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On the Product and Ratio of Gamma and Beta Random Variables

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Summary: The distributions of the product XY and the ratio X/Y are derived when X and Y are gamma and beta random variables distributed independently of each other. Tabulations of the associated percentage points and illustrations of their practical use are also provided.

Keywords: Linear combinations of random variables, products of random variables, ratios of random variables. JEL C100.

1. Introduction

For given random variables X and Y, the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences, engineering and medicine. Examples of XY include traditional portfolio selection models, relationship between attitudes and behavior, number of cancer cells in tumor biology and stream flow in hydrology. Examples of X/Y include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and safety factor in engineering.

The distributions of XY and X/Y have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to XY, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, AbuSalih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). With respect to X/Y, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Basu and Lochner (1971) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family. There is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions (see below for examples).

In this paper, we study the exact distributions of XY and X/Y when X and Y are independent gamma and beta random variables with pdfs

$$f_X(x) = \frac{\lambda^\beta x^{\beta-1} \exp(-\lambda x)}{\Gamma(\beta)} \tag{1}$$

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and

$$f_Y(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)} \tag{2}$$

respectively, for $x > 0$, $0 < y < 1$, $\beta > 0$, $\lambda > 0$, $a > 0$ and $b > 0$.

The paper is organized as follows. Exact expressions for both the pdf and the cdf of XY and X/Y are derived in Sections 2 and 3. The proofs are given for the main results but are omitted when particular cases are considered. Detailed proofs of all the results can be obtained from the first author. In Section 4 of the paper, an application of the results is provided by computing the associated percentage points and discussing four practical situations where these tables can be directly useful.

The calculations of this paper involve several special functions, including the confluent hypergeometric function defined by

$$F(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

the generalized hypergeometric function defined by

$$H(a,b;c,d;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{x^k}{k!},$$

the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x \exp(-t) t^{a-1} dt,$$

the complementary incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^{\infty} \exp(-t) t^{a-1} dt,$$

and, the Kummer function defined by

$$\Psi(a,b;x) = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-xt)\, t^{a-1} (1+t)^{b-a-1} dt,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1 *(Equation (2.10.2.3), Prudnikov et al., 1986, Volume 2) For $a > 0$, $\beta > 0$ and $c > 0$*

$$\begin{aligned}
&\int_a^{\infty} x^{\alpha-1}(x-a)^{\beta-1}\Gamma(\nu,cx)dx = \\
&a^{\alpha+\beta-1}\Gamma(\nu)B(\beta,1-\alpha-\beta) \\
&-\frac{a^{\alpha+\beta+\nu-1}c^{\nu}}{\nu}B(\beta,1-\alpha-\beta-\nu)H(\alpha+\nu,\nu;\nu+1,\alpha+\beta+\nu;-ac) \\
&-\frac{c^{1-\alpha-\beta}}{1-\alpha-\beta}\Gamma(\alpha+\beta+\nu-1)H(1-\alpha-\beta,1-\beta;2-\alpha-\beta-\nu,2-\alpha-\beta;-ac).
\end{aligned}$$

Lemma 2 *(Equation (2.10.2.2), Prudnikov et al., 1986, Volume 2) For $a > 0$, $\beta > 0$, $\alpha + \nu > 0$ and $\alpha > 0$,*

$$\int_0^a x^{\alpha-1}(a-x)^{\beta-1}\Gamma(\nu, cx)dx =$$

$$a^{\alpha+\beta-1}\Gamma(\nu)B(\alpha,\beta) - \frac{a^{\alpha+\beta+\nu-1}c^{\nu}}{\nu}B(\beta,\alpha+\nu)H\left(\nu,\alpha+\nu;\nu+1,\alpha+\beta+\nu;-ac\right).$$

Further properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. Product

Theorem 1 expresses the pdf and the cdf of XY in terms of the Kummer function and the generalized hypergeometric function, respectively.

Theorem 1 *Suppose X and Y are distributed according to (1) and (2), respectively. The cdf of $Z = XY$ can be expressed as:*

$$F(z) = \frac{1}{\Gamma(\beta)B(a,b)}\left[\frac{(\lambda z)^{\beta}}{\beta}B\left(b, a-\beta\right)H\left(1-a-b+\beta,\beta;\beta+1,1-a+\beta;-\lambda z\right)\right.$$
$$\left. + \frac{(\lambda z)^{a}}{a}\Gamma\left(\beta-a\right)H\left(a,1-b;a-\beta+1,1+a;-\lambda z\right)\right] \qquad (3)$$

for $z > 0$. The corresponding pdf of $Z = XY$ is

$$f(z) = \frac{\lambda^{\beta}\Gamma(b)}{\Gamma(\beta)B(a,b)}z^{\beta-1}\exp(-\lambda z)\Psi\left(b,1+\beta-a;\lambda z\right) \qquad (4)$$

for $z > 0$.

Proof The cdf corresponding to (1) is $1 - \Gamma(\beta,\lambda x)/\Gamma(\beta)$. Thus, one can write the cdf of XY as

$$\begin{aligned}
\Pr\left(XY \le z\right) &= \int_0^1 F_X(z/y)f_Y(y)dy \\
&= 1 - \frac{1}{\Gamma(\beta)B(a,b)}\int_0^1 \Gamma\left(\beta,\frac{\lambda z}{y}\right)y^{a-1}(1-y)^{b-1}dy \\
&= 1 - \frac{1}{\Gamma(\beta)B(a,b)}\int_1^{\infty}\Gamma\left(\beta,\lambda z w\right)w^{-(a+b)}(w-1)^{b-1}dw \\
&= 1 - \frac{1}{\Gamma(\beta)B(a,b)}I, \qquad (5)
\end{aligned}$$

which follows after setting $w = 1/y$. Application of Lemma 1 shows that the integral I can be calculated as

$$
\begin{aligned}
I = {} & \Gamma(\beta)B(a,b) - \frac{(\lambda z)^\beta}{\beta} B\left(b, a-\beta\right) H\left(1-a-b+\beta, \beta; \beta+1, 1-a+\beta; -\lambda z\right) \\
& - \frac{(\lambda z)^a}{a} \Gamma\left(\beta - a\right) H\left(a, 1-b; a-\beta+1, 1+a; -\lambda z\right). \qquad (6)
\end{aligned}
$$

The result in (3) follows by substituting (6) into (5). The pdf in (4) follows by differentiation and using properties of the hypergeometric function.

Springer and Thompson (1970) derived an exact expression (involving the MeijerG function) for the pdf of the product of m beta distributed random variables with parameters (a_k, b_k) and $n-m$ gamma distributed random variables with shape parameters c_k and scale parameters set to 1 (where all of the n random variables are assumed independent). The result of Theorem 1 is a particular case of this result for $m = 1$ and $n = 2$. If $a = q$, $b = p - q$ (for $0 < q < p$) and $\beta = p$ then it can be shown that (3) and (4) reduce to a gamma distribution. This corresponds to a result given in Stuart (1962). One can derive several other simpler forms of (3) when a, b and β take integer or half integer values. This is illustrated in the corollaries below.

Corollary 1 *If $\beta \geq 1$ is an integer then (3) can be reduced to the simpler form*

$$
F(z) = 1 - \frac{\Gamma(b)\exp(-\lambda z)}{B(a,b)} \sum_{k=0}^{\beta-1} \frac{(\lambda z)^k}{k!} \Psi\left(b, 1+k-a; \lambda z\right)
$$

for $z > 0$.

Corollary 2 *If $\beta - 1/2 = n \geq 1$ is an integer then (3) can be reduced to the simpler form*

$$
\begin{aligned}
F(z) = {} & -2\sqrt{\frac{\lambda z}{\pi}} \frac{B\left(b, a-1/2\right)}{B\left(a,b\right)} H\left(\frac{1}{2}, \frac{3}{2} - a - b; \frac{3}{2} - a, \frac{3}{2}; -\lambda z\right) \\
& - \frac{(\lambda z)^a}{\sqrt{\pi} a} \frac{\Gamma\left(1/2 - a\right)}{B\left(a,b\right)} H\left(a, 1-b; 1+a, a+\frac{1}{2}; -\lambda z\right) \\
& + \frac{(-1)^n \Gamma(b)\sqrt{\lambda z}\exp(-\lambda z)}{\Gamma(\beta)B(a,b)} \sum_{k=0}^{n-1} (-\lambda z)^k \left(\frac{1}{2} - n\right)_{n-k-1} \\
& \Psi\left(b, k + \frac{3}{2} - a; \lambda z\right)
\end{aligned}
$$

for $z > 0$.

COROLLARY 3 *If $a \geq 1$ is an integer then (3) can be reduced to the simpler form*

$$F(z) = 1 - \frac{b(\lambda z)^{\beta} \exp(-\lambda z)}{\Gamma(\beta)} \sum_{k=0}^{a} \frac{\Gamma(b+k-1)}{\Gamma(k)} \Psi\left(b+1, \beta - k + 2; \lambda z\right)$$

for $z > 0$.

COROLLARY 4 *If $b \geq 1$ is an integer then (3) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} + \frac{(\lambda z)^{\beta} \exp(-\lambda z)}{\Gamma(a)\Gamma(\beta)} \sum_{k=0}^{b} \Gamma(a+k-1) \Psi\left(k, \beta - a + 1; \lambda z\right)$$

for $z > 0$.

COROLLARY 5 *If $a = 1/2$ and $b = 1/2$ then (3) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)}$$

for $z > 0$, where

$$K = \int_{z}^{\infty} x^{\beta - 1} \exp(-\lambda x) \arctan\left(\frac{z}{x-z}\right) dx. \tag{7}$$

COROLLARY 6 *If $a + 1/2 = n \geq 1$ is an integer and $b = 1/2$ then (3) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} + \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)} - \frac{(\lambda z)^{\beta} \exp(-\lambda z)}{2\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(k)}{\Gamma\left(k + 1/2\right)} \Psi\left(\frac{3}{2}, \beta - k + \frac{3}{2}; \lambda z\right)$$

for $z > 0$, where K is given by (7).

COROLLARY 7 *If $b + 1/2 = n \geq 1$ is an integer and $a = 1/2$ then (3) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} + \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)} + \frac{(\lambda z)^{\beta} \exp(-\lambda z)}{\sqrt{\pi}\Gamma(\beta)} \sum_{k=1}^{n-1} \Gamma(k) \Psi\left(k + \frac{1}{2}, \beta + \frac{1}{2}; \lambda z\right)$$

for $z > 0$, where K is given by (7).

COROLLARY 8 *If $a + 1/2 = m \geq 1$ and $b + 1/2 = n \geq 1$ are integers then (3) can be reduced to the simpler form*

$$\begin{aligned} F(z) = & \frac{\gamma(\beta, z)}{\Gamma(z)} + \frac{2\lambda^\beta K}{\pi \Gamma(\beta)} \\ & - \frac{(\lambda z)^\beta \exp(-\lambda z)}{2\Gamma(\beta)} \sum_{k=1}^{m-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} \Psi\left(\frac{3}{2}, \beta - k + \frac{3}{2}; \lambda z\right) \\ & + \frac{(\lambda z)^\beta \exp(-\lambda z)}{\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(m+k-1)}{\Gamma(m-1/2)} \Psi\left(k + \frac{1}{2}, \beta - m + \frac{3}{2}; \lambda z\right) \end{aligned}$$

for $z > 0$, where K is given by (7).

The formulas for $F(z)$ in the corollaries above can be used to save computational time since the computation of the H in (3) can be more demanding. They can be computed by using the KummerU$(\cdot, \cdot, \cdot)$ function in MAPLE.

Figure 1 illustrates possible shapes of the pdf (4) for selected values of a, b and β (the pdf computed by using the KummerU$(\cdot, \cdot, \cdot)$ function in MAPLE). The four curves in each plot correspond to selected values of β. As expected, the densities are unimodal and the effect of the parameters is evident.

3. RATIO

Theorem 2 expresses the pdf and the cdf of X/Y in terms of the confluent hypergeometric function and the generalized hypergeometric function, respectively.

THEOREM 2 *Suppose X and Y are distributed according to (1) and (2), respectively. The cdf of $Z = X/Y$ can be expressed as:*

$$F(z) = \frac{B(b, a+\beta)(\lambda z)^\beta}{\Gamma(\beta+1) B(a,b)} H(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z) \quad (8)$$

for $z > 0$. The corresponding pdf of $Z = X/Y$ is

$$f(z) = \frac{\lambda^\beta B(\beta + a, b)}{\Gamma(\beta) B(a,b)} z^{\beta-1} F(\beta + a; \beta + a + b; -\lambda z) \quad (9)$$

for $z > 0$.

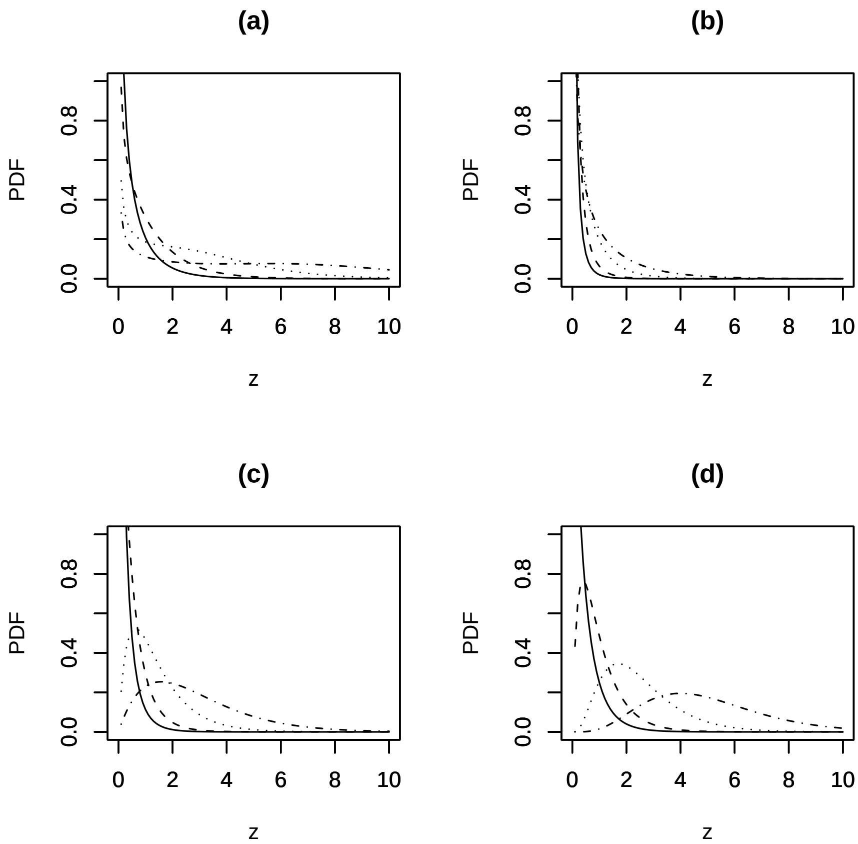

FIGURE 1. Plots of the pdf (4) for $\beta = 1, 2, 5, 10$ and (a): $a = 0.5$, $b = 0.5$; (b): $a = 0.5$, $b = 5$; (c): $a = 2$, $b = 5$; and, (d): $a = 5$, $b = 5$. The four curves in each plot are: the solid curve ($\beta = 1$), the curve of dots ($\beta = 2$), the curve of lines ($\beta = 5$), and the curve of dots and lines ($\beta = 10$).

PROOF The cdf corresponding to (1) is $1 - \Gamma(\beta, \lambda x)/\Gamma(\beta)$. Thus, one can write the cdf of X/Y as

$$\begin{aligned}
\Pr(X/Y \leq z) &= \int_0^1 F_X(zy) f_Y(y) dy \\
&= 1 - \frac{1}{\Gamma(\beta) B(a,b)} \int_0^1 \Gamma(\beta, \lambda y z) y^{a-1} (1-y)^{b-1} dy \\
&= 1 - \frac{1}{\Gamma(\beta) B(a,b)} I. \qquad (10)
\end{aligned}$$

Application of Lemma 2 shows that the integral I can be calculated as

$$I = \Gamma(\beta) B(a,b) - \frac{(\lambda z)^\beta}{\beta} B(b, a+\beta)$$

$$H\left(\beta, a+\beta; \beta+1, a+b+\beta; -\lambda z\right). \tag{11}$$

The result in (8) follows by substituting (11) into (10). The pdf in (9) follows by differentiation and using properties of the hypergeometric function.

Using special properties of the generalized hypergeometric function, one can derive several simpler forms for (8) when a, b and β take integer or half integer values. This is illustrated in the corollaries below.

COROLLARY 9 *If $\beta \geq 1$ is an integer then (8) can be reduced to the simpler form*

$$F(z) = 1 - \frac{1}{B(a,b)} \sum_{k=0}^{\beta-1} \frac{(\lambda z)^k}{k!} B(a+k, b) F\left(a+k; a+b+k; -\lambda z\right)$$

for $z > 0$.

COROLLARY 10 *If $\beta - 1/2 = n \geq 1$ is an integer then (8) can be reduced to the simpler form*

$$\begin{aligned} F(z) = {} & 2\sqrt{\frac{\lambda z}{\pi}} \frac{\Gamma\left(a+1/2, b\right)}{B\left(a,b\right)} H\left(a+\frac{1}{2}, \frac{1}{2}; a+b+\frac{1}{2}, \frac{3}{2}; -\lambda z\right) \\ & + \frac{(-1)^n \sqrt{\lambda z}}{\Gamma\left(n+1/2\right) B(a,b)} \sum_{k=0}^{n-1} (-\lambda z)^k \left(\frac{1}{2}-n\right)_{n-k-1} B\left(k+a+\frac{1}{2}, b\right) \\ & F\left(k+a+\frac{1}{2}; k+a+b+\frac{1}{2}; -\lambda z\right) \end{aligned}$$

for $z > 0$.

COROLLARY 11 *If $a \geq 1$ is an integer then (8) can be reduced to the simpler form*

$$\begin{aligned} F(z) = {} & \frac{(\lambda z)^\beta}{\Gamma(\beta)} \sum_{k=0}^{a} \frac{\Gamma(b+k-1) B(\beta+k-1, b+1)}{\Gamma(b)\Gamma(k)} \\ & F\left(\beta+k-1; \beta+b+k; -\lambda z\right) \end{aligned}$$

for $z > 0$.

COROLLARY 12 *If $b \geq 1$ is an integer then (8) can be reduced to the simpler form*

$$\begin{aligned} F(z) = {} & \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{(\lambda z)^\beta}{\Gamma(\beta)} \sum_{k=0}^{b} \frac{\Gamma(a+k-1) B(a+\beta, k)}{\Gamma(a)\Gamma(k)} \\ & F\left(a+\beta; a+\beta+k; -\lambda z\right) \end{aligned}$$

for $z > 0$.

COROLLARY 13 *If $a = 1/2$ and $b = 1/2$ then (8) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)}$$

for $z > 0$, where

$$K = \int_0^z x^{\beta-1} \exp(-\lambda x) \arctan\left(\frac{x}{z-x}\right) dx. \tag{12}$$

COROLLARY 14 *If $a + 1/2 = n \geq 1$ is an integer and $b = 1/2$ then (8) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)} + \frac{(\lambda z)^{\beta}}{\sqrt{\pi}\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} B\left(\beta + k - \frac{1}{2}, \frac{3}{2}\right)$$
$$F\left(\beta + k - \frac{1}{2}; \beta + k + 1; -\lambda z\right)$$

for $z > 0$, where K is given by (12).

COROLLARY 15 *If $b + 1/2 = n \geq 1$ is an integer and $a = 1/2$ then (8) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)} - \frac{(\lambda z)^{\beta}}{\sqrt{\pi}\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} B\left(\beta + \frac{1}{2}, k + \frac{1}{2}\right)$$
$$F\left(\beta + \frac{1}{2}; \beta + k + 1; -\lambda z\right)$$

for $z > 0$, where K is given by (12).

COROLLARY 16 *If $a + 1/2 = m \geq 1$ and $b + 1/2 = n \geq 1$ are integers then (8) can be reduced to the simpler form*

$$F(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^{\beta} K}{\pi \Gamma(\beta)} + \frac{(\lambda z)^{\beta}}{\sqrt{\pi}\Gamma(\beta)} \sum_{k=1}^{m-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} B\left(\beta + k - \frac{1}{2}, \frac{3}{2}\right)$$
$$F\left(\beta + k - \frac{1}{2}; \beta + k + 1; -\lambda z\right)$$
$$-\frac{(\lambda z)^{\beta}}{\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(m+k-1)}{\Gamma(m-1/2)\Gamma(k+1/2)} B\left(m + \beta - \frac{1}{2}, k + \frac{1}{2}\right)$$
$$F\left(m + \beta - \frac{1}{2}; k + m + \beta; -\lambda z\right)$$

for $z > 0$, where K is given by (12).

The formulas for $F(z)$ in the corollaries above can be used to save computational time since the computation of the H in (8) can be more demanding. They can be computed by using the hypergeom([·],[·],·) function in MAPLE.

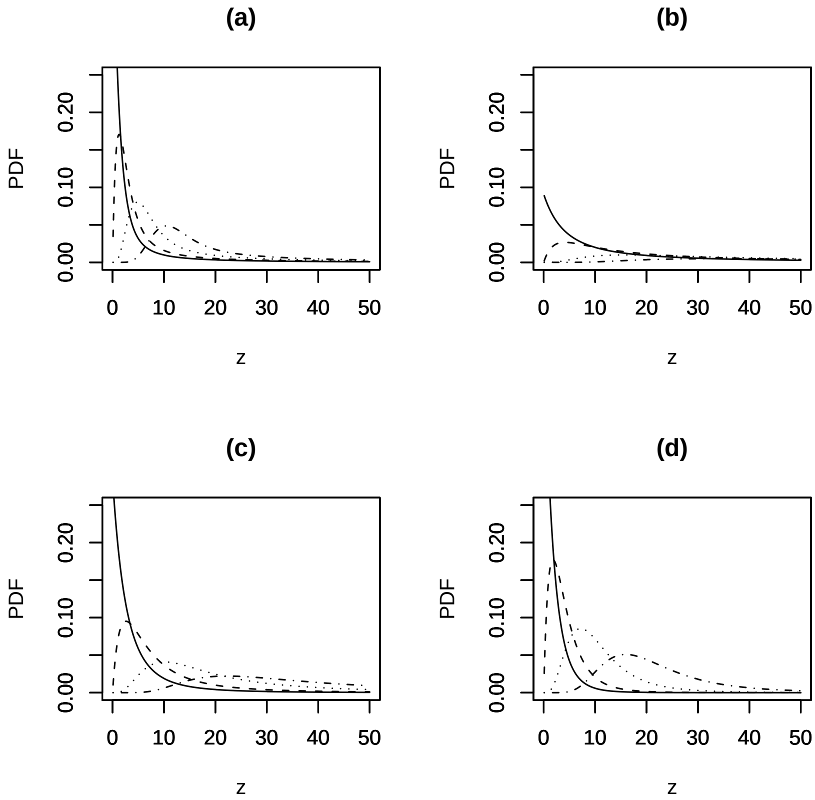

FIGURE 2. Plots of the pdf (9) for $\beta = 1, 2, 5, 10$ and (a): $a = 0.5$, $b = 0.5$; (b): $a = 0.5$, $b = 5$; (c): $a = 2$, $b = 5$; and (d): $a = 5$, $b = 5$. The four curves in each plot are the solid curve ($\beta = 1$), the curve of dots ($\beta = 2$), the curve of lines ($\beta = 5$), and the curve of dots and lines ($\beta = 10$).

Figure 2 illustrates possible shapes of the pdf (9) for selected values of a, b and β (the pdf computed using the hypergeom([·],[·],·) function in MAPLE). The four curves in each plot correspond to selected values of β. As expected, the densities are unimodal and the effect of the parameters is evident.

4. APPLICATION

In this section, we provide tabulations of percentage points z_p associated with the cdfs of XY and X/Y. We also illustrate the practical use of these tabulations by means of four examples. The percentage points are obtained by numerically solving the equations

$$\begin{aligned}\frac{1}{\Gamma(\beta)B(a,b)}\Bigg[&\frac{(\lambda z_p)^\beta}{\beta}B\left(b,a-\beta\right)\\ &H\left(1-a-b+\beta,\beta;\beta+1,1-a+\beta;-\lambda z_p\right)\\ &+\frac{(\lambda z_p)^a}{a}\Gamma\left(\beta-a\right)H\left(a,1-b;a-\beta+1,1+a;-\lambda z_p\right)\Bigg]=p\end{aligned}$$

and

$$\frac{B(b,a+\beta)(\lambda z_p)^\beta}{\Gamma(\beta+1)B(a,b)}H\left(\beta,a+\beta;\beta+1,a+b+\beta;-\lambda z_p\right)=p.$$

Evidently, this involves computation of the generalized hypergeometric function and routines for this are widely available. We used the function hypergeom($[\cdot,\ \cdot],[\cdot,\ \cdot],\ \cdot$) in MAPLE. The tables below provide the numerical values of z_p for $\lambda = 1$, $\beta = 1$, $a = 0.5, 1, \ldots, 4$ and $b = 0.5, 1, \ldots, 4$.

Similar tabulations could be easily derived for other values of λ, β, a, b and p by using the hypergeom($[\cdot,\ \cdot],[\cdot,\ \cdot],\ \cdot$) function in MAPLE. Let us now discuss four practical examples where these tabulations might be directly useful.

Firstly, in a portfolio of risks, let Y denote a random variable which gives the probability that the event that claim has occurred and let X denote the claim amount. Then the individual risk will be the product of the two random variables $Z = XY$. Since gamma distributions are popular models for claim amounts and since Y is a probability it will be most reasonable to assume that X and Y are distributed according to (1) and (2), respectively. The customer will be interested in extreme percentile points of the risk $Z = XY$ and these can read directly from Table 1 above once the estimates of λ, β, a and b are determined either from prior knowledge or by fitting (1) and (2) to some data. For example, if $\lambda = 1$, $\beta = 1$, $a = 1$ and $b = 1$ then the customer will be 99% confident that the risk will not exceed 3.050973. If $\lambda = 1$, $\beta = 1$, $a = 2$ and $b = 2$ then the customer will be 99% confident that the risk will not exceed 2.79073.

Secondly, let X and Y denote the rainfall intensity and the duration of a storm. Then $Z = XY$ will represent the amount of rainfall produced by that storm. Since gamma distributions are popular models for rainfall intensity and since Y will have a probable maximum it will be most reasonable to assume that X and Y are distributed according to (1) and (2), respectively, after suitable scaling. Hydrological purposes such as building of dams will require extreme percentile points of the amount of rainfall $Z = XY$ and these can read directly from Table 1. For example, if $\lambda = 1$, $\beta = 1$, $a = 1$ and $b = 1$ then the hydrologist will be 99% confident that the amount of rainfall will not exceed 3.050973. If $\lambda = 1$, $\beta = 1$, $a = 2$ and $b = 2$ then the

a	b	$p = 0.9$	$p = 0.95$	$p = 0.975$	$p = 0.99$	$p = 0.995$	$p = 0.999$
0.5	0.5	1.350807	1.921592	2.509859	3.305998	3.934966	5.405242
0.5	1	0.9246366	1.378242	1.880609	2.573808	3.138532	4.463549
0.5	1.5	0.6963203	1.070556	1.491125	2.097335	2.57746	3.796273
0.5	2	0.5552195	0.8705972	1.228533	1.759993	2.195485	3.287402
0.5	2.5	0.4607189	0.7343304	1.047790	1.511421	1.9035	2.893216
0.5	3	0.3931828	0.6322666	0.9111636	1.333517	1.691885	2.625940
0.5	3.5	0.3421972	0.5528579	0.7988305	1.177063	1.499041	2.356105
0.5	4	0.3052473	0.4955657	0.7234318	1.073549	1.368216	2.155246
1	1	1.273068	1.781816	2.318951	3.050973	3.625847	5.015368
1	1.5	1.034239	1.471915	1.940721	2.605049	3.126521	4.401046
1	2	0.864197	1.249660	1.667801	2.245203	2.716769	3.890489
1	2.5	0.7441922	1.085364	1.456746	1.990983	2.423308	3.500757
1	3	0.6504262	0.9565151	1.293933	1.779435	2.179003	3.172178
1	3.5	0.5782905	0.8567245	1.164784	1.624361	1.991466	2.93388
1	4	0.5193941	0.771616	1.052699	1.4687	1.796561	2.646618
1.5	1.5	1.237039	1.706439	2.207877	2.896656	3.427820	4.745187
1.5	2	1.067751	1.488456	1.939494	2.572969	3.070773	4.264825
1.5	2.5	0.935404	1.317194	1.723602	2.297983	2.761330	3.904707
1.5	3	0.835313	1.182335	1.557162	2.091378	2.518450	3.570648
1.5	3.5	0.753231	1.072771	1.422326	1.918213	2.319882	3.302508
1.5	4	0.6852923	0.9781752	1.30586	1.767379	2.143167	3.076237
2	2	1.217583	1.664904	2.142797	2.79073	3.301392	4.56797
2	2.5	1.085474	1.499118	1.936521	2.542024	3.035986	4.171192
2	3	0.978206	1.357887	1.760082	2.319559	2.766074	3.835108
2	3.5	0.8944201	1.244774	1.621197	2.151479	2.577904	3.617276
2	4	0.819903	1.146517	1.495670	1.992163	2.3802	3.383494
2.5	2.5	1.204960	1.639680	2.098883	2.732335	3.217662	4.416158
2.5	3	1.094035	1.495831	1.926453	2.521977	2.985598	4.110703
2.5	3.5	1.005817	1.382112	1.782983	2.339696	2.780643	3.864759
2.5	4	0.9346001	1.285397	1.656156	2.175758	2.582827	3.61519
3	3	1.19529	1.617942	2.062906	2.675902	3.165535	4.318287
3	3.5	1.103290	1.498549	1.913811	2.48278	2.927858	4.025325
3	4	1.028026	1.399356	1.790955	2.341286	2.766827	3.782245
3.5	3.5	1.188187	1.601505	2.033495	2.635952	3.102699	4.215792
3.5	4	1.110319	1.502538	1.911353	2.479353	2.914916	3.990049
4	4	1.18349	1.592512	2.020582	2.604536	3.069697	4.165839

TABLE 1. Percentage points of $Z = XY$.

hydrologist will be 99% confident that amount of rainfall will not exceed 2.79073.

Thirdly, let X and Y denote the areal precipitation and the annual stream flow. Then $1/Z = Y/X$ will represent the proportion of precipitation that ended up in stream flow. It is known on physical grounds that Y is finite valued (see, for example, Clarke (1979)); therefore, it will be most reasonable to assume that X and Y are distributed according to (1) and (2), respectively, after suitable scaling. The percentile points in Table 2 can be used to quantify the proportion of precipitation ended up in stream.

Finally, let X and Y denote the failure time of a component and the warning-time variable showing that the component will fail. Then $1/Z = Y/X$ will represent the efficiency of the warning-time system. Gamma distributions are popular models for failure time data and one would like the warning made within a fixed period of the time of operation; therefore, it will be most reasonable to assume that X and Y are distributed according

a	b	$p = 0.9$	$p = 0.95$	$p = 0.975$	$p = 0.99$	$p = 0.995$	$p = 0.999$
0.5	0.5	32.48203	127.8962	509.2384	3173.539	12697.78	292170.3
0.5	1	78.70447	315.0491	1284.370	7911.38	31357.91	794022.3
0.5	1.5	125.9188	507.0126	2004.461	12750.93	50646.68	1264481
0.5	2	173.7126	697.6159	2824.25	17414.26	70439.97	1880662
0.5	2.5	225.9965	902.395	3577.682	21911.66	87689.04	2050442
0.5	3	273.5791	1096.368	4396.529	27465.86	110620.4	3168239
0.5	3.5	323.5222	1294.409	5194.328	32231.75	127190.1	3088194
0.5	4	374.6507	1481.964	5887.041	38220.76	155826.0	3902570
1	1	10.02287	19.98618	40.11345	100.4015	202.0587	1019.199
1	1.5	14.45449	29.42847	59.18465	149.0476	299.514	1532.221
1	2	18.82975	38.74822	78.37019	198.9464	399.0623	2016.881
1	2.5	23.34507	48.33229	98.12294	244.4418	491.2418	2376.917
1	3	28.04906	58.17959	117.821	294.3444	589.7097	3027.578
1	3.5	32.61612	67.66834	137.5953	347.6574	685.3779	3304.145
1	4	36.78699	76.89603	157.3600	401.7975	800.5658	3891.656
1.5	1.5	7.36986	12.13781	19.55719	36.64305	58.42406	177.6259
1.5	2	9.184158	15.40633	25.13648	47.20579	75.20083	213.0504
1.5	2.5	10.99530	18.45023	30.23812	57.16297	92.51752	267.8566
1.5	3	12.84129	21.78999	35.63721	67.1393	107.4917	318.4216
1.5	3.5	14.62678	24.89312	41.29162	79.1167	126.8569	368.9398
1.5	4	16.41169	28.04203	46.4091	88.5084	140.6871	424.2156
2	2	6.443926	9.722802	14.26252	23.26482	33.5256	75.70047
2	2.5	7.503737	11.53403	17.08564	28.03295	40.10557	91.87897
2	3	8.60947	13.29108	19.76470	32.59954	47.18332	109.2675
2	3.5	9.65598	14.99951	22.49794	37.3086	53.9292	125.9157
2	4	10.75025	16.74129	25.10819	41.23939	59.5707	137.7901
2.5	2.5	5.955649	8.67964	12.15648	18.31139	24.68979	48.09819
2.5	3	6.711832	9.854807	13.83133	21.13947	28.3514	55.9593
2.5	3.5	7.486167	11.01804	15.58907	23.8649	32.13237	62.3496
2.5	4	8.244172	12.18589	17.26520	26.54475	36.49219	71.89607
3	3	5.69816	8.071177	10.97388	15.80631	20.65266	36.24219
3	3.5	6.291486	8.9642	12.22413	17.72351	23.20431	41.65589
3	4	6.852232	9.817119	13.40290	19.46428	25.46952	45.30394
3.5	3.5	5.501183	7.685005	10.25647	14.35553	18.22657	30.1805
3.5	4	5.984823	8.384275	11.24080	15.79199	20.14699	34.17545
4	4	5.38857	7.443689	9.796528	13.49142	16.76883	26.94032

TABLE 2. Percentage points of $Z = X/Y$.

to (1) and (2), respectively, after suitable scaling. The percentile points in Table 2 can be used to quantify the efficiency of the warning system.

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