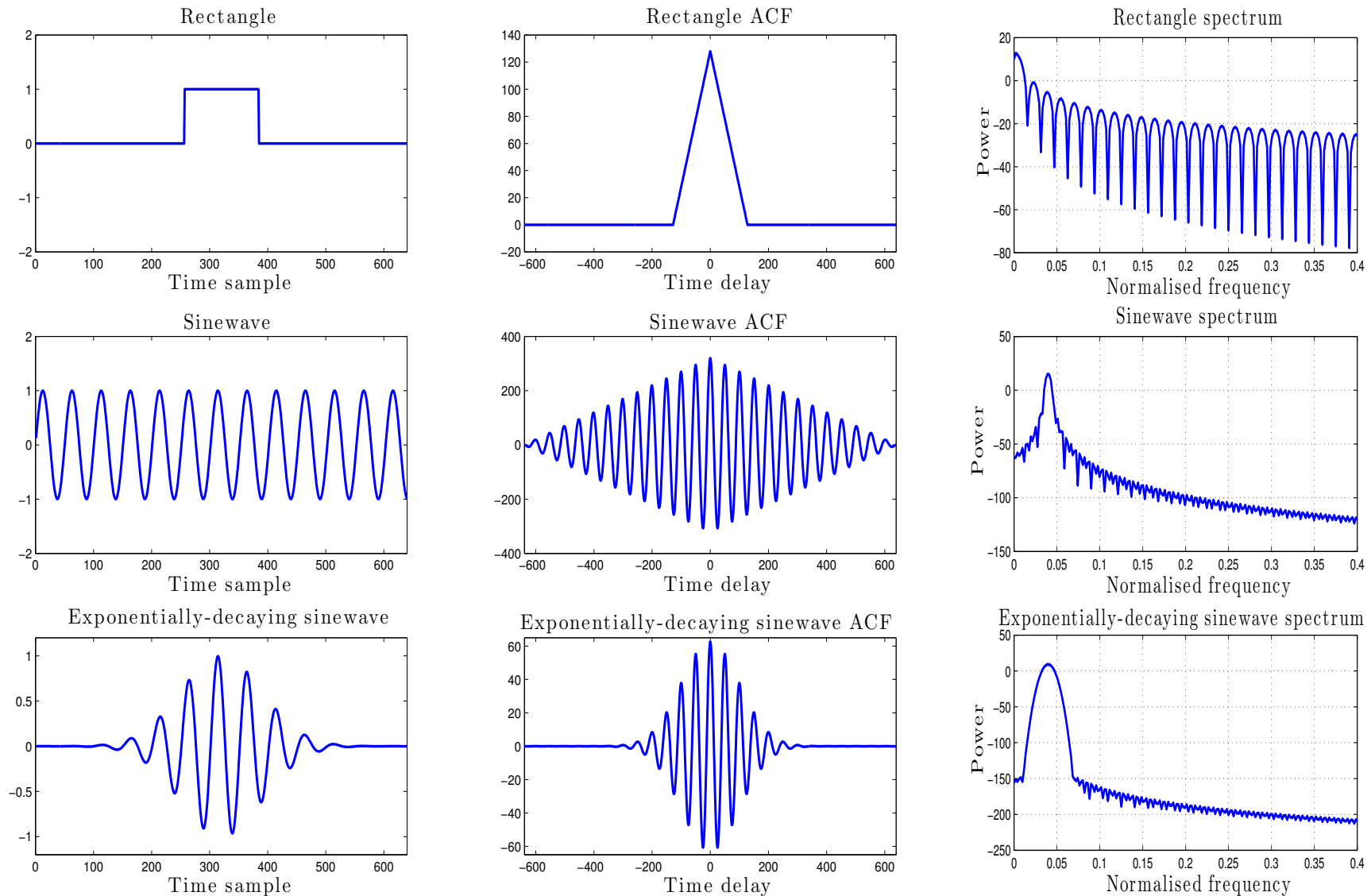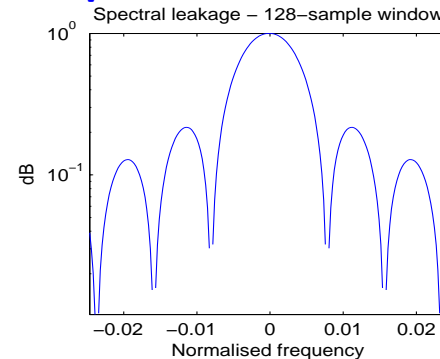
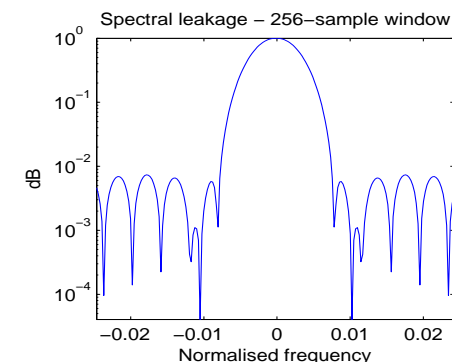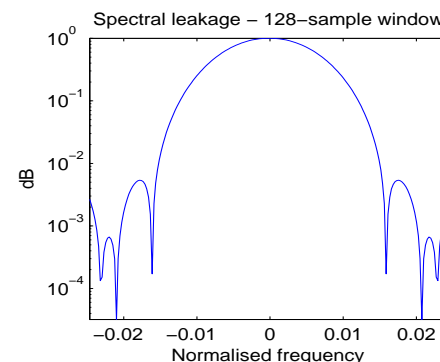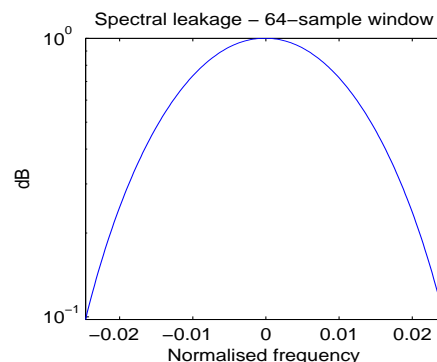# Some common windows for different window lengths

Time domain   Spectrum N=64   Spectrum N=128   Spectrum N=256
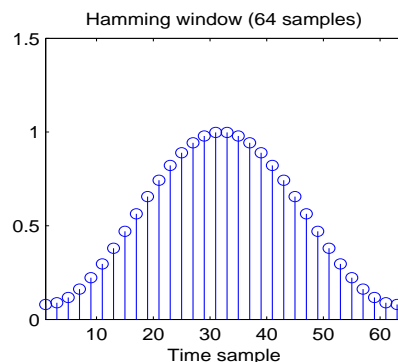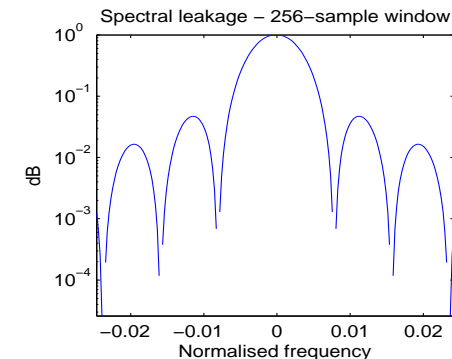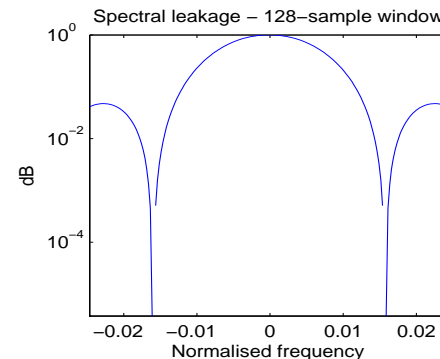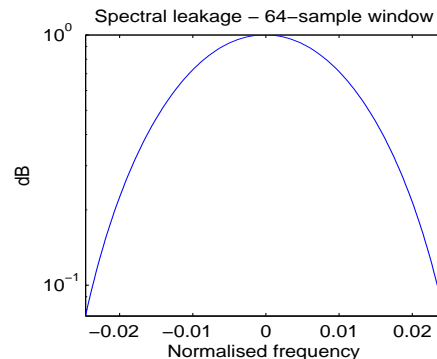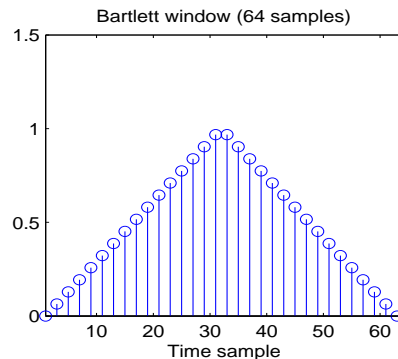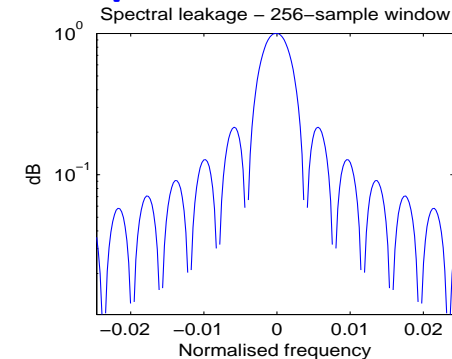
Example 1.3. Types of estimators
Window length matters in sequential estimation too!

Block based and sequential:

Example: block based - **sample mean:** $\hat{\theta}_{N-1} = \frac{1}{N} \sum_{k=0}^{N-1} x[k]$

It is block based since all the data are required to form an estimate.

Sequential estimator: A new estimate can be calculated as each new sample arrives.

we want to express $\hat{\theta}(N) = f\left(\hat{\theta}(N-1), x(N)\right)$

$$\hat{\theta}[N] = \frac{1}{N+1} \sum_{k=0}^{N} x[k] = \frac{1}{N+1} \left[\sum_{k=0}^{N-1} x[k] + x[N] \right]$$

$$= \frac{N}{N+1} \hat{\theta}[N-1] + \frac{1}{N+1} x[n]$$

$$= \left(1 - \frac{1}{N+1}\right) \hat{\theta}[N-1] + \frac{1}{N+1} x[n]$$

$$= \hat{\theta}[N-1] + \frac{1}{N+1} \left[x[N] - \hat{\theta}[N-1] \right]$$

New quantity = old quantity + Gain × Error

$\frac{1}{N+1} = \text{gain}, \quad x[N] - \hat{\theta}[N-1] = \text{error}$

This update equation is the common form of all adaptive algorithms: the LMS (Least mean square), NLMS (Normalised LMS), Kalman filtering.

Example 1.4. Convergence of sequential estimators
for simplicity we consider a Gaussian $\sim \mathcal{N}(0,1)$

Sequential DC level estimation.
(mean of a Gaussian signal)

Signal: uncorrelated Gaussian noise

Sample and recursive variance estimates

— Sample mean

$\hat{\theta}[N] = \hat{\theta}[N-1] + \frac{1}{N}\left[x[N-1] - \hat{\theta}[N-1]\right]$
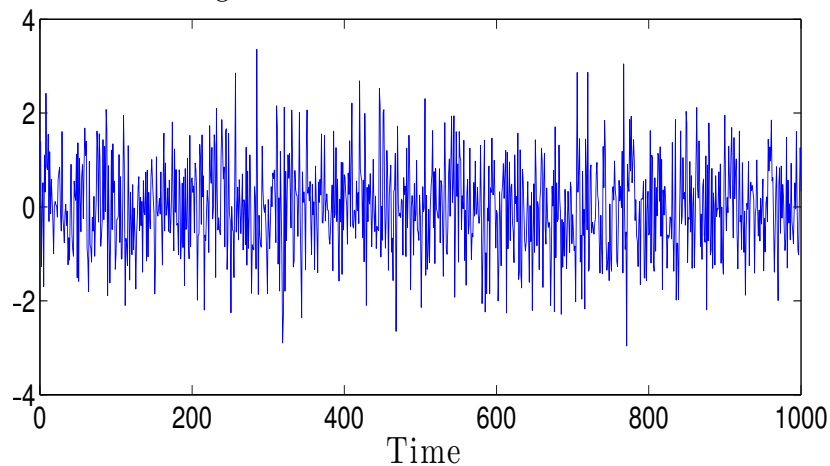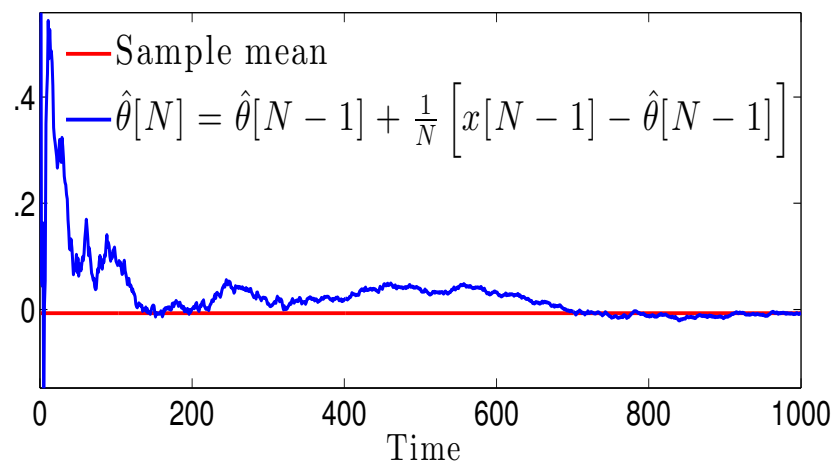
Sequential variance estimation.
(variance of a Gaussian)

Signal: uncorrelated Gaussian noise

Sample and recursive variance estimates

— Sample variance

$c_n = \frac{n-1}{n}c_{n-1} + \frac{y_n^2}{n}$

$c_n = c_{n-1} + \frac{y_n^2 - y_{n-M+1}^2}{M}; \ M = 50$

$c_n = \lambda * c_{n-1} + (1-\lambda)y_n^2, \ \lambda = 0.9$

$c_n = \lambda * c_{n-1} + (1-\lambda)y_n^2, \ \lambda = 0.99$
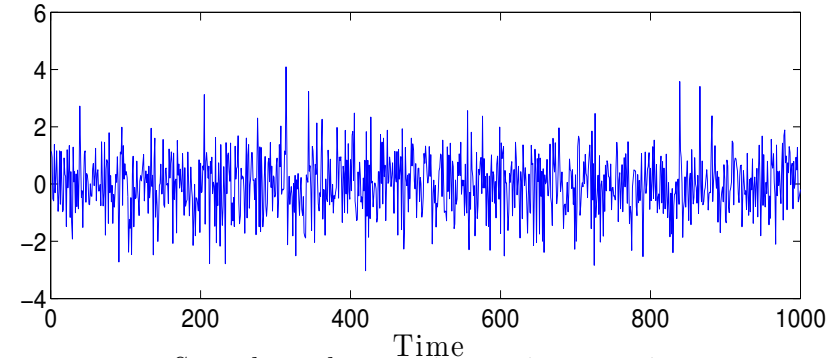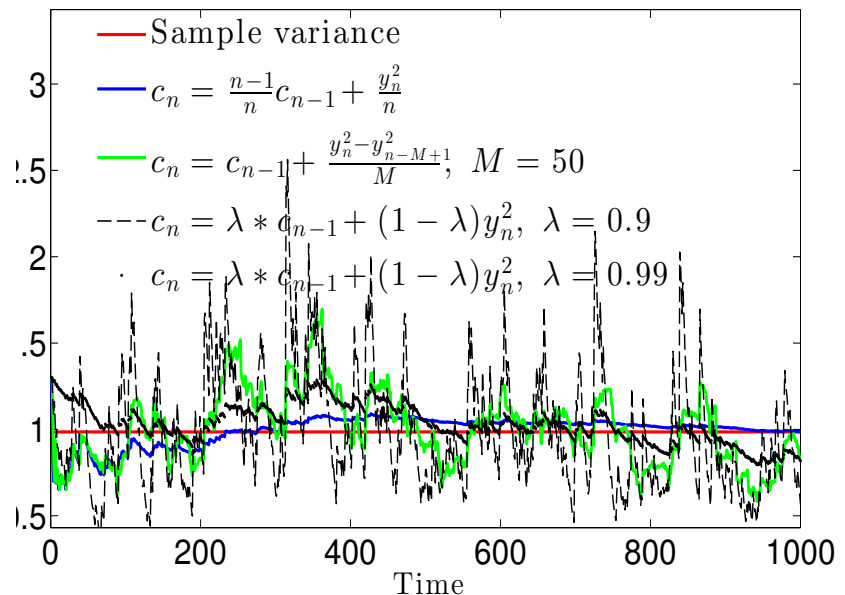
Second order data modelling

AR(p) model: $x(n) = a_1 x(n-1) + \cdots + a_p x(n-p) = \mathbf{a}^T \mathbf{x} = <\mathbf{a}, \mathbf{x}>$

We start from: $y(n) = a_1(n)x(n-1) + a_2(n)x(n-2) + \cdots + a_p(n)x(n-p)$

Teaching signal: $d(n)$ **Output error:** $e(n) = d(n) - y(n)$

Fixed coeff. $\mathbf{a}$ & $x(n) = y(n)$
Autoregressive modelling

$$r_{xx}(1) = a_1 r_{xx}(0) + \cdots + a_p r_{xx}(p-1)$$
$$r_{xx}(2) = a_1 r_{xx}(1) + \cdots + a_p r_{xx}(p-2)$$
$$\vdots \;\; = \;\; \vdots$$
$$r_{xx}(p) = a_1 r_{xx}(p-1) + \cdots + a_p r_{xx}(0)$$

$$\cdots \qquad \cdots$$

$$\mathbf{r}_{xx} = \mathbf{R}_{xx}\mathbf{a}$$

Solution: $\mathbf{a} = \mathbf{R}_{xx}^{-1}\mathbf{r}_{xx}$

Yule–Walker equation

Fixed optimal coeff. $\mathbf{w}_o = \mathbf{a}_{opt}$

$$J = E\{\frac{1}{2}e^2(n)\} = \sigma_d^2 - 2\mathbf{w}^T\mathbf{p} + \mathbf{w}^T\mathbf{R}\mathbf{w}$$

is quadratic in $\mathbf{w}$ and for a full rank $\mathbf{R}$, it has **one unique minimum.**

Now:

$$\frac{\partial J}{\partial \mathbf{w}} = -\mathbf{p} + \mathbf{R}\cdot\mathbf{w} = \mathbf{0}$$

Solution: $\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p}$

Wiener–Hopf equation

Inverse matrix

we need to calculate those inverses for our main models

The structure of an inverse matrix is also important!

- A symmetric matrix has a symmetric inverse
- A **Toeplitz** matrix has a persymmetric inverse (symmetric about the cross diagonal)
- A Hankel matrix has a symmetric inverse (useful in harmonic retrieval)

A useful formula for matrix inversion is **Woodbury's identity** (matrix inversion Lemma)

$$\left(\mathbf{A} + \mathbf{U}\mathbf{V^T}\right)^{-1} = \mathbf{A^{-1}} - \left[\mathbf{A^{-1}}\mathbf{U}\left(\mathbf{I} + \mathbf{V^T}\mathbf{A^{-1}}\mathbf{U}\right)^{-1}\mathbf{V^T}\mathbf{A^{-1}}\right]$$

If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then this simplifies into

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}$$

which will be important in the derivation of adaptive algorithms.

In a special case $\mathbf{A} = \mathbf{I}$ and we have

$$(\mathbf{I} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{I} - \frac{\mathbf{u}\mathbf{v}^T}{1 + \mathbf{v}^T\mathbf{u}}$$

Example 1.5. Matrix inversion lemma $\looparrowright$ another approach

Construct an 'augmented' matrix $[A, B; C, D]$ and its inverse $[E, F; G, H]$, so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Consider the following two products

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + HC & CF + DH \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} EA + FC & EB + FD \\ GA + HC & GB + HD \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Combine the corresponding terms to get another form

$$(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

Matrix inversion lemma ↦ A handy derivation

General form:

$$
\begin{aligned}
\left(\mathbf{A}+\mathbf{B}\mathbf{C}\mathbf{D}\right)^{-1} &= \left(\mathbf{A}\left[\mathbf{I}+\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\right]\right)^{-1} = \left[\mathbf{I}+\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\right]^{-1}\mathbf{A}^{-1} \\
&= \left[\mathbf{I}-\left(\mathbf{I}+\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\right)^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\right]\mathbf{A}^{-1} \qquad \text{(using P1)} \\
&= \mathbf{A}^{-1}-\left(\mathbf{I}+\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\right)^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\mathbf{A}^{-1}
\end{aligned}
$$

where the identity P1:

$$
\left(\mathbf{I}+\mathbf{P}\right)^{-1} = \left(\mathbf{I}+\mathbf{P}\right)^{-1}\left(\mathbf{I}+\mathbf{P}-\mathbf{P}\right) = \mathbf{I}-\left(\mathbf{I}+\mathbf{P}\right)^{-1}\mathbf{P}
$$

We can also use identity P2:

$$
\mathbf{P}+\mathbf{P}\mathbf{Q}\mathbf{P} = \mathbf{P}(\mathbf{I}+\mathbf{Q}\mathbf{P}) = (\mathbf{I}+\mathbf{P}\mathbf{Q})\mathbf{P} \quad \Rightarrow \quad (\mathbf{I}+\mathbf{P}\mathbf{Q})^{-1}\mathbf{P} = \mathbf{P}(\mathbf{I}+\mathbf{Q}\mathbf{P})^{-1}
$$

to have another matrix inverse:

$$
(\mathbf{A}+\mathbf{B}\mathbf{C}\mathbf{D})^{-1} = \mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\mathbf{C}\mathbf{D}\mathbf{A}^{-1}(\mathbf{I}+\mathbf{B}\mathbf{C}\mathbf{D}\mathbf{A}^{-1})^{-1}
$$

There are many other equivalent forms - you can derive them easily when needed

What is that a matrix does to a vector?

matrix–vector products

A ⇝ any general matrix

R ⇝ a rotation matrix ($\mathbf{R}^T = \mathbf{R}^{-1}$ and $\det \mathbf{R} = 1$)

$\mathbf{E}\mathbf{x} = \lambda \mathbf{x}$ ⇝ eigenanalysis

P ⇝ projection matrix

An example of a rotation matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What can we say about the properties of the matrix $\mathbf{A}$, matrix $\mathbf{E}$ and the projection matrix $\mathbf{P}$ (rank, invertibility, ...)?

Is the projection matrix invertible?
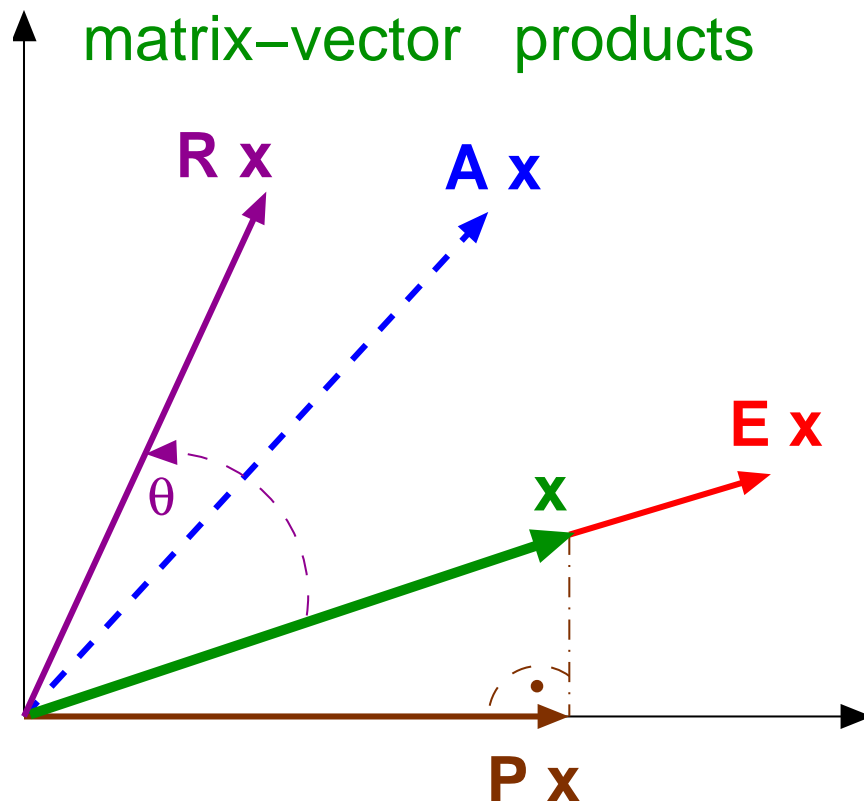
Ampli-twist: a matrix $\mathbf{A}$ which multiplies a vector $\mathbf{x}$
(i) stretches or shortents the vector
(ii) rotates the vector

The meaning of eigenanalysis

Let $\mathbf{A}$ be an $n \times n$ matrix, where $\mathbf{A}$ is a linear operator on vectors in $\mathbb{R}^n$, such that $\mathbf{A}\,\mathbf{x} = \mathbf{b}$

An **eigenvector** of $\mathbf{A}$ is a vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{A}\,\mathbf{v} = \lambda\mathbf{v}$, where λ is called the corresponding eigenvalue.

Matrix $\mathbf{A}$ only changes the length of $\mathbf{v}$, not its direction!
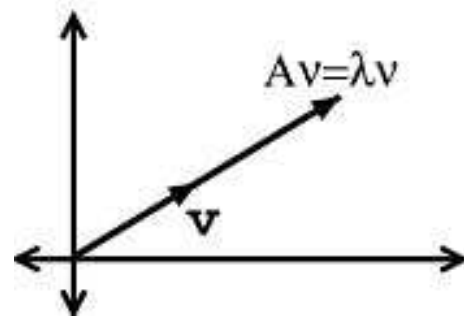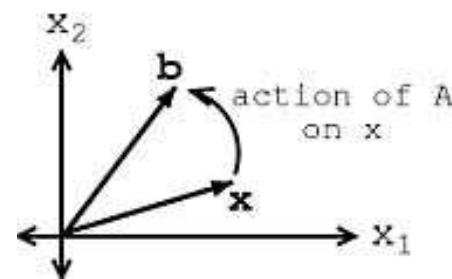
Equation $\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x}$ **Equation $\mathbf{A}\,\mathbf{x} = \mathbf{b}$.**

Eigenvalues

For an $n \times n$ matrix $\mathbf{A}$, its **eigenvalues** are found from the n-th order polynomial in λ defined by

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad \Rightarrow (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \text{nontrivial solution} \quad det\,(\mathbf{A} - \lambda \mathbf{I}) = 0$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\lambda_1, \ldots, \lambda_n$ the eigenvalues.

The corresponding n **eigenvectors**, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, satisfy $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ and are generally normalised to have unit norm $\parallel \mathbf{v} \parallel_2 = 1$.

For distinct eigenvalues, these eigenvectors are **linearly independent**.

A symmetric matrix is positive definite iff all its eigenvalues are positive

The **Spectral Theorem** allows for a symmetric matrix to be written as
$$\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T$$

and the
$$\text{Trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i \qquad \text{Any connection with signal power?}$$

Example 1.6. More eigenanalysis

Let $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$. The characteristic polynomial $det(\mathbf{A} - \lambda\mathbf{I})$ is

$$
\begin{aligned}
p(\lambda) &= det\left(\begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix}\right) \\
&= (2-\lambda)(-1-\lambda) - (-4)(-1) = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)
\end{aligned}
$$

Thus the **eigenvalues** of $\mathbf{A}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$.

To find **eigenvectors** $\mathbf{v} = [v_1, \ldots, v_n]^T$ corresponding to an eigenvalue λ

$$
solve \qquad (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = 0 \qquad for \ \mathbf{v}
$$

For $\lambda_1 = 3$ we thus have

$$
\begin{bmatrix} 2-3 & -4 \\ -1 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}
$$

Similarly, for $\lambda_2 = -2$ we have $\mathbf{v}_2 = [1,\ 1]^T$, and thus $\mathbf{v}_1 \perp \mathbf{v}_2$.

$\mathbf{v}_1$ and $\mathbf{v}_2$ are **bases of the eigenspace spanned by these vectors**

Example 1.7. Eigenanalysis ↦ An intuitive example

A sphere of unit radius is positioned at the centre of a three–dimensional coordinate system. It is rotating about the x_3 axis. This rotation[1] can be described by the matrix

$$\mathbf{C} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & \\ 0 & 0 & 1 \end{bmatrix}$$
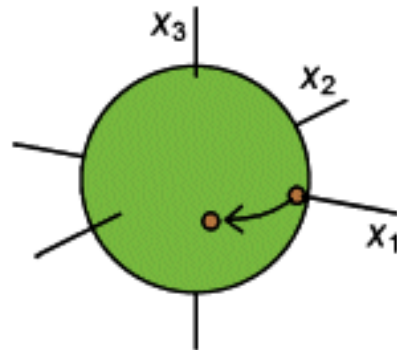
Intuitively, what is the eigenvector of $\mathbf{C}$? **Is there a point on the unit sphere that a 45° rotation transforms into a multiple of itself?**

This is the north pole $[0, 0, 1] \Rightarrow$ an eigenvector of $\mathbf{C}$ is the vector $[0, 0, 1]$.

---

[1] By 45° in this case. For example, multiplying $\mathbf{C}$ by the vector $[1, 0, 0]$ yields the vector $[0.707, 0.707.0]$, which is rotated 45°.

Example 1.8. Eigenanalysis – Image representation

Eigenvector of a transformation $\Rightarrow$ a vector which, in the transformation, is multiplied by a constant factor, called the **eigenvalue of that vector**
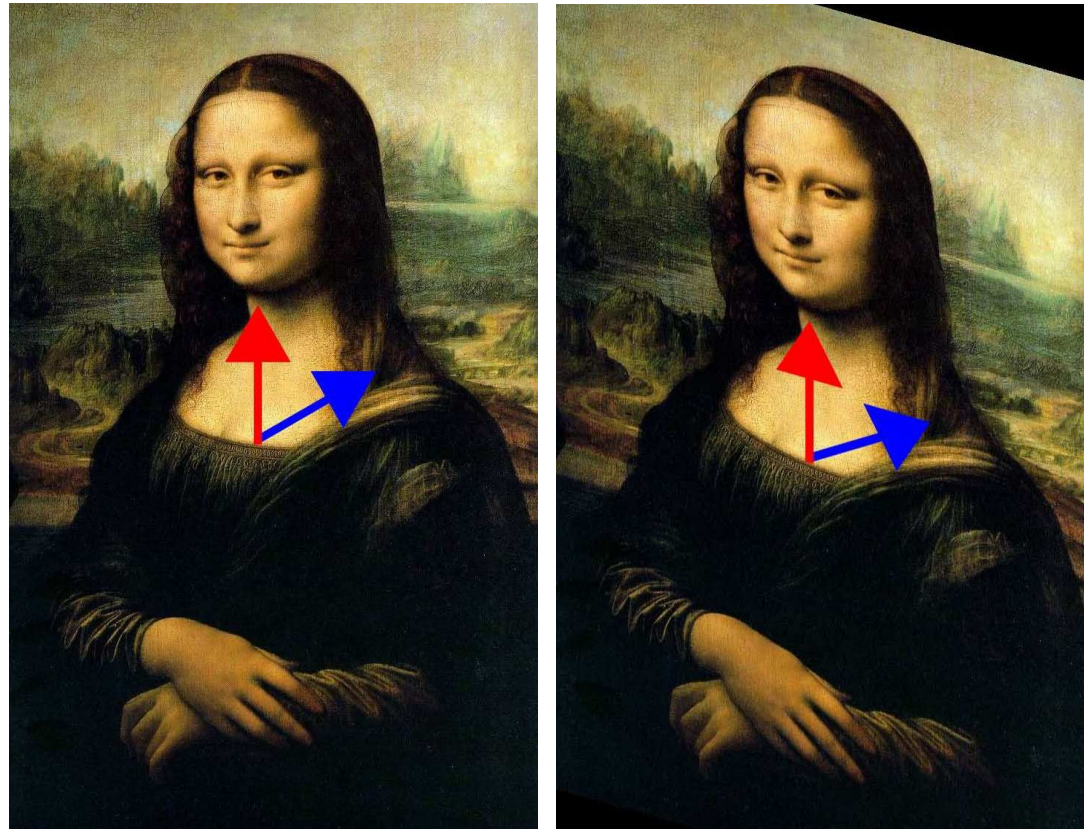
The red vector is an eigenvector of the transformation, and the blue is not

Example 1.8. Eigenanalysis – Eigenvectors of the Mona Lisa image

The red vector was neither stretched nor compressed $\Rightarrow$ its eigenvalue is 1.

All vectors with the same vertical direction (parallel) are also eigenvectors, with the same eigenvalue.

Together with the zero vector, they form the eigenspace for this eigenvalue.

The eigenvalue problem in linear algebra:

☐ Given a matrix $\mathbf{A}$, are there nonzero vectors $\mathbf{x}$ such that product $\mathbf{A}\,\mathbf{x}$ is a multiple of $\mathbf{x}$, that is $\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x}$?

☐ If so, then the factor λ by which $\mathbf{x}$ is multiplied is called an **eigenvalue** of the matrix $\mathbf{A}$.

☐ The corresponding vector $\mathbf{x}$ is called an **eigenvector** of the matrix $\mathbf{A}$ corresponding to this eigenvalue.

Example 1.8. Back to Mona Lisa
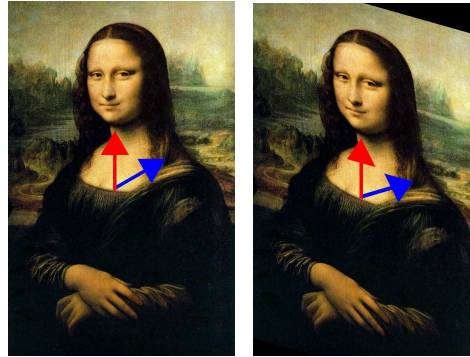
$$\text{Transformation} \qquad A = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Eigenvectors:

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \Rightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

For non–trivial solutions $\rightarrow det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \Rightarrow$

$$det\left(\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = det \begin{bmatrix} 1-\lambda & 0 \\ -\frac{1}{2} & 1-\lambda \end{bmatrix} = 0 \Rightarrow \lambda = 1$$

Example 1.8. Eigenvectors for Mona Lisa

We have found $\lambda = 1$, the eigenvalue of matrix $\mathbf{A}$.

We can now solve for eigenvectors

$$\begin{bmatrix} 1 - \lambda & 0 \\ -\frac{1}{2} & 1 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0}$$

Substituting $\lambda = 1$ we have

$$\mathbf{v} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

where c is an arbitrary constant.

All vectors of this form, pointing straight up or down, are eigenvectors of $\mathbf{A}$

In general $\mathbf{A}$ will have two distinct eigenvalues, and thus two distinct eigenvectors.

Most vectors will have both their lengths and direction changed by $\mathbf{A}$ whereas eigenvectors will have only their lengths changed.

Example 1.9. Dynamical systems and eigenanalysis

South Ken has two pizza places and $N \to \infty$ of pizza–loving students.

• Suppose that 5000 people buy one pizza each every week.

Tony's Pizza place has the better pizza and 80% of people who buy pizza each week at Tony's return the following week. Mike's Pizza does not have a good sauce and only 40% of the customers return the following week.

We can represent this situation by a discrete dynamical system

$$\mathbf{x}_{n+1} = \mathbf{A}\,\mathbf{x}_n \quad where \quad \mathbf{A} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

Let us start from $\mathbf{x}_0 = [2500,\, 2500]^T$, then we have

$$\mathbf{x}_1 = \begin{bmatrix} 3500 \\ 1500 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 3700 \\ 1300 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3740 \\ 1260 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 3748 \\ 1252 \end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix} 3750 \\ 1250 \end{bmatrix}$$

Also $\mathbf{x}_6 = \cdots = \mathbf{x}^* = \cdots = x_\infty = [3750, 1250]^T$!

Clearly $\mathbf{x}^* = [3750, 1250]^T$ is the eigenvector of $\mathbf{A}$, and $\mathbf{A}\,\mathbf{x}^* = \mathbf{x}^*$

Example 1.9. Will Mike have to close down?

In Matlab

```
[v,lambda] = eig(A)              %also look at the demo 'eigshow(A)'


          0.9487    -0.7071
v =
          0.3162     0.7071



          1.0000          0
lambda =
          0          0.2000
```

$\hookrightarrow \mathbf{v}_1 = [0.9487, 0.3162]^T, \ \mathbf{v}_2 = [-0.7071, 0.7071]^T, \quad \lambda_1 = 1, \ \lambda_2 = 0.2.$

Notice that the elements of $\mathbf{v}_1$ are related as $3 \div 1$, the same as the ratio of Tony's and Mike's customers

Since $\mathbf{x}_n = \mathbf{A}^n \mathbf{x}_0 \rightsquigarrow \mathbf{A}^n_{n \to \infty} = [\mathbf{v}_1 \vdots \mathbf{v}_2] \quad \hookrightarrow \quad$ equilibrium!

This is closely related to fixed point theory since $\mathbf{A}$ is a Markov matrix

Estimation theory: Recap of basic notions

Problem: To estimate one or more parameters from some given discrete-time signal $\{\mathbf{x}[n]\}$.

Mathematical Context:

Given an N-point data set which depends upon an unknown parameter θ (scalar), define an "estimator" as some function g of the dataset, that is

$$\hat{\theta} = g\big(x[0], x[1], \ldots, x[N-1]\big)$$

Note: difference between estimator and estimate

- Estimate $=$ particular value of $\hat{\theta}$ for one observation data set
- Estimator $=$ rule that assigns a value of $\hat{\theta}$ for a given realisation of

$$\mathbf{x} = \big[x[0], x[1], \ldots, x[N-1]\big]$$

There are two types of estimators

- Classical: the unknown parameter(s) is(are) fixed

- Bayesian: the unknown parameter(s) is(are) random

The bias–variance dilemma

The mean square error (MSE) of an estimate $\hat{\theta}$ of a parameter θ is given by

$$MSE(\hat{\theta}) = E\{(\hat{\theta} - \theta)^2\} \quad \text{average deviation from the true value}$$

For every estimator: **Bias:** $B = E\{\hat{\theta}\} - \theta$ **Variance:** $\text{var} = E\{(\hat{\theta} - E\{\hat{\theta}\})^2\}$

Therefore:

$$
\begin{aligned}
\text{MSE} \;=\;& E\{(\hat{\theta} - \theta)^2\} = E\Big\{\big[\hat{\theta} - E\{\hat{\theta}\} + \underbrace{E\{\hat{\theta}\} - \theta}_{\text{bias} \;\; B(\hat{\theta})}\big]^2\Big\} \\[2mm]
=\;& E\big\{\big[\hat{\theta} - E\{\hat{\theta}\}\big]^2\big\} + E\{B^2(\hat{\theta})\} + 2E\big\{\big[\hat{\theta} - E\{\hat{\theta}\}\big]B(\hat{\theta})\big\} \\[2mm]
& \text{due to the linearity of the } E\{\cdot\} \text{ operator and that } E\{B(\hat{\theta})\} = B(\hat{\theta}) \\[2mm]
=\;& E\big\{\big[\hat{\theta} - E\{\hat{\theta}\}\big]^2\big\} + B^2(\hat{\theta}) + \underbrace{2E\big\{\big[\hat{\theta} - E\{\hat{\theta}\}\big]\big\}}_{=0, \text{ the } E\{\hat{\theta}\} \text{ are equal}} B(\hat{\theta}) \\[2mm]
=\;& \text{var}(\hat{\theta}) + B^2(\hat{\theta}) \qquad \text{MSE depends both on bias and variance}
\end{aligned}
$$

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Data model: Gaussianity

Justification: Central Limit Theorem

If we form a sum of independent measurements

$\Rightarrow$ the distribution of the sum tends to a Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \qquad x \sim \mathcal{N}(\mu_x, \sigma_x^2)$$

$\Rightarrow$ **distribution defined by its mean and variance!!! SOS**

If $x \sim \mathcal{N}(0, \sigma_x^2)$ then $E\{x^{2n-1}\} = 1, 3, \ldots, (2n-1)\sigma_x^{2n}, \quad \forall n$

In the vector case (N Gaussian random variables)

$$p\big(x[0], x[1], \ldots, x[N-1]\big) = \frac{1}{(2\pi)^{N/2} det(\mathbf{C}_{xx})^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu_x)^T \mathbf{C}_{xx}^{-1}(\mathbf{x}-\mu_x)}$$

where $\mathbf{C}_{xx} = E\{(\mathbf{x}-\mu_x)(\mathbf{x}-\mu_x)^T\}$ is the **covariance matrix.**

Gaussian random variables: Properties

Note

$$cov(\mathbf{x}_i, \mathbf{x}_j) = E\left[(\mathbf{x}_i - E(\mathbf{x}_i))(\mathbf{x}_j - E(\mathbf{x}_j))\right] = E\left[\mathbf{x}_i \mathbf{x}_j\right] - E[\mathbf{x}_i]E[\mathbf{x}_j]$$

If $\mathbf{x}_i$ and $\mathbf{x}_j$ are **independent** $\Rightarrow \mathbf{C}_{xx}$ is **diagonal**.

For mathematical tractability, we will frequently assume that the component of $\mathbf{x}$ are statistically independent, such that:
$$\mathbf{C}_{xx} = diag\left(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{N-1}^2\right)$$

and if all the variances and mean values are identical, then the elements of $\mathbf{x}$ are said to be **iid**, that is **Independent Identically Distributed.**

$$\Rightarrow \quad \mathbf{C}_{xx} = \sigma_x^2 \mathbf{I}, \quad \text{where } \mathbf{I} = N \times N \text{ identity matrix}$$

$$For \quad \mathbf{x} \quad IID \quad \Rightarrow \quad p(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\sigma_x^N} e^{-\frac{1}{2\sigma_x^2}\sum_{i=0}^{N-1}(x[i]-\mu_x)^2}$$

Example 1.10. Properties of a Gaussian

$p(x)$

$\text{sqrt}^{-1}(2\pi\sigma^2)$

68%

$\mu-\sigma$ μ $\mu+\sigma$

95%

$\mu-2\sigma$ $\mu+2\sigma$

$\mu-3\sigma$ $\mu+3\sigma$

>99%

x

For $\mu = 0$, $\sigma = 1$, the inflection points are ±1.
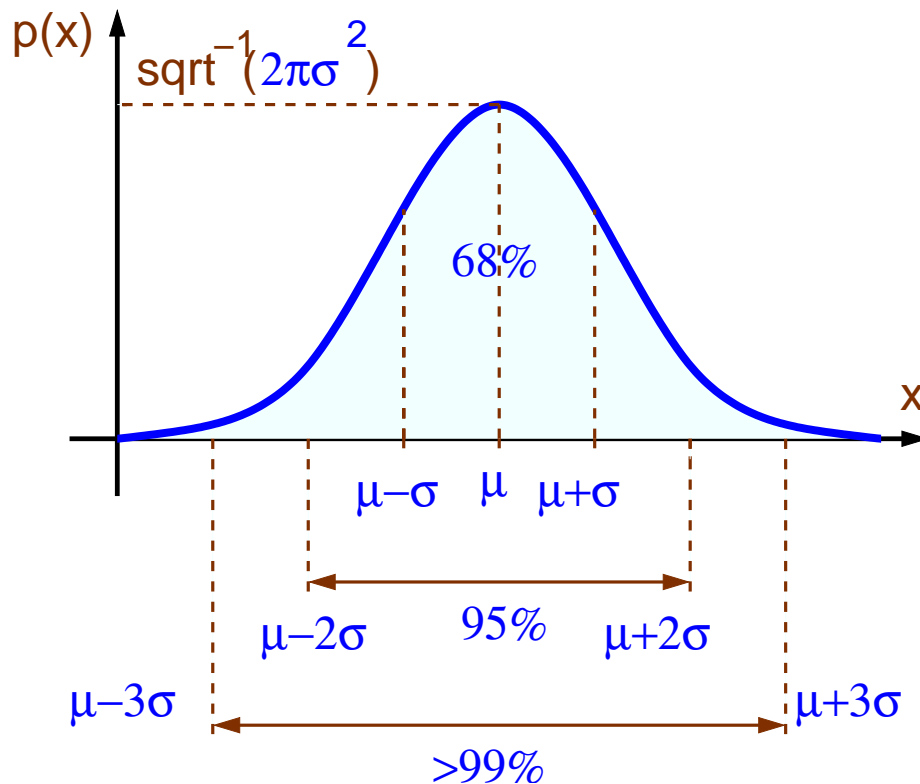
1) If x and y are jointly Gaussian, then for any constants a and b the random variable

$$z = ax + by$$

is Gaussian with mean

$$m_z = am_x + bm_y$$

and variance

$$\sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y\rho_{xy}$$
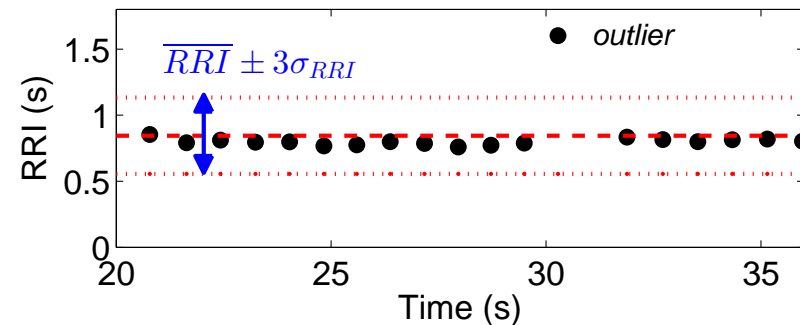
2) If two jointly Gaussian random variables are *uncorrelated* ($\rho_{xy} = 0$) then they are also statistically independent, that is

$$f_{x,y} = f(x)f(y)$$

Rejecting outliers from cardiac data

○ Failed detection of R peaks in ECG [left] causes outliers in R-R interval (RRI, time difference between consecutive R peaks) [right]
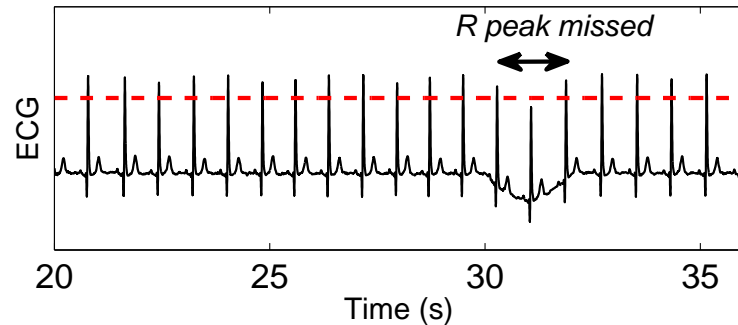
○ No clear outcome from PSD analysis of outlier-compromised RRI [left], but PSD of RRI with outliers removed reveals respiration rate [right]
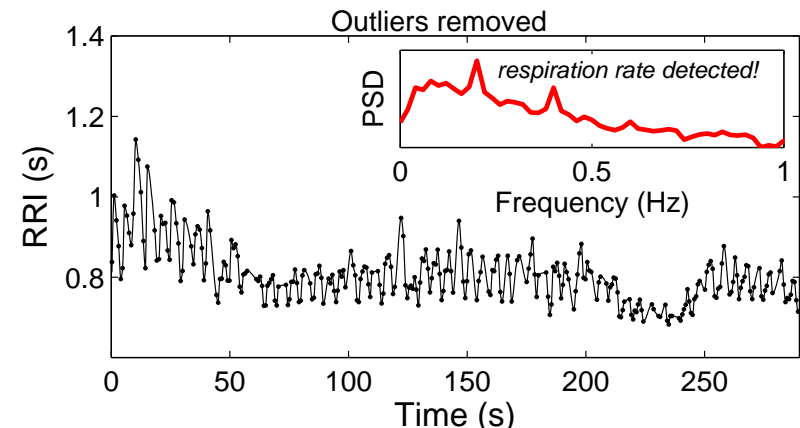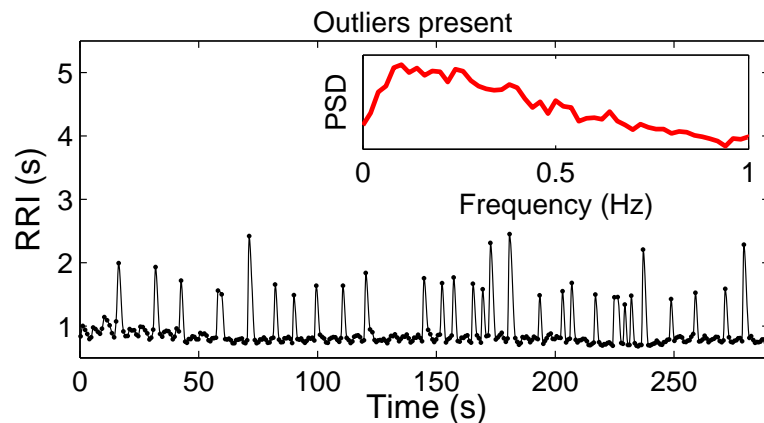
Linear algebra

Solving a set of linear equations is necessary in the formulation of many algorithms for spectral estimation and adaptive signal processing!

A set of linear equations is most conveniently represented in matrix form:-

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where the dimensions are

$$\bullet \; \mathbf{A}: \; m \times n \qquad \mathbf{x}: \; n \times 1 \qquad \mathbf{b}: \; m \times 1$$

Depending on the values of m and n we have three special cases:

$$m = n \quad \hookleftarrow \quad \textbf{exactly determined case}$$

$$m < n \quad \hookleftarrow \quad \textbf{over-determined case}$$
$$m > n \quad \hookleftarrow \quad \textbf{under-determined case}$$

The three cases: Possible solutions

- $m = n \hookrightarrow$ **Exactly determined set of equations** If $\mathbf{A}$ is invertible or nonsingular, i.e. it has full column rank, with m linearly independent columns, then
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$
If $\mathbf{A}$ is singular, then there may be either no solution or many solutions.

- $m < n \hookrightarrow$ **There are more equations than unknowns (over-determined)** Therefore, the additional degrees of freedom can be used for enhanced accuracy in the presence of noise. A **least squares solution** that minimises $\| \mathbf{Ax} - \mathbf{b} \|_2^2$ yields
$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$
If $\mathbf{A^TA}$ is invertible, then
$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \qquad \text{The } l\text{east squares solution}$$

with minimum least squares error $\mathbf{b}^T\mathbf{b} - \mathbf{b}^T\mathbf{Ax}$.

- $m > n \hookrightarrow$ **The under-determined case** (under-constrained system)
A solution: the **minimum norm solution which finds** $\min \| \mathbf{x} \|_2^2$ such that $\mathbf{Ax} = \mathbf{b}$.
Provided that the rows of $\mathbf{A}$ are linearly independent (and $\mathbf{AA}^T$ is invertible), we have
$$\mathbf{x} = \mathbf{A}^T \left(\mathbf{AA}^T \right)^{-1} \mathbf{b}$$
where $\mathbf{A}^T \left(\mathbf{AA}^T \right)^{-1}$ is the pseudo-inverse of $\mathbf{A}$. (we can also use Lagrange multipliers)

Special matrix forms

The exchange matrix (or reflection matrix)

$$\mathbf{J} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \text{ - notice } \mathbf{J} \times \mathbf{J} = \mathbf{I}$$

$\Rightarrow \mathbf{J}$ is its own inverse; and $\mathbf{J}^T \mathbf{A} \mathbf{J}$ reflects each element of the square matrix $\mathbf{A}$ about its central element.

Toeplitz matrix ↬tremendous amount of structure (constant diagonals, $a_{i,j} = a_{i+1,j+1}$)

Examples: Toeplitz $\mathbf{T} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 6 & 4 & 2 & 1 \end{bmatrix}$ Hankel $\mathbf{H} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 4 \\ 5 & 7 & 4 & 2 \\ 7 & 4 & 2 & 1 \end{bmatrix}$

Notice that all entries are defined once the first column and first row have been specified. A **convolution** matrix is an example of a Toeplitz matrix.

Hankel matrix ↬ similar to Toeplitz for which $a_{i,j} = a_{i-1,j+1}$ (upside-down Toeplitz). An example of a Hankel matrix is the exchange matrix. Also, $\mathbf{H} = \mathbf{T}\mathbf{J}$.

Taylor series expansion (TSE)

Most 'smooth' functions can be expanded into their Taylor Series Expansion (TSE)

$$f(x) = f(x_0) + \frac{f'(x_0)}{1}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}$$

To show this consider the polynomial

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

1. To get a_0 $\looparrowright$ choose $x = x_0$ $\Rightarrow$ $a_0 = f(x_0)$
2. To get a_1 $\looparrowright$ take derivative of the polynomial above to have

$$\frac{d}{dx}f(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4a_4(x - x_0)^4 + \cdots$$

choose $x = x_0$ $\Rightarrow$ $a_1 = \frac{df(x)}{dx}\big|_{x=x_0}$ and so on ... $a_k = \frac{1}{k!}\frac{d^k f(x)}{dx^k}\big|_{x=x_0}$

Power series - contd.

Consider

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \;\Rightarrow\; f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \;\text{ and }\; \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

1. Exponential function, cosh, sinh, sin, cos, ...

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \;\text{ and }\; e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \;\Rightarrow\; \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

2. other useful formulas

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \;\Rightarrow\; \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \;\text{ and }\; \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n n x^2$$

Integrate to obtain $atan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. For $x = 1$ we have
$\frac{\pi}{4} = 1 = 1/3 + 1/5 - 1/7 + \cdots$

Numerical differentiation ↦ Some examples

- $f' = \frac{f(1) - f(0)}{h}$

- $f' = \frac{f(0) - f(1)}{h}$

- $f' = \frac{f(1) - 2f(0) + f(-1)}{2h}$

- $f'' = \frac{f(1) - 2f(0) + f(-1)}{h^2}$

- $f' = \frac{f(-2) - 8f(-1) + 8f(1) - f(2)}{12h}$

- $f'' = \frac{-f(-2) + 16f(-1) - 30f(0) + 16f(1) - f(2)}{12h^2}$

Vectorial scalar function

$f(\mathbf{x} = f(x_1, \ldots, x_N)$

$$\text{Gradient } \nabla_x f(\mathbf{x})) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_N} \end{bmatrix} = \mathbf{0} \text{ and the Hessian matrix } \mathbf{H}_x > \mathbf{0}.$$

where the elements of the Hessian matrix are $\{H_x\}_{i,j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$

Theorem: If $f(\mathbf{z}, \mathbf{z}^*)$ is a real-valued function of the complex vectors $\mathbf{z}$ and $\mathbf{z}^*$, the vector pointing in the direction of the maximum rate of change of $f(\mathbf{z}, \mathbf{z}^*)$ is $\nabla_{\mathbf{z}} f(\mathbf{z}, \mathbf{z}^*)$, the derivative of $f(\mathbf{z}, \mathbf{z}^*)$ wrt $\mathbf{z}^*$. [Hayes 1996].

Thus, the turning points of $f(\mathbf{z}, \mathbf{z}^*)$ are solutions to $\nabla_{\mathbf{z}^*} f(\mathbf{z}, \mathbf{z}^*) = \mathbf{0}$,

$$\text{where } \nabla_{\mathbf{z}^*} = \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial x_1} - \jmath\frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial x_n} - \jmath\frac{\partial}{\partial y_n} \end{bmatrix} \quad , \nabla_{\mathbf{z}} \mathbf{a}^H \mathbf{z} = \mathbf{a}^*, \quad , \nabla_{\mathbf{z}^*} \mathbf{a}^H \mathbf{z} = \mathbf{0}$$

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Summary

○ Several background concepts in one place

○ The course is self–contained, most of the background should be found in this Lecture

○ For more detail about spectrum estimation, refer to Hayes' book

○ For more detail about adaptive filtering, refer to Haykin's book

○ We will frequently come back to this lecture, when using these techniques in spectral estimation and adaptive signal processing

○ Towards the end of the course, we will combine spectrum estimation and adaptive signal processing towards an application in brain computer interface (BCI)

We will next "get our hands dirty" with real world spectrum estimation

Notes:

○

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