Spectral Estimation and ASP

Lecture 3 - Nonparametric Spectrum Estimation

Danilo Mandic

room 813, ext: 46271

Department of Electrical and Electronic Engineering
Imperial College London, UK

d.mandic@imperial.ac.uk, URL: www.commsp.ee.ic.ac.uk/∼mandic

Outline

Part 1: Background

- History of Spectrum Estimation
- The Discrete Fourier Transform
- Four Practical Issues with DFT: Aliasing, Frequency Resolution Incoherent Sampling, Leakage

Part 2: Nonparametric Spectral Estimation

- From Fourier Transform to the Periodogram
- Bias-variance dilemma

Part 3: Periodogram Modifications

- Windowing
- Averaging
- Blackman-Tukey Method

Part 1: Background

Problem Statement

From a **finite** record of stationary data sequence, **estimate** how the total power is distributed over frequency.

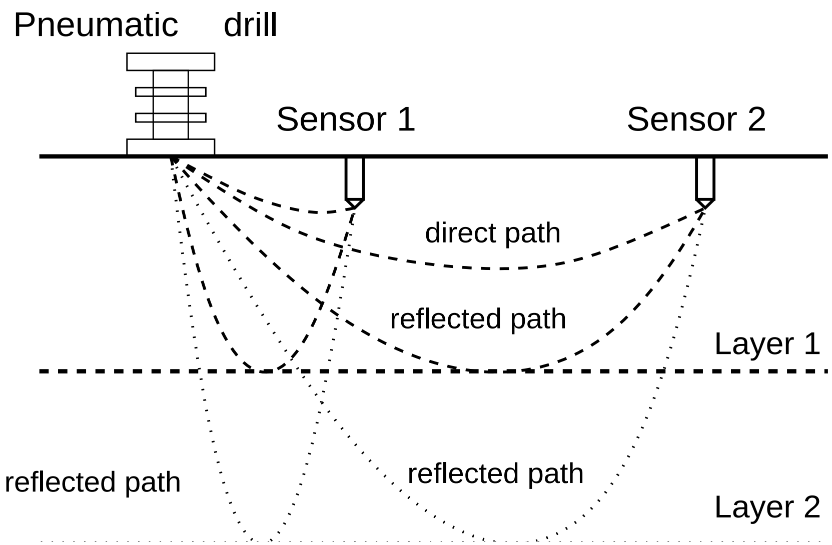
Has found a tremendous number of applications:-

- Seismology $\rightarrow$ oil exploration, earthquake
- Radar and sonar $\rightarrow$ location of sources
- Speech and audio $\rightarrow$ recognition
- Astronomy $\rightarrow$ periodicities
- Economy $\rightarrow$ seasonal and periodic components
- Medicine $\rightarrow$ EEG, ECG, fMRI
- Circuit theory, control systems

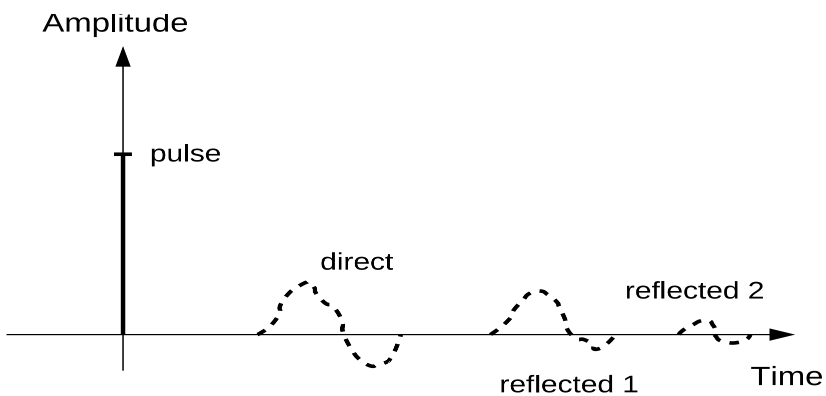
Some examples

Seismic estimation — Speech processing

periodic pulse excitation

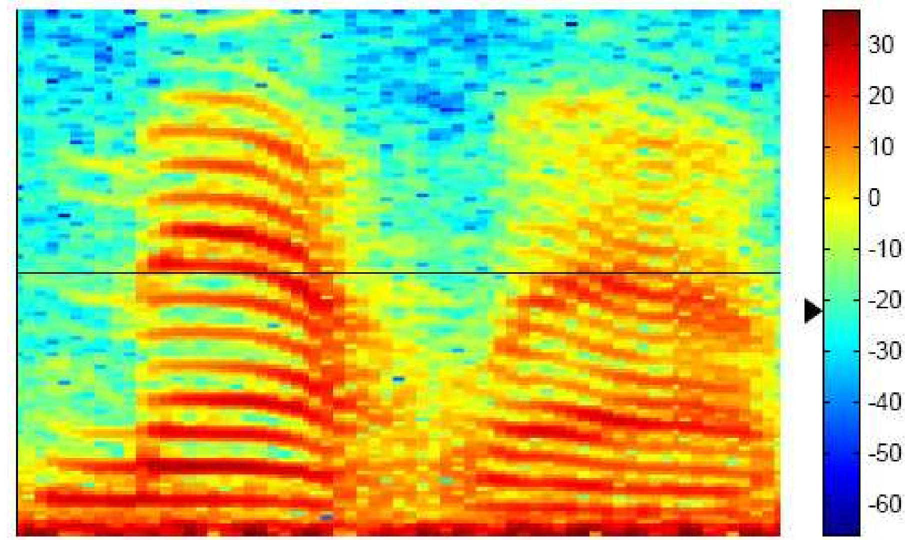


(a) Simplified seismic paths.



(b) Seismic impulse response.

frequency



time

M aaa t l aaa b

For every time segment 'Δt', the PSD is plotted along the vertical axis. Observe the harmonics in 'a'

Darker areas: higher magnitude of PSD (magnitude encoded in color)

Use Matlab function 'specgram'

Historical perspective

1772 **Lagrange** proposes use of rational functions to identify multiple periodic components;

1840 **Buys–Ballot**, tabular method;

1860 **Thomson**, harmonic analyser;

1897 **Schuster**, periodogram, periods not necessarily known;

1914 **Einstein**, smoothed periodogram;

1920-1940 Probabilistic theory of time series, Concept of spectrum;

1946 **Daniell**, smoothed periodogram;

1949 **Hamming & Tukey** transformed ACF;

1959 **Blackman & Tukey**, B–T method;

1965 **Cooley & Tukey**, FFT;

1976 **Lomb**, periodogram of unevenly spaced data;

1970– Modern spectrum estimation!

Discrete Fourier Transform as a Least Squares Problem

Problem: Fitting data $x[n]$ with a linear model with $[N-1]$ complex sinusoids:

$$\hat{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} w[k]e^{j\frac{2\pi}{N}nk} \tag{1}$$

Eq (1) can be formulated in vector notation as $\hat{\mathbf{x}} = \frac{1}{N}\mathbf{F}\mathbf{w}$, where

$$\underbrace{\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \vdots \\ \hat{x}[N-1] \end{bmatrix}}_{\hat{\mathbf{x}}} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}(1)(1)} & \cdots & e^{j\frac{2\pi}{N}(1)(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2\pi}{N}(N-1)(1)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} w[0] \\ w[1] \\ \vdots \\ w[N-1] \end{bmatrix}}_{\mathbf{w}}$$

The least squares solution to $\mathbf{w}$ is found by (CW question):

$$\hat{\mathbf{w}} = \operatorname*{argmin}_{\mathbf{w}} \|\mathbf{x} - \mathbf{F}\mathbf{w}\|^2 = \mathbf{F}^H\mathbf{x}$$

$\Longrightarrow$ DFT coefficient at bin k is $w[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}nk}$

FT basics

Periodic signal $\leftrightsquigarrow$ **Discrete** FT
Discrete signal $\leftrightsquigarrow$ **Periodic** FT
Periodic and **Discrete** signal $\leftrightsquigarrow$ **Discrete** and **Periodic** FT
Discrete and **Periodic** signal $\leftrightsquigarrow$ **Periodic** and **Discrete** FT

- **Sampling** yields a new signal ($f_s = \frac{2\pi}{T}$) (poor approximation)

$$g[n] = T\ f(nT) \quad \Leftrightarrow G(\jmath\omega) = \sum_{k=-\infty}^{\infty} F(\jmath\omega + \jmath k\Omega_0)$$

- **Limiting** the length to N samples effectively introduces rectangular windowing (Leakage)

$$W(\jmath\omega) = \frac{\sin(N\omega T/2)}{\sin(\omega T/2)} e^{-\jmath\frac{N-1}{2}\omega T}$$

$\Rrightarrow$ **Estimated Spectrum $=$ True spectrum $*$ Sinc**

Convolution with Sinc

Issues with finite duration measurements

To analyse the effects of a finite signal duration, consider a rectangular window

$$\underbrace{\Big|\;\;\Big|\;\;\cdots\;\;\Big|}_{0,\ldots,N-1} \quad \xrightarrow{\mathcal{F}} \quad \sum_{k=0}^{N-1} e^{-\jmath 2\pi f k}$$

$$W(f) = \sum_{k=0}^{N-1} e^{-\jmath 2\pi f k} = \frac{1-e^{-\jmath 2\pi f N}}{1-e^{-\jmath 2\pi f}} = \frac{e^{-\jmath \frac{2\pi f N}{2}}}{e^{-\jmath \frac{2\pi f}{2}}} \frac{2\jmath \sin(\pi f N)}{2\jmath sin(\pi f)} =$$

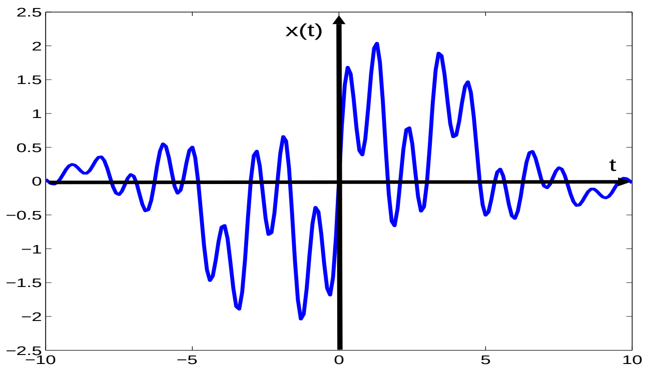
$$e^{-\jmath \pi f (N-1)} \frac{\sin(\pi f N)}{\pi f N} \times \frac{\pi f N}{\sin(\pi f)} = e^{-\jmath \pi f (N-1)} \frac{sinc(\pi f N)}{sinc(\pi f)} \times N$$

If the sampling is **coherent**, zeroes of the sinc functions all lie at multiples of $1/N$, and hence the outputs of DFT are all zero except at $f = \pm\frac{1}{N}$.

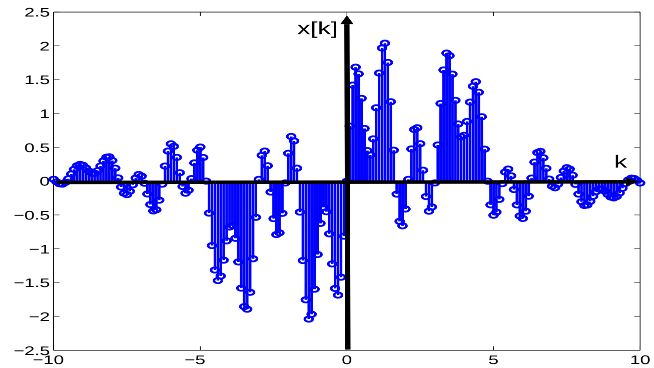
Practical Issue #1: Aliasing

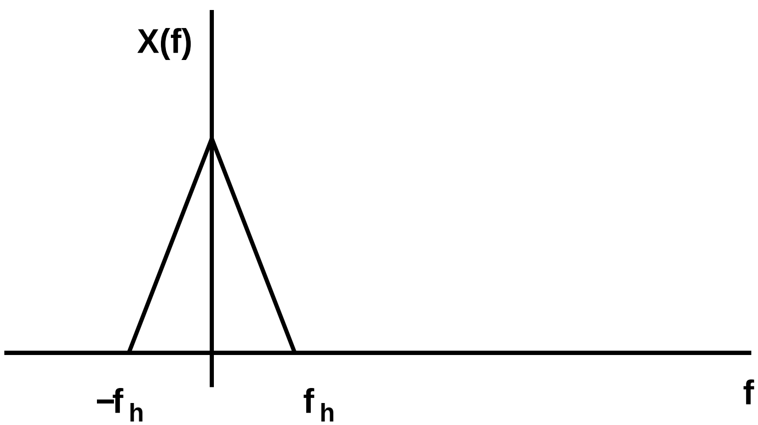
Sampling Theorem Revisited

Original signal

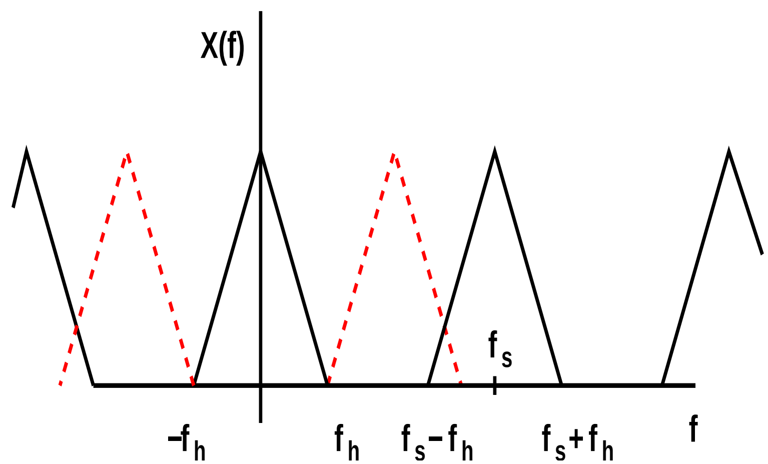


Sampled original signal





Original spectrum

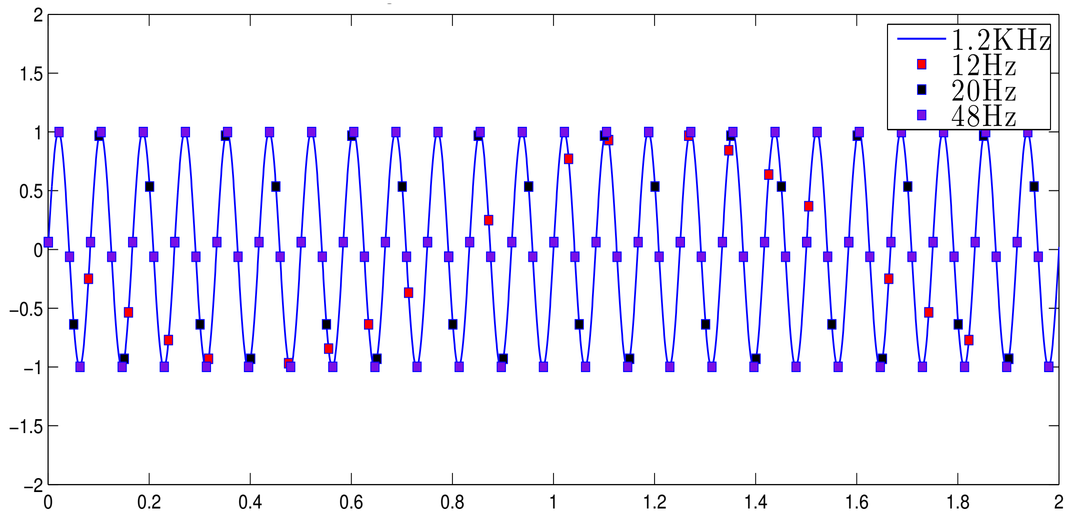


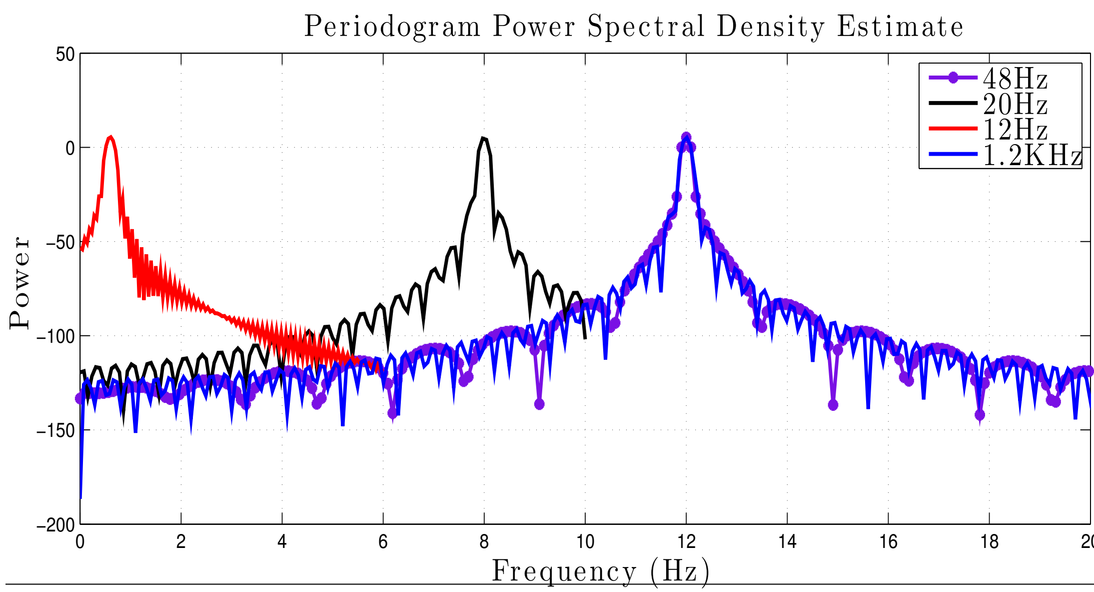
Spectrum of sampled signal

For sampling period T and sampling frequency $f_s = 1/T \Rightarrow f_s \geq 2f_h$

Practical Issue #1: Aliasing

Sampling theorem: An example





- Sub-Nyquist sampling causes aliasing
- This distorts physical meaning of information
- In signal processing, we require faithful data representation
- Problem: the noise model is always all-pass
- The easiest and most logical remedy is to low-pass filter the data so that the Nyquist criterion is satisfied.

Practical Issue #2: Frequency Resolution

Def: Frequency resolution is the minimum separation between two sinusoids, resolvable in frequency.

Ideally, we want an excellent resolution for a very few data samples (genomic SP)

However,

i) Due to the wide mainlobe of the SINC function (spectrum of the rectangular window), the convolution between the true spectrum and the sinc function **smears** the spectrum;

ii) **For two impulses in frequency to be resolvable, at least one frequency bin must separate them, that is**

$$\frac{2\pi}{NT} \qquad \Rightarrow \qquad T \;\; fixed \;\rightarrow\; N \;\; increase$$

Practical Issue #2: Frequency Resolution

Time-bandwidth Product

- Suppose we know the **maximum frequency** in the signal ω_{max}, and the required resolution $\Delta\omega$. Then

$$\Delta\omega > 2\frac{2\pi}{NT} = 2\frac{\omega_s}{N} \qquad \Rightarrow \qquad N > \frac{4\omega_{max}}{\Delta\omega}$$

- For both the **prescribed resolution and bandwidth**, then

$$\omega_s = \frac{2\pi}{T} > 2\omega_{max} \qquad \& \qquad 2\omega_s < \Delta\omega N$$

$$hence$$

$$\frac{f_s}{2} = \frac{\pi}{T} > \omega_{max} \quad that \;\; is \quad T < \frac{\pi}{\omega_{max}} \quad \Leftrightarrow \quad N > \frac{4\omega_{max}}{\Delta\omega}$$

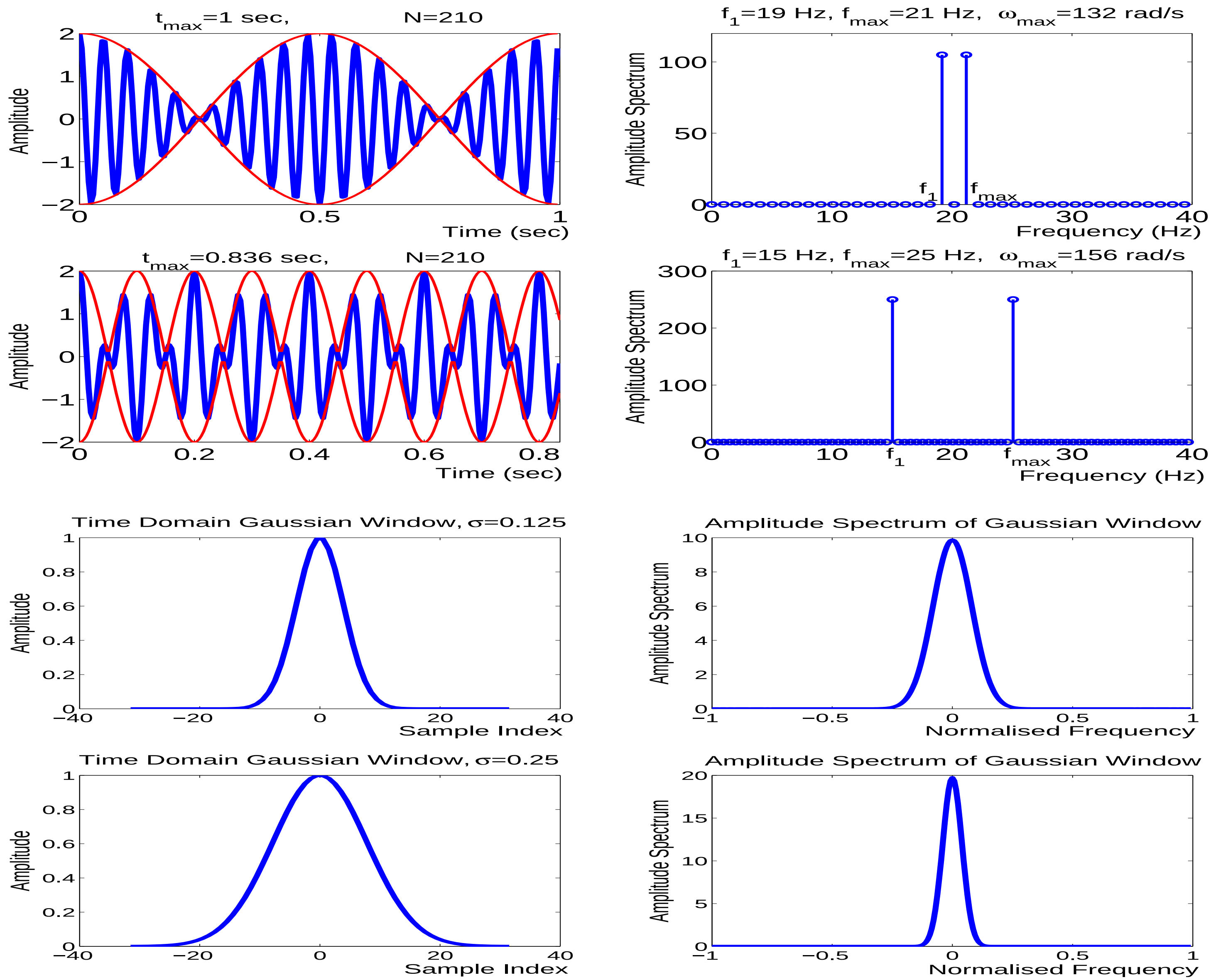
- If we know **signal duration** ($f_s \geq 2f_{max} \Rightarrow \frac{2\pi}{T} \geq 2\omega_{max} \Rightarrow T < \frac{\pi}{\omega_{max}}$)

$$N > \frac{2t_{max}}{T} \qquad \Rightarrow \qquad N > \frac{2t_{max}\omega_{max}}{\pi}$$

$t_{max} \times \omega_{max} \rightarrow$ **time–bandwidth product of a signal.**

Example: the time–bandwidth product

Top: AM signals Bottom: Gaussian signals



Practical Issue #3: Spectral Leakage

Two sines with close frequencies

Top: A 32-point DFT of an $N = 32$ long sampled ($f_s = 64\text{Hz}$) mixed sinewave

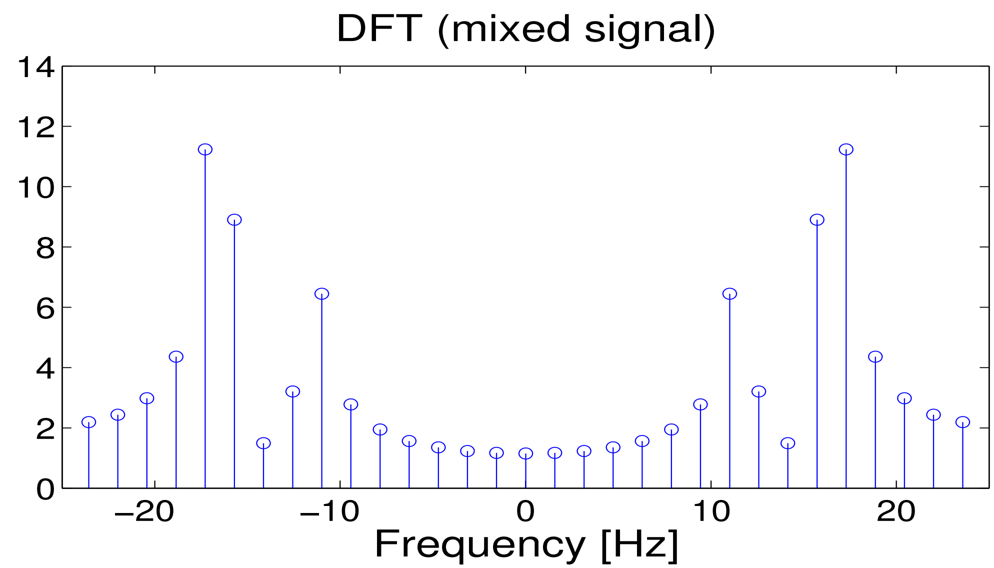
$$x(k) = \sin(2\pi 11k) + \sin(2\pi 17k)$$

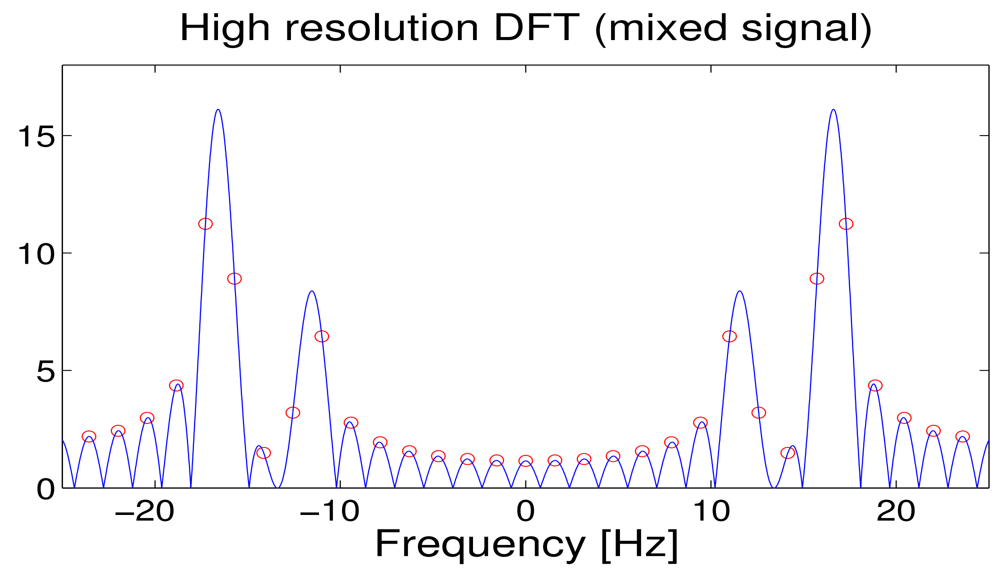
It is difficult to determine how many distinct sinewawes we have.

Bottom: A 3200-point DFT of an $N = 32$ long sampled ($f_s = 64\text{Hz}$) sine

$$x(k) = \sin(2\pi 11k) + \sin(2\pi 17k)$$

- Both the $f = 11\text{Hz}$ and $f = 17\text{Hz}$ sinewaves appear quite sharp
- This is a consequence of a high-resolution ($N = 3200$) DFT
- The overlay plot compares it with the top diagram



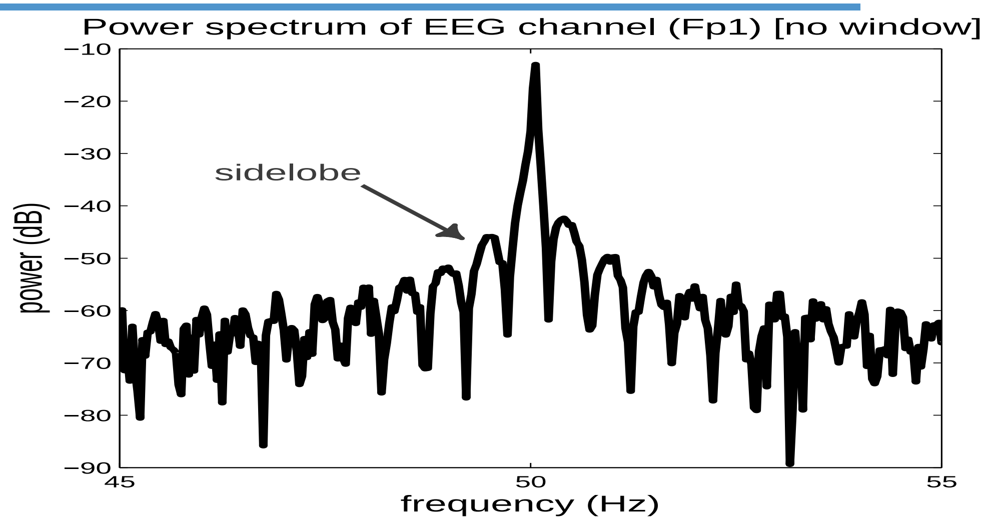


Example: FFT leakage $\leadsto$ EEG power spectrum

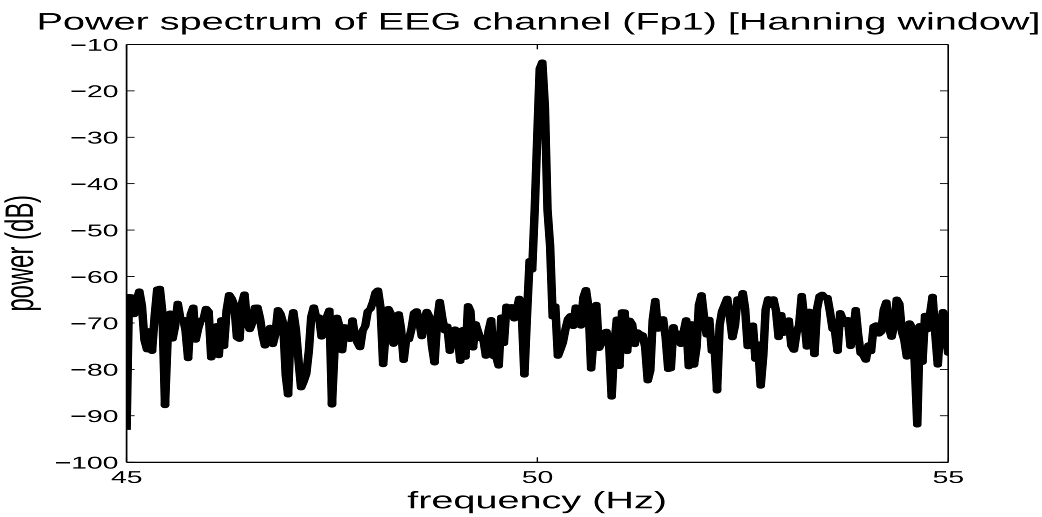
we record $\approx 10\mu V$ signals in the presence of external noise

Problem: estimate power of the 50Hz artefact picked up by EEG leads

- Using the standard periodogram - the resolution is good but the artefact is partially masked
- **Remedy:** Use a windowing function (e.g. Hanning window).
  - Note that sidelobes are reduced, energy over narrow frequency range around 50Hz.
- Window value is zero at the beginning and end of a segment
  - Multiply with the signal with a window that has small sidelobes to reduce leakage
- **Windows reduce, but do not eliminate leakage completely!**



`periodogram(x,[],N,Fs,‘onesided’);`



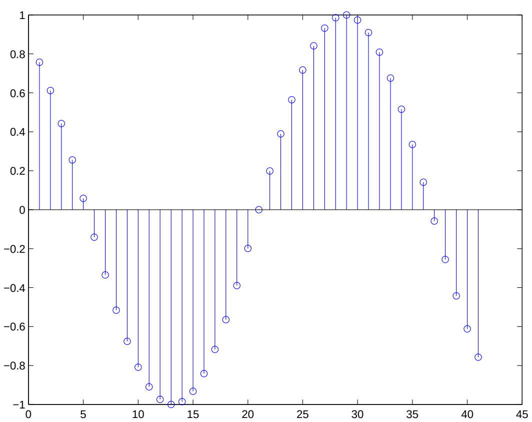
`periodogram(x,hann(length(x)),N,Fs,‘onesided’);`

Practical Issue #4: Incoherent sampling

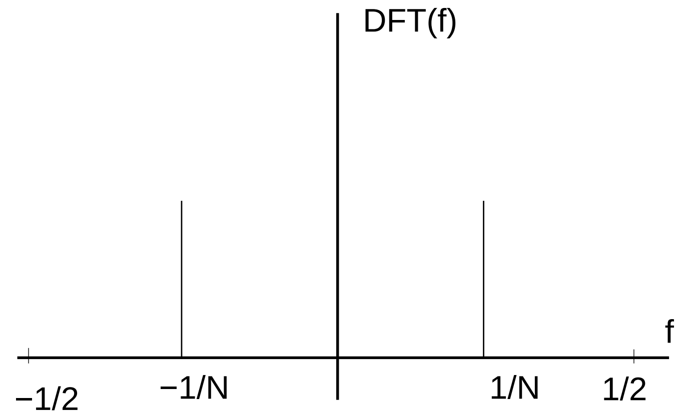
Before introducing the conventional methods for spectral estimation, it would be useful to re–examine the effects of coherent and incoherent sampling upon the Discrete Fourier Transform (DFT).

Consider a sampled sinusoid and its corresponding DTFT magnitude

$$X(f) = \sum_{k=0}^{N-1} x[k] e^{-\jmath 2\pi f k} \quad \leftrightarrow \quad X[m] = \sum_{k=0}^{N-1} x[k] e^{-\jmath \frac{2\pi}{N} mk}$$



A sinewave with N samples



Spectrum of sinewave

Practical Issue #4: Incoherent sampling

Are the signal frequencies, $f = k\frac{f_s}{N}$?

Top: A 32-point DFT of an $N = 32$ long sampled ($f_s = 64$Hz) sinewave of $f = 10$Hz

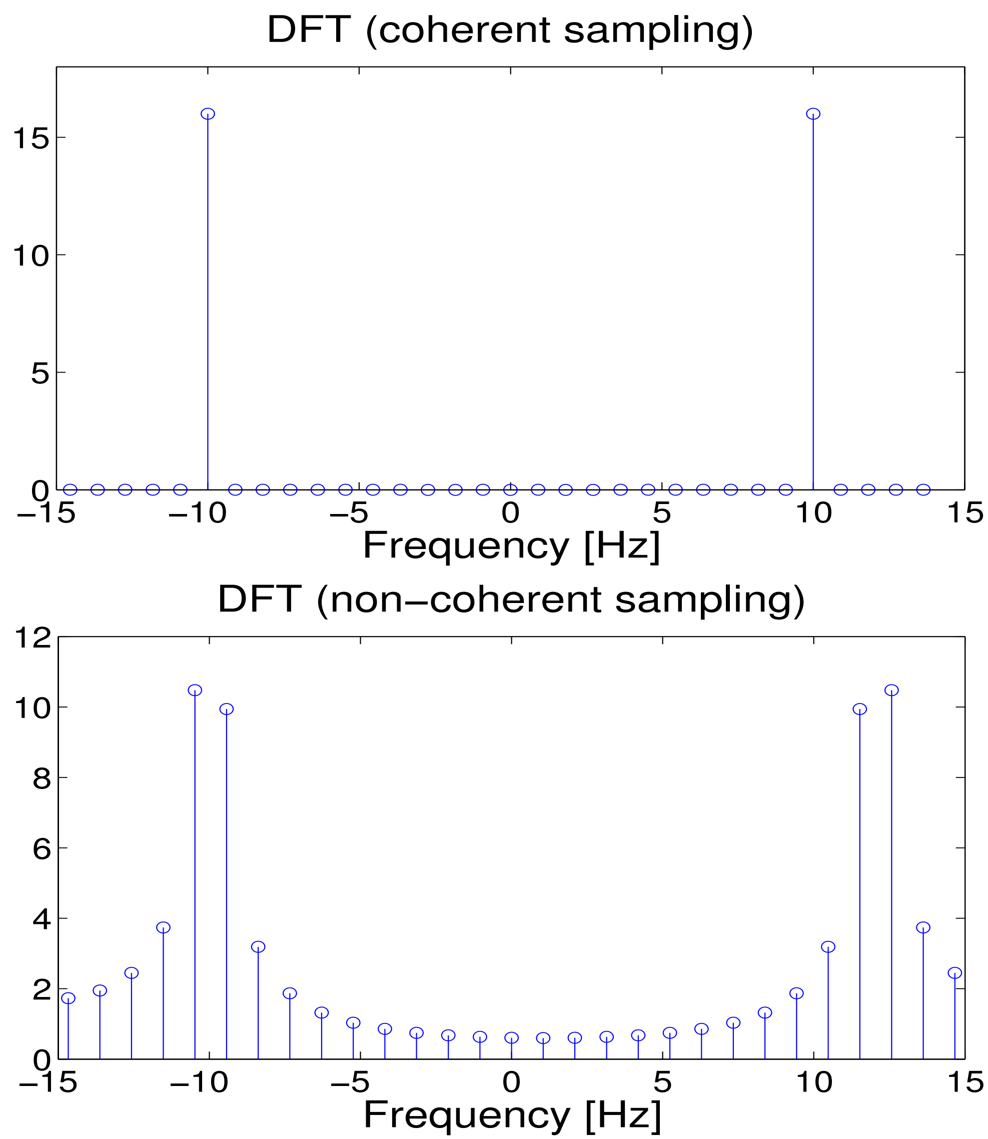
- For $f_s = 64$ Hz, the DFT bins will be located in Hz at $k/NT = 2k$, $k = 0, 1, 2, \ldots, 63$
- One of these points is at given signal frequency of 10 Hz

Bottom: A 32-point DFT of an $N = 32$ long sampled ($f_s = 64$Hz) sine of $f = 11$Hz

- Since

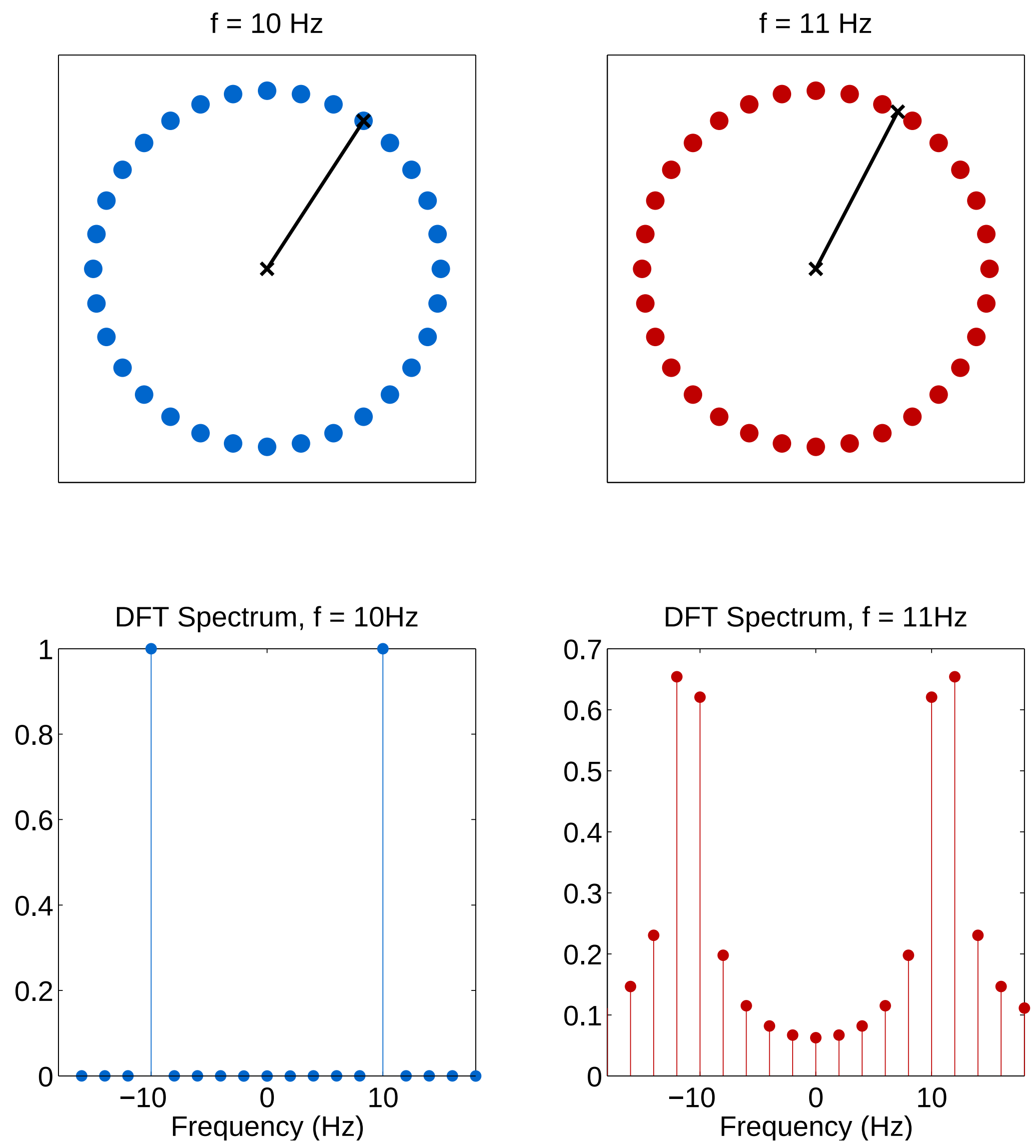
$$\frac{f_R}{f_s} = \frac{f \times N}{f_s} = \frac{11 \times 32}{64} = 5.5$$

  the impulse at $f = 11$ Hz appears between the DFT bins $k = 5$ and $k = 6$
- The impulse at $f = -11$ Hz appears between DFT bins $k = 26$ and $k = 27$ (10 and 11 Hz)



Practical Issue #4: Incoherent sampling

Visual Representation



What to look for next?

- We must examine the statistical properties of the periodogram estimator
- For the general case, the statistical analysis of the periodogram is intractable
- We can, however, derive the mean of the periodogram estimator for any real process
- The variance can only be derived for the special case of a real zero–mean WGN process with $P_{xx}(f) = \sigma_x^2$
- Can this can be used as indication of the variance of the periodogram estimator for other random signals
- Can we use our knowledge about the analysis of various estimators, to treat the periodogram in the same light (is it an MVU estimator, does it attain the CRLB)
- Can we make a compromise between the bias and variance in order to obtain a mean squared error (MSE) estimator of power spectrum?

This is all covered in the next part

Why do not you think a little about ...

- ⊛ The resolution for zero-padded spectra is higher, what can we tell about the variance of such a periodogram?
- ⊛ If the samples at the start and end of a finite-length data sequence have significantly different amplitudes, how does this affect the spectrum?
- ⊛ What uncertainties are associated with the concept of "frequency bin"?
- ⊛ Why happens with high frequencies in tapered periodograms?
- ⊛ What would be the ideal properties of a "data window"?
- ⊛ How frequently do we experience incoherent sampling in real life applications and what is a most pragmatic way to deal with the frequency resolution when calculating spectra of such signals?
- ⊛ How can we use the time–bandwidth product to ensure physical meaning of spectral estimates?
- ⊛ The "double summation" formula that uses progressively fewer samples to estimate the ACF is very elegant, but does it come with some problems too, especially for larger lags?

Part 2: Nonparametric Spectral Estimation

Aims

- To introduce methods for **nonparametric** spectrum estimation, which are based on the Fourier transform

- To analyse the periodogram as an estimator and to understand its properties in terms of the mean squared error (MSE) performance

- To derive expressions for the **bias** and **variance** of the periodogram

- To introduce variants of the periodogram and their analyse their properties as estimators of power spectrum

- To understand the trade-off between the periodogram resolution, bias, variance, window function, and data length

- To illustrate practical applications of periodogram (narrowband signal estimation, multiple harmonic components, brain computer interface)

Problem Statement

Estimate Power Spectral Density (PSD) of a wide-sense stationary signal

Recall that PSD $= \mathcal{F}(ACF)$.

Therefore, estimating the power spectrum is equivalent to estimating the autocorrelation.

Recall that for an autocorrelation ergodic process,

$$\lim_{N\to\infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x(n) \right\} = r_{xx}(k)$$

If $x(n)$ is known for all n, estimating the power spectrum is straightforward

- **Difficulty 1:** the amount of data is **always limited**, and may be very small (genomics, biomedical)
- **Difficulty 2:** real world data is **almost invariably corrupted by noise**, or contaminated with an interfering signal

Classical spectral estimation

The **power spectrum** or **power spectral density** $P_{xx}(f)$ of a process $\{x[n]\}$ is defined as (Wiener–Khintchine Theorem)

$$P_{xx}(f) = \mathcal{F}\{r_{xx}(m)\} = \sum_{m=-\infty}^{\infty} r_{xx}(m) e^{-\jmath 2\pi m f} \qquad f \in (-1/2, 1/2], \omega \in (-\pi, \pi]$$

The sampling period T is assumed to be unity, thus f is the *normalised frequency*.
From the inversion formula (Fourier), we can write

$$r_{xx}(m) = \int_{-1/2}^{1/2} P_{xx}(f) e^{\jmath 2\pi m f} df$$

$\Rightarrow$ ACF and PSD tell us about the power within the signal (average)
For example, $r(0) = \int_{-1/2}^{1/2} P_{xx}(f) df = E\{x^2[n]\} \geq 0$.

$\Rightarrow$ **the area below the PSD (power spectral density) curve = Signal Power**

Power spectrum – some insights

We shall now show that the PSD can be written in an equivalent form:-

$$P_{xx}(f) = \lim_{M\to+\infty} \frac{1}{2M+1} E\left\{ \left| \sum_{k=-M}^{+M} x[k] e^{-\jmath 2\pi f k} \right|^2 \right\}$$

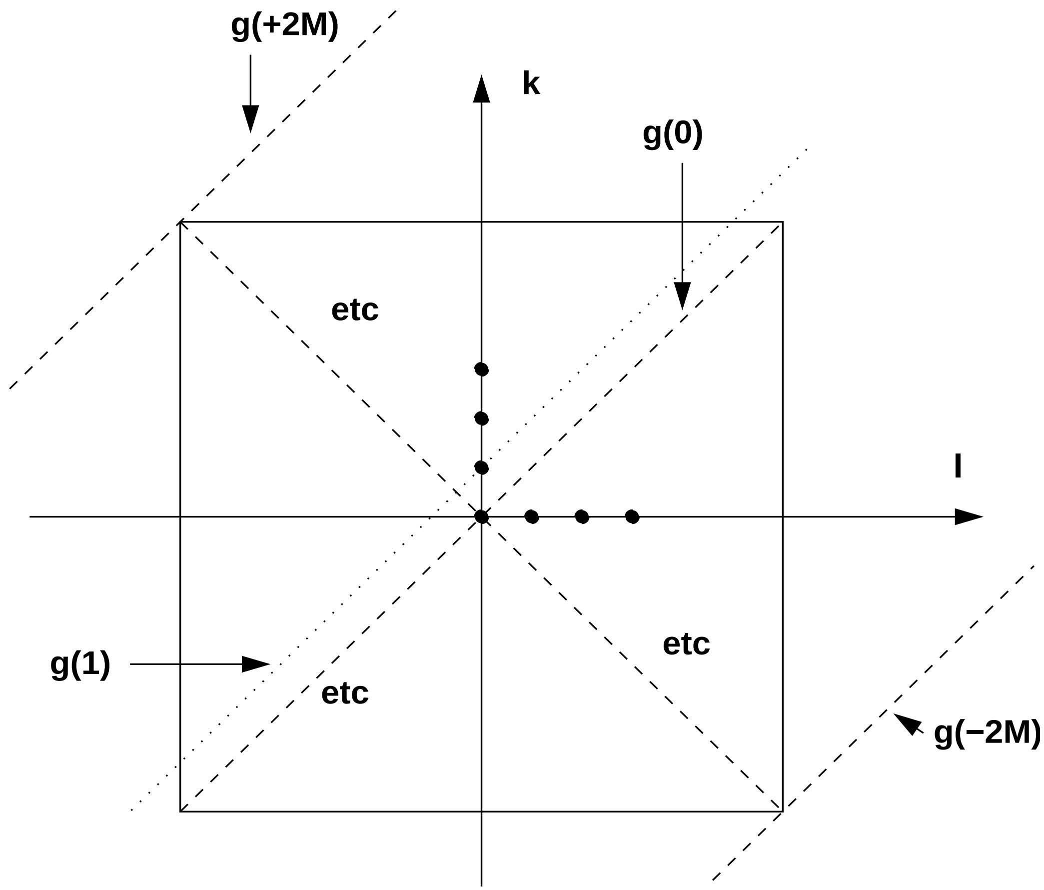
Let us begin by expanding

$$\begin{aligned} P_{xx}(f) &= \lim_{M\to+\infty} \frac{1}{2M+1} E\left\{ \sum_{k=-M}^{+M} \sum_{l=-M}^{M} x[k]x[l] e^{-\jmath 2\pi f (k-l)} \right\} \\ &= \lim_{M\to+\infty} \frac{1}{2M+1} \sum_{k=-M}^{+M} \sum_{l=-M}^{M} \underbrace{E\{x[k]x[l]\}}_{\mathbf{r}_{xx}(k-l)} e^{-\jmath 2\pi f (k-l)} \\ &= \lim_{M\to+\infty} \frac{1}{2M+1} \sum_{k=-M}^{+M} \sum_{l=-M}^{M} g(k-l) \end{aligned}$$

Note that $\left(\sum_i\right)^2 = \sum_j \times \sum_k$

Converting double into a single summation

$$\sum_{k=-M}^{+M} \sum_{l=-M}^{M} g(k-l) = \sum_{\tau=-2M}^{2M} (2M+1-|\tau|)g(\tau)$$



(2M+1) points $\leftrightsquigarrow$ $g = g(0)$
2M points $\leftrightsquigarrow$ $g = g(1)$
$\vdots$ $\vdots$ $\vdots$
1 point $\leftrightsquigarrow$ $g = g(2M)$

Reminds you of a triangle?

Recall: the autocorrelation of two rectangles of width 2M is a triangle of width 4M!

This underpins our first practical power spectrum estimator

Schuster's periodogram (1898)

Hence

$$P_{xx}(f) = \lim_{M \rightarrow +\infty} \sum_{\tau=-2M}^{2M} \underbrace{\left(\frac{2M+1-|\tau|}{2M+1} \right)}_{=\left(1-\frac{|\tau|}{2M+1}\right)} \mathbf{r}_{xx}(\tau) e^{-\jmath 2\pi f \tau}$$

Provided the autocorrelation function decays fast enough, we have

$$P_{xx}(f) = \sum_{\tau=-\infty}^{+\infty} \mathbf{r}_{xx}(\tau) e^{-\jmath 2\pi f \tau}$$

Note $\mathbf{r}_{xx}(\tau) = \mathbf{r}_{xx}(-\tau) \Rightarrow P_{xx}(f)$ is real!

In practice, we only have access to $[x(0), \ldots, x(N-1)]$ data points (we drop the expectation), then

$$\hat{\mathbf{P}}_{per}(f) = \frac{1}{N} \left| \sum_{k=0}^{N-1} x[k] e^{-\jmath 2\pi f k} \right|^2$$

Symbol $\hat{}$ denotes an estimate, since due to the finite N the ACF is imperfect

Physical intuition: Connecting PSD and ACF
positive (semi)-definiteness

Recall: $\mathbf{R}_{xx} = E\{\mathbf{x}\mathbf{x}^T\} = \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r(1) & r(0) & \cdots & r(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & \cdots & r(0) \end{bmatrix}$

Then, for a linear system with input sequence $\{x\}$, output $\{y\}$, and the vector of coefficients $\mathbf{a}$, the output has the form

$$y(n) = \sum_{k=0}^{N-1} a(k)x(n-k) = \mathbf{x}^T\mathbf{a} = \mathbf{a}^T\mathbf{x} \quad \text{where} \quad \mathbf{a} = [a(0), \ldots, a(N-1)]^T$$

The power $P_y = E\{y^2\}$ is **always** positive, and thus $\left((\mathbf{a}^T\mathbf{b})^T = \mathbf{b}^T\mathbf{a}^T\right)$

$$E\Big\{y^2[n]\Big\} = E\Big\{y[n]y^T[n]\Big\} = E\Big\{\mathbf{a}^T\mathbf{x}\mathbf{x}^T\mathbf{a}\Big\} = \mathbf{a}^T E\Big\{\mathbf{x}\mathbf{x}^T\Big\}\mathbf{a} = \mathbf{a}^T\mathbf{R}_{xx}\mathbf{a}$$

$\Rightarrow$ **to maintain positive power, the autocorrelation matrix $\mathbf{R}_{xx}$ must be positive semidefinite**

In other words: a positive semidefinite $\mathbf{R}_{xx}$ will alway produce positive power spectrum!

But, is our estimate of ACF always positive definite?

Two ways to estimate the ACF

For an **autocorrelation ergodic** process with an unlimited amount of data, the ACF may be determined:

1) Using the time–average

$$r_{xx}(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+k)x(n)$$

If $x(n)$ is measured over a finite time interval, $n = 0, 1, \ldots, N-1$ then we need to *estimate* the ACF from a finite sum

$$\hat{r}_{xx}(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+k)x(n)$$

2) In order to ensure that the values of $x(n)$ that fall outside interval $[0, N-1]$ are excluded from the sum, we have (**biased estimator**)

$$\hat{r}_{xx}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x(n), \quad k = 0, 1, \ldots, N-1$$

Cases 1) and 2) are equivalent for small lags and a fast decaying ACF

Case 1) gives positive semidefinite ACF, this is not guaranteed for Case 2)

Periodogram based estimation of power spectrum
more intuition $\looparrowright$ connection with DFT

A nonparametric estimator the power spectrum – **the periodogram**

$$\hat{P}_{per}(e^{\jmath\omega}) = \sum_{k=-N+1}^{N+1} \hat{r}_{xx}(k)e^{-\jmath k\omega}$$

It is, however, more convenient to express the periodogram in terms of the process $x[n]$ (alternative derivation):

- Notice that $\hat{r}_{xx}(k) = \frac{1}{N}x(k) * x(-k)$
- Apply the FT to obtain

$$\hat{P}_{per}(e^{\jmath\omega}) = \frac{1}{N}X(e^{\jmath\omega})X^*(e^{\jmath\omega}) = \frac{1}{N}\left|X(e^{\jmath\omega})\right|^2$$

where $X(e^{\jmath\omega}) = \sum_{n=0}^{N-1} x(n)e^{-\jmath\omega n}$. (this is a DTFT of $x(n)$).

Periodogram and Matlab

```
Px=abs(fft(x(n1:n2))).^2/(n2-n1-1)
```

or the direct command '**periodogram**'

- `Pxx = PERIODOGRAM(X)`
  returns the PSD estimate of the signal specified by vector X in the vector Pxx. By default, the signal X is windowed with a BOXCAR window of the same length as X;

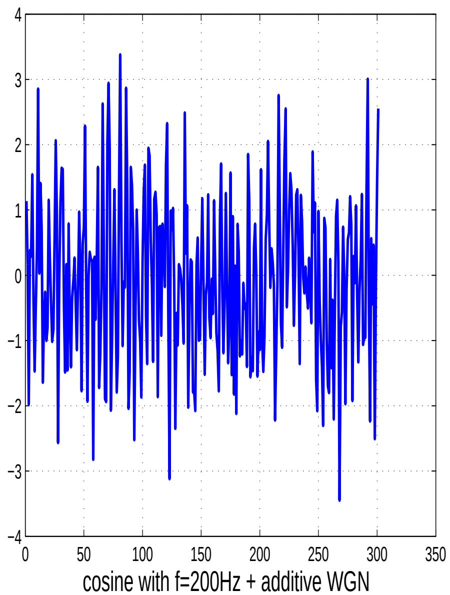
- `PERIODOGRAM(X,WINDOW)`
  specifies a window to be applied to X. WINDOW must be a vector of the same length as X;

- `[Pxx,W] = PERIODOGRAM(X,WINDOW,NFFT)`
  specifies the number of FFT points used to calculate the PSD estimate.

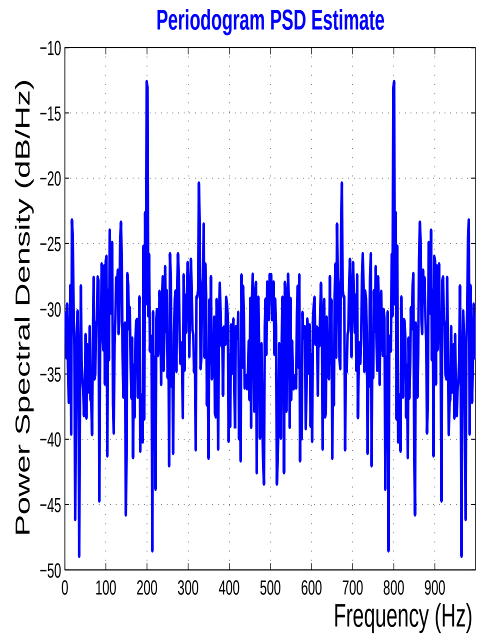
Example: PSD of a cosine in WGN

Calculate the PSD of a signal given by

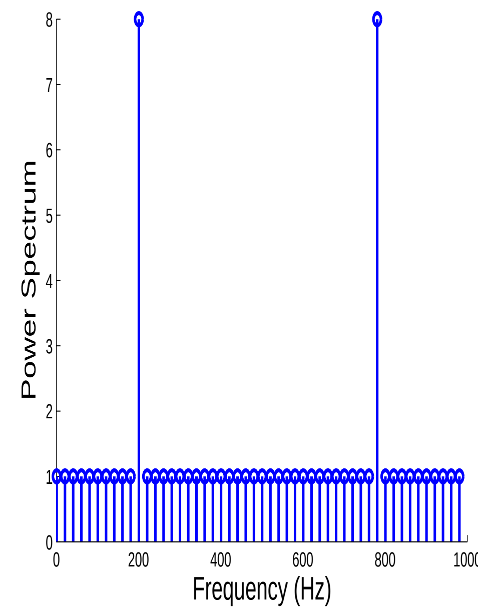
```
x = cos(2*pi*t*200)+randn(size(t))
```



Signal x



Periodogram of x



Ideal PSD

Homework: i) Plot PSDs for varios signal to noise ratios (SNR)

ii) Use both the linear and logarithmic scale (in [dB])

The bias–variance dilemma
a tool for the performance evaluation of periodograms

The mean square error (MSE) of an estimate $\hat{\theta}$ of a parameter θ is given by

$$MSE(\hat{\theta}) = E\{(\hat{\theta} - \theta)^2\} \quad \text{average deviation from the true value}$$

For every estimator: **Bias:** $B = E\{\hat{\theta}\} - \theta$ **Variance:** $\text{var} = E\{(\hat{\theta} - E\{\hat{\theta}\})^2\}$

Therefore:

$$\begin{aligned}
\text{MSE} &= E\{(\hat{\theta} - \theta)^2\} = E\{[\hat{\theta} - E\{\hat{\theta}\} + \underbrace{E\{\hat{\theta}\} - \theta}_{\text{bias} \quad \text{B}(\hat{\theta})}]^2\} \\
&= E\{[\hat{\theta} - E\{\hat{\theta}\}]^2\} + E\{B^2(\hat{\theta})\} + 2E\{[\hat{\theta} - E\{\hat{\theta}\}]B(\hat{\theta})\} \\
&\quad \text{due to the linearity of the } E\{\cdot\} \text{ operator and that } E\{B(\hat{\theta})\} = B(\hat{\theta}) \\
&= E\{[\hat{\theta} - E\{\hat{\theta}\}]^2\} + B^2(\hat{\theta}) + \underbrace{2E\{[\hat{\theta} - E\{\hat{\theta}\}]\}}_{=0, \text{ the } E\{\hat{\theta}\} \text{ are equal}} B(\hat{\theta}) \\
&= \text{var}(\hat{\theta}) + B^2(\hat{\theta})
\end{aligned}$$

Performance of the periodogram

We desire a minimum variance unbiased (MVU) est.

Its performance is analysed in the same was as the performance of any other estimator:

- **Bias**, whether it is unbiased, that is,

$$\lim_{N\to\infty} E\left\{\hat{P}_{per}(f)\right\} = P_x(f)$$

- **Variance** whether it is consistent, that is,

$$\lim_{N\to\infty} Var\left\{\hat{P}_{per}(f)\right\} = 0$$

- **Mean square convergence**

$$\mathrm{MSE} = \mathrm{bias}^2 + \mathrm{variance} = E\left\{\left[\hat{P}_{per}(f) - P_x(f)\right]^2\right\}$$

$$\text{we desire} \quad \lim_{N\to\infty} E\left\{\left[\hat{P}_{per}(f) - P_x(f)\right]^2\right\} = 0$$

☞ we need to check $\hat{P}_{per}(f)$ is a **consistent** estimator of the true PSD.

Bias of the periodogram as an estimator

We can calculate this by finding the expected value of $\hat{\mathbf{r}}_{xx}(k) = \frac{1}{N}\sum_{n=0}^{N-1-|k|} x(n)x(n+|k|)$. Thus (**biased estimate**)

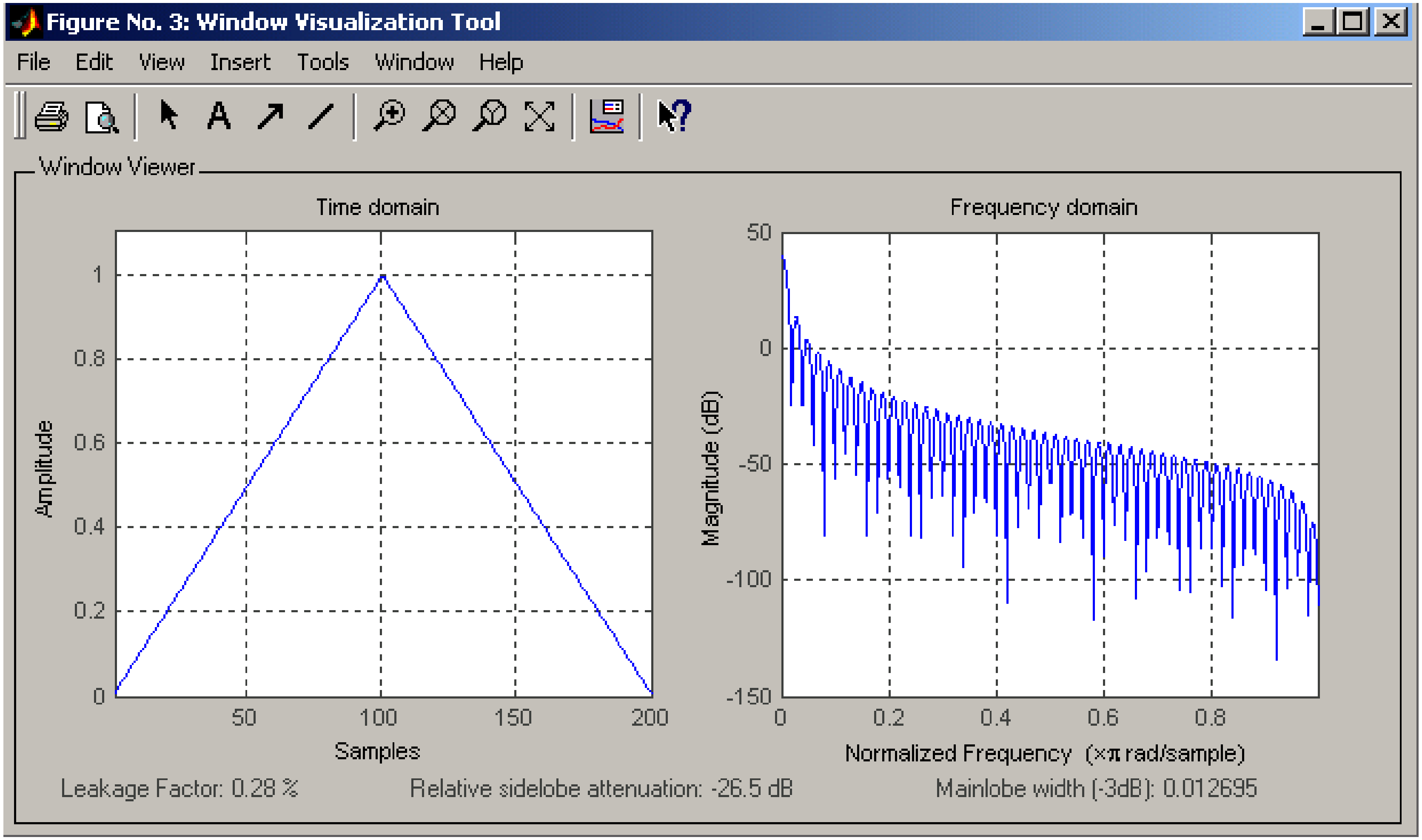
$$\begin{aligned} E\left\{P_{per}(f)\right\} &= \sum_{k=-(N-1)}^{N-1} E\{\hat{\mathbf{r}}_{xx}(k)\}e^{-\jmath 2\pi fk} \\ &= \sum_{k=-(N-1)}^{N-1} \frac{N-|k|}{N}\,\mathbf{r}_{xx}(k)e^{-\jmath 2\pi fk} = \text{``}\mathbf{w}_B(k) \;\times\; \mathbf{r}_{xx}(k)'' \end{aligned}$$

where $\mathbf{r}_{xx}$ is the **true ACF** and the Bartlett (triangular) window is defined by

$$\mathbf{w}_B(k) = \begin{cases} 1 - \frac{|k|}{N}; & |k| \leq N \\ 0; & |k| > N-1 \end{cases}$$

Notice the maximum at n=0, and a slow decay towards the end of the sequence

Effects of the Bartlett window on resolution



Behaves as $sinc^2$

Periodogram bias – continued

From the previous observation, we have

$$E\left\{\hat{P}_{per}(f)\right\} \;=\; \sum_{k=-\infty}^{\infty} \mathbf{r}_{xx}(k)\mathbf{w}_B(k)e^{-\jmath 2\pi k f} \Leftrightarrow W_B(f) \;*\; P_{xx}(f)$$

where

$$W_B(f) = \frac{1}{N}\left[\frac{\sin \pi f N}{\sin \pi f}\right]^2.$$

In words, the expected value of the periodogram is the **convolution** of the power spectrum $P_{xx}(f)$ with the Fourier transform of the Bartlett window, and therefore, the periodogram is a **biased** estimate.

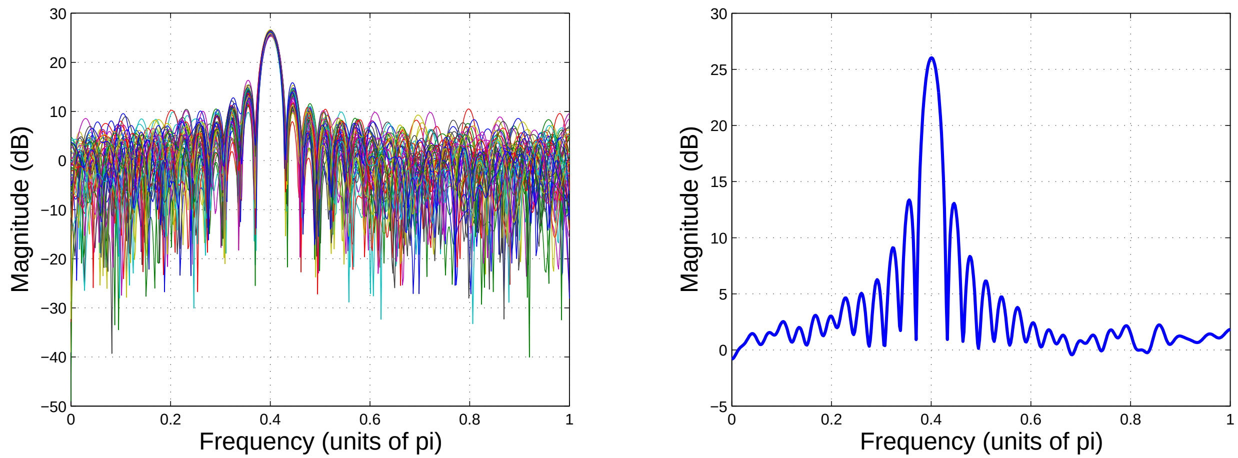
Since when $N \to \infty$, $W_B(f) \to \delta(0)$, the periodogram is **asymptotically unbiased**

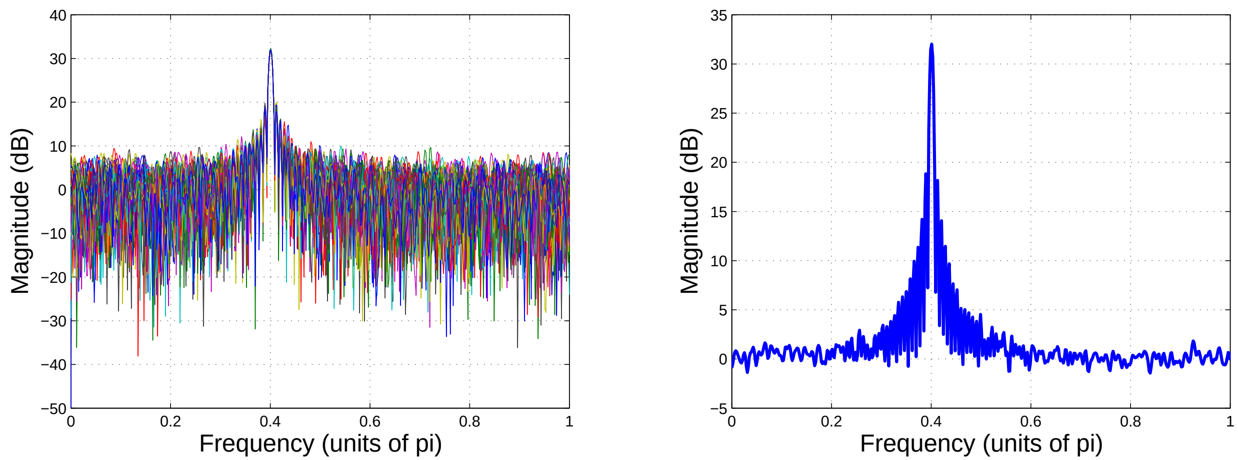
$$\lim_{N\to\infty} E\left\{\hat{P}_{per}(f)\right\} = P_{xx}(f)$$

Example: Sinusoid in WGN

$x(n) = A\sin(n\omega_0 + \Phi) + w(n)$, $A = 5, \omega_0 = 0.4\pi$

N=64: Overlay of 50 periodograms   periodogram average





N=256: Overlay of 50 periodograms   periodogram average

Periodogram resolution: Two sinusoids in white noise

This is a random process ($\Phi_1 \perp \Phi_2$, $w(n) \sim \mathcal{U}(0, \sigma_w^2)$) described by :

$$x(n) = A_1 \sin(n\omega_1 + \Phi_1) + A_2 \sin(n\omega_2 + \Phi_2) + w(n)$$

The true PSD is

$$P_{xx}(\omega) = \sigma_w^2 + \frac{1}{2}\pi A_1^2 \left[\delta(\omega - \omega_1) + \delta(\omega + \omega_1)\right] + \frac{1}{2}\pi A_2^2 \left[\delta(\omega - \omega_2) + \delta(\omega + \omega_2)\right]$$

The expected PSD $E\left\{\hat{P}_{per}(\omega)\right\}$ $(P_x * W_B)$ becomes

$$\sigma_w^2 + \frac{1}{4}A_1^2 \left[W_B(\omega - \omega_1) + W_B(\omega + \omega_1)\right] + \frac{1}{4}A_2^2 \left[W_B(\omega - \omega) + W_B(\omega + \omega_2)\right]$$

☞ **there is a limit on how closely two sinusoids or two narrowband processes may be located before they can no longer be resolved.**

Example: Estimation of two sinusoids in WGN

Based on previous example, try to generate these yourselves

$$x(n) = A_1 \sin(n\omega_1 + \Phi_1) + A_2 \sin(n\omega_2 + \Phi_2) + w(n)$$

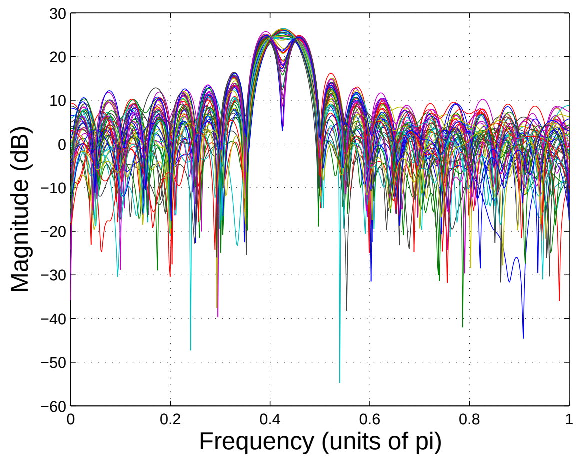
where

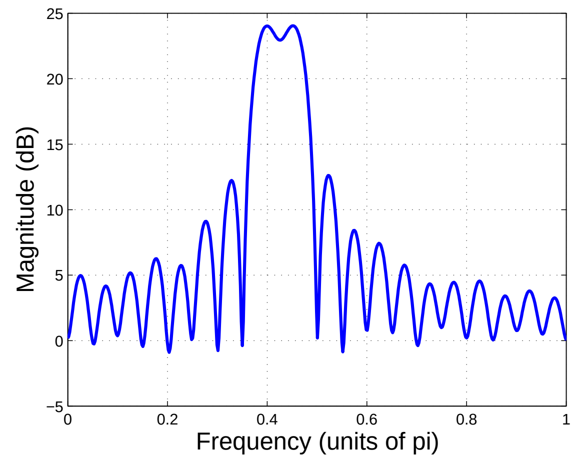
- datalength $N = 40$, $N = 64$, $N = 256$
- $A_1 = A_2$, $\omega_1 = 0.4\pi$, $\omega_2 = 0.45\pi$
- $A_1 \neq A_2$, $\omega_1 = 0.4\pi$, $\omega_2 = 0.45\pi$
- produce overlay plots of 50 periodograms and also averaged periodograms

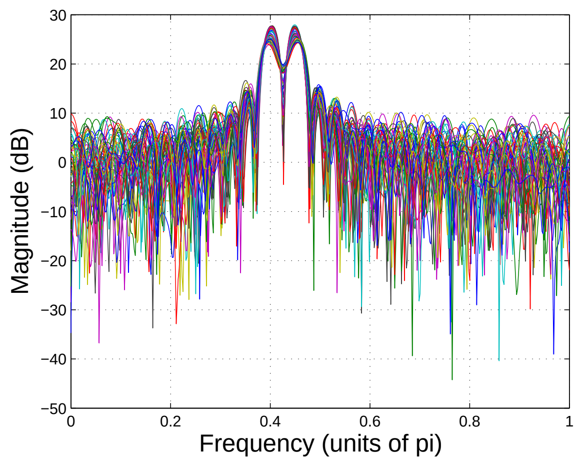
Example: Periodogram resolution ↬ two sinusoids

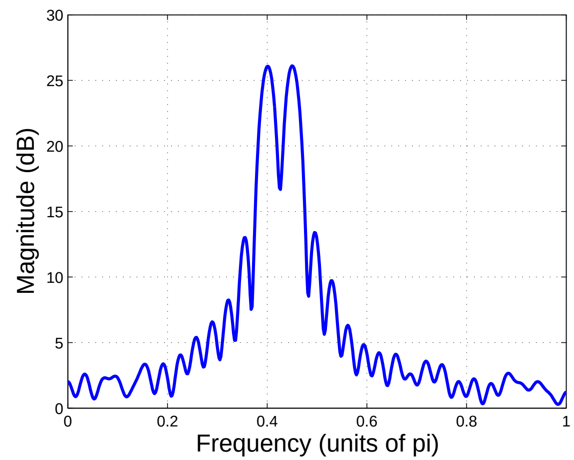
see also Problem 4.6 in your Problem/Answer set

N=40: Overlay of 50 periodograms   periodogram average









N=64: Overlay of 50 periodograms   periodogram average

Effects of the Window Choice

Recall: The spectrum of the (rectangular) window is a ***sinc*** which has a main lobe and sidelobes

All the other window functions (addressed later) also have the mainlobe and sidelobes.

- The effect of the main lobe (its width) is to **smear** or **smooth** the estimated spectrum shape
- From the previous slide: the width of the mainlobe causes the next peak in the spectrum to be masked if the two peaks are not separated by $1/N$ - the spectral resolution
- The sidelobes cause **spectral leakage** $\looparrowright$ transferring power from the correct frequency bin into the frequency bins which contain no signal power

These effects are dangerous, e.g. when estimating peaky spectra

Some observations

- The Bartlett window **biases** the periodogram;
- It also introduces **smoothing**, which **limits** the ability of the periodogram to resolve closely–spaced narrowband components in $x(n)$;
- This is due to the width of the main lobe of $W_B(f)$;
- Periodogram **averaging would reduce the variance** (remember MVU estimators!)
- **Resolution of the periodogram**
  - set $\Delta\omega =$ width of the main lobe of spectral window, at its "half power"
  - for Bartlett window $\Delta\omega \sim 0.89(2\pi/N) =$ periodogram resolution!
  - notice that the resolution is inversely proportional to the amount of data N

Variance of the periodogram

- It is difficult to evaluate the variance of the periodogram of an arbitrary process $x(n)$ since the variance depends on the fourth–order moments of the process.
- However, the variance may be evaluated in the special case of WGN $\longrightarrow$

$$E\left\{\hat{P}_{per}(f_1)\hat{P}_{per}(f_2)\right\} = \left(\frac{1}{N}\right)^2 \sum_k \sum_l \sum_m \sum_n E\left\{x(k)x(l)x(m)x(n)\right\} \quad \times$$

$$\times \quad e^{-\jmath 2\pi[f_1(k-l)+f_2(m-n)]}$$

For WGN, these fourth–order moments become

$$E\left\{x(k)x(l)x(m)x(n)\right\} \quad =$$

$$E\{x(k)x(l)\}E\{x(m)x(n)\} + E\{x(k)x(m)\}E\{x(l)x(n)\} + E\{x(k)x(n)\}E\{x(l)x(m)\}$$

$$= \sigma_x^4\left[\delta(k-l)\delta(m-n) + \delta(k-m)\delta(l-n) + \delta(k-n)\delta(l-m)\right]$$

This is = σ_x^4 if k=l, m=n, or k=m, l=n, or k=n, l=m, or otherwise 0

Variance of the periodogram – contd.

After some simplifications, and recognising

$$\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}\sigma_x^4 = \sigma_x^4$$

we have the variance of the periodogram for a given frequency:

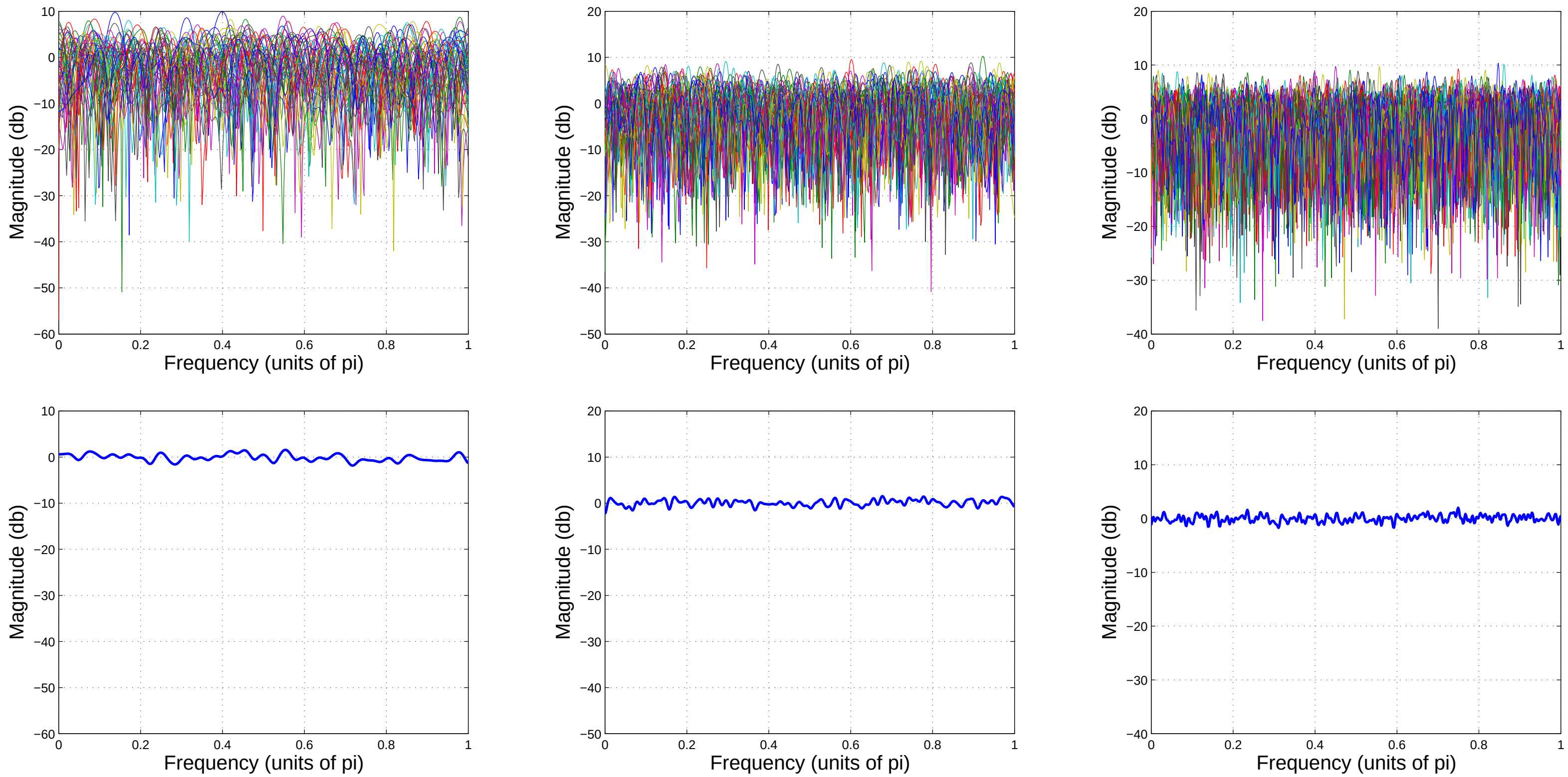
$$var\left\{\hat{P}_{per}(f)\right\} = P_{xx}^2(f)\left[1 + \left(\frac{\sin 2\pi N f}{N \sin 2\pi f}\right)^2\right]$$

For the periodogram to be consistent, $var(P_{per}) \rightarrow 0 \; as \; N \rightarrow \infty$.

From the above, this is **not** the case $\Rightarrow$ the **periodogram estimator is inconsistent.** In fact, $var(P_{per}(f)) = P_x^2(f) \quad \looparrowright$ quite large

Example: Periodogram of white noise

N=64 **N = 128** **N=256**



$$P_{xx} = 1, \quad E\{\hat{P}_{per}(e^{\jmath\omega})\} = 1, \quad var\left[\hat{P}_{per}(e^{\jmath\omega})\right] = 1$$

Although the periodogram is unbiased, the variance is equal to a constant, that is, independent of the data length N

Bias vs variance – recap

- **Bias** pertains to the question: "**Does the estimate approach the correct value as $N \to \infty$**".

  - If yes then the estimator is unbiased, else it is biased
  - Notice that the main lobe of the window has a width of $2\pi/N$ and hence when $N \to \infty$ we have $\lim_{N\to\infty} \hat{P}_{per}(f) = P_{xx}(f) \Rightarrow$ periodogram is an **asymptotically unbiased** estimator of true PSD.
  - **For the window to yield an unbiased estimator:**

$$\sum_{n=0}^{N-1} w^2(n) = N \quad \& \quad \text{the mainlobe width} \sim \quad \frac{1}{N}$$

- **Variance** refers to the "goodness" of the estimate, that is, whether the power of the estimation error tend to zero when $N \to \infty$.

  - We have shown that even for a very large window the variance of the estimate is as large as the true PSD
  - This means that the periodogram **is not a consistent** estimator of true PSD.

Properties of the standard periodogram

Functional relationship:

$$\hat{P}_{per}(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-\jmath n\omega}\right|^2$$

- **Bias**

$$E\left\{\hat{P}_{per}(\omega)\right\} = \frac{1}{2\pi}P_x(\omega) \ * \ W_B(\omega)$$

- **Resolution**

$$\Delta\omega = 0.89\frac{2\pi}{N}$$

- **Variance**

$$Var\left\{\hat{P}_{per}(\omega)\right\} \approx P_x^2(\omega)$$

Part 3: Periodogram Modifications

Periodogram modifications $\looparrowright$ some intuition

Clearly, we need to reduce the variance of the periodogram, since in general they are not adequate for precise estimation of PSD.

We can think of several modifications:

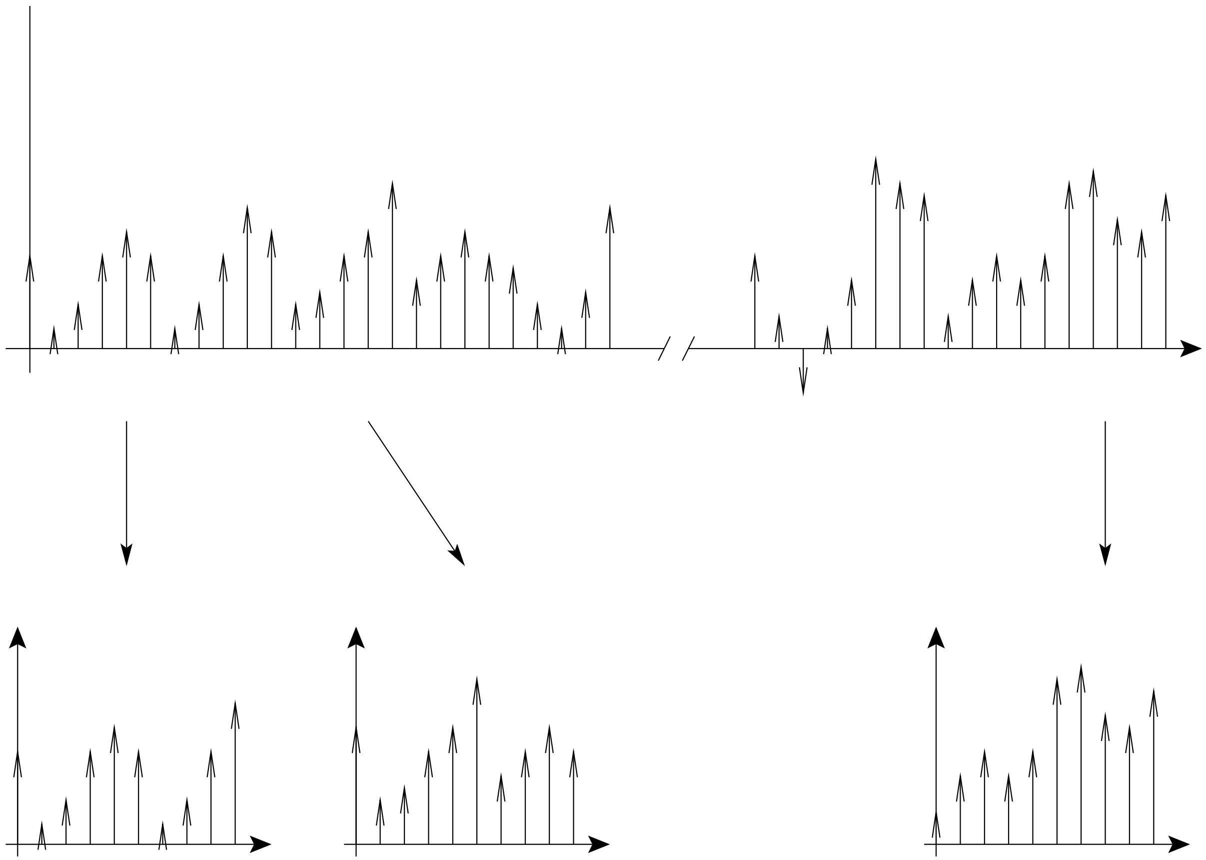
1) **averaging over a set of periodograms** (we have already seen the effect of this in some simulations).

   Recall that from the general estimation theory, by averaging M times we have the effect of $var \rightarrow var/M$.

2) **applying different windows** $\looparrowright$ it is possible to choose or design a window which will have a narrow mainlobe

3) **overlapping windowed segments** for additional variance reduction $\looparrowright$ averaging periodograms along one realisation of a random process (instead of across the ensemble)

Partitioning the data set (K segments of length L each)



Partitioning $x(n)$ into K non–overlapping segments

This way, the total length $N = K \times L$

Bartlett's method: Averaging periodograms

The **averaged** periodogram can be expressed as:

$$\hat{P}_{aver,per}(f) = \frac{1}{K}\sum_{m=1}^{K}\hat{P}_{per}^{(m)}(f)$$

where for each of the K segments, the segment-wise PSD estimate $P_{per}^{(i)},\ i = 1, \ldots, K$ is given by

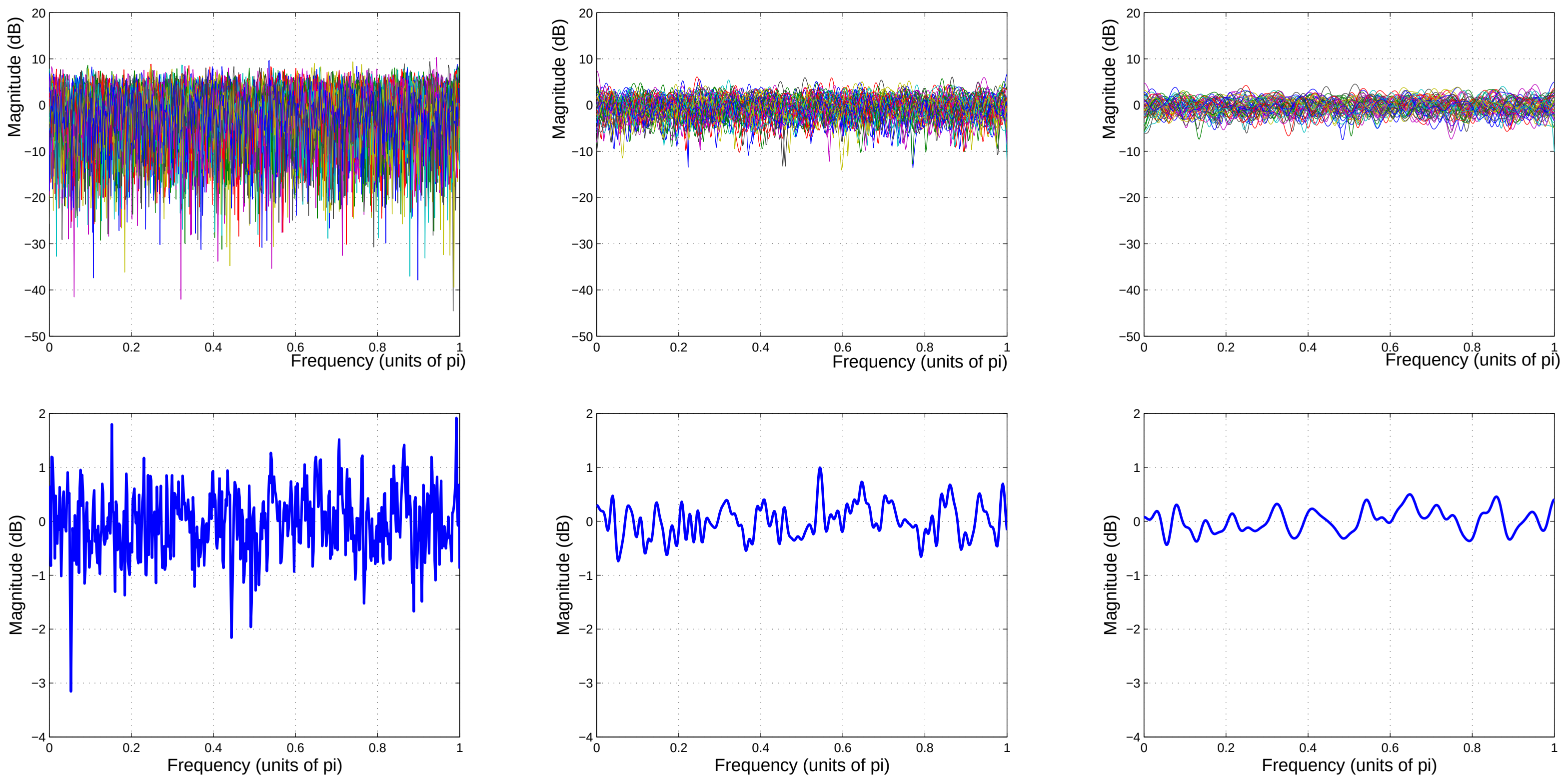
$$P_{per}^{(i)}(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x_i[n]e^{-\jmath n\omega}\right|^2$$

- Idea: to reduce the variance by the factor of "K" $=$ total number of blocks
- Therefore: provided that the blocks are statistically independent (not often the case in practice) we desire to have

$$var\left\{\hat{P}_{aver,per}(f)\right\} = \frac{1}{K} var\left\{\hat{P}_{per}(f)\right\}$$

Example: Estimation of WGN spectrum using Bartlett's method

50 periodograms with $N = 512$ | **50 Bartlett estimates $K = 4,\ L = 128$** | **50 Bartlett estimates $K = 8,\ L = 64$**



Ensemble averages

Performance evaluation of Bartlett's method

- **Bias:** The expected value of Bartlett's estimate

$$E\left\{\hat{P}_B(\omega)\right\} = \frac{1}{2\pi} P_x(\omega) \; * \; W_B(\omega)$$

$\Rightarrow$ **asymptotically unbiased**.

- **Resolution:** Due to K segments of length L, as a consequence we have that Res(P_B) $<$ Res(P_{per}), that is

$$Res\left[\hat{P}_B(\omega)\right] = 0.89 \, \frac{2\pi}{L} = 0.89 \; K \, \frac{2\pi}{N}$$

- **Variance:**

$$Var\left\{\hat{P}_B(\omega)\right\} \approx \frac{1}{K} \, Var\left\{\hat{P}^{(i)}_{per}(\omega)\right\} \approx \frac{1}{K} \, P_x^2(\omega)$$

For non–white data, variance reduction is not as large as K times!

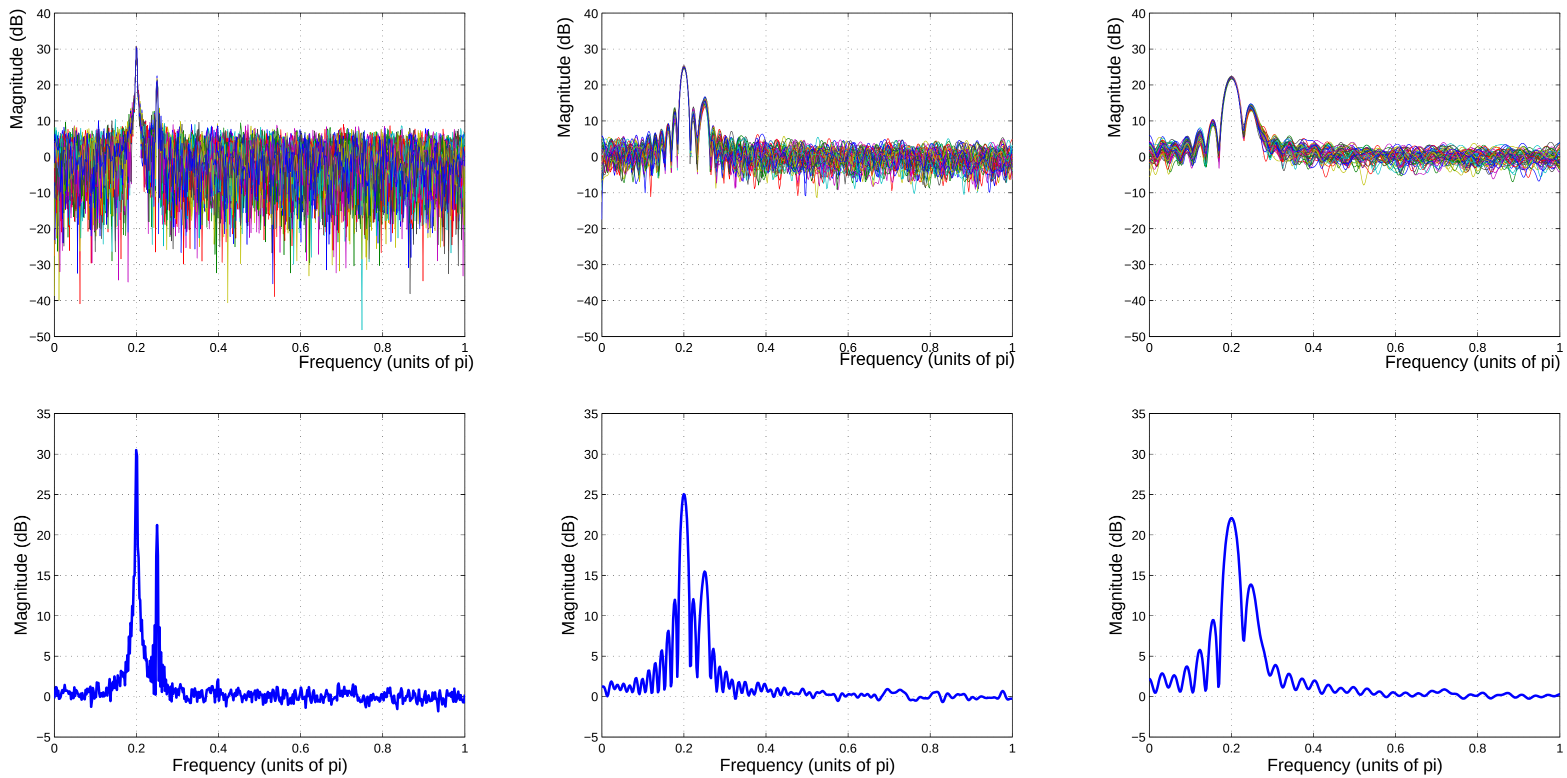
By changing the values of L and K, Bartlett's method allows us to:

trade a reduction in spectral resolution for a reduction in variance

Example: Estimation of two sinewaves in white noise

$$x[n] = \sqrt{10} sin(n * 0.2\pi + \Phi_1) + sin(n * 0.25\pi + \Phi_2) + w[n]$$

50 periodograms with $N = 512$ | **50 Bartlett estimates $K = 4,\ L = 128$** | **50 Bartlett estimates $K = 8,\ L = 64$**



Ensemble averages

Notice the variance – resolution trade–off!

The Modified Periodogram

The periodogram of a process that is windowed with a suitable general window $w[n]$ is called a **modified periodogram** and is given by:

$$\hat{P}_M(\omega) = \frac{1}{NU} \left| \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-\jmath n\omega} \right|^2$$

where N is the window length and $U = \frac{1}{N}\sum_{n=0}^{N-1} |w[n]|^2$ is a constant, **and is defined so that $\hat{P}_M(\omega)$ is asymptotically unbiased.**

In Matlab:

```
xw=x(n1:n2).*w/norm(w);
Pm=N * periodogram(xw);
```

where, for different windows

```
w=hanning(N); w=bartlett(N);w=blackman(n);
```

The Modified Periodogram – "Windowing"

Recall that

$$Periodogram \quad \sim \quad \mathcal{F}\left(|x[n]w_r[n]|^2\right)$$

Therefore: The amount of smoothing in the periodogram is determined by the window that is applied to the data. For instance, a rectangular window has a narrow main lobe (and hence least amount of spectral smoothing), but its relatively large sidelobes may lead to masking of weak narrowband components.

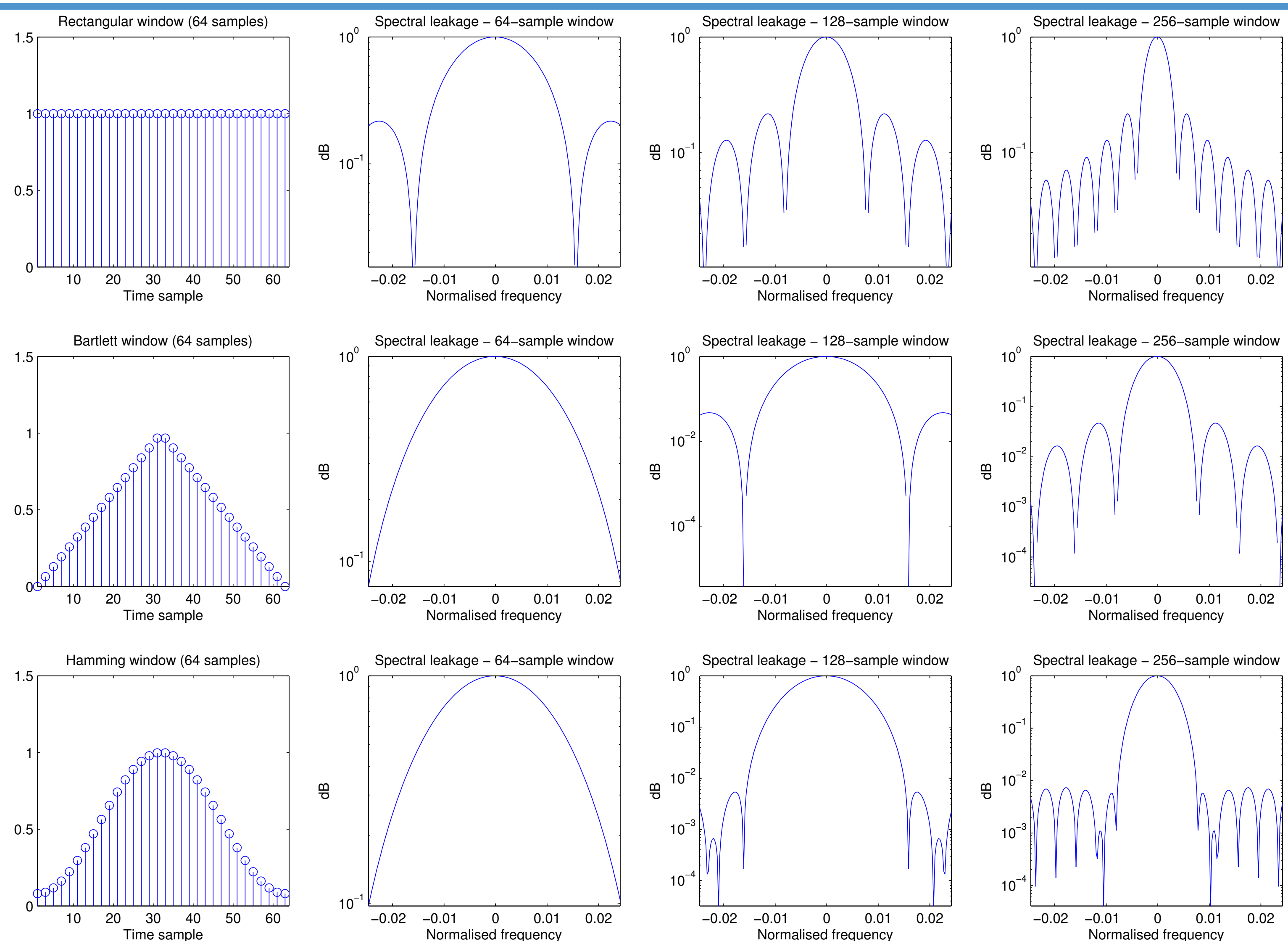
Question: Would there be any benefit of using a different data window on the bias and resolution of the periodogram.

Example: can we differentiate between the following two sinusoids for $\omega_1 = 0.2\pi, \omega_2 = 0.3\pi, N = 128$

$$x[n] = 0.1\sin(n\omega_1 + \Phi_1) + \sin(n\omega_2 + \Phi_2) + w[n]$$

Some common windows for different window lengths:

Time domain **Spectrum N=64** **Spectrum N=128** **Spectrum N=256**

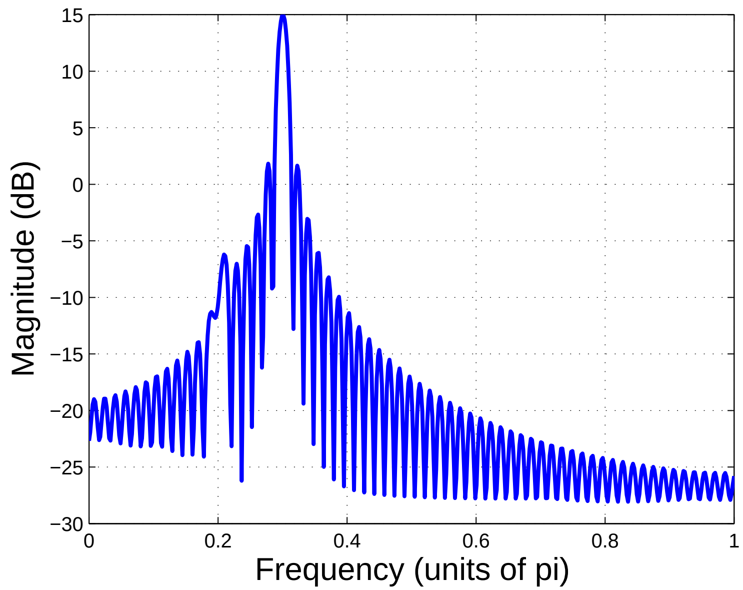


Example: Estimation of two sinusoids in WGN
Modified periodogram using Hamming window

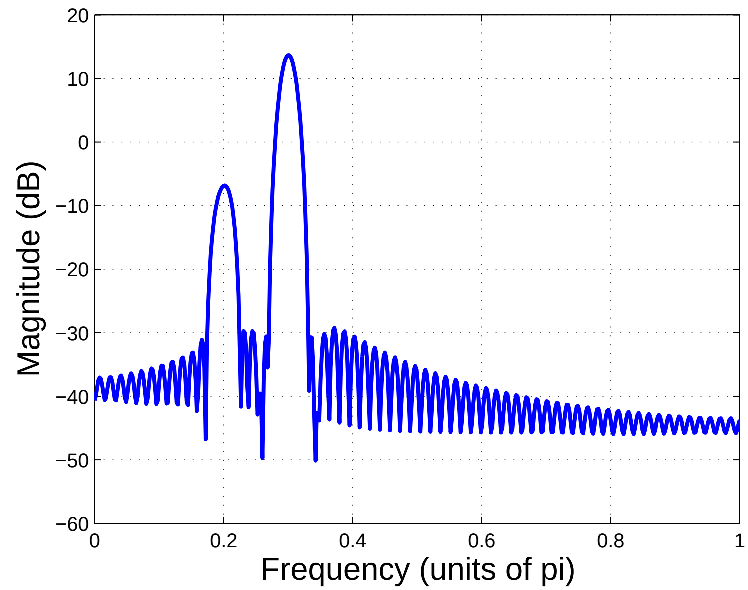
Problem: Estimate spectra of the following two sinusoids using: (a) The standard periodogram; (b) Hamming-windowed periodogram

$$x[n] = 0.1\sin(n * 0.2\pi + \Phi_1) + \sin(n * 0.3\pi + \Phi_2) + w[n] \qquad N = 128$$

Hamming window $\quad w[n] = 0.54 - 0.46\cos\left(2\pi\frac{n}{N}\right)$



Expected value of periodogram



Periodogram using Hamming window

Properties of an ideal window function

Consider a window sequence $w(n)$ whose DFT is a **squared magnitude of another sequence** $v(n)$, that is

$$V(\omega) = \sum_{k=0}^{M-1} v(k)e^{-\jmath\omega k} \quad \looparrowright \quad W(\omega) = |V(\omega)|^2 \quad \textbf{(positive definite)}$$

Then

$$\sum_{k=-(M-1)}^{M-1} w(k)e^{-\jmath\omega k} = \sum_{n=0}^{M-1}\sum_{p=0}^{M-1} v(n)v(p)e^{-\jmath(n-p)}$$

$$= \sum_{k=-(M-1)}^{M-1}\Big[\sum_{n=0}^{M-1} v(n)v(n-k)\Big]e^{-\jmath k}, \quad \text{for } v(k)=0,\ k\notin[0, M-1]$$

This gives

$$w(k) = \sum_{n=0}^{M-1} v(n)v(n-k) = v(k)*v(k) \quad \Leftrightarrow \quad W(\omega)\geq 0 \quad \textbf{pos. semidefinit.}$$

A window design should trade-off between smearing and leakage

For instance: weak sinewave + strong narrowband interference $\rightarrow$ leakage more detrimental than smearing

Homework: can we use optimisation to balance between smearing and leakage

Several frequently used "cosine–type windows"

Idea: suppress sidelobes, perhaps sacrifice the width of mainlobe

- **Hann** window

```
w = 0.5 * (1 - cos(2*pi*(0:m-1)'/(n-1)));
```

- **Hamming** window

```
w = (54 - 46*cos(2*pi*(0:m-1)'/(n-1)))/100;
```

- **Blackman** window

```
w = (42 - 50*cos(2*pi*(0:m-1)/(n-1)) +
                  + 8*cos(4*pi*(0:m-1)/(n-1)))'/100;
```

Performance of the modified periodogram

- **Bias:** Since

$$U = \frac{1}{N}\sum_{n=0}^{N-1}|w[n]|^2 = \frac{1}{N}\int_{-\pi}^{\pi}\left|W(e^{\jmath\omega})\right|^2 d\omega \quad \Rightarrow \quad \frac{1}{2\pi NU}\int_{-\pi}^{\pi}\left|W(e^{\jmath\omega})\right|^2 d\omega = 1$$

for $N \rightarrow \infty$ the modified periodogram is asymptotically unbiased.

- **Variance:** Since $\hat{P}_M$ is simply $\hat{P}_{per}$ of a windowed data sequence

$$Var\left\{\hat{P}_M(\omega)\right\} \approx P_{xx}^2(\omega)$$

$\Rightarrow$ **not a consistent estimate** of the power spectrum, and the data window offers no benefit in terms of reducing the variance

- **Resolution:** Data window provides a trade–off between spectral resolution (main lobe width) and spectral masking (sidelobe amplitude).

Periodogram modifications: Effects of different windows

Properties of several commonly used windows with length N:

- **Rectangular** – Sidelobe level = -13 [dB], $3\ dB$ BW $\rightarrow 0.89(2\pi/N)$
- **Bartlett** – Sidelobe level = -27 [dB], $3\ dB$ BW $\rightarrow 1.28(2\pi/N)$
- **Hanning** – Sidelobe level = -32 [dB], $3\ dB$ BW $\rightarrow 1.44(2\pi/N)$
- **Hamming** – Sidelobe level = -43 [dB], $3\ dB$ BW $\rightarrow 1.30(2\pi/N)$
- **Blackman** – Sidelobe level = -58 [dB], $3\ dB$ BW $\rightarrow 1.68(2\pi/N)$

Notice the relationship between the sidelobe level and bandwidth!

Welch's method: Averaging modified periodograms

In 1967, Welch proposed two modifications to Bartlett's method:

- allow the sequences $x_i[n]$ to overlap
- to allow data window $w[n]$ to be applied to each sequence $\Rightarrow$ averaging modified periodograms

This way, successive segments are offset by D points and each segment is L points long

$$x_i[n] = x[n + iD] \qquad n = 0, 1, \ldots, L-1$$

The amount of overlap between $x_i[n]$ and $x_{i+1}[n]$ is $L - D$ points and

$$N = L + D(K-1)$$

N- total number of points, L- length of segments, D- amount of overlap, K- number of sequences

Variations on the theme

We may vary between **no overlap D=L** and say 50 % overlap **D = L/2** or anything else.

☺ we can trade a reduction in the variance for a reduction in the resolution, since

$$\hat{P}_W(\omega) = \frac{1}{KLU} \sum_{i=0}^{K-1} \left| \sum_{n=0}^{L-1} w[n]x[n+iD]e^{-\jmath n\omega} \right|^2$$

or in terms of modified periodograms

$$\hat{P}_W(\omega) = \frac{1}{K} \sum_{i=0}^{K-1} \hat{P}_M^{(i)}(\omega)$$

$\rightsquigarrow$ **asymptotically unbiased** (follows from the bias of the modified periodogram)

Welch vs. Bartlett

- the amount of overlap between $x_i[n]$ and $x_{i+1}[n]$ is $L-D$ points, and if K sequences cover the entire N data points, then

$$N = L + D(K+1)$$

- If there is no overlap, $(D = L)$ we have $K = \frac{N}{L}$ sections of length L as in Bartlett's method
- Of the sequences are overlapping by 50 % $D = \frac{L}{2}$ then we may form $K = 2\frac{N}{L} - 1$ sections of length L. thus maintaining the same resolution as Bartlett's method while doubling the number of modified periodograms that are averaged, thereby reducing the variance.
- With 50% overlap we could also form $K = \frac{N}{L} - 1$ sequences of length $2L$, thus increasing the resolution while maintaining the same variance as Bartlett's method.

Therefore, by allowing sequences to overlap, it is possible to increase the number and/or length of the sequences that are averaged, thereby trading a reduction in variance for a reduction in resolution.

Properties of Welch's method

- **Functional relationship:**

$$\hat{P}_W(\omega) = \frac{1}{KLU}\sum_{i=0}^{K-1}\left|\sum_{n=0}^{L-1} w[n]x[n+iD]e^{-\jmath n\omega}\right|^2 \qquad U = \frac{1}{L}\sum_{n=0}^{L-1}|w[n]|^2$$

- **Bias**

$$E\left\{\hat{P}_W(\omega)\right\} = \frac{1}{2\pi LU}P_x(\omega) \; * \; |W(\omega)|^2$$

- **Resolution** $\looparrowright$ window dependent

- **Variance** (assuming 50 % overlap and Bartlett window)

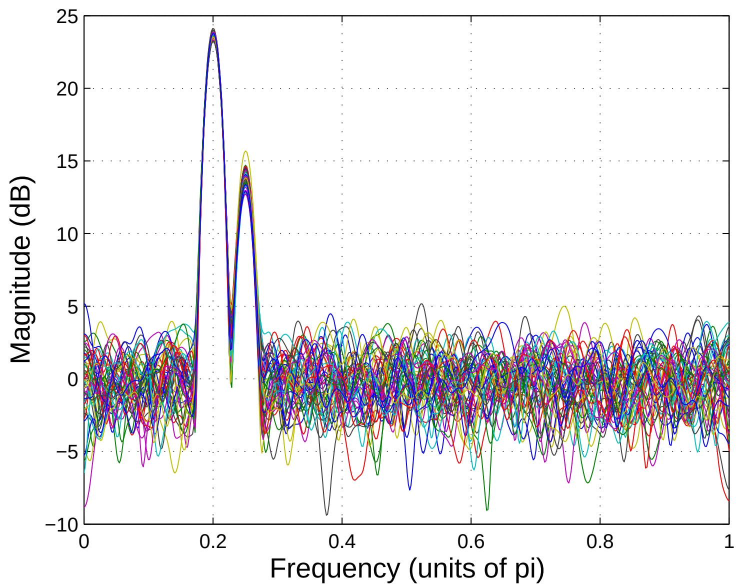
$$Var\left\{\hat{P}_W(\omega)\right\} \approx \frac{9}{16}\frac{L}{N}P_x^2(\omega)$$

Example: Two sinusoids in noise $\looparrowright$ Welch estimates

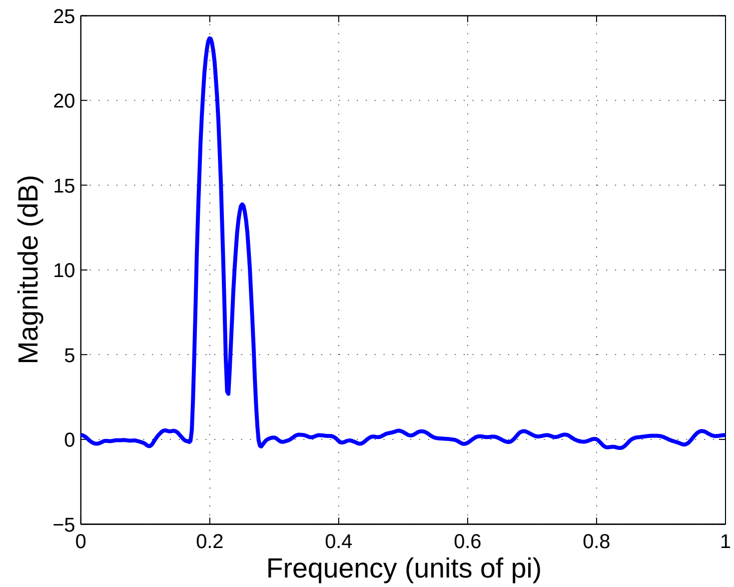
Problem: Estimate the spectra of the following two sinewaves using Welch's method

$$x[n] = \sqrt{10}\sin(n * 0.2\pi + \Phi_1) + \sin(n * 0.3\pi + \Phi_2) + w[n]$$

Unit noise variance, $N = 512$, $L = 128$, 50 % overlap (7 sections)



Overlay of 50 estimates

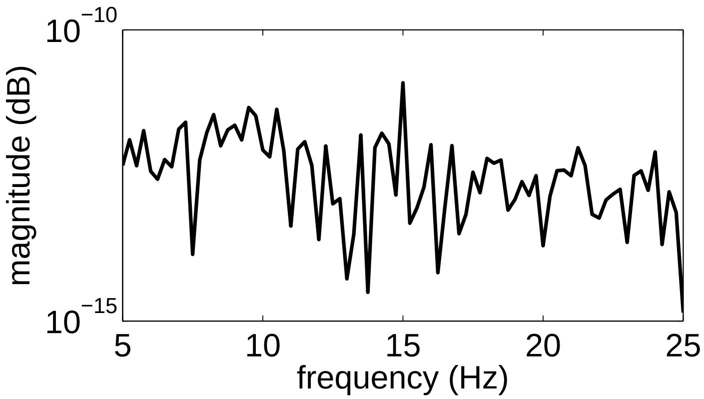


Periodogram using Welch's method

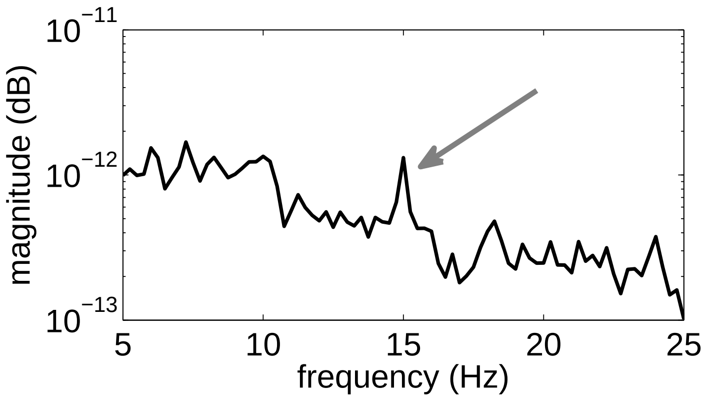
EEG feature estimation ssvep movie.wmv

Subject presented with flashing visual stimulus (15 Hz) which causes a response in EEG at same frequency

- Applying the periodogram to data gives a noisy estimate of PSD
  - **stimulus response not clearly visible**
- The averaged periodogram reduces the level of noise
  - **stimulus response at 15 Hz !**
  - The total signal length was $N = 48000$, the averaged periodogram used a window length of $L = 3600$, and (with overlap) the total number of averaged windows was $K = 102$



Periodogram.



Averaged Periodogram.

More on EEG spectral estimation: EEG propagation

only for illustration, not examinable

EEG recording principle

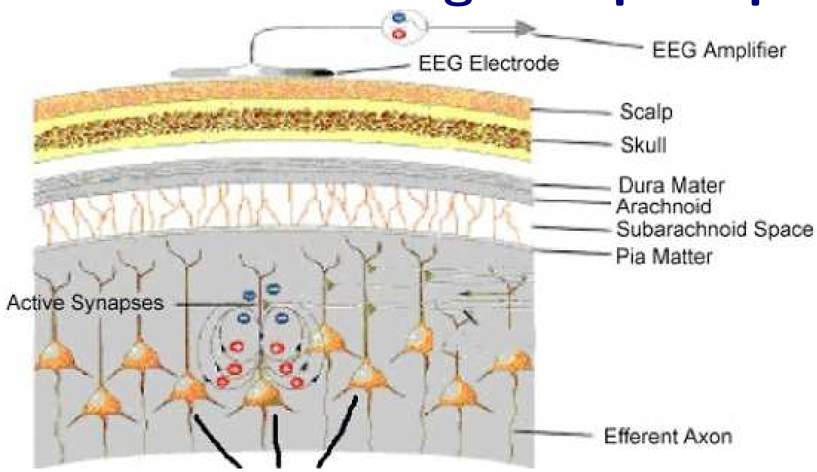


Figure 1: [**Left**] The left Ear-EEG earplug with electrode positions visible (grey dots) and an arrow indicating the direction in which it enters the ear canal. [**Right**] Recording setup: joint recording of Ear-EEG and on-scalp EEG for comparative analysis.

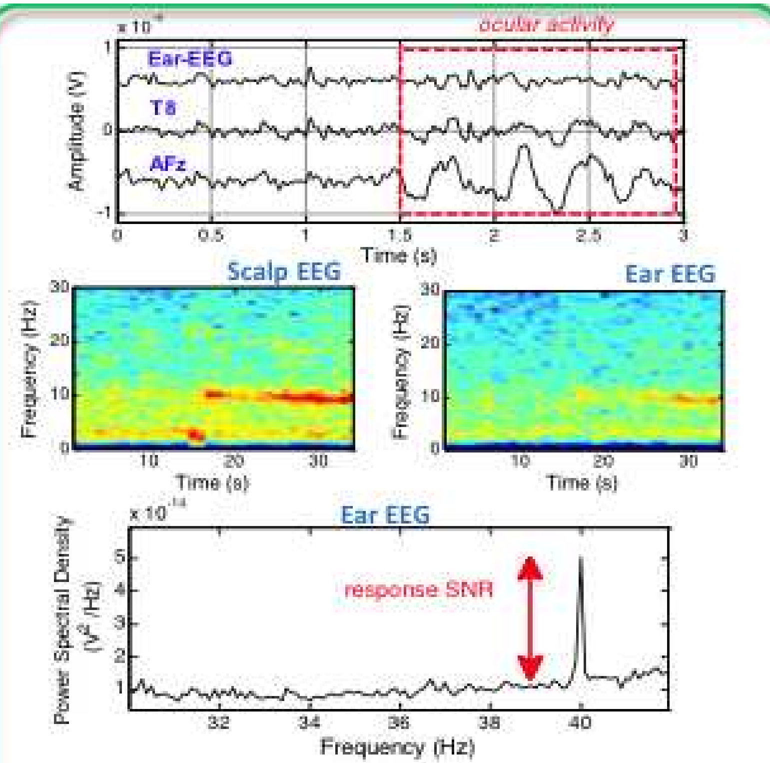


Figure 2: [**Upper**] Time waveforms for scalp and Ear-EEG over 3s with consecutive eye blinking starting at 1.5s, Ear-EEG exhibits a suppression of ocular artifacts. [**Centre**] Time-frequency plots as subject closes eyes from 15-35s, with increased activity visible for scalp [**Centre, left**] and Ear-EEG [**Centre, right**] in the alpha range (8-12 Hz). [**Lower**] The auditory steady state response for Ear EEG (40 Hz stimulus). The SNR (ratio of the response peak to background EEG) matches that of temporal scalp electrodes [4].

Principle of in-the-ear EEG

Ear-EEG vs on-scalp EEG

More on EEG spectral estimation: Some applications

only for illustration, not examinable

EEG brain computer interface

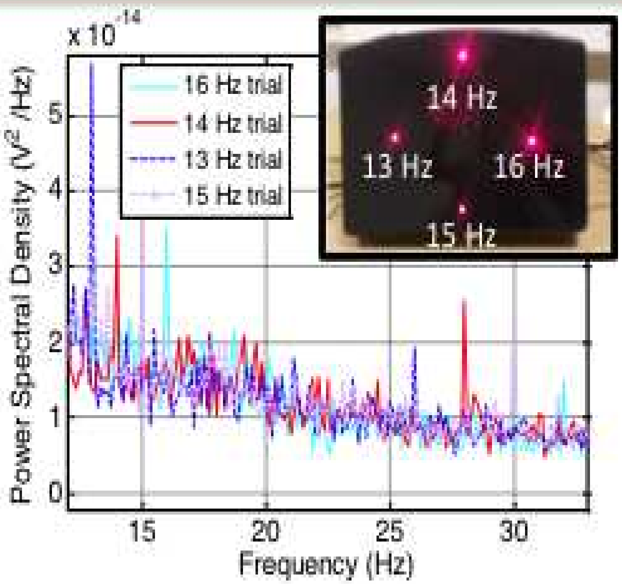
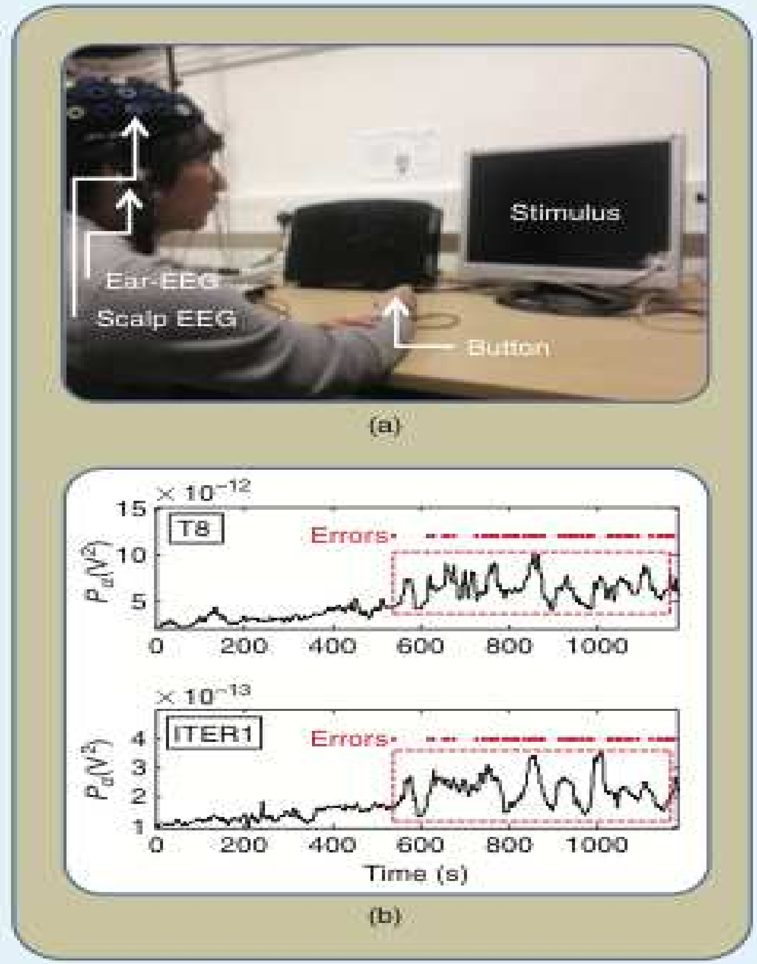


Figure 3: The average Ear-EEG power spectral density (PSD) for four 120s trials as the user attended 13, 14, 15 and 16 Hz stimuli. [**Insert: upper right**] The SSVEP stimulus interface with four LEDs.

The subject focuses on flashing diodes of different frequencies (13, 14, 15, 16 Hz), and this is reflected in EEG. Our task is to perform good spectrum estimation.

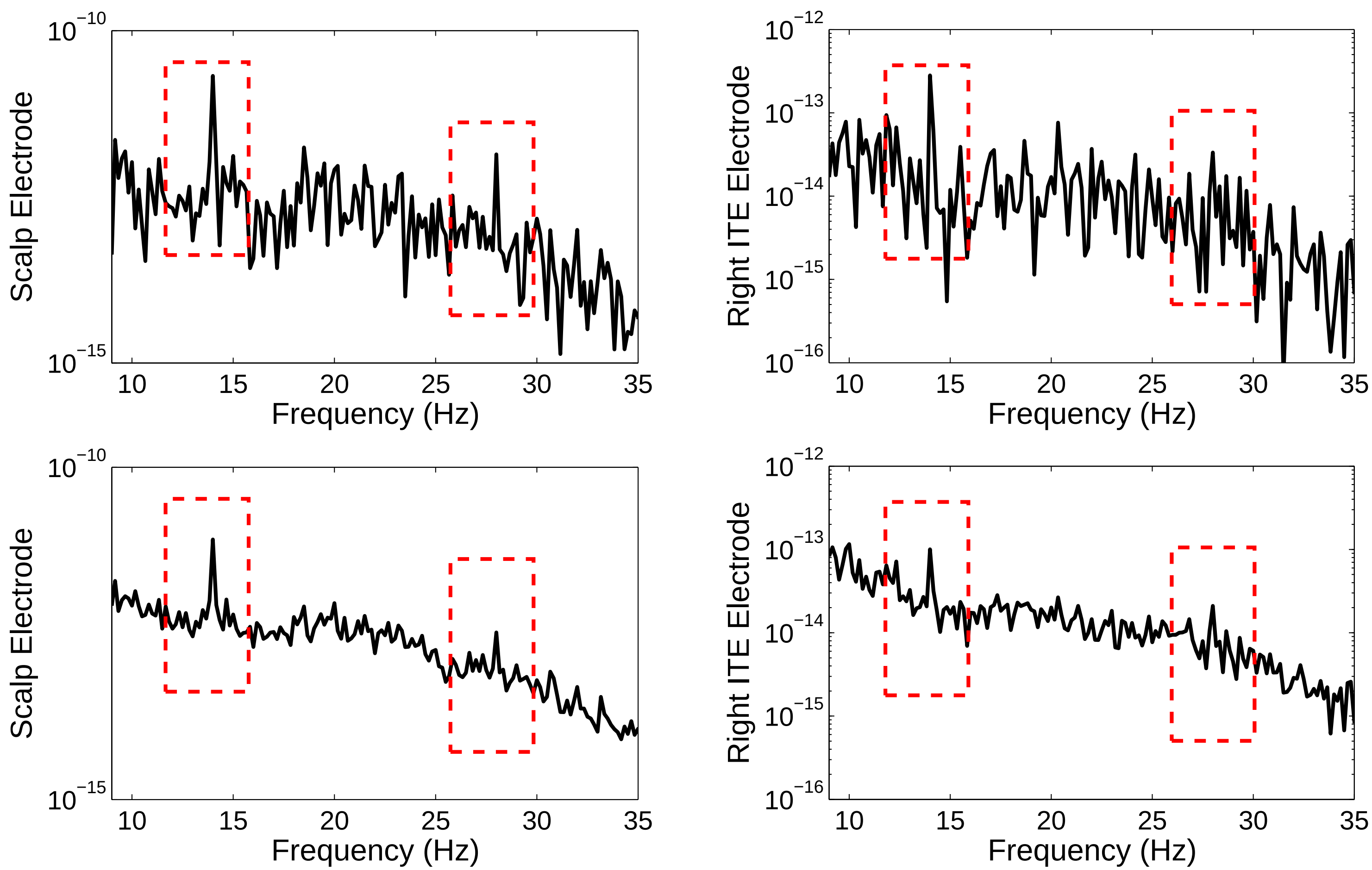


An interactive vigilance test

SSVEP - we look for a 14 Hz stimulus in a 50s recording using Welch's method

Standard: A 50s EEG from scalp (Oz) and right ear (ITE). Averaged: 27 segments of 12s.

Top: no window **Bottom: Hann window**



Blackman–Tukey method: Periodogram smoothing

Recall that the methods by Bartlett and Welch are designed to reduce the variance of the periodogram by averaging periodograms and modified periodograms, respectively.

Another possibility is "periodogram smoothing" often called the Blackman–Tukey method.

Let us identify the problem ☹

$$\hat{r}_x[N-1] = \frac{1}{N} x[N-1]x[0]$$

$\Rightarrow$ there is little averaging when calculating the estimates of $\hat{r}_x[k]$ for $|k| \approx N$.

These estimates will be **unreliable** no matter how large N. We have two choices:

- reduce the variance of those unreliable estimates
- reduce the contribution these unreliable estimates make to the periodogram

Blackman–Tukey Method: Resolution vs. Variance

The variance of the periodogram is decreased by reducing the variance of the ACF estimate by calculating more robust ACF estimates over fewer data points ($M < N$).

$\Rightarrow$ Apply a window to $\hat{r}_x[k]$ to decrease the contribution of unreliable estimates and obtain the Blackman–Tukey estimate:

$$\hat{P}_{BT}(\omega) = \sum_{k=-M}^{M} \hat{r}_x[k] w[k] e^{-\jmath k\omega}$$

where $w[k]$ is a **lag window** applied to the ACF estimate.

$$\hat{P}_{BT}(\omega) = \frac{1}{2\pi} \hat{P}_{per}(\omega) \ * \ W(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{P}_{per}(e^{\jmath u}) W(e^{\jmath(\omega - u)}) du$$

that is, **we trade the reduction in the variance for a reduction in the resolution** (smaller number of ACF estimates used to calculate the PSD)

Properties of the Blackman–Tukey method

- **Functional relationship:**

$$\hat{P}_{BT}(\omega) = \sum_{k=-M}^{M} \hat{r}_x[k]w[k]e^{-\jmath k\omega}$$

- **Bias**

$$E\left\{\hat{P}_{BT}(\omega)\right\} \approx \frac{1}{2\pi}P_x(\omega) \;*\; W(\omega)$$

- **Resolution**– window dependent (window – conjugate symmetric and with non–negative FT)
- **Variance:** Generally, it is recommended $M < N/5$.

$$Var\left\{\hat{P}_{BT}(\omega)\right\} \approx P_x^2(\omega)\frac{1}{N}\sum_{k=-M}^{M} w^2[k]$$

Trade–off: for a small bias M needs to be large to minimize the width of the mainlobe of $W(\omega)$, whereas M should be small in order to minimize the variance.

Non-negative definiteness of the BT spectrum estimator

see also Problem 4.9 in your Problem/Answer set

The main problem with periodogram is its high statistical variability. This arises from:

- Poor accuracy of the autocorrelation estimate for large lags m
- Accumulating of these errors in the spectrum estimate

These effects can be mitigated by taking fewer points (M instead of N) in ACF estimation.

Observe that the Blackman–Tukey spectral estimator corresponds to a locally weighted average of the periodogram.

Roughly speaking:

- the resolution of the BT estimator is $\sim 1/M$
- the variance of the BT estimator is $\sim M/N$

Performance comparison of periodogram–based methods

Let us introduce criteria for performance comparison:

- **Variability of the estimate**

$$\nu = \frac{var\left\{\hat{P}_x(\omega)\right\}}{E^2\left\{\hat{P}_x(\omega)\right\}}$$

which is effectively **normalised variance**

- **Figure of merit**

$$\mathcal{M} = \nu \times \Delta\omega$$

that is, **product of variability and resolution.**

$\mathcal{M}$ **should be as small as possible.**

Performance measures for the Nonparametric methods of Spectrum Estimation

Method	Variability ν	Resolution $\Delta\omega$	Figure of merit $\mathcal{M}$
Periodogram	1	$0.89\frac{2\pi}{N}$	$0.89\frac{2\pi}{N}$
Bartlett	$\frac{1}{K}$	$0.89K\frac{2\pi}{N}$	$0.89\frac{2\pi}{N}$
Welch	$\frac{9}{8}\frac{1}{K}$	$1.28\frac{2\pi}{L}$	$0.72\frac{2\pi}{N}$
Blackman–Tukey	$\frac{2}{3}\frac{M}{N}$	$0.64\frac{2\pi}{M}$	$0.43\frac{2\pi}{N}$

- Observe that each method has a Figure of Merit which is approximately the same
- Figure of merit are inversely proportional to N
- Although each method differs in its resolution and variance, **the overall performance is fundamentally limited by the amount of data that is available.**

Conclusions

FFT based spectral estimation is limited by:

- correlation assumed to be zero beyond N - biased/unbiased estimates
- resolution limited by the DFT "baggage"
- if two frequencies are separated by Δf, then we need $N \geq \frac{1}{\Delta f}$ data points to separate them
- limitations for spectra with narrow peaks (resonances, speech, sonar)
- limit on the resolution imposed by N also causes bias
- variance of the periodogram is almost independent of data length
- the derived variance formulae are only illustrative for real–world signals

But also many opportunities: spectral coherency, spectral entropy, TF, ...

Next time: model based spectral estimation for discrete spectral lines

Appendix: Spectral Coherence and LS Periodogram see also Problem 4.7 in your P/A sets

The **spectral coherence** shows similarity between two spectra

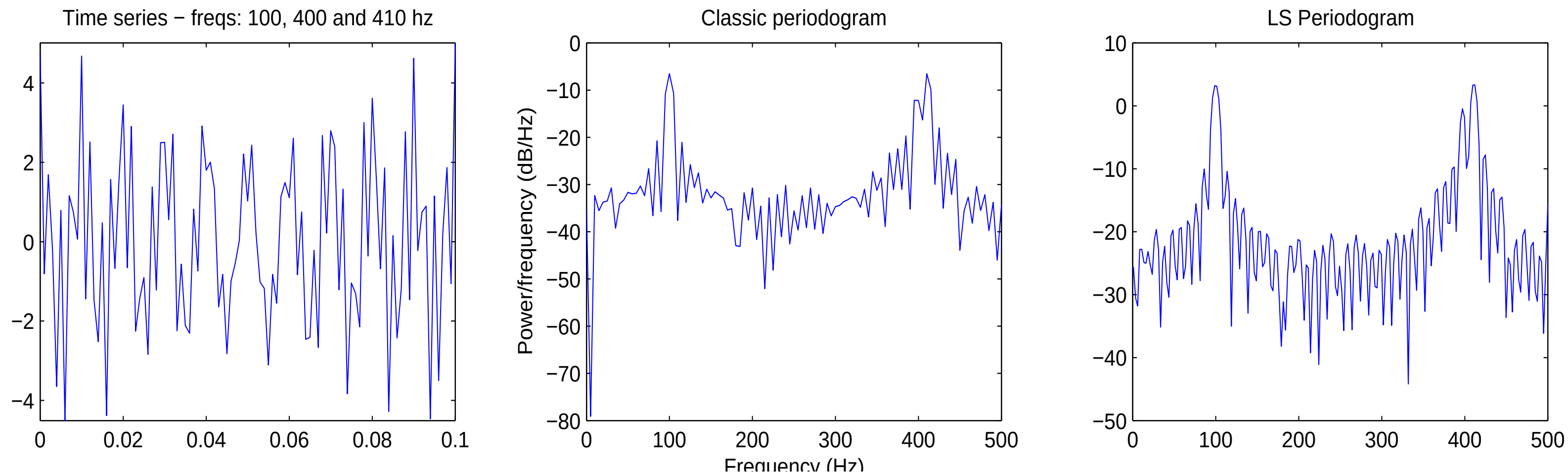
$$C_{xy}(\omega) = \frac{P_{xy}(\omega)}{\left[P_{xx}(\omega)P_{yy}(\omega)\right]^{1/2}}$$

It is invariant to linear filtering of x and y (even with different filters)

The periodogram $P_{per}(\omega)$ can be seen as a **Least Squares** solution to

$$P_{per}(\omega) = \|\hat{\beta}(\omega)\|^2, \;\; \hat{\beta} = \underset{\beta(\omega)}{\operatorname{argmin}} \sum_{n=1}^{N} \|y(n) - \beta e^{j\omega n}\|^2,$$

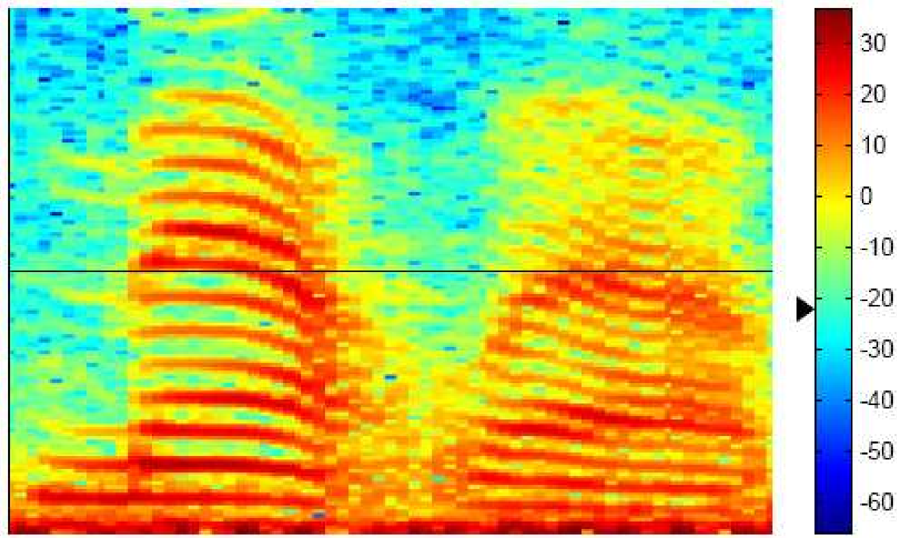
Periodogram and LS periodog. for a sinewave mixture (100, 400, 410) Hz



Appendix: Time-Frequency estimation

time–frequency spectrogram of "Matlab" ↬ 'specgramdemo'

Frequency



time

M aaa t l aaa b

For every time instant "t", the PSD is plotted along the vertical axis

Darker areas: higher magnitude of PSD

Appendix: Time-Frequency (TF) analysis - Principles

Assume $x(n)$ has a Fourier transform $X(\omega)$ and power spectrum $|X(\omega)|^2$.

The function $TF(n,\omega)$ determines how the energy is distributed in time-frequency, and it satisfies the following **marginal properties:**

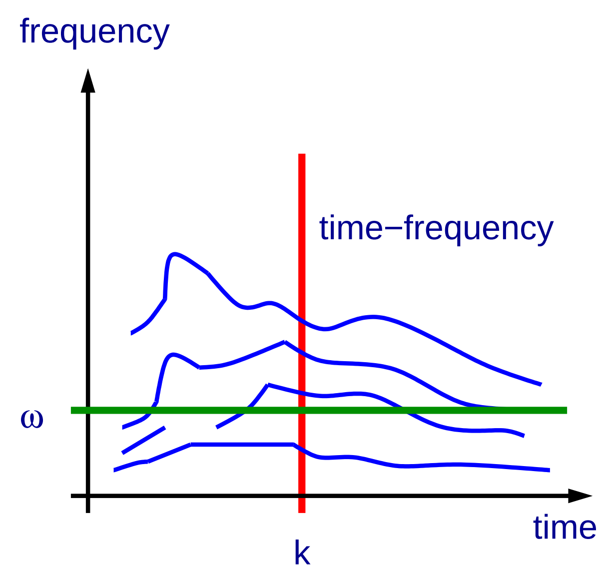
$$\sum_{n=-\infty}^{\infty} TF(n,\omega) = |X(\omega)|^2 \quad \text{energy in the signal at frequency } \omega$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} TF(n,\omega)d\omega = |x(n)|^2 \quad \text{energy at time instant `k' due to all } \omega$$

Then

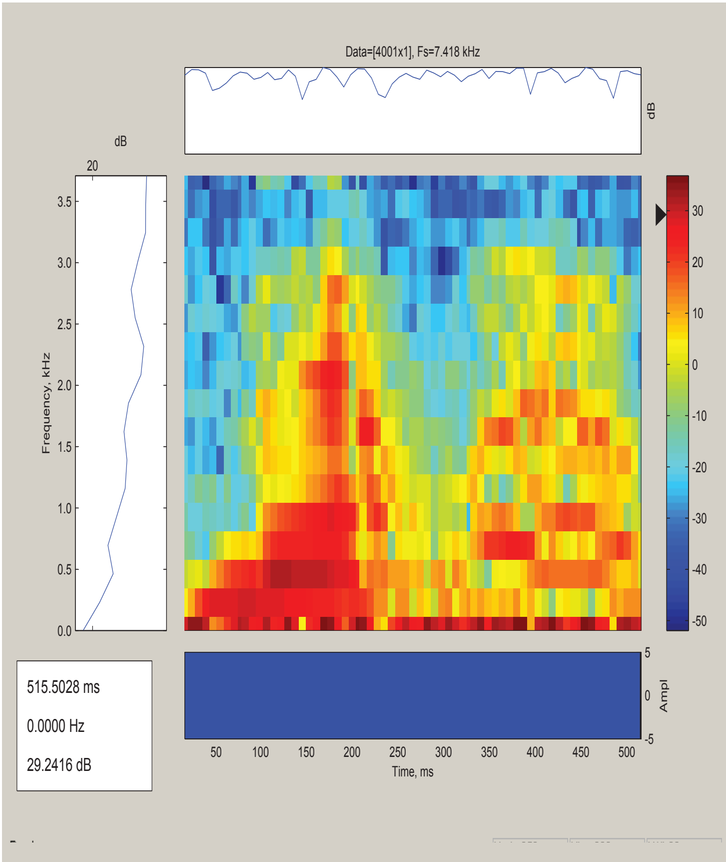
$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\infty}^{\infty} TF(n,\omega)d\omega = \sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega$$

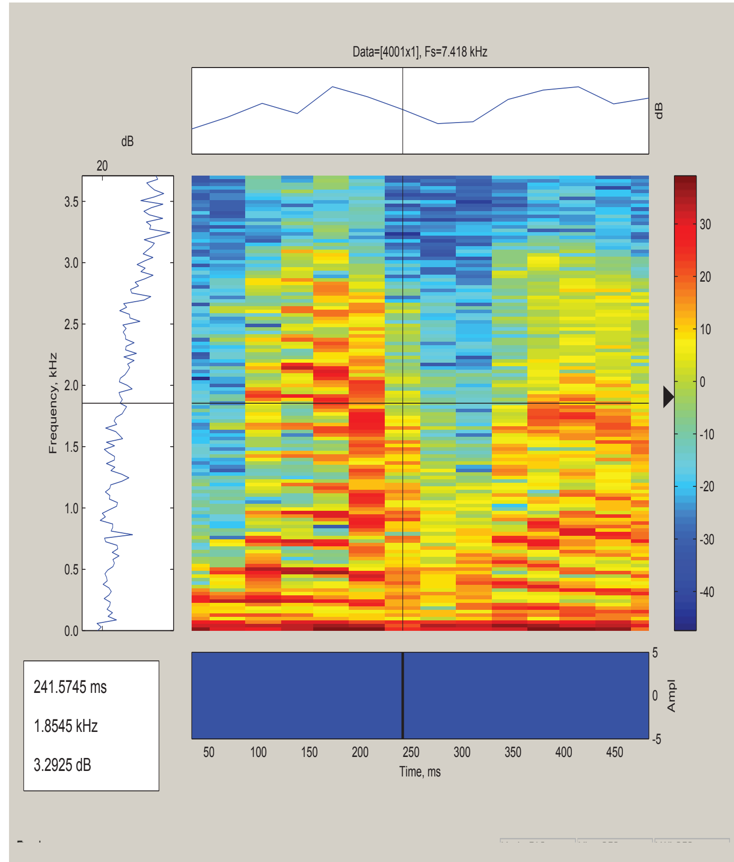
giving the **total energy** (all frequencies and samples) of a signal.



Time–frequency spectrogram of a speech signal

(wide band spectrogram) (narrow band spectrogram)





(win-len=256, overlap=200, ftt-len=32) (win-len=512, overlap=200, ftt-len=256)

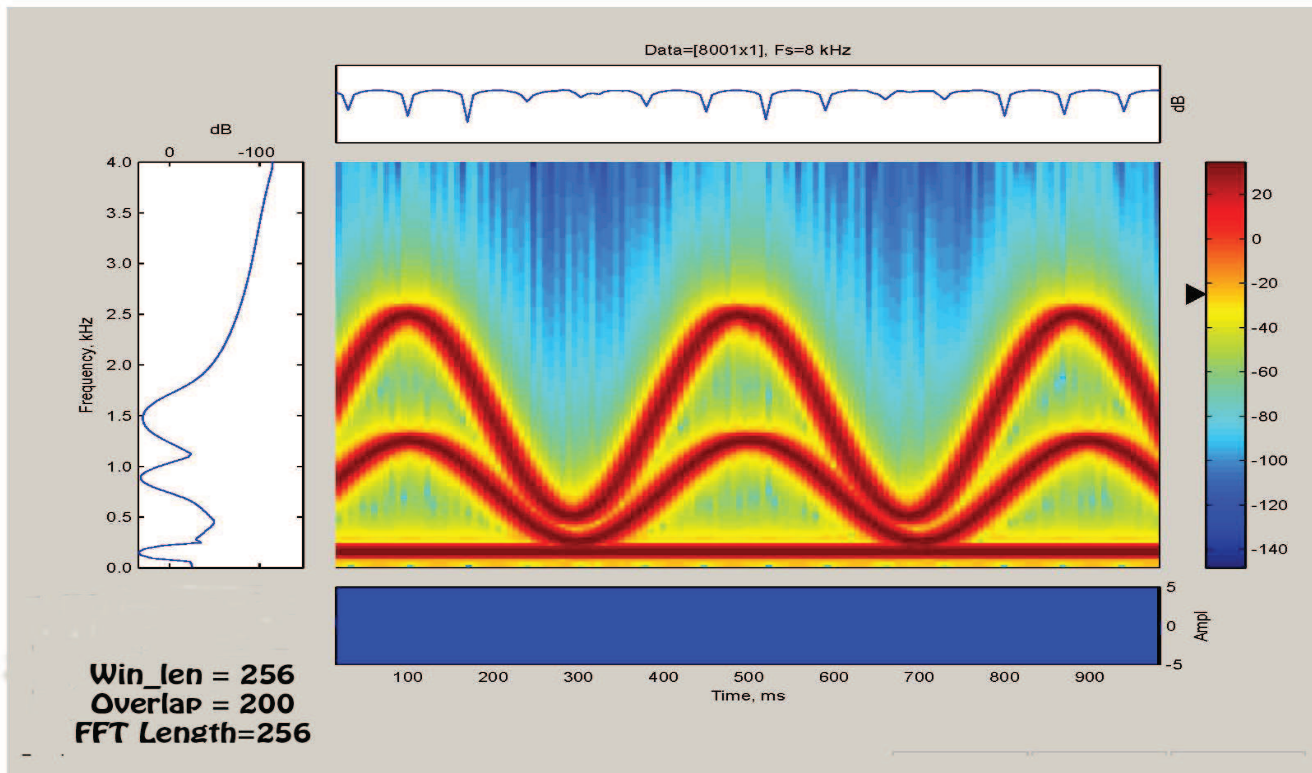
Homework: evaluate all the methods from the lecture for this T-F spectrogram

TF spectrogram of a frequency-modulated signal
(check also your coursework)

The time-frequency spectrogram of a frequency modulated (FM) signal

$$y(t) = A\cos\left[\omega_0 t + k_f \int_{-\infty}^{t} x(\alpha)d\alpha\right]$$

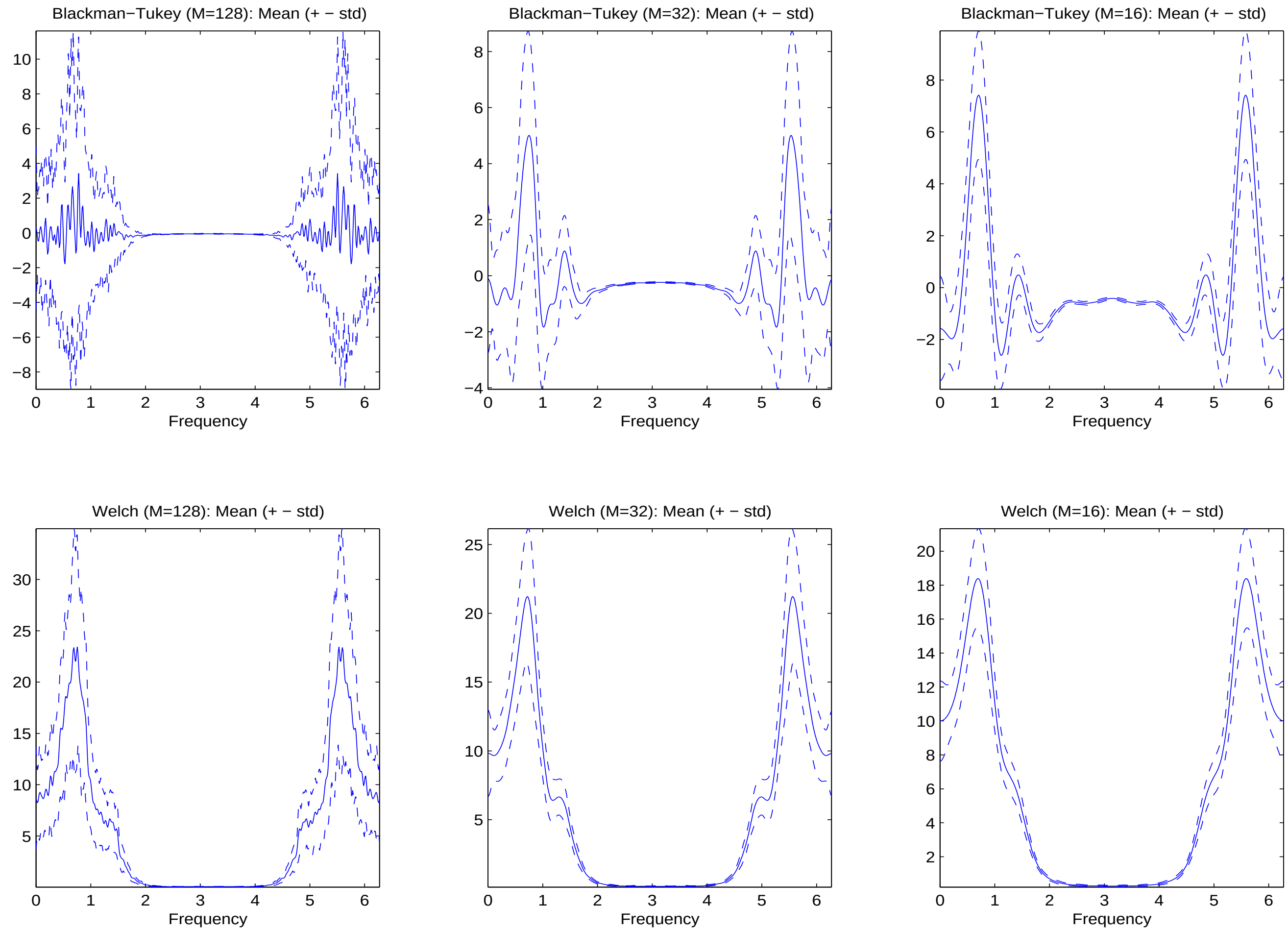
frequency



time

Opportunities: ARMA spectrum

N=512 samples, freq. res=1/500



Signal: ARMA(4,4), b=[1, 0.3544, 0.3508, 0.1736, 0.2401] a=[1, -1.3817, 1.5632, -0.8843, 0.4096]

Sometimes we only desire the correct position of the peaks $\looparrowright$ **ARMA Spectrum Estimation**

A note on positive-semidefiniteness of the $\mathbf{R}_{xx}$

The autocorrelation matrix $\quad \mathbf{R}_{xx} = E\big[\mathbf{x}\mathbf{x}^T\big]$

where $\mathbf{x} = \big[x[0], \ldots, x[N-1]\big]^T$. It is symmetric and of size $N \times N$.

There are four ways to define positive semidefiniteness: (see also your Problem-Answer sets)

1. All the eigenvalues of the autocorrelation matrix $\mathbf{R}$ are such that $\lambda_i \geq 0$, for i=1,. . . ,N
2. For any nonzero vector $\mathbf{a} \in \mathbb{R}^{N \times 1}$ we have $\mathbf{a}^T\mathbf{R}\mathbf{a} \geq 0$. For complex valued matrices, the condition becomes $\mathbf{a}^H\mathbf{R}\mathbf{a}$
3. There exists a matrix $\mathbf{U}$ such that $\mathbf{R} = \mathbf{U}\mathbf{U}^T$, where the matrix $\mathbf{U}$ is called a root of $\mathbf{R}$
4. All the principal submatrices of $\mathbf{R}$ are positive semidefinite. A principal submatrix is formed by removing $i = 1, \ldots, N$ rows and columns of $\mathbf{R}$

For positive definiteness conditions, replace $\geq$ with $>$

Opportunities: Spectral Entropy

Spectral entropy can be used to measure the peakiness of the spectrum.

This is achieved via the probability mass function (PMF) (normalised PSD) given by

$$\eta[i] = \frac{P_{per}[i]}{\sum_{l=0}^{N-1} P_{per}[l]} \quad \rightarrow \quad H_{sp} = -\sum_{i=0}^{N-1} \eta[i] \log_2 \eta[i] = \sum_{i=0}^{N-1} \eta[i] \log_2 \frac{1}{\eta[i]}$$

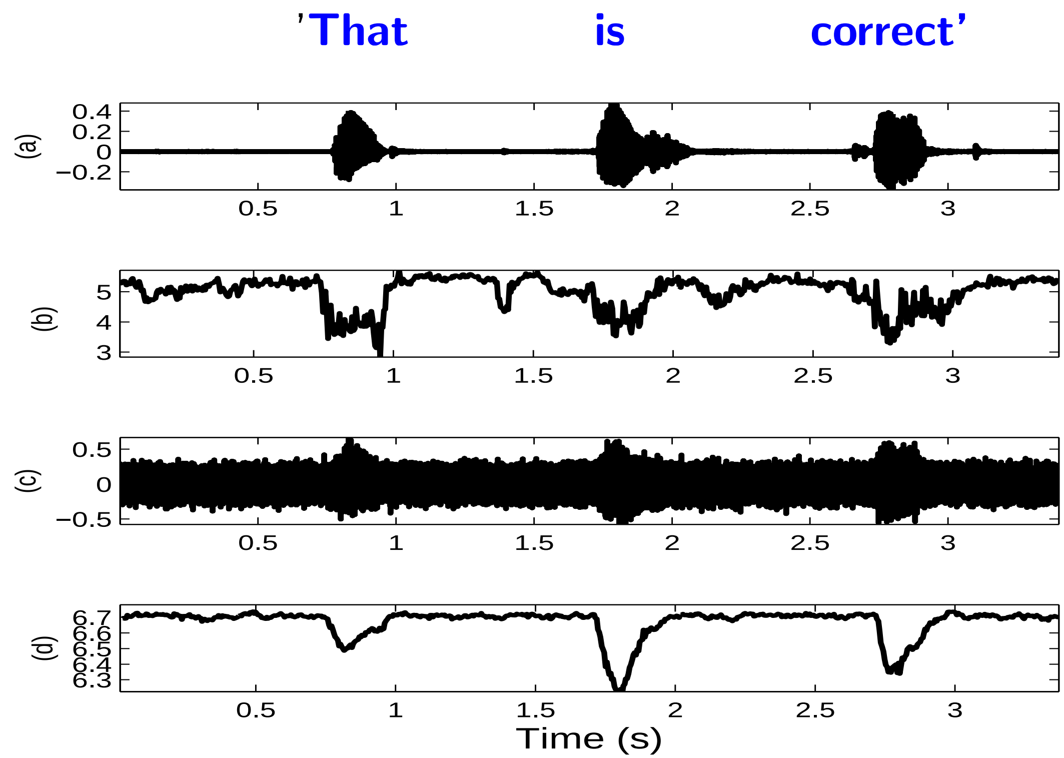
Intuition:

- peaky spectrum (e.g. $\sin(x)$) $\looparrowright$ low spectral entropy

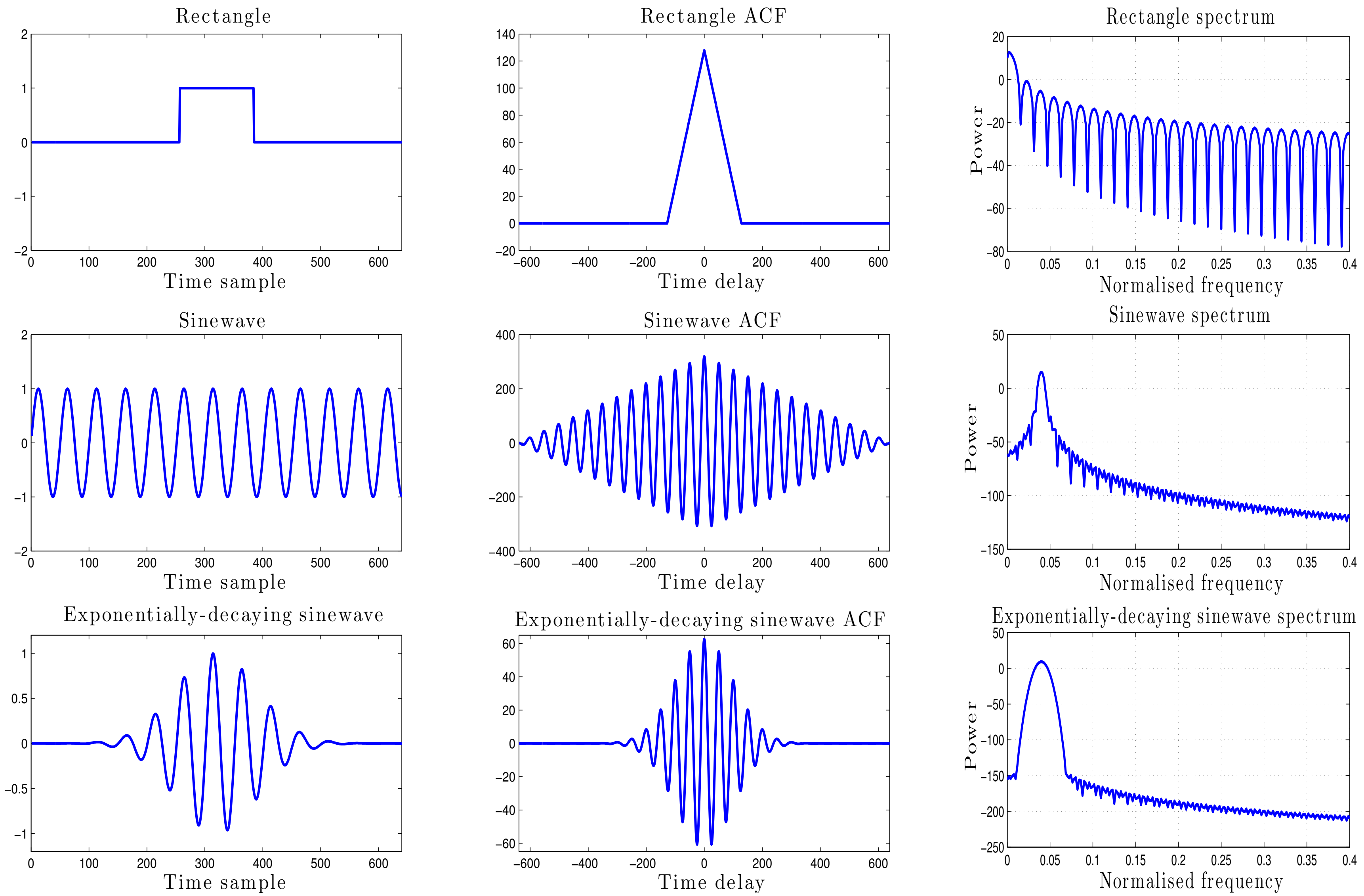
- flat spectrum (e.g. WGN) $\looparrowright$ high spectral entropy

Figure on the right:

From top to bottom: a) clean speech, b) spectral entropy, c) speech + noise, d)spectral entropy of (speech+noise)



Appendix: Practical issues in correlation and spectrum estimation



Appendix: Trade-off in window design

window length $\looparrowright$ trade-off between spectral resolution and statistical variance

- most windows take non-negative values in both time and frequency
  - They also peak at origin in both domains

For this type of window we can define:

- An **equivalent time width** N_x $\quad (N_x \approx 2M$ for rectangular and $N_x \approx M$ for triangular window)
- An **equivalent bandwidth** B_x ($\approx$ determined by window's length), as

$$N_w = \frac{\sum_{k=-(M-1)}^{M-1} w(k)}{w(0)} \qquad B_w = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)d\omega}{W(0)}$$

We also know that

$$W(0) = \sum_{k=-\infty}^{\infty} w(k) = \sum_{k=-(M-1)}^{M-1} w(k) \qquad \text{and} \qquad w(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)d\omega$$

It then follows that $\qquad N_w \times B_w = 1$

A window cannot be both time-limited and band-limited, usually $M \le N/10$

Appendix: More on time–bandwidth products

The previous slide assumes that both $w(n)$ and $W(\omega)$ peak at the origin $\hookrightarrow$ most energy concentrated in the main lobe, whose width should be $\sim 1/M$.

For a general signal: $x(n)$ and $X(\omega)$ can be negative or complex

If x(n) peaks at n_0 $(cf.\ X(\omega)\ at\ \omega_0) \hookrightarrow\ N_x = \dfrac{\sum_{n=-\infty}^{\infty} |x(n)|}{|x(n_0)|},\ \ B_x = \dfrac{\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega_0)| d\omega}{|X(\omega_0)|}$

Because $x(n)$ and $X(\omega)$ are Fourier transform pairs:

$$
\begin{aligned}
|X(\omega_0)| &= \left| \sum_{n=-\infty}^{\infty} x(n) e^{-\jmath \omega_0 n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n)| \\
|x(n_0)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{\jmath \omega n_0} d\omega \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)| d\omega
\end{aligned}
$$

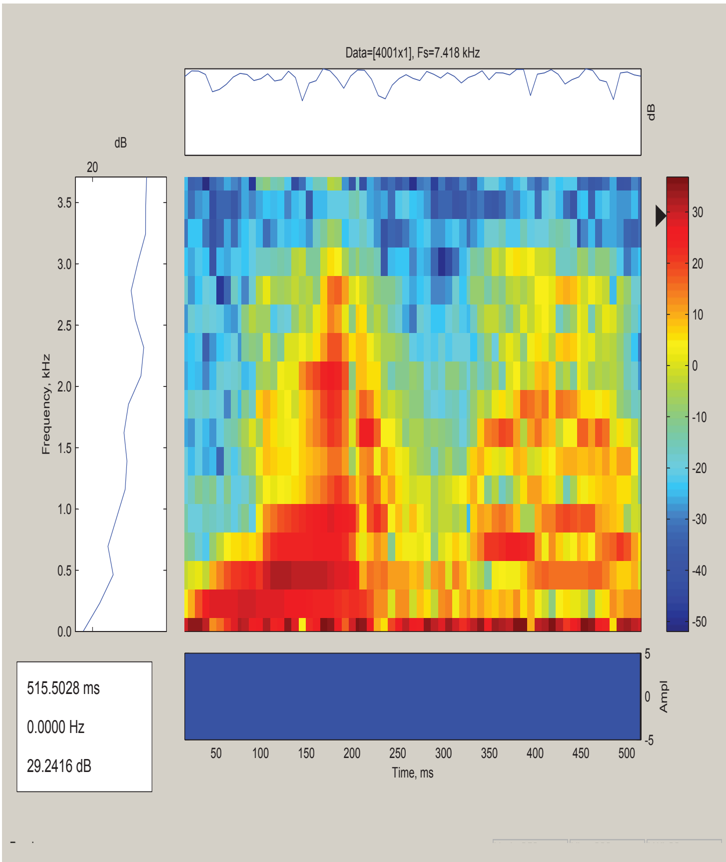
This implies

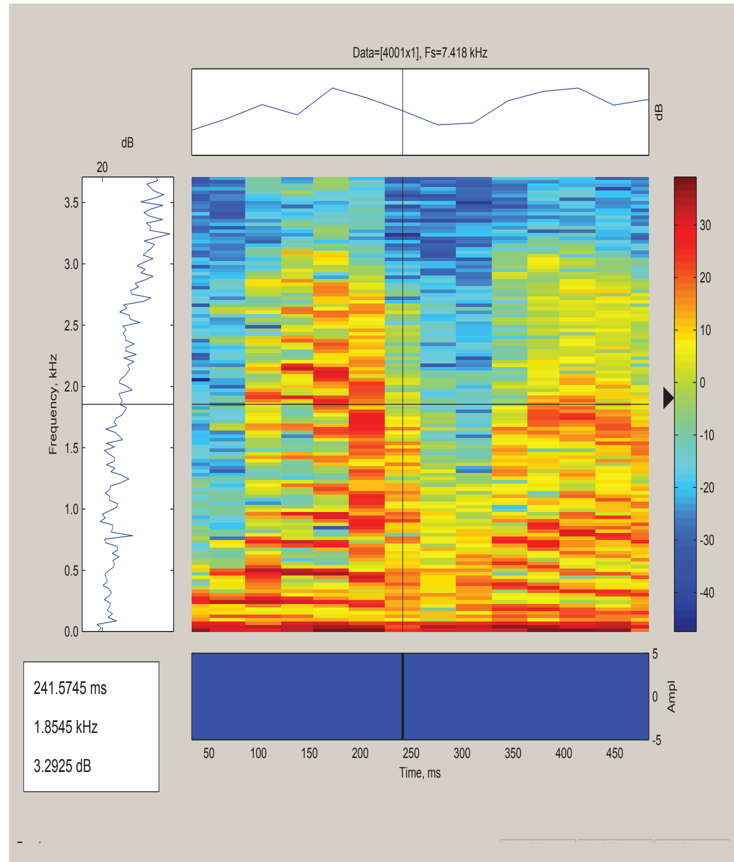
$N_x \times B_x \geq 1$ (a sequence cannot be narrow in both time and frequency)

More precisely: if the sequence is narrow in one domain then it must be wide in the other domain (uncertainty principle)

Appendix: STFT of a speech signal

wide band spectrogram **narrow band spectrogram**





(win-len=256, overlap=200, ftt-len=32) **(win-len=512, overlap=200, ftt-len=256)**

Notes

Notes

Notes