A Method to Resolve Ambiguities for Sparse Array Using Multiple Joint MUSIC Algorithm

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Abstract

Compared with conventional Uniform Linear Array (ULA), sparse linear array could provide better direction-finding performance with fewer sensors and lower cost. However, the applications of sparse array is vulnerable on account of its manifold ambiguities. Aiming to resolve this problem, a method based on MUltiple SIgnal Classification (MUSIC) algorithm employing two crossly placed indentical sparse ULAs is proposed. Firstly, after modeling and examining the ambiguities variation with the included angle within the crossly placed arrays, a criteria is proven to indicate the condition that makes mismatch spurious peak sets. Next, based on the criteria and analyzing the linear independent characteristic, a multiple joint MUSIC (MJ-MUSIC) algorithm is proposed. Finally, the numerical simulations demonstrate the effectiveness and feasibility of this method.

Index Terms

DOA, Direction Finding Ambiguities, MUSIC Algorithm.

I. Introduction

Direction-finding resolution has great influences on the performance of Direction-of-Arrival (DOA) estimation. In general, the direction-finding resolution is not only depending on the signal-to-noise ratio (SNR) and the estimation precision of the estimated vector signal correlation matrix, but also significantly relating to the effective array aperture. To enlarge the aperture, the conventional approach is to increase the number of array element while keeping the element spacing less than a half of wavelength of incident signal. But this approach may bring excessive components to the processing system and certainly rises the computations and costs. Another method is to widen the element spacing uniformly or non-uniformly instead of increasing the element number. The arrays constituted by the latter method are usually called as sparse array. However, as Schmidt deliberated in [1], the sparse arrays usually cause direction-finding ambiguous problem due to the deficiency of linear independent among their steering vectors.

Nevertheless, considering the advantages of sparse array, the study on the direction-finding ambiguity still has aroused wide concern and research. The ambiguities in linear array was studied to identify the ambiguous sets associated with a given geometry in [2]. A general framework for the analysis of ambiguities is proposed in [3]. A theorem for characterizing inherent ambiguities associated with cross arrays was derived in [4]. The types of ambiguities as trivial and non-trivial and their classification definitions are introduced in [5]. The further mathematical connotations of ambiguities were studied in [6]–[9].

Besides, many methods and theories to avoid or resolve the ambiguities have been proposed. Athley F. and Gzaazh H. proposed a method that generating many random located elements and employing the Weiss-Weinstein Bound or Cramer-Rao Bound as optimization criteria, the elements which make less contribution to estimation are continually rejected from the original array to obtain an optimal array [10], [11]. These two methods usually need a proper cost function and occupy large scale of computations. Xiumin S. employing spatial up sampling technique to insert virtual sensor to resolve the ambiguities. This method needs prior information to establish a proper interpolation algorithm, which limits its feasibility in practice [12]. The main drawback of this method is that the power estimation algorithm normally performing after all the DOAs obtained. In some time pursuing circumstance, it may be unsuitable. Hai-Lang S.s method is achieved a set of DOAs in original array, then slide the sensor slightly to form a new array to perform a second estimation, finally combined the two DOAs to resolve ambiguity [13]. The sensor sliding operation in this method usually needs time and the precision of sliding influence the finally estimation expressively. Ziyuan H. et al. offered an algorithm to distinguish the real DOAs from ambiguities based on estimating the power at every DOA [14]. In this paper, we propose a new method to solve the direction-finding ambiguity.

The essence of this method is to construct an X-shaped array from two identical sparse Uniform Linear Arrays (ULAs) and utilize the relationship between the included angle in this array and the DOAs of each sub-array to counteract the spurious peaks.

The reminder of this paper is organized as follows. In Section II, the data model, MUtiple SIgnal Classification algorithem (MUSIC) and the classification of ambiguities are discussed in details. The new method and a criterion are derived in Section III. Numerical simulations and their results are shown in Section IV to demonstrate the effectiveness and feasibility of the proposed method. Conclusions are repsented in Section V. The detailed derivation of proposed criterion is shown in Appendix A.

II. Problem Formulation and Preliminaries

A. Array and Signal Model

Consider an N-element Uniform Linear Array (ULA) with spacing of each sensor ρ measured in a wavelength unit. Let D denotes the array aperture, the relationship of D,N and ρ is as shown as

$$D = (N-1)\rho \tag{1}$$

Assume $M(M < N)$ narrowband signals are intruding on the array from far field with angles $\theta_1, \theta_2, \ldots, \theta_M; \theta_i \in [0, 180^\circ), i = 1 \sim M$ respectively. Let $\mathbf{X}(n) = [x_1(n), x_2(n), \ldots, x_N(n)]^T$ denotes the $N \times 1$ vector output of this array and the $x_i(n), i = 1 \sim N$ represents the output of the ith sensor at time n correspondingly. The superscript $[\bullet]^T$ denotes a transpose. In this case, $\mathbf{X}(n)$ can be expressed as

$$\mathbf{X}(n) = \mathbf{AS}(n) + \mathbf{N}(n) \tag{2}$$

where $\mathbf{N}(n)$ is an $N \times 1$ vector defined by $\mathbf{N}(n) = [N_1(n), N_2(n), \ldots, N_N(n)]^T$ and $N_i(n), i = 1 \sim N$ represents the noise of the ith sensor at time n. Let the noise sources of each sensor are uncorrelated independent and identically distributed Gaussian sources with zero mean. The array manifold matrix $\mathbf{A} \in \mathbb{C}^{N \times M}$ is a combanation of M steering vectors in columns which is expressed as Eq.(3). The steering vector denotes the delayed phases at each sensor of each intruded signal from angle θ as shown as Eq.(4).

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}(\theta_1) & \mathbf{a}(\theta_2) & \ldots & \mathbf{a}(\theta_M) \end{bmatrix} \tag{3}$$

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 & e^{-j2\pi\rho\cos\theta} & \ldots & e^{-j2\pi(N-1)\rho\cos\theta} \end{bmatrix}^T \tag{4}$$

$\mathbf{S}(n) \in \mathbb{R}^{N \times 1}$ in Eq.(2) is a $N \times 1$ real vector which denotes the incident signals at time n. It can be expanded as Eq.(5). Here we assume these M incident signals are uncorrelated to each other.

$$\mathbf{S}(n) = \begin{bmatrix} s_1(n) & s_2(n) & \ldots & s_M(n) \end{bmatrix} \tag{5}$$

Let $E[\bullet]$ and the superscript $[\bullet]^H$ denote the expection operator and the conjugate transpose operator repectively, the correlation matrix $\mathbf{R_{XX}}$ of $\mathbf{X}(n)$ is

$$\begin{aligned} \mathbf{R_{XX}} &= E[\mathbf{X}(n)\mathbf{X}^H(n)] \\ &= \mathbf{A}E[\mathbf{SS}^H]\mathbf{A}^H + E[\mathbf{NN}^H] \\ &= \mathbf{AR_{SS}A}^H + \sigma^2\mathbf{I} \end{aligned} \tag{6}$$

where $\mathbf{R_{SS}}$ is the correlation matrix of incident signal vector $\mathbf{S}(n)$, σ^2 is the average power of noise $N_i(n), i = 1 \sim N$ and $\mathbf{I}$ is an $N \times N$ unit matrix.

B. MUSIC Algorithm

Noticed that both $\mathbf{R_{XX}}$ and $\mathbf{R_{SS}}$ in Eq.(6) are positive definite matrixes. In general, the array manifold $\mathbf{A}$ is Vandermonde matrix. Thus $rank(\mathbf{AR_{SS}A}^H) = rank(\mathbf{R_{SS}}) = M$ and $rank(\mathbf{R_{XX}}) = N$. Assume the eigenvalues of $\mathbf{AR_{SS}A}^H$ are $\xi_1, \xi_2, \ldots, \xi_M; 0, \ldots, 0$ where $\xi_i > 0, i = 1 \sim M$ and the zero is an $N - M$ repeated eigenvalue. Therefore, the eigenvalues of $\mathbf{R_{XX}}$ are $\xi_1 + \sigma^2, \xi_2 + \sigma^2, \ldots, \xi_M + \sigma^2; \sigma^2, \ldots, \sigma^2$ correspondingly, where the σ^2 is an $N - M$ repeated eigenvalue.

Assume $N \times 1$ vectors $\mathbf{u}_1, \ldots, \mathbf{u}_M; \mathbf{u}_{M+1}, \ldots, \mathbf{u}_N$ are the eigenvectors corresponding to the eigenvalues $\xi_1 + \sigma^2, \xi_2 + \sigma^2, \ldots, \xi_M + \sigma^2; \sigma^2, \ldots, \sigma^2$ repectively. Those eigenvectors corresponds to the larger M eigenvalues could compose a space called signal subspace. Similarly, the remaining $M - N$ eigenvectors, which only relate to the noise in each sensor, could compose a space called noise subspace. Hence, the $span\{\mathbf{u}_1, \ldots, \mathbf{u}_M\} \perp span\{\mathbf{u}_{M+1}, \ldots, \mathbf{u}_N\}$ on account of the differences in eigenvalues.

$$P_{MUSIC}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{UU}^H\mathbf{a}(\theta)}, \theta \in [0, 180^\circ) \tag{7}$$

Therefore, the MUSIC specturm, which is a scalar function of θ representing the estimation of DOA, can be obtained from Eq.(7), where $\mathbf{U} = [\mathbf{u}_{M+1}, \ldots, \mathbf{u}_N]$ is known as noise base. In general, the peaks in MUSIC spectrum represent the estimated DOAs. However, some peaks may appear at false angles, also known as spurious peaks. This phenomenon usually states that there has a direction finding ambiguity problem.

C. Causation and Classification of the Ambiguities

From Eq.(7), the MUSIC spectrum is varying with the steering vector $\mathbf{a}(\theta)$ which is an array-geometry-related mapping from a variable $\theta \in [0, 180^\circ)$ to $\mathbf{a}(\theta) \in \mathbb{C}^{N\times 1}$. In order to obtain a trustable DOA estimation, it is improtant to guarantee the steering vectors at every θ are distinguishable. Obviously, if the above mapping is one-to-one, the steering vectors are unique to each other and this direction finding problem based on array measurement has an unique solution.

As a rule, the one-to-one mapping implying the sensor spacing of an ULA should meets the requirement as

$$\rho \leq 0.5 \tag{8}$$

However, once the mapping violated the one-to-one relationship, like the sparse ULA couldn't satisfy Eq.(8), the ambiguities occures. The ambiguities can be classified into two categories: trivial ambiguities and non-trivial ambiguities based on their linear combination properties.

A trivial ambiguity means that there at least exists a couple of angles θ_r and $\tilde{\theta}$ with $\theta_r \neq \tilde{\theta}$. The relationship between the steering vectors obtained by θ_r and $\tilde{\theta}$ satisfing

$$\mathbf{a}(\tilde{\theta}) = \kappa_r \mathbf{a}(\theta_r) \tag{9}$$

where $\kappa_p \in \mathbb{C}, \kappa_p \neq 0$, κ_p at spurious peak and κ_r at real peak. Eq.(9) indicates that the $\mathbf{a}(\theta_r)$ and $\mathbf{a}(\tilde{\theta})$ are collinear vectors in $\mathbb{C}^{N\times 1}$.

An non-trivial ambiguity was defined as that there exists a set of angles $\{\theta_i\}, i > 2$ and their corresponding steering vectors are $\{\mathbf{a}(\theta_i)\}$. Those steering vectors satisfy

$$\mathbf{a}(\tilde{\theta}) = \sum_{r\neq p} \kappa_r \mathbf{a}(\theta_r) \tag{10}$$

where $\theta_r, \tilde{\theta} \in \{\theta_i\}$, $\kappa_p \in \mathbb{C}, \kappa_p \neq 0$, κ_p at spurious peak and κ_r at real peaks. Eq.(10) indicates that a steering vector obtained by any angle in $\{\theta_i\}$ could be linear expressed by the steering vectors obtained by the rest angles in $\{\theta_i\}$.

From above, no matter what kind an ambiguity is, it's essence can be interpreted as the steering vector at spurious peaks being a linear combination of other steering vectors at real peaks after all.

III. Resolving Ambiguities Using Multiple Joint MUSIC Algorithm

A. Non-linear Angle Variation of Spurious Peaks

Consider the same model mentioned above except element spacing $\rho > 0.5$, known as a sparse ULA. Because of the existence of minifold ambiguities problem, assume the steering vectors of the DOAs obtained in this array can be divided into $\{\mathbf{a}(\theta_i)\}, i = 1 \sim M, \theta_i \in [0, 180^\circ)$ and $\{\mathbf{a}(\tilde{\theta})\}, \tilde{\theta} \in \{\tilde{\Theta}\} \subset [0, 180^\circ) | \tilde{\theta} \notin \{\theta_1, \ldots, \theta_M\}$. Where $\{\mathbf{a}(\theta_i)\}$ represents the steering vectors at real peaks $\{\theta_1, \ldots, \theta_M\}$ and $\{\mathbf{a}(\tilde{\theta})\}$ represents the steering vectors at spurious peaks. From Eq.(9) and Eq.(10), an ambiguous steering vector should be a linear combination of the real incident steering vectors as

$$\begin{aligned} \mathbf{a}(\tilde{\theta}) &= \sum_{i=1}^{M} \kappa_i \mathbf{a}(\theta_i) \\ &= \begin{bmatrix} \sum_{i=1}^{M} \kappa_i \\ \sum_{i=1}^{M} \kappa_i e^{-j2\pi\rho\cos\theta_i} \\ \vdots \\ \sum_{i=1}^{M} \kappa_i e^{-j2\pi(N-1)\rho\cos\theta_i} \end{bmatrix} \end{aligned} \tag{11}$$

Eq.(11) implies a trivial ambiguity if and only if there was only one $\kappa_i \neq 0$. Correspondingly, Eq.(11) represents an non-trivial ambiguity if there were more than one $\kappa_i \neq 0$.

Let $Array_{original}$ denotes this sparse ULA. Imaging only $Array_{original}$ rotates colckwisely around its geometry center with $\angle\varphi < 90^\circ$ to form a new array called $Array_{+\varphi}$ and the incident signal sources maintain the same. Noticed that for $Array_{+\varphi}$, the M signals would propagate from $\{\theta_1 + \varphi, \ldots, \theta_M + \varphi\}$ to it and change along with $\angle\varphi$ linearly. Denote the

DOAs' spurious peaks of $Array_{+\varphi}$ as $\hat{\theta}_{+\varphi} \in \{\hat{\Theta}_{+\varphi}\}$ and the complex coefficient as $\gamma_{i,+\varphi}$. Similarly to Eq.(11), an ambiguous steering vector of $Array_{+\varphi}$ is

$$\begin{aligned}
\mathbf{a}(\hat{\theta}_{+\varphi}) &= \sum_{i=1}^{M} \gamma_{i,+\varphi}\mathbf{a}(\theta_i + \varphi) \\
&= \begin{bmatrix} \sum_{i=1}^{M} \gamma_{i,+\varphi} \\ \sum_{i=1}^{M} \gamma_{i,+\varphi} e^{-j2\pi\rho\cos(\theta_i+\varphi)} \\ \vdots \\ \sum_{i=1}^{M} \gamma_{i,+\varphi} e^{-j2\pi(N-1)\rho\cos(\theta_i+\varphi)} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{M} \gamma_{i,+\varphi} \\ \sum_{i=1}^{M} \gamma_{i,+\varphi} e^{-j2\pi\rho\cos\theta_i\cos\varphi} e^{j2\pi\rho\sin\theta_i\sin\varphi} \\ \vdots \\ \sum_{i=1}^{M} \gamma_{i,+\varphi} e^{-j2\pi(N-1)\rho\cos\theta_i\cos\varphi} e^{j2\pi(N-1)\rho\sin\theta_i\sin\varphi} \end{bmatrix}
\end{aligned} \tag{12}$$

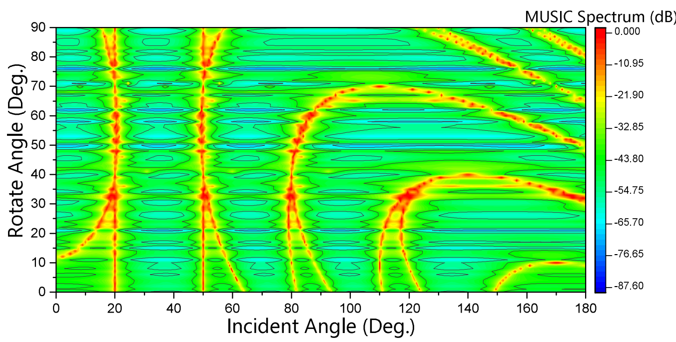

Fig. 1. The fixed DOAs of $Array_{+\varphi}$ varying with $\angle\varphi$ from $0°$ to $90°$. $N = 3$, $\rho = 2$, $M = 2$, Incident Angle $= [20°, 50°]$, SNR $= 20dB$, Snapshots $= 5000$.

From the expaned vector expression in Eq.(12), it is obvious that the $\hat{\theta}$ related to spurious peak by no means changing linearly like the signal-incident angles do. Imaging the $Array_{+\varphi}$ keeps clockwisely rotating from $\angle\varphi = 0°$ (the same as $Array_{original}$) to $\angle\varphi = 90°$ (orthogonal to $Array_{original}$) and perform an DOA estimation at each $\angle\varphi$.

Fig.1 illustrates two signals intruding to a rotating 2-element ULA with $\rho = 2$ from $20°$ and $50°$. The X-axis represents the incident angle $\angle\theta$ in degree, the Y-axis represents the ULA rotated angle $\angle\varphi$ in degree and the color represents the MUSIC spectrum magnitude in dB. By transforming or fixing the DOAs at every $\angle\varphi$ by minusing its corresponding rotatory angle, the real peaks maintain and spurious peaks change non-linearly. In this paper, we call this phenomenon the non-linear angle variation of spurious peaks.

B. Rotatory Symmetric Ambiguity Cancelling Criteria

As shown in Fig.2, consider two identical sparse ULAs, $Array_{+\varphi}$ and $Array_{-\varphi}$ center-symmetrically crossly placed to form a new X-shaped array and their common geometry center becomes the point of intersection. $[\bullet]_{\pm\varphi}$ denotes the sub-array with this subscript rotated clockwisely or anti-clockwisely in $\angle\varphi$ refering to the horizontal one of their symmetry axises. Where the soild lines and soild circles denotes the N-element sub-arrays like $Array_{+\varphi}$ and $Array_{-\varphi}$, the dash line and dashed circles on the horizontal symmetry axis represents a virtual reference sub-array $Array_{original}$ and the arrowed dash lines with different color indicate signals intruding to this X-shaped array from different angles, only two of M signals has been marked for convenience. Refering to $Array_{original}$, signals intruding to this X-shaped array from $\theta_1, \theta_2, \ldots, \theta_M$. Apostrophe marks stand in the abbreviatory array sensors.

Denotes the DOAs' spurious peaks of $Array_{-\varphi}$ as $\hat{\theta}_{-\varphi} \in \{\hat{\Theta}_{-\varphi}\}$ and the complex coefficient as $\gamma_{i,-\varphi}$. Based on the non-linear angle variation of spurious peaks property of rotated arrays, to resolve the ambiguities occured in those two sub-arrays is to try to find a proper $\angle\varphi$ to combin the DOAs of each sub-array, which could counteract the spurious peaks in MUSIC spectrum. We obtained a criteria (Detailed proof is shown in Appendix A) that reveals the relationship between element spacing ρ and the rotated angle φ:

Criteria 1 (Rotatory Symmetric Ambiguity Cancellation Criteria): For the maintaining incident signals, $\{\hat{\Theta}_{-\varphi}\}$ is the spurious peak set of a sparse ULA. $\{\hat{\Theta}_{+\varphi}\}$ is the spurious peak set of this sparse ULA after rotating clockwisely in $\angle 2\varphi$. ρ is the element spacing. When

$$\varphi \in (\arccos\frac{0.25}{\rho}, 90°), \rho > 0.5 \tag{13}$$

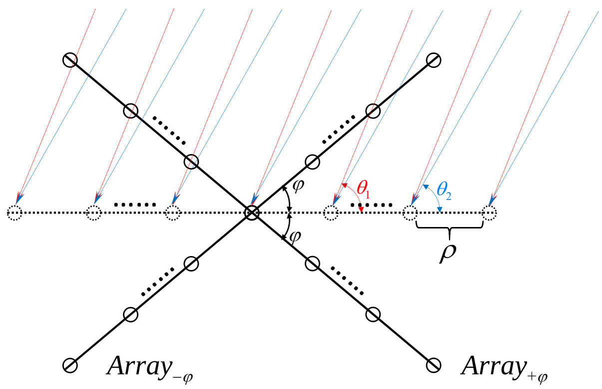

Fig. 2. An array configuration illustration of $Array_{original}$, $Array_{+\varphi}$ and $Array_{-\varphi}$, $\rho > 0.5$, each sub array has N sensors and the total sensors are $2N-1$. M signals incident to this array from differnet angles and only 2 are shown in this figure.

those two spurious peak sets have the relationship

$$\{\hat{\Theta}_{+\varphi} - \varphi\} \bigcap \{\hat{\Theta}_{-\varphi} + \varphi\} = \varnothing \tag{14}$$

C. Mutiple Joint MUSIC Algorithm

Denote $P_{\pm}(\theta, \varphi)$ as the fixed MUSIC spectrum of $Array_{\pm\varphi}$ refering to $Array_{original}$. According to Criteria 1 and Eq.(7) the Rotated MUSIC spectrum $P_r(\theta, \varphi)$ defined as

$$\begin{aligned} P_r(\theta, \varphi) &= \left[P_+(\theta, \varphi) P_-(\theta, \varphi)\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{\mathbf{a}_+^H(\theta,\varphi)\mathbf{U}_+\mathbf{U}_+^H\mathbf{a}_+(\theta,\varphi)} \frac{1}{\mathbf{a}_-^H(\theta,\varphi)\mathbf{U}_-\mathbf{U}_-^H\mathbf{a}_-(\theta,\varphi)}\right]^{\frac{1}{2}} \end{aligned} \tag{15}$$

where $\mathbf{a}_{\pm}(\theta, \varphi) = \mathbf{a}(\theta \pm \varphi)$ and $\mathbf{U}_{\pm}$ denote the fixed steering vector and the noise base of $Array_{\pm\varphi}$.

To further ensure the reliability and improve the ambiguity cancelling performance of the DOA estimation in the array configuration shown in Fig.2, the employment of redundant information is necessary. Denote $\mathbf{X}_{\pm\varphi}$, $\mathbf{N}_{\pm\varphi}$ and $\mathbf{A}_{\pm\varphi}$ as the output vector signal, vector noise and the mainifold matrix of $Array_{\pm\varphi}$ respectively. The combined array output vector signal $\mathbf{X}_c$ can be obtained by adding up the as shown as

$$\mathbf{X}_c = \mathbf{X}_{+\varphi} + \mathbf{X}_{-\varphi} = (\mathbf{A}_{+\varphi} + \mathbf{A}_{-\varphi})\mathbf{S} + \mathbf{N}_{+\varphi} + \mathbf{N}_{-\varphi} \tag{16}$$

Furthermore, the correlation matrix $\mathbf{R}_c$ of $\mathbf{X}_c$ is similar as Eq.(6) as

$$\begin{aligned} \mathbf{R}_c &= E[\mathbf{X}_c\mathbf{X}_c^H] \\ &= (\mathbf{A}_{+\varphi} + \mathbf{A}_{-\varphi})E[\mathbf{S}\mathbf{S}^H](\mathbf{A}_{+\varphi} + \mathbf{A}_{-\varphi})^H + 2\sigma^2\mathbf{I} \\ &= \mathbf{A}_c\mathbf{R}_{SS}\mathbf{A}_c^H + 2\sigma^2\mathbf{I} \end{aligned} \tag{17}$$

$$\begin{aligned} \mathbf{A}_c &= \begin{bmatrix}\mathbf{a}_c(\theta_1, \varphi) & \dots & \mathbf{a}_c(\theta_M, \varphi)\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{a}(\theta_1 + \varphi) + \mathbf{a}(\theta_1 - \varphi) & \dots & \mathbf{a}(\theta_M + \varphi) + \mathbf{a}(\theta_M - \varphi)\end{bmatrix} \end{aligned} \tag{18}$$

where $\mathbf{A}_c$ is called combined manifold matrix. It is consist of combined incident steering vectors. Examining the combined steering vector $\mathbf{a}(\theta + \varphi)$ could arrive at

$$\begin{aligned} \mathbf{a}_c(\theta, \varphi) &= \begin{bmatrix} 1 \\ e^{-j2\pi\rho\cos(\theta+\varphi)} \\ \vdots \\ e^{-j2\pi(N-1)\rho\cos(\theta+\varphi)} \end{bmatrix} + \begin{bmatrix} 1 \\ e^{-j2\pi\rho\cos(\theta-\varphi)} \\ \vdots \\ e^{-j2\pi(N-1)\rho\cos(\theta-\varphi)} \end{bmatrix} \\ &= 2\begin{bmatrix} 1 \\ \cos[2\pi\rho\sin\varphi\sin\theta]e^{-j2\pi\rho\cos\varphi\cos\theta} \\ \vdots \\ \cos[2\pi(N-1)\rho\sin\varphi\sin\theta]e^{-j2\pi(N-1)\rho\cos\varphi\cos\theta} \end{bmatrix} \end{aligned} \tag{19}$$

Explain the non one-to-one mapping in Eq.(19)

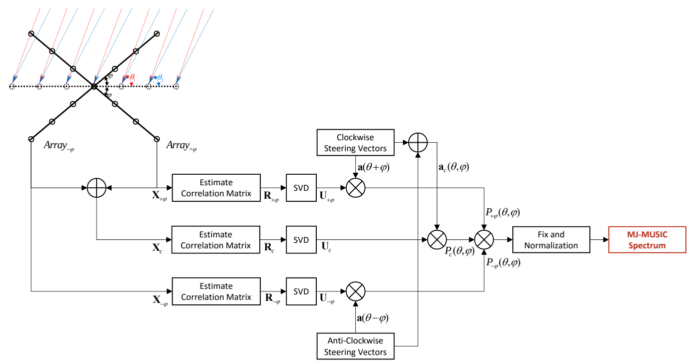

Fig. 3. Block illustration of MJ-MUSIC algorithm. $\varphi \in (\arccos \frac{0.25}{\rho}, 90°)$, $\rho > 0.5$

Decompose the $\mathbf{R}_c$ from Eq.(17) to obtain a combined noise base $\mathbf{U}_c$, a combined MUSIC spectrum function could be express as:

$$P_c(\theta, \varphi) = \frac{1}{\mathbf{a}_c^H(\theta, \varphi)\mathbf{U}_c\mathbf{U}_c^H\mathbf{a}_c(\theta, \varphi)} \tag{20}$$

Merge this function with the Rotated MUSIC in Eq.(15), the Multiple Joint MUSIC (MJ-MUSIC) could be obtained as

$$P_{MJ-MUSIC}(\theta, \varphi) = \left[P_+(\theta, \varphi)P_-(\theta, \varphi)P_c(\theta, \varphi)\right]^{\frac{1}{3}} \tag{21}$$

where ρ and φ should satisfy $\varphi \in (\arccos \frac{0.25}{\rho}, 90°)$ to resolve the ambiguities.

Fig.3 illustrates the entire procedure of MJ-MUSIC algorithm for resolving the ambiguities. Noticed that the paths obtaining $P_+(\theta, \varphi)$, $P_-(\theta, \varphi)$ and $P_c(\theta, \varphi)$ could process simultaneously. Table I briefly compares the MUSIC and MJ-MUSIC from array aperture, computation costs and time cost these 3 aspects respectively.

TABLE I
COMPARISON BETWEEN MUSIC AND MJ-MUSIC

	MUSIC	**MJ-MUSIC**
Aperture	$(N-1)\rho, \rho \leq 0.5$	$(\frac{(N+1)}{2} - 1)\rho, \rho > 0.5$
Computations	EVD, 1-D Search	3×EVD, 3×1-D Search
Time Cost	EVD, 1-D Search	EVD, 1-D Search in parallel computing

IV. NUMERICAL SIMULATION AND DISCUSSION

In this section, several numerical simulations were conducted to demonstrate the effiency of proposed method.

1) Experiment 1: Consider a normal sparse ULA and an X-shaped sparse array. Both arrays have the same aperture $D = 4\times\lambda$, where λ denotes the wavelength of incident signal. The normal sparse array has 5 sensors and the element spacing $\Delta d_m = 0.8\lambda$, the X-shaped sparse array has 5 sensors in total and the element spacing $\Delta d_j = 2\lambda$ and with an included angle $\angle\varphi = 85°$. Two signals with same λ intruding to these arrays from $34.6°$ and $48°$. Both signals have the same power and the signal-to-noise ratio (SNR) equals to $10dB$. The normal sparse ULA employs the conventional MUSIC algorithm and the X-shaped sparse array employs the MJ-MUSIC. The number of total snapshots of each algorithm is 200.

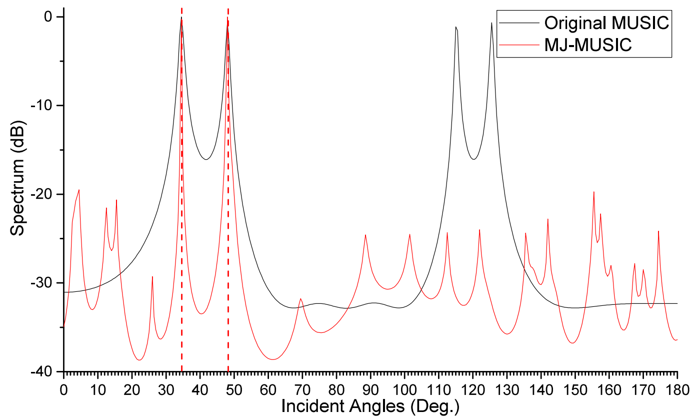

Fig. 4. DOAs of experiment 1. Element Number of each array is 5, $\Delta d_m = 0.8\lambda$, $\Delta d_j = 2\lambda$, Incident Angle $= [34.6°, 48°]$, SNR $= 10dB$, Snapshots $= 200$, $\angle\varphi = 85°$.

Simulation result shown in Fig.4

2) Experiment 2: Consider

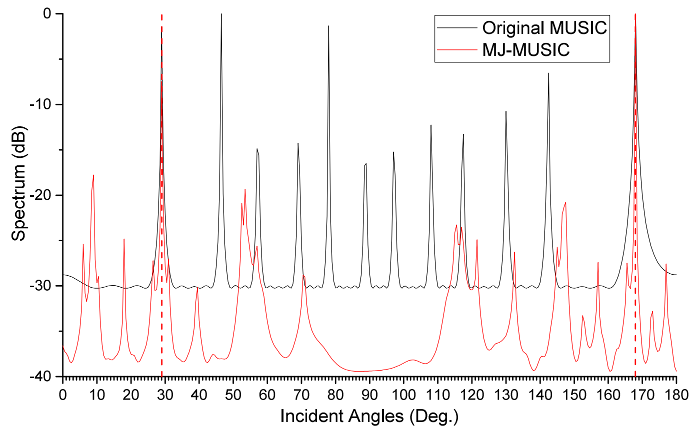

Fig. 5. Block illustration of MJ-MUSIC algorithm. $\varphi \in (\arccos\frac{0.25}{\rho}, 90°)$, $\rho > 0.5$

3) Experiment 3: Consider

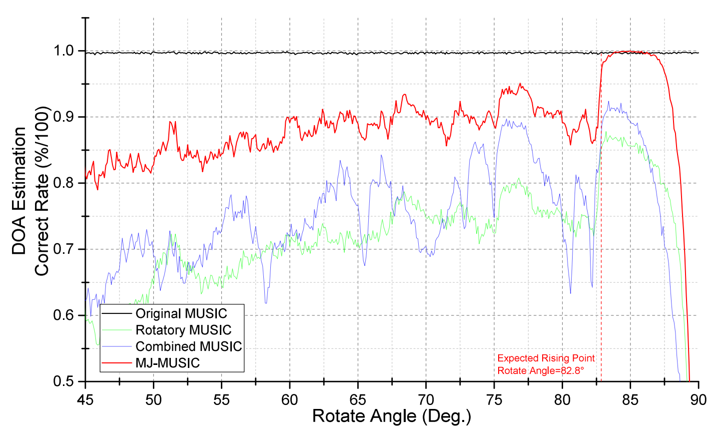

Fig. 6. Block illustration of MJ-MUSIC algorithm. $\varphi \in (\arccos\frac{0.25}{\rho}, 90°)$, $\rho > 0.5$

V. CONCLUSION

The conclusion goes here.

Appendix A
Proof of Rotatory Symmetric Ambiguity Cancellation Criteria

Appendix one text goes here.

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Ao Cui Biography text here.

Pei-Yu He Biography text here.