Smoothing Algorithms for Variable Rate Models

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***Abstract*—Standard state-space methods assume that the latent state evolves uniformly over time, and can be modelled with a discrete-time process synchronous with the observations. This may be a poor representation of some systems in which the state evolution displays discontinuities in its behaviour. For such cases, a variable rate model may be more appropriate; the system dynamics are conditioned on a set of random changepoints which constitute a marked point process. In this paper, new particle smoothing algorithms are presented for use with conditionally linear-Gaussian and conditionally deterministic dynamics. These are demonstrated on problems of financial modelling and target tracking. Results indicate that the smoothing approximations provide more accurate and more diverse representations of the state posterior distributions.**

***Index Terms*—Bayesian inference, state-space model, variable rate, particle filter, filtering, smoothing**

I. Introduction

THE objective of sequential Bayesian inference is to estimate an imperfectly observed quantity as it varies over time. This is accomplished through the use of probabilistic models for the state evolution and measurement processes. Often, the latent state is a continuously varying quantity, whereas the observations are made at a discrete set of times. In these circumstances, it is simplest to discretise the state onto the same set of times as the observations. When the system is also Markovian, this leads to the standard "fixed rate" hidden Markov model (HMM). The standard HMM is poorly suited to systems where the state evolution contains discontinuities; for example, the price of a financial asset which may display large jumps at random times between periods of diffusion-like behaviour, or the kinematic state of a manoeuvring vehicle which may have sudden changes in the acceleration when turns begin or end. Such problems can be handled more naturally using a "variable rate" model, in which the state dynamics are conditioned upon a set of changepoints which characterise transitions in behaviour.

In a variable rate model, the set of changepoints and associated parameters are modelled as a marked point process (MPP), the mathematical properties of which are set out thoroughly in [1]. Conditional upon this MPP the state evolves according to some benign dynamics. In [2], [3], the conditional state evolution is treated as deterministic, while in [4], [5] a conditionally linear-Gaussian state model is considered.

The posterior distribution for the changepoint MPP is inherently nonlinear, and cannot be calculated analytically. Instead, inference may be conducted using numerical approximations. The particle filter (introduced by [6]) and smoother (see [7], [8]) are schemes which approximate a posterior distribution using a set of samples, or "particles", drawn sequentially from it. A thorough introduction to particle filtering and smoothing methods can be found in [9], [10]. In [2]–[4], the particle filter was adapted for use with variable rate models, resulting in the variable rate particle filter (VRPF).

The VRPF allows the changepoint sequence – and hence the current state – to be estimated sequentially as observations are received. However, estimates can often be improved later once further observations have been made. In this paper, we address the problem of smoothing in variable models, i.e. the estimation of the changepoint sequence and latent state given all the observations. This is achieved with an efficient backward sweep through the observations, in a similar manner to the method for standard HMMs described in [8]. Two new schemes are introduced: one for conditionally linear-Gaussian models which exploits the method of Rao-Blackwellisation; and a second for use with conditionally-deterministic models which uses an augmented target distribution in the style of an SMC sampler [11].

We introduce the general structure of variable rate models in section II. The VRPF is reviewed in section III, and new variable rate smoothing algorithms are described in section IV. In section V, particular examples of variable rate models are presented and algorithm performance is demonstrated in a series of simulations.

II. Variable Rate Models

We consider a general model from time 0 to T, between which observations, $\{y_1 \dots y_N\}$, are made at times $\{t_1 \dots t_N = T\}$. During this period, an unknown number of changepoints, K, occur at times $\{\tau_0 = 0, \tau_1 \dots \tau_K\}$, each with associated changepoint parameters, $\{u_0, u_1 \dots u_K\}$. The pairs $\{\tau_k, u_k\}$ are the elements of an marked point process (MPP). The latent state is a continuous-time process denoted $x(t)$. Discrete sets containing multiple values over time will be written as, e.g. $y_{1:n} = \{y_1 \dots y_n\}$.

The objective for inference will be to estimate the changepoint sequence. This will be denoted as $\theta = \{\tau_{0:K}, u_{0:K}\}$. At a particular time t_n, the sequence will be divided into past $\theta_n = \{\tau_j, u_j \forall j : 0 \leq \tau_j < t_n\}$, and future $\theta_n^+ = \{\tau_j, u_j \forall j : t_n \leq \tau_j < T\}$. It will also be useful to define a variable for the changepoints which occur in the interval $[t_{n-1}, t_n)$, $\theta_{n \setminus n-1} = \{\tau_j, u_j \forall j : t_{n-1} \leq \tau_j < t_n\}$.

For notational simplicity, the following counting variables are introduced to keep track of the most recent changepoint to have occurred,

$$K(t) = \max(k : \tau_k < t) \tag{1}$$
$$K_n = K(t_n). \tag{2}$$

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The changepoint sequence is assumed to be a Markov process.

$$\{\tau_k, u_k\} \sim p(\tau_k, u_k|\tau_{k-1}, u_{k-1}) \tag{3}$$

This density will be constructed such that $P(\tau_k < \tau_{k-1}) = 0$.

In the manner of [3], a survivor function is defined as the probability that no new changepoint occurs before a given time,

$$\begin{aligned} S(\tau_k, u_k, t) &= P(\tau_{k+1} > t|\tau_k, u_k) \\ &= 1 - \int_{\tau_k}^{t} p(\xi|\tau_k, u_k) d\xi. \end{aligned} \tag{4}$$

It is now possible to write down a prior for the changepoint sequence, where we use the convention that $\tau_0 = 0$. The existence of such a density for a MPP is addressed in [1].

$$p(\theta_n) = S(\tau_{K_n}, t_n) p(u_0) \prod_{k=1}^{K_n} p(\tau_k, u_k|\tau_{k-1}, u_{k-1}) \tag{5}$$

A. Conditionally Linear-Gaussian Models

The first class of variable rate models to be considered is those whose state dynamics are linear-Gaussian conditional on the changepoint sequence. Such a model may be discretised onto the set of observation times in exactly the same manner as a standard the standard HMM.

$$x_n = A_n(\theta_n) x_{n-1} + w_n \tag{6}$$
$$y_n = C_n(\theta_n) x_n + v_n, \tag{7}$$

where,

$$w_n \sim \mathcal{N}(w_n|0, Q_n(\theta_n)) \tag{8}$$
$$v_n \sim \mathcal{N}(v_n|0, R_n(\theta_n)). \tag{9}$$

In addition, the prior state distribution should be Gaussian, with known mean and variance.

$$x_0 \sim \mathcal{N}(x_0|m_0, P_0). \tag{10}$$

If the changepoint sequence is known, or has been estimated, then the state values, x_n may be inferred using optimal Kalman filtering and smoothing recursions. As well as the basic Kalman filter [12], the Rauch-Tung-Striebel (RTS) smoother [13] and two-filter smoother [14] will prove useful for this step.

Conditionally linear-Gaussian variable rate models were introduced in [4] for a financial inference algorithm. Changepoints correspond to jumps in the value or trend of a security, at which points the process covariance, $Q_n(\theta_n)$, is inflated.

B. Conditionally Deterministic Models

The second class of variable rate models for consideration is those in which the state is completely specified by the changepoint sequence, with no additional random components. Such a process is commonly referred to as "piecewise-deterministic", as the latent state follows a deterministic path between changepoints. In this case, it is not necessary to discretise the state – it may be kept as a continuous variable.

In general, the state dynamics will be governed by a differential equation which may depend on the entire changepoint sequence. Here we assume that only the most recent changepoint is significant.

$$dx(t) = \mathrm{f}(x(t), \tau_{K(t)}, u_{K(t)}). \tag{11}$$

By introducing a new sequence, $\{x_0, x_1 \dots x_K\}$, which denotes the value of the state at each changepoint (i.e. $x(\tau_k)$), and assuming that an analytic solution exists, a state transition function may be found,

$$x(t) = f(x_{K_n}, u_{K_n}, \tau_{K_n}, t), \tau_{K_n} < t \le \tau_{K_n+1}. \tag{12}$$

By choosing $t = \tau_{K_n+1}$, this equation specifies the state at the next changepoint time. Similarly, by choosing $t = t_n$, the state at the observation times may be calculated — these points will be denoted $\hat{x}_n$. To complete the model, a probabilistic measurement model must be devised for the observation process, $p(y_n|\hat{x}_n)$.

For convenience, we assume that x_0 is known in the following sections. This means that $x(t)$ may be calculated deterministically for all t given θ. This condition is easily relaxed by including x_0 as a random variable in the posterior distribution.

Target tracking algorithms are commonly based upon fixed rate models (see, e.g. [15] for a thorough survey), in which the target kinematics (position, velocity, etc.) are estimated at a set of fixed times at which observations (e.g. radar measurements) are made. In [2]–[4], variable rate models were introduced for tracking, in which the state trajectory is divided up by a set of changepoints between which the motion follows a deterministic path governed by motion parameters (accelerations, etc.) which are fixed for that division.

III. The Variable Rate Particle Filter

The variable rate particle filter (VRPF) is described in [2]–[4]. The objective of the algorithm is to sequentially estimate the posterior distribution of the changepoint sequence, $p(\theta_n|y_{1:n})$, at each time t_n. This distribution may be expanded using Bayes' rule,

$$\begin{aligned} p(\theta_n|y_{1:n}) \propto\, & p(y_n|\theta_n, y_{1:n-1}) \\ & \times p(\theta_{n\backslash n-1}|\theta_{n-1}) p(\theta_{n-1}|y_{1:n-1}). \end{aligned} \tag{13}$$

The transition term, $p(\theta_{n\backslash n-1}|\theta_{n-1})$, has a similar form to the changepoint prior of (5) [1], comprising a product of density terms (one for each new changepoint) and a survivor function term.

$$\begin{aligned} & p(\theta_{n\backslash n-1}|\theta_{n-1}) \\ & = p(\tau_{K_{n-1}+1:K_n}, u_{K_{n-1}+1:K_n}, \tau_{K_n+1} > t_n|\tau_{K_{n-1}+1} > t_{n-1}, \tau_{K_{n-1}}, u_{K_n} \\ & = \begin{cases} S(\tau_{K_n}, t_n) \prod\limits_{j: t_{n-1} \le \tau_j < t_n} p(\tau_j, u_j|\tau_{j-1}, u_{j-1}, \tau_j > t_{n-1}) & K_n > K_{n} \\ S(\tau_{K_n}, t_n)/S(\tau_{K_n}, t_{n-1}) & K_n = K_{n} \end{cases} \end{aligned}$$

For all but the first changepoint in the interval, the density is given by the prior model of (3). For the first changepoint, indexed by $k = K_{n-1} + 1$, we must account for the fact that it cannot occur before t_{n-1},

$$p(\tau_k, u_k|\tau_{k-1}, u_{k-1}, \tau_k > t_{n-1})$$
$$= \frac{1}{S(\tau_{k-1}, t_{n-1})} \begin{cases} p(\tau_k, u_k|\tau_{k-1}, u_{k-1}) & \tau_k > t_{n-1} \\ 0 & \tau_k < t_{n-1} \end{cases} \tag{15}$$

Practically, because changepoints will be relatively rare events, it is not likely that more than one new changepoint will occur between t_{n-1} and t_n.

The target distribution of (13) cannot be calculated analytically, but may be approximated numerically. A particle filter is an algorithm for approximating a probability distribution using a set of weighted samples (or "particles") drawn from that distribution. In this case, each particle will be a set of changepoint times and parameters.

$$\hat{p}(\theta_n|y_{1:n}) = \sum_j w_n^{(j)} \delta_{\theta_n^{(j)}}(\theta_n) \tag{16}$$

where $\delta_x(X)$ is a dirac probability mass at $X = x$.

The particle filter works recursively. At the nth step, a particle, $\theta_{n-1}^{(i)}$, is first resampled from those approximating the filtering distribution at the $(n-1)$th step, using an appropriately chosen set of proposal weights.

$$q(\theta_{n-1}) = \sum_j v_{n-1}^{(j)} \delta_{\theta_{n-1}^{(j)}}(\theta_{n-1}) \tag{17}$$

The choice of weights determines the type of resampling used. The simplest choice, $v_{n-1}^{(j)} = 1/N_F$ (where N_F is the number of filter particles) may be achieved by simply omitting this step all together and using the particles of $\hat{p}(\theta_{n-1}|y_{1:n-1})$. This, however, leads to degeneracy of the particle weights over time. Conventional resampling is achieved by using $v_{n-1}^{(j)} = w_{n-1}^{(j)}$. Any other choice results in an auxiliary particle filter [16]. For further discussion of resampling, see [9], [10], [17].

Next, an extension to the changepoint sequence, $\theta_{n\backslash n-1}^{(i)}$, is proposed from an importance distribution, $q(\theta_{n\backslash n-1}|\theta_{n-1}^{(i)}, y_n)$, and concatenated with θ_{n-1} to create an estimate of θ_n. Finally, the particle is weighted according to the ratio of the target and proposal densities.

$$\begin{aligned} w_n^{(i)} &= \frac{p(\theta_n^{(i)}|y_{1:n})}{q(\theta_n)} \\ &\propto \frac{p(y_n|\theta_n, y_{1:n-1})p(\theta_{n\backslash n-1}|\theta_{n-1})p(\theta_{n-1}|y_{1:n-1})}{q(\theta_{n-1})q(\theta_{n\backslash n-1}|\theta_{n-1}^{(i)}, y_n)} \\ &= \frac{w_{n-1}^{(i)}}{v_{n-1}^{(i)}} \times \frac{p(y_n|\theta_n, y_{1:n-1})p(\theta_{n\backslash n-1}|\theta_{n-1})}{q(\theta_{n\backslash n-1}|\theta_{n-1}^{(i)}, y_n)} \end{aligned} \tag{18}$$

The normalisation may be enforced by scaling the weights so that they sum to 1.

For the most basic "bootstrap" [6] form of the VRPF, $\theta_{n\backslash n-1}$ may be proposed from the prior transition density (14). This can be achieved by sampling new changepoints sequentially from the transition model (3) (apart from the first which is sampled from (15)) until one falls after the current time, t_n. This final future changepoint is discarded. (This process can be thought of as sampling the entire future changepoint sequence from t_{n-1} onwards, and then marginalising those which fall after t_n.) The bootstrap proposal leads to the usual simplification of the weight formula.

$$w_n^{(i)} = \frac{w_{n-1}^{(i)}}{v_{n-1}^{(i)}} \times p(y_n|\hat{x}_n) \tag{19}$$

In [2], the following choice of proposal weights was found to work well, as it preserves low probability particles which might turn out to be good estimates at later time steps,

$$v_{n-1}^{(i)} \propto \max(1, N_F w_{n-1}^{(i)}). \tag{20}$$

It only remains to consider the likelihood term required for evaluation of the importance weights, $p(y_n|\theta_n, y_{1:n-1})$. The form of this term depends on the particular model under consideration. In the following sections, the likelihood expressions for the conditionally linear-Gaussian and deterministic cases are considered.

A. Conditionally Linear-Gaussian Likelihoods

For a conditionally linear-Gaussian model, the required likelihood term, $p(y_n|\theta_n, y_{1:n-1})$ is the predictive distribution estimated by the Kalman filter. Conveniently, the Kalman filter also provides us with an estimate of the current state given the changepoint sequence and all the preceding observations. It follows from the the Gaussian dynamics and prior that these distributions are all Gaussian as well, [18],

$$p(x_n|\theta_n, y_{1:n-1}) = \mathcal{N}(x_n|m_n^-, P_n^-) \tag{21}$$
$$p(x_n|\theta_n, y_{1:n}) = \mathcal{N}(x_n|m_n, P_n) \tag{22}$$
$$p(y_n|\theta_n, y_{1:n-1}) = \mathcal{N}(y_n|\mu_n, S_n), \tag{23}$$

with means and variances given by the following standard recursions (dependence on θ_n suppressed for clarity),

$$m_n^- = A_n m_{n-1} \tag{24}$$
$$P_n^- = A_n P_{n-1} A_n^T + Q_n \tag{25}$$
$$\mu_n = C_n m_n^- \tag{26}$$
$$S_n = C_n P_n^- C_n^T + R_n \tag{27}$$
$$K_n = P_n^- C_n^T S_n^{-1} \tag{28}$$
$$m_n = m_n^- + K_n(y_n - \mu_n) \tag{29}$$
$$P_n = P_n^- - K_n S_n K_n^T. \tag{30}$$

This completes the requirements for the particle filter, resulting in the final algorithm shown in Fig. 1.

B. Conditionally Deterministic Likelihoods

When a conditionally deterministic model is used, the state at observation time t_n is specified by the changepoint sequence θ_n (plus the initial state, x_0), using (12). Thus, the required likelihood term is simply given by,

$$p(y_n|\theta_n, y_{1:n-1}) = p(y_n|\hat{x}_n). \tag{31}$$

This leads to a particle filter for variable rate models with conditionally deterministic dynamics, summarised in Fig. 2.

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For each i, initialise particle changepoint sequence with {θ_0^(i)} ← {τ_0^(i), u_0^(i)}, where τ_0^(i) = 0 and u_0^(i) ∼ p(u_0).
For each i, initialise particle sufficient statistics, m_0^(i) and P_0^(i) with prior values.
for n = 1 . . . N do
    for i = 1 . . . N_F do
        Sample a history θ_{n−1}^(i) ∼ Σ_j v_{n−1}^(j) δ_{θ_{n−1}^(j)}(θ_{n−1}).
        Propose an extension θ_{n\n−1}^(i) ∼ q(θ_{n\n−1} | θ_{n−1}^(i)).
        Add extension to sequence θ_n^(i) ← θ_{n−1}^(i) ∪ θ_{n\n−1}^(i).
        Predict observation mean and covariance μ_n^(i) and S_n^(i) using (24) to (27).
        Calculate weight w_n^(i) using (18).
        Update state mean and covariance m_n^(i) and P_n^(i) using (28) to (30).
    end for
    Scale weights such that Σ_i w_n^(i) = 1.
end for
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Fig. 1. Rao-Blackwellised variable rate particle filter

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For each i, initialise particle changepoint sequence with {θ_0^(i)} ← {τ_0^(i), u_0^(i)}, where τ_0^(i) = 0 and u_0^(i) ∼ p(u_0).
for n = 1 . . . N do
    for i = 1 . . . N_F do
        Sample a history θ_{n−1}^(i) ∼ Σ_j v_{n−1}^(j) δ_{θ_{n−1}^(j)}(θ_{n−1}).
        Propose an extension θ_{n\n−1}^(i) ∼ q(θ_{n\n−1} | θ_{n−1}^(i), y_n).
        Add extension to sequence θ_n^(i) ← θ_{n−1}^(i) ∪ θ_{n\n−1}^(i).
        Calculate state x̂_n using (12).
        Calculate weight w_n^(i) using (18).
    end for
    Scale weights such that Σ_i w_n^(i) = 1.
end for
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Fig. 2. Piecewise deterministic variable rate particle filter

C. Improving the Variable Rate Particle Filter

The bootstrap versions of the VRPF may perform poorly if changepoints are not obvious until significantly after they occur. For example, in a tracking example, if a jump occurs in the acceleration, but only the position is observed, then this change may not be obvious until a number of observations have arrived. In this case, the estimation may be improved by the introduction of resample-move (RM) steps [19]. In an RM scheme, optional Metropolis-Hastings (MH) moves are conducted to alter the particle states after the importance sampling has taken place. For variable rate models, any one of the previous changepoints, τ_k, or associated parameters, u_k, could be adjusted. Because more observations are available than when the changepoint was first proposed, it may be possible to construct more informed proposals and so move the changepoints towards regions with higher posterior probability. It is even possible to retrospectively add or remove changepoints, using reversible jump MH moves [20]. Variable rate particle filters using RM with piecewise deterministic models are described in [3], [21].

Rather than conducting the IS and MH steps separately, it is possible to combine them using the framework of SMC samplers [11]. This was suggested in [3], again for piecewise deterministic dynamics, but the extension to conditionally linear-Gaussian models is straightforward.

IV. Variable Rate Particle Smoothing

A filter conducts inference sequentially as new observations are introduced. The purpose of a smoother is to produce a revised estimate once all the observations have been made, using future values to improve upon the filter performance. Estimating changepoints online is a challenging task because the presence of a change may not be obvious until after it has happened. Thus, a smoothing algorithm is expected to achieve significantly improved performance at changepoint estimation.

The target distribution for a variable rate smoothing algorithm is the posterior distribution over the entire changepoint sequence, $p(\theta|y_{1:N})$. This is the distribution approximated by the final step of the VRPF. However, in the same manner as the fixed rate filter-smoother of [22], this approximation is likely to lack path-space diversity — because of the necessary resampling step in the filtering algorithm, the particles all share the same set of changepoints before a particular time, with variation only appearing for changepoints closer to T. For a better characterisation of the smoothing distribution, it is necessary to rejuvenate the set of particles. This is achieved with a backward pass through the observations in a similar manner to the forward-backward method described in [8].

A. Conditionally Linear-Gaussian Smoothing

At time t_n, the target distribution may be factorised as follows,

$$p(\theta|y_{1:N}) = p(\theta_n^+|y_{1:N})p(\theta_n|\theta_n^+, y_{1:N}). \tag{32}$$

This suggests a sequential strategy for particle smoothing. Starting with a low diversity set of particles from $p(\theta|y_{1:N})$, we first marginalise θ_n by simply discarding the changepoints before t_n. New values of θ_n are then sampled from the conditional distribution, $p(\theta_n|\theta_n^+, y_{1:N})$, which is approximated using the filter particles, and the past and future sequences are concatenated. By iterating backwards from $n = N \dots 1$, a diverse set of particles approximating $p(\theta|y_{1:N})$ will be generated.

It remains to devise a scheme for sampling the conditional distribution, $p(\theta_n|\theta_n^+, y_{1:N})$. The simplifications used by the fixed rate particle smoother which exploit the Markovian nature of the state sequence [8] are of no help, because past changepoints are not independent of future observations (i.e. $p(\theta_n|\theta_n^+, y_{1:N}) \neq p(\theta_n|\theta_n^+, y_{1:n})$). However, when the state dynamics are linear-Gaussian conditional on the changepoint sequence, it is possible split the dependence on the observations into past and future by introducing the current state as an additional variable. This trick was used in [23] in the derivation of the Rao-Blackwellised particle filter for fixed rate conditionally linear-Gaussian models.

$$\begin{aligned}
&p(\theta_n|\theta_n^+, y_{1:N}) \\
&\quad \propto p(\theta_n, \theta_n^+|y_{1:N}) \\
&\quad = \int p(x_n, \theta_n, \theta_n^+|y_{1:N}) dx_n \\
&\quad \propto \int p(y_{n+1:N}|x_n, \theta_n, \theta_n^+, y_{1:n}) p(x_n, \theta_n, \theta_n^+|y_{1:n}) dx_n \\
&\quad = \int p(y_{n+1:N}|x_n, \theta_n^+) p(x_n|\theta_n, y_{1:n}) dx_n \\
&\qquad \times p(\theta_n^+|\theta_n) p(\theta_n|y_{1:n})
\end{aligned} \tag{33}$$

Finally, the particle approximation is substituted for the filtering distribution.

$$\hat{p}(\theta_n|\theta_n^+, y_{1:N}) = \sum_i \tilde{w}_n^{(i)} \delta_{\theta_n^{(i)}}(\theta_n) \tag{34}$$

where the backwards conditional weights are given by

$$\begin{aligned}
\tilde{w}_n \propto w_n &\int p(y_{n+1:N}|x_n, \theta_n^+) \\
&\times p(x_n|\theta_n^{(i)}, y_{1:n}) dx_n p(\theta_n^+|\theta_n^{(i)})
\end{aligned} \tag{35}$$

As before, normalisation is enforced by scaling the weights so that they sum to 1.

The changepoint transition density may be expressed as,

$$\begin{aligned}
p(\theta_n^+|\theta_n) &= p(\tau_{K_n+1:K}, u_{K_n+1:K}|\tau_{K_n}, u_{K_n}, \tau_{K_n+1} > t_n) \\
&\propto p(\tau_{K_n+1}, u_{K_n+1}|\tau_{K_n}, u_{K_n}, \tau_{K_n+1} > t_n).
\end{aligned} \tag{36}$$

The integral in (35) contains two terms involving the current state, x_n. The first is the familiar state posterior generated by the Kalman filter using (24–30).

$$p(x_n|\theta_n^{(i)}, y_{1:n}) = \mathcal{N}(x_n|m_n^{(i)}, P_n^{(i)}) \tag{37}$$

The mean and covariance of this distribution will have been calculated during the filtering stage, and can be stored for use now in the smoother.

The second state term in (35) is the likelihood, $p(y_{n+1:N}|x_n, \tilde{\theta}_n^+)$. This may be considered to be an improper density over x_n, and may be calculated analytically using a backwards Kalman filter, in a similar manner to that used in the two-filter smoother [14], [23]–[25]. Such a backwards Kalman filter uses the following recursions. Derivations may be found in the aforesaid references.

$$p(y_{n+1:N}|x_n, \theta_n^+) \propto \mathcal{N}(x_n|\tilde{m}_n^-, \tilde{P}_n^-) \tag{38}$$
$$p(y_{n:N}|x_n, \theta_n^+) \propto \mathcal{N}(x_n|\tilde{m}_n, \tilde{P}_n) \tag{39}$$

$$\tilde{m}_n^- = A_{n+1}^{-1} \tilde{m}_{n+1} \tag{40}$$
$$\tilde{P}_n^- = A_{n+1}^{-1} (\tilde{P}_{n+1} + Q_{n+1}) A_{n+1}^{-T} \tag{41}$$
$$\tilde{\mu}_n = C_n \tilde{m}_n^- \tag{42}$$
$$\tilde{S}_n = C_n \tilde{P}_n^- C_n^T + R_n \tag{43}$$
$$\tilde{K}_n = \tilde{P}_n^- C_n^T \tilde{S}_n^{-1} \tag{44}$$
$$\tilde{m}_n = \tilde{m}_n^- + \tilde{K}_n (y_n - \tilde{\mu}_n) \tag{45}$$
$$\tilde{P}_n = \tilde{P}_n^- - \tilde{K}_n \tilde{S}_n \tilde{K}_n^T \tag{46}$$

Exact methods for initialising this recursion are discussed in [24]. However, it is often sufficient to use an approximation, for example $\tilde{m}_N^{(i)} = m_N^{(i)}$ and $\tilde{P}_N^{(i)} = P_N^{(i)}$.

Substituting into (35), the backwards conditional weights are given by,

$$\tilde{w}_n \propto w_n p(\theta_n^+|\theta_n^{(i)}) \mathcal{N}(\tilde{m}_n^-|m_n, \tilde{P}_n^- + P_n). \tag{47}$$

Using these weights, a sample of θ_n may be drawn from the particle distribution of (34), completing the smoothing algorithm. Once sampling has progressed backwards from $n = N \dots 1$, a particle from the smoothing distribution will have been generated. This procedure may then be repeated until sufficient particles have been obtained. The algorithm is summarised in Fig. 3.

The algorithmic complexity of this variable rate particle smoother for conditionally linear-Gaussian models is $O(N_F \times N_S \times N)$. It is possible to reduce this by using an MCMC sampling scheme in the style of [26], which avoids the necessity of calculating the sampling weights for every filter particle.

B. Conditionally Deterministic Smoothing

In a conditionally deterministic system, the entire continuous time state trajectory, $x(t)$, is a function of the changepoint sequence θ and the initial state, x_0. When the changepoint sequence is split into past, θ_n, and future θ_n^+, and the past is altered, this will alter the state trajectory in the future. This is undesirable, as each sampling operation will require the recalculation of state values and observations likelihoods for all future observation times.

In order to achieve conditional independence between past and future observations, consider the augmented changepoint sequence, $\tilde{\theta} = \{\tau_{0:K}, u_{0:K}, x_{0:K}\}$ ($\tilde{\theta}_n$ and $\tilde{\theta}_n^+$ defined in the same manner as before). The changepoint state values, $x_{0:K}$ may be calculated deterministically from θ for each particle. The smoother is then formulated by factorising the augmented sequence posterior distribution as before,

$$p(\tilde{\theta}|y_{1:N}) = p(\tilde{\theta_n}^+|y_{1:N}) p(\tilde{\theta}_n|\tilde{\theta}_n^+, y_{1:N}). \tag{48}$$

Run Rao-Blackwellised variable rate particle filter to approximate $p(\theta_n|y_{1:n})$ with weighted particles $\{\theta_n^{(i)}, w_n^{(i)}\}$ and $\{p(x_n|\theta_n^{(i)}, y_{1:n})\}$ as normal distributions with moments $\{m_n^{(i)}\}$ and $\{P_n^{(i)}\}$. Store all results.
for $i = 1 \dots N_S$ **do**
  Initialise particle using $\theta^{(i)} \sim \sum_j w_N^{(j)} \delta_{\theta^{(j)}}(\theta)$.
  Initialise sufficient statistics $\tilde{m}_N^{(i)}$ and $\tilde{P}_N^{(i)}$ (see text).
  for $n = N-1 \dots 1$ **do**
    Discard $\theta_n^{(i)}$.
    Predict state mean and covariance $\tilde{m}_n^{-(i)}$ and $\tilde{P}_n^{-(i)}$ using 40 to 41.
    for $j = 1 \dots N_P$ **do**
      Calculate weight $\tilde{w}_n^{(j)}$ using (47).
    end for
    Sample $\theta_n^{(i)} \sim \sum_j \tilde{w}_n^{(j)} \delta_{\theta_n^{(j)}}(\theta_n)$.
    Join $\theta^{(i)} \leftarrow \theta_n^{(i)} \cup \theta_n^{+(i)}$.
    Update state mean and covariance $\tilde{m}_n^{(i)}$ and $\tilde{P}_n^{(i)}$ using 42 to 46.
  end for
end for

Fig. 3. Rao-Blackwellised Variable Rate Particle Smoother

Again, this suggests a backwards sequential sampling algorithm for smoothing. The conditional term may be expanded as,

$$p(\tilde{\theta}_n|\tilde{\theta}_n^+, y_{1:N}) \propto p(y_{n+1:N}|\tilde{\theta}_n, \tilde{\theta}_n^+) \times p(\tilde{\theta}_n^+|\tilde{\theta}_n) p(\tilde{\theta}_n|y_{1:n}). \quad (49)$$

This presents a problem. We cannot directly substitute the particle approximation for the filtering distribution because the transition density term for the augmented sequences is given by,

$$p(\tilde{\theta}_n^+|\tilde{\theta}_n) \propto p(\tau_{K_n+1}, u_{K_n+1}|\tau_{K_n}, u_{K_n}) \times \delta_{f(x_{K_n}, u_{K_n}, \tau_{K_n}, \tau_{K_n+1})}(x_{K_n+1}), \quad (50)$$

which will be 0 for almost all filtering particles. Intuitively, we are trying to join two deterministic state trajectories together, but they do not meet up in the middle.

The solution to this problem is provided by an idea from the Sequential Monte Carlo (SMC) samplers of [11]. Rather than sampling $p(\tilde{\theta}_n|\tilde{\theta}_n^+, y_{1:N})$ directly, a value of $\tilde{\theta}_n$ is proposed from the filtering particles, after which a replacement, u'_{K_n}, is proposed for the existing final changepoint parameter, with a value chosen such that the past and future paths meet up. The modified sequence is denoted $\tilde{\theta}'_n$. Such a proposal density may be written as,

$$p(\tilde{\theta}_n|y_{1:n}) q(u_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^+, y_{1:N}) \propto p(y_{1:n}|\tilde{\theta}_n) p(\tilde{\theta}_n) q(u'_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^+, y_{1:N}). \quad (51)$$

The target distribution is augmented by introduction of an artificial density to cover the new changepoint sequence and the discarded parameter.

$$p(\tilde{\theta}'_n|\tilde{\theta}_n^+, y_{1:N}) L(u_{K_n}|\tilde{\theta}'_n, \tilde{\theta}_n^+, y_{1:N}) \propto p(y_{1:N}|\tilde{\theta}'_n, \tilde{\theta}_n^+) p(\tilde{\theta}_n^+|\tilde{\theta}'_n) p(\tilde{\theta}'_n) L(u_{K_n}|\tilde{\theta}'_n, \tilde{\theta}_n^+, y_{1:N}). \quad (52)$$

Clearly this new target distribution admits the desired posterior as a marginal.

For this method to work, a new condition must be imposed on the state dynamics. For any pair of adjacent changepoints, (τ_k, x_k), and (τ_{k+1}, x_{k+1}) it must be possible to calculate a parameter value which results in the transition from the former to the latter. As a rule of thumb, this requires the number of dimensions of u_k to be greater or equal to the number of dimensions of x_k.

The augmented target distribution may be sampled using Markov chain Monte Carlo (MCMC) [27], in a similar manner to the fixed rate smoother of [26]. Metropolis-Hastings (MH) steps are conducted to draw samples, $\{\tilde{\theta}_n, u'_{K_n}\}$, from the target distribution by sampling the proposal and accepting the new values with a given probability.

The ratio of target (52) and proposal (51) densities is given by,

$$\beta_n \propto \frac{p(y_{r_n^-:r_n^+}|\tilde{\theta}'_n)}{p(y_{r_n^-:n}|\tilde{\theta}_n)} \times \frac{p(u'_{K_n}|\tau_{K_n}, \tau_{K_n-1}, u_{K_n-1})}{p(u_{K_n}|\tau_{K_n}, \tau_{K_n-1}, u_{K_n-1})} \times p(\tilde{\theta}_n^+|\tilde{\theta}'_n) \times \frac{L(u_{K_n}|\tilde{\theta}'_n, \tilde{\theta}_n^+, y_{1:N})}{q(u'_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^+, y_{1:N})}, \quad (53)$$

where $r_n^- = \min(m : t_m > \tau_{K_n})$ and $r_n^+ = \max(m : t_m < \tau_{K_n+1})$ are the indexes of the earliest and latest observations to fall between the changepoints each side of t_n.

The proposal density $q(u'_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^+, y_{1:N})$ is constructed so as to be non-zero only where $p(\tilde{\theta}_n^+|\tilde{\theta}'_n)$ is non-zero. This may require it to be a delta function if there is only one feasible value. The artificial conditional term, $L(u_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^+, y_{1:N})$, may be any arbitrary density, but the simplest choice is simply as a uniform distribution so that it cancels out in the MH acceptance probability. The remaining terms are likelihoods and parameter transition density terms, defined by the system equations.

If $\beta_n^{(m-1)}$ is the target-proposal ratio for the current state in the chain, and β_n^* that for the new state, then the acceptance probability is given by,

$$\alpha_n^{(m)} = \min\left(1, \frac{\beta_n^*}{\beta_n^{(m-1)}}\right). \quad (54)$$

Each Markov chain may be initialised with the output from the previous stage. This value is itself a particle from the target distribution (albeit one from an approximation with poor diversity), meaning that no burn-in period is required.

Run variable rate particle filter to approximate $p(\tilde{\theta}_n|y_{1:n})$ with weighted particles $\{\tilde{\theta}_n^{(i)}, w_n^{(i)}\}$. Store all results.
for $i = 1 \dots N_S$ **do**
  Initialise particle using $\tilde{\theta}^{(i)} \sim \sum_j w_N^{(j)} \delta_{\tilde{\theta}^{(j)}}(\tilde{\theta})$.
  for $n = N \dots 1$ **do**
    Initialise chain with $\tilde{\theta}_n^{(i)(0)} \leftarrow \tilde{\theta}_n^{(i)}$.
    for $m = 1 \dots M$ **do**
      Propose a new history $\tilde{\theta}_n^{(i)*} \sim \sum_j w_n^{(j)} \delta_{\tilde{\theta}_n^{(j)}}(\tilde{\theta}_n)$.
      Propose a parameter $u'^{*}_{K_n} \sim q(u_{K_n}|\tilde{\theta}_n, \tilde{\theta}_n^{+}, y_{1:N})$.
      Replace $u^{*}_{K_n}$ with $u'^{*}_{K_n}$ to form $\tilde{\theta}_n'^{(i)*}$.
      Calculate $\alpha_n^{(m)}$ using (53) and (54).
      With probability $\alpha_n^{(m)}$, $\tilde{\theta}_n^{(i)(m)} \leftarrow \tilde{\theta}_n^{(i)*}$.
      Otherwise, $\tilde{\theta}_n^{(i)(m)} \leftarrow \tilde{\theta}_n^{(i)(m-1)}$.
    end for
    Store final sample $\tilde{\theta}_n^{(i)} \leftarrow \tilde{\theta}_n^{(i)(M)}$.
  end for
end for

Fig. 4. MCMC piecewise-deterministic variable rate particle smoother

The algorithm is summarised in Fig. 4.

The number of MH steps, M, at each observation time, n, allows a trade-off of performance against time. Larger values of M will ensure more unique samples in the smoothing approximation, but will also take longer to execute. See [26] for analysis of this trade-off on a fixed rate model. In the simulations discussed in this report, $M = 1$ was used throughout.

V. Simulations

A. Conditionally Linear-Gaussian Model

The Rao-Blackwellised VRPS algorithm was tested on the financial time series model of [4], [5], in which prices of an asset are treated as noisy observations of a latent state, which evolves according to a drift-diffusion with occasional jumps.

The latent state is a vector with two elements, the underlying value of the asset, and the trend followed by this value.

$$\mathbf{x}(t) = [x(t), \dot{x}(t)]^T \tag{55}$$

This evolves continuously according to a jump-diffusion model:

$$d\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\lambda \end{bmatrix} \mathbf{x}(t)dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} d\beta(t) + d\mathbf{J}(t) \tag{56}$$

where λ introduces a mean regression effect on the trend and $\beta(t)$ is standard Brownian motion (with unit diffusion constant). The jump term, $d\mathbf{J}(t)$, is zero everywhere except where jumps occur.

$$d\mathbf{J}(t) = \begin{cases} \mathbf{J}_k & t \in \{\tau_k\} \\ 0 & \text{elsewhere} \end{cases} \tag{57}$$

$$\mathbf{J}_k \sim \mathcal{N}(\mathbf{J}_k|\mathbf{0}, Q_{J,u_k}) \tag{58}$$

Jumps occur at a random set of times, $\{\tau_k\}$, and are one of two types: value jumps, indicated by $u_k = 1$, and trend jumps, indicated by $u_k = 2$.

The jump covariance matrices are,

$$Q_{J,u_k} = \begin{cases} \begin{bmatrix} \sigma_{J1}^2 & 0 \\ 0 & 0 \end{bmatrix} & u_k = 1 \\ \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{J2}^2 \end{bmatrix} & u_k = 2 \end{cases} . \tag{59}$$

This model may be discretised at the observation times using standard methods (see e.g. [18]). Assuming observations of value only and Gaussian observation noise with standard deviation σ_y^2, the resulting discrete time dynamics are described by the following equations:

$$\mathbf{x}_n = A\mathbf{x}_{n-1} + \mathbf{w}_n \tag{60}$$

$$y_n = C\mathbf{x}_n + v_n \tag{61}$$

where the $\mathbf{w}_n$ and v_n are Gaussian random variables with covariance matrixes Q_n and R respectively. The time between observations times is denotes $\Delta t = t_n - t_{n-1}$.

$$A = \begin{bmatrix} 1 & \frac{1}{\lambda}(1 - e^{(-\lambda\Delta t)}) \\ 0 & e^{(-\lambda\Delta t)} \end{bmatrix} \tag{62}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{63}$$

$$Q_n = \begin{cases} Q_D + Q_{J,u_k} & \exists k : \tau_k \in [t_{n-1}, t_n] \\ Q_D & \text{otherwise} \end{cases} \tag{64}$$

$$Q_D = \frac{\sigma^2}{2\lambda} \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \tag{65}$$

$$q_1 = \frac{1}{\lambda^2}(2\lambda\Delta t - (3 - e^{(-\lambda\Delta t)})(1 - e^{(-\lambda\Delta t)})\ ) \tag{66}$$

$$q_2 = \frac{1}{\lambda}(1 - e^{(-\lambda\Delta t)})^2 \tag{67}$$

$$q_3 = 1 - e^{(-2\lambda\Delta t)} \tag{68}$$

$$R = [\sigma_y^2] \tag{69}$$

The times between changepoints were assumed to be exponentially distributed, with equal probability of value and trend jumps.

The algorithms were first tested on artificial data simulated from this model. The following parameters were used: $\Delta t = 0.0017$, $N = 1000$, $\alpha = 20$, $\lambda = 5$, $\sigma = 0.05$, $\sigma_{J1} = 0.005$, $\sigma_{J2} = 0.05$, $\sigma_y = 0.001$.

The filter used $N_P = 100$ particles, and the smoother resampled $N_S = 100$ sequences. Bootstrap proposals were used for the filter.

An example realisation simulated from the model is shown in figure 5.

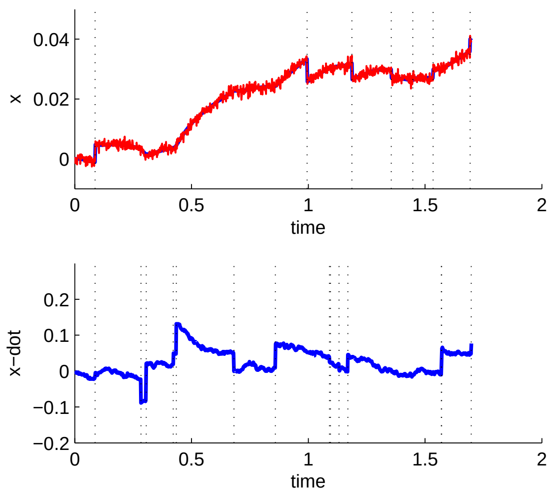


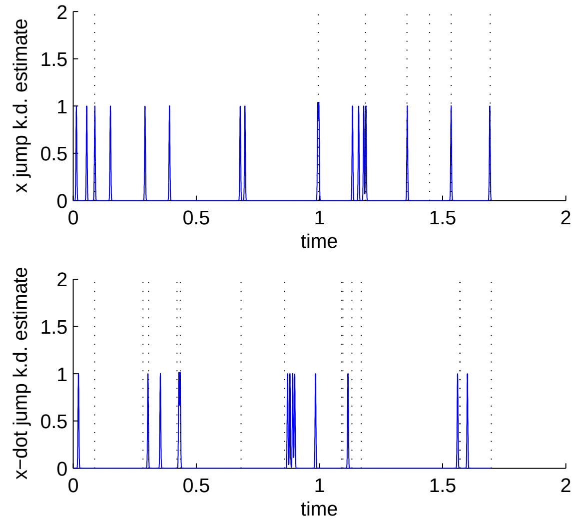
Fig. 5. An example simulated data set, showing value (top) and trend (bottom). Value observations are overlayed. Jump times are shown as dotted vertical lines.

Both the smoother and the final processing step of the filter produce a particle approximation to the changepoint sequence posterior, $p(\theta|y_{1:N})$. It is actually no simple task to compare the quality of these approximations. Each particle consists of a list of changepoint times and types, and there is no obvious way to take an average over the set to produce a single estimate. Furthermore, once a best estimate has been calculated, it is not obvious how the error should be assessed, as it is not trivial to define a distance metric between lists of varying lengths.

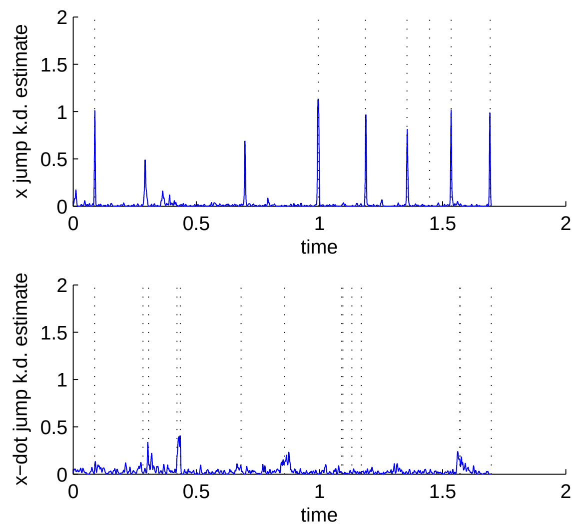
For a first, qualitative comparison of the changepoint sequence distributions, it is possible to generate kernel density estimates for the changepoint times. These are constructed by simply adding a (unit amplitude) Gaussian kernel for each changepoint present in each particle and scaling by the number of particles. The resulting function provides an approximate measure of the likelihood of finding a jump at a given time. These are shown in figure 6 for the example data set.

It is immediately apparent that the approximation produced by the filter lacks diversity. The jump time kernel density estimate comprises peaks with magnitude 1, with regions of 0 between. This is a result of every particle containing an almost identical list of changepoints, as a result of resampling. In contrast, the smoother is clearly providing a less degenerate particle representation, and generally contains larger kernel density peaks at the times of larger jumps. Furthermore, the kernel density for the trend jump times contains broader peaks, capturing the fact that because the trend is not observed directly, the jumps are harder to localise precisely.

A further comparison of the algorithms is possible via the state estimates they generate. Here we can compare three options: the filtering results, using the VRPF and a Kalman filter for the state estimates; the filter-smoother results, using the final VRPF approximation for the changepoint sequence and a Rauch-Tung-Striebel (RTS) smoother for the state estimates; and the full smoothing results, using the VRPS



(a)



(b)

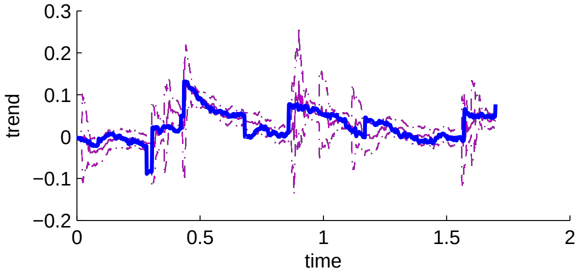
Fig. 6. Filter (a) and smoother (b) kernel density estimates for value (top) and trend (bottom) jump times. Correct times overlayed as dotted vertical lines.

followed by an RTS smoother. For the example data set, the trend estimates are shown in figure 7 (The estimates of value are less informative, as this quantity is observed). Again, the improved particle diversity of the smoother is apparent.

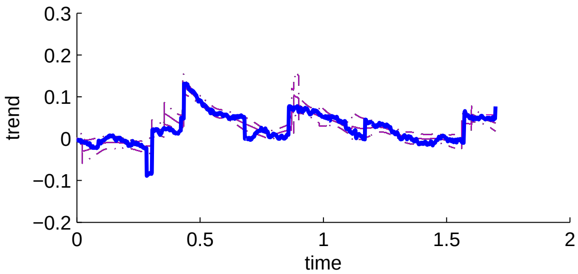
For a quantitative comparison, the algorithms were tested on 10 realisations from the model, each of 1000 time steps, and the following statistics were calculated for each:

- The number of unique changepoint sequences. This is a measure of particle diversity of the approximation.
- The number of unique changepoint times. Another measure of particle diversity.
- The root-mean-square error (RMSE) of the mean state estimate. The mean state estimate is the average of the Gaussian means from all the particles.

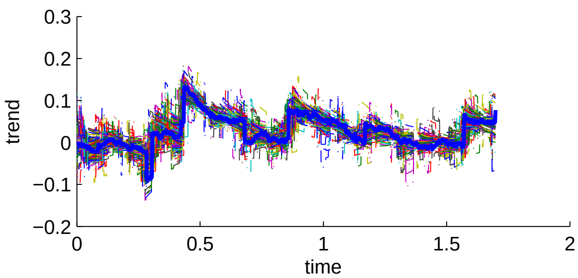
The results are shown in tables I and II.



(a)



(b)



(c)

Fig. 7. Trend estimates using RBVRPF + Kalman filter (a) RBVRPF + RTS smoother (b) and RBVRPS + RTS smoother. Solid lines show means of each particle. Dashed lines show mean ± 2 standard deviations.

TABLE I
CHANGEPOINT SEQUENCE ESTIMATION PERFORMANCE.

	RBVRPF	RBVRPS
mean unique no. sequences	7.8	100
mean unique no. jump times	47.8	1076.7

The new RBVRPS algorithm outperforms the filter in terms of both state estimation accuracy and particle diversity.

TABLE II
STATE ESTIMATION PERFORMANCE.

	RBVRPF and KF	RBVRPF and RTS	RBVRPS and RTS
mean value estimate RMSE	5.31×10^{-4}	4.48×10^{-4}	4.16×10^{-4}
mean trend estimate RMSE	2.56×10^{-2}	1.79×10^{-2}	1.49×10^{-2}

B. Conditionally Deterministic Model

The piecewise-deterministic VRPS was tested using a 2-dimensional dynamic model for manoeuvring vehicles [28]. The target is modelled as a particle subject to perpendicular forces tangential and normal to the direction of motion. These forces are held constant between changepoint times, $\tau_{1:K}$, which represent the beginnings and ends of manoeuvres.

The state vector consists of cartesian position, $x(t)$ and $y(t)$, plus bearing, $\psi(t)$, and speed, $\dot{s}(t)$.

$$\mathbf{x}(t) = [x(t), y(t), \psi(t), \dot{s}(t)]^T \tag{70}$$

In addition to the tangential and normal accelerations, $a_{T,k}$ and $a_{N,k}$, we introduce two linear velocity terms, $d_{X,k}$ and $d_{Y,k}$. Without these, the model could not be used for smoothing, because for an arbitrary pair of adjacent changepoints (τ_k, $\mathbf{x}_k$) and (τ_{k+1}, $\mathbf{x}_{k+1}$), there would not in general be a pair of accelerations which achieved the required transition. These linear velocity terms may be considered to be merely relaxation terms, allowing arbitrary past and future trajectories to be joined, or they may represent real physical effects, such as wind (for aircraft) or currents (for boats). Together, these four variables make up the vector of motion parameters,

$$\mathbf{u}_k = [a_{T,k}, a_{N,k}, d_{X,k}, d_{Y,k}]^T \tag{71}$$

The target dynamics are described by four differential equations.

$$\ddot{s}_t = a_{T,K(t)} \tag{72}$$
$$\dot{s}_t \dot{\psi}_t = a_{N,K(t)} \tag{73}$$
$$\dot{x}_t = \dot{s}_t \cos(\psi_t) + d_{X,K(t)} \tag{74}$$
$$\dot{y}_t = \dot{s}_t \sin(\psi_t) + d_{Y,K(t)}. \tag{75}$$

Solving these yields the following state equations (where $\Delta\ t = t - \tau_{K(t)}$):

$$\dot{s}(t) = \dot{s}_{K(t)} + a_{T,k}\Delta t \tag{76}$$

$$\psi(t) = \psi_{K(t)} + \frac{a_{N,k}}{a_{T,k}} \log\left(\frac{\dot{s}(t)}{\dot{s}_{K(t)}}\right) \tag{77}$$

$$\begin{aligned} x(t) = {} & x_{K(t)} + d_{X,K(t)}\Delta t \\ & + \frac{\dot{s}(t)^2}{4a_{T,k}^2 + a_{N,k}^2}\left[a_{N,k}\sin(\psi(t)) + 2a_{T,k}\cos(\psi(t))\right] \\ & - \frac{\dot{s}_{K(t)}^2}{4a_{T,k}^2 + a_{N,k}^2}\left[a_{N,k}\sin(\psi_{K(t)}) + 2a_{T,k}\cos(\psi_{K(t)})\right] \end{aligned} \tag{78}$$

$$\begin{aligned} y(t) = {} & y_{K(t)} + d_{Y,K(t)}\Delta t \\ & + \frac{\dot{s}(t)^2}{4a_{T,k}^2 + a_{N,k}^2}\left[-a_{N,k}\cos(\psi(t)) + 2a_{T,k}\sin(\psi(t))\right] \\ & - \frac{\dot{s}_{K(t)}^2}{4a_{T,k}^2 + a_{N,k}^2}\left[-a_{N,k}\cos(\psi_{K(t)}) + 2a_{T,k}\sin(\psi_{K(t)})\right]. \end{aligned} \tag{79}$$

The changepoint motion parameters are assumed to be independent and zero-mean Gaussian distributed. The times between changepoints are modelled as gamma distributed. Observations are made via a radar-style bearing and range measurement model with Gaussian noise.

The filtering and smoothing algorithms were tested on artificial data simulated from the model. Observations were

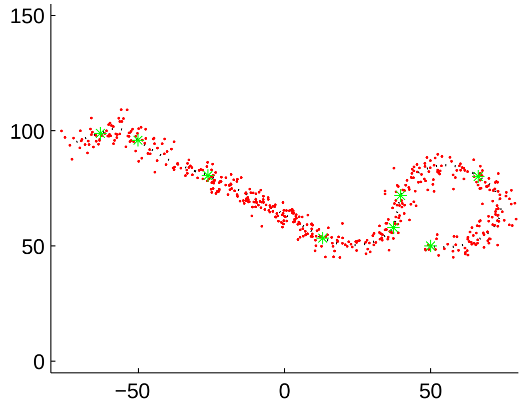


Fig. 8. An example simulated data set, showing true position (dashed line), observations (dots) and changepoint positions (crosses).

TABLE III
STATE ESTIMATION PERFORMANCE.

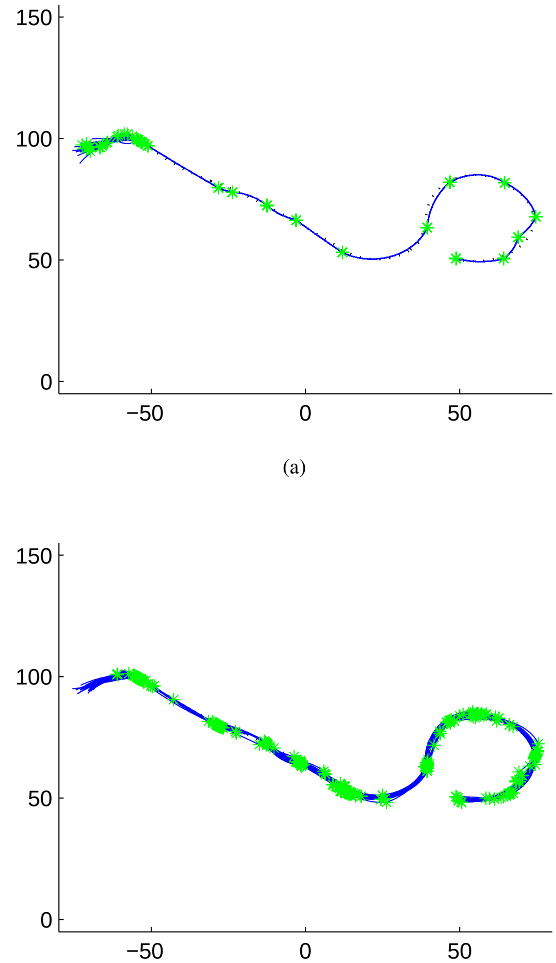
	VRPF	Final frame VRPF	VRPS
mean position estimate RMSE	2.14	1.36	1.12
mean velocity estimate RMSE	1.89	1.03	0.85

generated every 0.1. The standard deviations of tangential and normal accelerations were set to 0.01 and 1 respectively, and those for the drift velocities to 1. The parameters of the Gamma distribution for inter-changepoint times were 5 and 1 for the shape and scale respectively. Observation noise standard deviations were 0.1 and 0.25° respectively.

The filter and smoother were each used to generate 50 particles, with the filter employing resample-move steps with proposals based on an unscented transform [29] approximation to the optimal proposal density.

An example trajectory is shown in Fig. 8, with the results of the VRPF and VRPS algorithms in Fig.9. The improved particle diversity achieved by the smoothing algorithm is apparent.

Performance was evaluated by testing on 10 scenarios, each with 500 observations. At each observation time, an estimate of the position and velocity is made by taking the average of values from the array of particles. The resulting root mean square errors (RMSEs) are displayed in table III, demonstrating an improvement in accuracy from the smoother. The smoother also provides a more accurate estimate of the number of changepoints in each test, with an average error of 0.6 compared to 2.0 from the final frame filter approximation. In addition, the mean number of unique particle estimates at each time step is shown in Fig. 10. This illustrates the main advantage of the smoother algorithm — the increased number of unique particles in the approximation means a better characterisation of the state distribution, allowing, for example, calculation of state covariance measures.



(a)

(b)

Fig. 9. Filter (a) and smoother (b) particle estimates of position (solid lines), with particle changepoint positions (crosses).

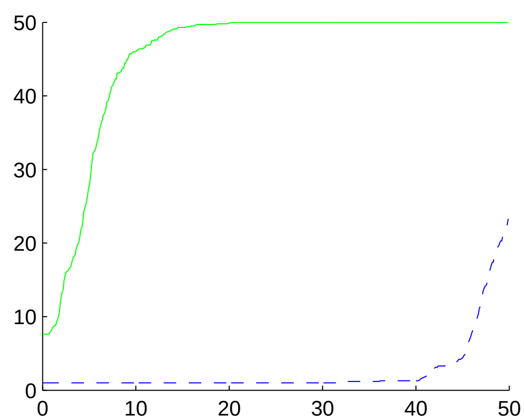


Fig. 10. Mean number of unique particles in the final filter (dashed) and smoother (solid) estimates.

VI. CONCLUSIONS

New smoothing algorithms have been introduced for variable rate models which have either linear-Gaussian or deterministic dynamics conditional on a set of unknown changepoints. For the linear-Gaussian case, the algorithm employs Rao-Blackwellisation, using particle methods to estimate the nonlinear changepoint sequence, and Kalman filtering/smoothing to estimate the linear state components. For the deterministic case, smoothing is achieved by sampling an augmented target distribution using MCMC. Simulations demonstrate that the smoothers generate approximation with improved particle diversity and accuracy when compared to the filters.

References

[1] M. Jacobsen, *Point process theory and applications: marked point and piecewise deterministic processes*, C. Newman and S. Resnick, Eds. Birkhauser, 2006.

[2] S. Godsill, J. Vermaak, W. Ng, and J. Li, "Models and algorithms for tracking of maneuvering objects using variable rate particle filters," *Proceedings of the IEEE*, vol. 95, no. 5, pp. 925 –952, May 2007.

[3] N. Whiteley, A. M. Johansen, and S. Godsill, "Monte carlo filtering of piecewise deterministic processes," *Journal of Computational and Graphical Statistics*, vol. 20, no. 1, pp. 119–139, 2011. [Online]. Available: http://pubs.amstat.org/doi/abs/10.1198/jcgs.2009.08052

[4] S. Godsill, "Particle filters for continuous-time jump models in tracking applications," *ESAIM: Proc.*, vol. 19, pp. 39–52, 2007. [Online]. Available: http://dx.doi.org/10.1051/proc:071907

[5] H. L. Christensen, J. Murphy, and S. J. Godsill, "Forecasting high-frequency futures returns using online langevin dynamics," *submitted to IEEE journal of selected topics in signal processing*, vol. under review, p. under review, 2012.

[6] N. Gordon, D. Salmond, and A. Smith, "Novel approach to nonlinear/non-Gaussian Bayesian state estimation," *Radar and Signal Processing, IEE Proceedings F*, vol. 140, no. 2, pp. 107 –113, April 1993.

[7] A. Doucet, S. Godsill, and C. Andrieu, "On sequential Monte Carlo sampling methods for Bayesian filtering," *Statistics and Computing*, vol. 10, pp. 197–208, 2000, 10.1023/A:1008935410038. [Online]. Available: http://dx.doi.org/10.1023/A:1008935410038

[8] S. J. Godsill, A. Doucet, and M. West, "Monte Carlo smoothing for nonlinear time series," *Journal of the American Statistical Association*, vol. 99, no. 465, pp. 156–168, 2004. [Online]. Available: http://pubs.amstat.org/doi/abs/10.1198/016214504000000151

[9] O. Cappé, S. Godsill, and E. Moulines, "An overview of existing methods and recent advances in sequential Monte Carlo," *Proceedings of the IEEE*, vol. 95, no. 5, pp. 899–924, 2007.

[10] A. Doucet and A. M. Johansen, "A tutorial on particle filtering and smoothing: Fifteen years later," in *The Oxford Handbook of Nonlinear Filtering*, D. Crisan and B. Rozovsky, Eds. Oxford University Press, 2009. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.157.772&rep=rep1&type=pdf

[11] P. Del Moral, A. Doucet, and A. Jasra, "Sequential Monte Carlo samplers," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 68, no. 3, pp. 411–436, 2006.

[12] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal Of Basic Engineering*, vol. 82, no. Series D, pp. 35–45, 1960. [Online]. Available: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.129.6247&rep=rep1&type=pdf

[13] H. Rauch, F. Tung, and C. Striebel, "Maximum likelihood estimates of linear dynamic systems," *AIAA journal*, vol. 3, no. 8, pp. 1445–1450, 1965.

[14] D. Fraser and J. Potter, "The optimum linear smoother as a combination of two optimum linear filters," *Automatic Control, IEEE Transactions on*, vol. 14, no. 4, pp. 387 – 390, aug 1969.

[15] X. Li and V. Jilkov, "Survey of maneuvering target tracking. part i: Dynamic models," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1333–1364, 2003.

[16] M. K. Pitt and N. Shephard, "Filtering via simulation: Auxiliary particle filters," *Journal of the American Statistical Association*, vol. 94, no. 446, pp. pp. 590–599, 1999. [Online]. Available: http://www.jstor.org/stable/2670179

[17] R. Douc, O. Cappe, and E. Moulines, "Comparison of resampling schemes for particle filtering," in *Image and Signal Processing and Analysis, 2005. ISPA 2005. Proceedings of the 4th International Symposium on*, sept. 2005, pp. 64 – 69.

[18] M. S. Grewal and A. P. Andrews, *Kalman Filtering: Theory and Practice Using MATLAB*. John Wiley & Sons, Inc., 2002. [Online]. Available: http://dx.doi.org/10.1002/0471266388

[19] W. R. Gilks and C. Berzuini, "Following a moving target-Monte Carlo inference for dynamic Bayesian models," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 63, no. 1, pp. 127–146, 2001. [Online]. Available: http://dx.doi.org/10.1111/1467-9868.00280

[20] P. Green, "Reversible jump Markov chain Monte Carlo computation and bayesian model determination," *Biometrika*, vol. 82, no. 4, p. 711, 1995.

[21] K. Gilholm and N. O. Everett, "A novel algorithm for tracking targets using manoeuvre models," in *Target Tracking and Data Fusion: Algorithms and Applications, 2008 IET Seminar on*, april 2008, pp. 143 –148.

[22] G. Kitagawa, "Monte Carlo filter and smoother for non-Gaussian nonlinear state space models," *Journal of Computational and Graphical Statistics*, vol. 5, no. 1, pp. pp. 1–25, 1996. [Online]. Available: http://www.jstor.org/stable/1390750

[23] S. Sarkka, P. Bunch, and S. Godsill, "A backward-simulation based rao-blackwellized particle smoother for conditionally linear gaussian models," in *Preprint submitted to 16th IFAC Symposium on System Identification.*, 2012.

[24] G. Kitagawa, "The two-filter formula for smoothing and an implementation of the gaussian-sum smoother," *Annals of the Institute of Statistical Mathematics*, vol. 46, pp. 605–623, 1994, 10.1007/BF00773470. [Online]. Available: http://dx.doi.org/10.1007/BF00773470

[25] A. Gelb, Ed., *Applied optimal estimation*. MIT Press, 1974, good intro to RTS and two-filter linear-Gaussian smoothers.

[26] P. Bunch and S. Godsill, "Improved particle approximations to the joint smoothing distribution using markov chain monte carlo," *submitted for publication*, 2012.

[27] W. Gilks, S. Richardson, and D. Spiegelhalter, *Markov chain Monte Carlo in practice*. Chapman and Hall, 1996.

[28] P. Bunch and S. Godsill, "Dynamical models for tracking with the variable rate particle filter," in *submitted for publication*, 2012.

[29] S. Julier and J. Uhlmann, "Unscented filtering and nonlinear estimation," *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401 – 422, Mar. 2004.

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