MATH 4281 — Prof. N. Madras
NOTES ON INVERSE GAMMA AND SIMULATION

The Inverse Gamma Distribution

A random variable X is said to have the *inverse Gamma distribution with parameters α and θ* if $1/X$ has the Gamma$(\alpha, 1/\theta)$ distribution.

Properties: Assume that X has the inverse Gamma distribution with parameters α and θ. Then its p.d.f. is

$$f(x) \;=\; \frac{\theta^\alpha \, e^{-\theta/x}}{x^{\alpha+1}\,\Gamma(\alpha)}\,. \tag{1}$$

We also have

$$E(X) \;=\; \frac{\theta}{\alpha-1}, \qquad E(X^2) \;=\; \frac{\theta^2}{(\alpha-1)(\alpha-2)},$$
$$\text{and} \qquad \mathrm{Var}(X) \;=\; \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}\,.$$

To verify Equation (1), we start with the fact that $1/X$ has p.d.f. $g(t) = t^{\alpha-1}\,\theta^\alpha\,e^{-\theta t}/\Gamma(\alpha)$. Let F be the cumulative distribution function (c.d.f.) of X, and let G be the c.d.f. of $1/X$. Then (since $X > 0$),

$$F(t) \;=\; \Pr\left(\frac{1}{X} \le t\right) \;=\; \Pr\left(\frac{1}{t} \le X\right) \;=\; 1 - G\left(\frac{1}{t}\right)\,.$$

Therefore the p.d.f. f of X satisfies

$$f(t) \;=\; \frac{d}{dt}\,F(t) \;=\; \frac{d}{dt}\left(1 - G\left(\frac{1}{t}\right)\right) \;=\; \frac{1}{t^2}\,g\left(\frac{1}{t}\right),$$

from which we can obtain Equation (1).

The values of $E(X)$, $E(X^2)$, and $\mathrm{Var}(X)$ are obtained by calculating the moments of $1/Y$ and $1/Y^2$ where Y has the Gamma$(\alpha, 1/\theta)$ distribution.

Simulation of Random Variables

Many simulations call for random numbers with specified distributions. Typical pseudo-random number generators (such as RAND in Excel) give sequences of numbers that statistically look like independent sequences of random variables with the $U[0,1]$ distribution (i.e. uniformly distributed on the interval $[0,1]$). For our discussion, we shall assume that we have an independent sequence $U_1, U_2, \ldots$ of $U[0,1]$ random numbers. We shall show how one can use the U_n's to generate random numbers with other specified distributions.

Inversion Method: This is one of the most direct methods, and can be used for a number of common distributions. The method is based on the following theorem.

Theorem SIM.1: *Suppose that X is a random variable, and let F be its* cumulative distribution function (c.d.f.)*:*

$$F(t) = \Pr\{X \leq t\} \qquad (x \in \mathbf{R}).$$

Assume that F is continuous and strictly increasing on the interval $[a,b]$, with $F(a) = 0$ and $F(b) = 1$. (We allow $a = -\infty$ or $b = +\infty$ or both.) Then F has a well-defined inverse function, F^{-1}, from $[0,1]$ to $[a,b]$. Let U have the distribution $U[0,1]$. Then $F^{-1}(U)$ has the same distribution as X.

Remark: The conditions of Theorem SIM.1 are satisfied if X is a continuous random variable with a probability density function (p.d.f.) $f(x)$ that is strictly positive for every x in $[a,b]$, and is 0 for every other x.

Proof of Theorem SIM.1: We need to show that $\Pr\{F^{-1}(U) \leq t\} = \Pr\{X \leq t\}$ for every t in $[a,b]$. For any t in $[a,b]$, we have

$$\begin{aligned}
\Pr\{F^{-1}(U) \leq t\} &= \Pr\{F(F^{-1}(U)) \leq F(t)\} \\
&\qquad \text{(since } F \text{ is strictly increasing)} \\
&= \Pr\{U \leq F(t)\} \\
&= \Pr\{0 \leq U \leq F(t)\} \\
&= F(t) \\
&= \Pr\{X \leq t\}. \qquad\qquad \text{Q.E.D.}
\end{aligned}$$

Examples: (a) Uniform distribution on the interval $[a,b]$: the p.d.f. is $f(x) = 1/(b-a)$ for $a \leq x \leq b$, and the c.d.f. is $F(t) = (t-a)/(b-a)$ for $a \leq t \leq b$. Then $F^{-1}(u) = (b-a)u + a$ (to get this, solve $F(t) = u$ for t). So if $U \sim U[0,1]$, then $(b-a)U + a \sim U[a,b]$.

(b) Exponential distribution with mean β: the p.d.f. is $f(x) = \beta^{-1}e^{-x/\beta}$ for $0 \leq x < \infty$, and the c.d.f. is $F(t) = 1 - e^{-t/\beta}$ for $0 \leq t < \infty$. Then $F^{-1}(u) = -\beta \ln(1-u)$. Thus, if $U \sim U[0,1]$ then $-\beta \ln(1-U)$ has the desired distribution. *Note*: Since $1-U$ is also uniformly distributed on $[0,1]$, it follows that $-\beta \ln U$ is also exponentially distributed.

(c) Gamma(α, θ) distribution, in the special case that α is a positive integer: It is not easy to calculate F^{-1}, but we can take a different approach. Recall that the Gamma$(1,\theta)$ distribution is the exponential distribution with mean θ. We know that if $Z_1, \ldots, Z_\alpha$ are i.i.d. random variables with Exponential(θ) distribution, then $Z_1 + \cdots + Z_\alpha$ has the Gamma(α,θ) distribution. Therefore, by part (b), we see that $-\theta(\sum_{j=1}^{\alpha} \ln U_j)$ has the Gamma(α,θ) distribution.