Efficient MIMO Channel Estimation With Optimal Training Sequences

Oomke Weikert and Udo Zölzer
Department of Signal Processing and Communications
Helmut Schmidt University - University of the Federal Armed Forces Hamburg
Holstenhofweg 85, 22043 Hamburg, Germany
Email: {oomke.weikert, udo.zoelzer}@hsu-hamburg.de

***Abstract*—In a real MIMO communication system an exact MIMO channel estimate is required. In comparison to a single antenna system with only one channel to be estimated, in a MIMO system with four transmit and four receive antennas 16 channels have to be estimated. The increased number of required training symbols may reduce the higher data rate of a MIMO system. The objective of this paper is to present an efficient MIMO channel estimation. On the basis of the required accuracy of the estimate, optimal training sequences of minimum length are determined.**

I. Introduction

Multiple-input multiple-output (MIMO) systems provide higher data rates that data-demanding applications require. A real MIMO communication system [1] requires an exact knowledge of the MIMO channel. To make a statement about the achievable MIMO channel capacity and the maximal possible data rate, the MIMO channel estimate is required. On the basis of the MIMO channel capacity the configuration of the transmitter and receiver can be evaluated and optimized [2]. Furthermore the MIMO channel estimate is a requirement for the equalization at the receiver. The derivation of a MIMO channel model for a specified environment implies an exact knowledge of the channel impulse responses of the MIMO channel.

For all these applications the channels from every transmit to every receive antenna have to be estimated simultaneously and not consecutively. Consecutive measurements of these channels do not meet the objectives. By consecutive measurements only correlations introduced by the antennas might be measured. More important are correlations between different propagation paths which are caused by the environment. These correlations are introduced during the transmission. Only by simultaneous measurement of all channels these correlations can be measured.

The estimation of frequency selective MIMO channels is based on the transmission of training sequences. For a classical system with one transmit and one receive antenna one channel has to be estimated. For a MIMO system with four transmit and four receive antennas 16 channels have to be estimated. Due to the increased number of channels to be estimated, an increased number of training symbols is required. Assuming a frame of constant length, consisting of the payload and the training symbols, in a classical one antenna system the training symbols only require a small fraction of the frame. In the MIMO case, only a small part of the frame may remain for the payload. Especially by consecutive channel measurements most of the frame is needed for training symbols. In this case the potential higher data rate of a MIMO system can not be achieved.

The objective of this paper is to present an efficient MIMO channel estimation with optimal training sequences of minimum length. The simultaneous channel estimation reduces the required number of training symbols. To reduce the complexity of the estimation, the MIMO channel is considered as a superposition of several multiple-input single-output (MISO) channels, as proposed by [3]. With optimal training sequences the minimum mean square error of the channel estimate is achieved. Depending on the required accuracy of the channel estimate, an equation to determine the required number of optimal training symbols is derived.

The paper is organized as follows. The fundamentals of MIMO systems are described in section II. In Section III the Minimum Mean Square Error (MMSE) channel estimation of MIMO frequency selective channels is shown. A criterion for optimal training sequences is derived in section IV and optimal training sequence design is presented. On the basis of the required accuracy of the estimate, optimal training sequences of minimum length are determined. Section V covers the normalization of measured MIMO channels. A summary and conclusion marks can be found in Section VI.

II. Fundamentals

We consider a frequency selective MIMO wireless system with N_T transmit and N_R receive antennas. The symbol transmitted by antenna m at time instant k is denoted by $s_m(k)$. The transmitted symbols are arranged in the vector

$$\mathbf{s}(k) = [s_1(k),\ \ldots,\ s_{N_T}(k)]^T \tag{1}$$

of length N_T, where $(\cdot)^T$ denotes the transpose operation.

Between every transmit antenna m and every receive antenna n there is a complex single-input single-output (SISO) channel impulse response $h_{n,m}(k)$ of length $L+1$, described by the vector

$$\mathbf{h}_{n,m} = [h_{n,m}(0),\ \ldots,\ h_{n,m}(L)]^T. \tag{2}$$

Assuming the same channel order L for all channels, the frequency selective MIMO channel can be described by $L+1$ complex channel matrices

$$\mathbf{H}(k) = \begin{bmatrix} h_{1,1}(k) & \cdots & h_{1,N_T}(k) \\ \vdots & \ddots & \vdots \\ h_{N_R,1}(k) & \cdots & h_{N_R,N_T}(k) \end{bmatrix}, \quad k = 0, ..., L \quad (3)$$

of the dimension $N_R \times N_T$. The channel energy is normalized by the condition [4]

$$\sum_{k=0}^{L} E\left\{ |h_{n,m}(k)|^2 \right\} := 1 \quad \forall\, n, m. \quad (4)$$

The symbol received by antenna n at time instant k is denoted by $x_n(k)$. The symbols received by the N_R antennas are arranged in a vector

$$\mathbf{x}(k) = [x_1(k),\ \ldots,\ x_{N_R}(k)]^T \quad (5)$$

of length N_R, which can be expressed with (1), (3) and $\mathbf{n}(k)$ as noise vector of length N_R as

$$\mathbf{x}(k) = \sum_{i=0}^{L} \mathbf{H}(i)\ \mathbf{s}(k-i) + \mathbf{n}(k). \quad (6)$$

We assume additive white Gaussian noise (AWGN) with zero mean and variance σ_n^2 per receive antenna, i.e. the spatial correlation matrix of the noise is given by

$$\mathbf{R}_{nn} = E\{\mathbf{n}(k)\ \mathbf{n}^H(k)\} = \sigma_n^2\ \mathbf{I}_{N_R} \quad (7)$$

where $\mathbf{I}$ is the identity matrix and $(\cdot)^H$ denotes the complex-conjugate (Hermitian) transpose.

III. Channel Estimation

To reduce the complexity of the estimation, the channel is modeled as a superposition of N_R multiple-input single-output (MISO) channels, as proposed in [3]. Each MISO channel has $N_T(L+1)$ unknowns.

The receive signal $x_n(k)$ of the n-th MISO channel can be expressed as

$$x_n(k) = \sum_{m=1}^{N_T} \sum_{i=0}^{L} h_{n,m}(i)\ s_m(k-i) + n(k). \quad (8)$$

The number of training symbols per transmit antenna and per frame is N_P. As the first L training symbols are influenced by unknown previous symbols, only the last $N_P - L$ received training symbols can be used for channel estimation.

With the matrix $\mathbf{S}_m$ of dimension $(N_P - L) \times (L+1)$, which contains the transmitted training symbols of transmitantenna m, given by

$$\mathbf{S}_m = \begin{bmatrix} s_m(L) & \cdots & s_m(0) \\ s_m(L+1) & & s_m(1) \\ \vdots & & \vdots \\ s_m(N_P-1) & \cdots & s_m(N_P-(L+1)) \end{bmatrix}, \quad (9)$$

the receive vector

$$\mathbf{x}_n = [x_n(L),\ \ldots,\ x_n(N_P-1)]^T \quad (10)$$

of length $N_P - L$, containing the last $N_P - L$ received training symbols $x_n(k)$ of receive antenna n, can be written as

$$\mathbf{x}_n = \mathbf{S}_1\ \mathbf{h}_{n,1} + \ \ldots\ + \mathbf{S}_{N_T}\ \mathbf{h}_{n,N_T} + \mathbf{n} \quad (11)$$

$$= [\mathbf{S}_1,\ \ldots,\ \mathbf{S}_{N_T}] \begin{bmatrix} \mathbf{h}_{n,1} \\ \vdots \\ \mathbf{h}_{n,N_T} \end{bmatrix} + \mathbf{n} \quad (12)$$

with the additive noise $\mathbf{n}$.

The matrix

$$\mathbf{S} = [\mathbf{S}_1,\ \ldots,\ \mathbf{S}_{N_T}] \quad (13)$$

of dimension $(N_P - L) \times N_T(L+1)$ contains the transmitted training symbols of all transmit antennas. The matrix $\mathbf{S}$ as well as all matrices $\mathbf{S}_m$ are cyclic. The vector

$$\mathbf{h}_n = [\mathbf{h}_{n,1}^T,\ \ldots,\ \mathbf{h}_{n,N_T}^T]^T \quad (14)$$

of length $N_T(L+1)$ contains the elements of the channel impulse responses $h_{n,m}(k)$, $k = 0, ..., L$, between all transmit antennas and the n-th receive antenna. The elements of the channel impulse responses $h_{n,m}(k)$, $k = 0, ..., L$, have to be estimated. The receive vector $\mathbf{x}_n$ follows to

$$\mathbf{x}_n = \mathbf{S}\ \mathbf{h}_n + \mathbf{n}. \quad (15)$$

A. *Least Square Channel Estimation*

The linear Least Square (LS) channel estimate is given by [5]

$$\hat{\mathbf{h}}_{n,\text{LS}} = \left[\mathbf{S}^H\ \mathbf{S}\right]^{-1}\ \mathbf{S}^H\ \mathbf{x}_n \quad (16)$$

or generally for non-white noise

$$\hat{\mathbf{h}}_{n,\text{LS}} = \left[\mathbf{S}^H\ \mathbf{R}_{\mathbf{nn}}^{-1}\ \mathbf{S}\right]^{-1}\ \mathbf{S}^H\ \mathbf{R}_{\mathbf{nn}}^{-1}\ \mathbf{x}_n \quad (17)$$

where $(\cdot)^H$ and $(\cdot)^{-1}$ denote the complex-conjugate (Hermitian) transpose and the inverse, respectively.

B. *Minimum Mean Square Error Channel Estimation*

The Minimum Mean Square Error (MMSE) channel estimate is given by [5]

$$\hat{\mathbf{h}}_{n,\text{MMSE}} = \left[\mathbf{S}^H\ \mathbf{R}_{\mathbf{nn}}^{-1}\ \mathbf{S} + \mathbf{R}_{\mathbf{hh}}^{-1}\right]^{-1}\ \mathbf{S}^H\ \mathbf{R}_{\mathbf{nn}}^{-1}\ \mathbf{x}_n, \quad (18)$$

where $\mathbf{R}_{\mathbf{hh}} = E\{\mathbf{h}_n\ \mathbf{h}_n^H\}$ is the correlation matrix of the channel impulse response vector $\mathbf{h}_n$. At the receiver the correlation matrix of the channel $\mathbf{R}_{\mathbf{hh}}$ is unknown. As the channel may change from measurement to measurement past estimates of $\mathbf{h}_n$ cannot be used to calculate $\mathbf{R}_{\mathbf{hh}}$. It is assumed that the SISO channel impulse responses of the MIMO channel

$$E\{h_{n_1,m_1}(k_1)\ h_{n_2,m_2}^*(k_2)\} = 0 \quad \text{for } n_1 \neq n_2 \text{ or } m_1 \neq m_2\ \forall\, k_1, k_2 \quad (19)$$

as well as the elements of the channel impulse responses

$$E\{h_{n_1,m_1}(k_1)\ h^*_{n_2,m_2}(k_2)\} = 0 \quad \text{for } n_1 = n_2\ \&\ m_1 = m_2\ \&\ k_1 \neq k_2 \tag{20}$$

are uncorrelated from each other. It follows that

$$\mathbf{R_{hh}} = E\{\mathbf{h}_n\ \mathbf{h}_n^H\} \tag{21}$$

$$= \text{diag}\left(\sigma^2_{n,1}(0), ..., \sigma^2_{n,1}(L), \sigma^2_{n,2}(0), ..., \sigma^2_{n,N_T}(L)\right) \tag{22}$$

is a diagonal matrix with

$$E\left\{|h_{n,m}(k)|^2\right\} = \sigma^2_{n,m}(k). \tag{23}$$

The power delay profile (PDP) describes by the values $\sigma^2_{n,m}(k), k = 0, \ldots, L$ how the power is distributed over the taps of the channel impulse response. The PDP may change over time and is unknown at the receiver. It is supposed that all SISO channels of the MIMO channel have the same channel order and the same PDP. Supposing a constant PDP with equal values $\sigma^2_{n,m}(k) = \sigma^2_h\ \forall k$ and taking into consideration the normalization condition (4) it follows that

$$E\left\{|h_{n,m}(k)|^2\right\} = \sigma^2_{n,m}(k) = \sigma^2_h = \frac{1}{L+1} \quad \forall\ n, m, k. \tag{24}$$

This leads to the approximation

$$\mathbf{R_{hh}} = \sigma^2_h\ \mathbf{I}_{N_T(L+1)} = \frac{1}{L+1}\ \mathbf{I}_{N_T(L+1)}. \tag{25}$$

Given $\mathbf{R_{hh}}$ specified above, the MMSE channel estimate is given by

$$\hat{\mathbf{h}}_{n,\text{MMSE}} = \left[\mathbf{S}^H\ \mathbf{R}^{-1}_{\mathbf{nn}}\ \mathbf{S} + \frac{1}{\sigma^2_h}\ \mathbf{I}\right]^{-1} \mathbf{S}^H\ \mathbf{R}^{-1}_{\mathbf{nn}}\ \mathbf{x}_n, \tag{26}$$

and in the case of additive white noise the MMSE channel estimate follows to

$$\hat{\mathbf{h}}_{n,\text{MMSE}} = \left[\mathbf{S}^H\ \mathbf{S} + \frac{\sigma^2_n}{\sigma^2_h}\ \mathbf{I}\right]^{-1} \mathbf{S}^H\ \ \mathbf{x}_n. \tag{27}$$

Setting the term σ^2_n/σ^2_h in (27) to zero yields the Least Square channel estimate in the case of additive white noise.

The term $\left[\mathbf{S}^H\ \mathbf{S} + \frac{\sigma^2_n}{\sigma^2_h}\ \mathbf{I}\right]^{-1}\ \mathbf{S}^H$ can be calculated one-time in advance as it only depends on the known training symbols and the noise power σ^2_n.

C. Mean Square Error of the Channel Estimation

The correlation matrix of the error of the channel estimation is given by

$$\mathbf{R_{ee}} = E\left\{[\mathbf{h}_n - \hat{\mathbf{h}}_n][\mathbf{h}_n - \hat{\mathbf{h}}_n]^H\right\}. \tag{28}$$

For the Minimum Mean Square Error channel estimation it follows to [5]

$$\mathbf{R_{ee}} = \left[\mathbf{S}^H\ \mathbf{R}^{-1}_{\mathbf{nn}}\ \mathbf{S} + \mathbf{R}^{-1}_{\mathbf{hh}}\right]^{-1}. \tag{29}$$

The Mean Square Error (MSE) of a MISO channel

$$\text{MSE}_{\text{MISO}} = E\left\{[\hat{\mathbf{h}}_n - \mathbf{h}_n]^H[\hat{\mathbf{h}}_n - \mathbf{h}_n]\right\} = \text{tr}\left(\mathbf{R_{ee}}\right) \tag{30}$$

is the trace of the error correlation matrix $\mathbf{R_{ee}}$. The trace of a matrix denoted by $\text{tr}\left(\cdot\right)$ is the sum of the diagonal elements. Given $\mathbf{R_{hh}}$ specified in (25) and for additive white noise the Mean Square Error follows to

$$\text{MSE}_{\text{MISO}} = \sigma^2_n\ \text{tr}\left(\left[\mathbf{S}^H\ \mathbf{S} + \frac{\sigma^2_n}{\sigma^2_h}\ \mathbf{I}\right]^{-1}\right). \tag{31}$$

For the Least Square channel estimation the term σ^2_n/σ^2_h has to be set to zero.

IV. Training Sequences

A. Minimum Number of Training Symbols

For channel estimation there are $N_T(L+1)$ unknowns in every MISO channel, as remarked before. There have to be leastwise as many training symbols as unknowns to estimate the channel and as only the last $N_P - L$ training symbols may be used for the estimation, the number of training symbols N_P per transmit antenna and per frame has to be at least

$$N_P \geq N_T(L+1) + L. \tag{32}$$

B. Criterion for Optimal Training Sequences

The training sequences should be designed so that the Mean Square Error of the channel estimation is minimized. Such training sequences are called optimal. The energy of the training sequences

$$s^2_m(k) = a^2 = \text{constant} > 0\ \forall k \tag{33}$$

should be constant and equal. The SISO channels of the MIMO channel as well as the elements of the channel impulse responses are assumed to be uncorrelated from each other. Additive white Gaussian noise is assumed. If the matrix $\mathbf{S}$, which contains the training sequences, holds the criterion

$$\left[\mathbf{S}^H\ \mathbf{S} + \frac{\sigma^2_n}{\sigma^2_h}\ \mathbf{I}\right] = \lambda\ \mathbf{I}, \tag{34}$$

with

$$\lambda = \left((N_P - L)\ a^2 + \frac{\sigma^2_n}{\sigma^2_h}\right) \tag{35}$$

the Mean Square Error (MSE) is minimized. The proof is given in [4]. The Minimum Mean Square Error (MMSE) follows to [4]

$$\text{MMSE}_{\text{MISO}} = \sigma^2_n\ \frac{N_T(L+1)}{(N_P - L)\cdot a^2 + \frac{\sigma^2_n}{\sigma^2_h}}. \tag{36}$$

For the Least Square channel estimation the term σ^2_n/σ^2_h has to be set to zero.

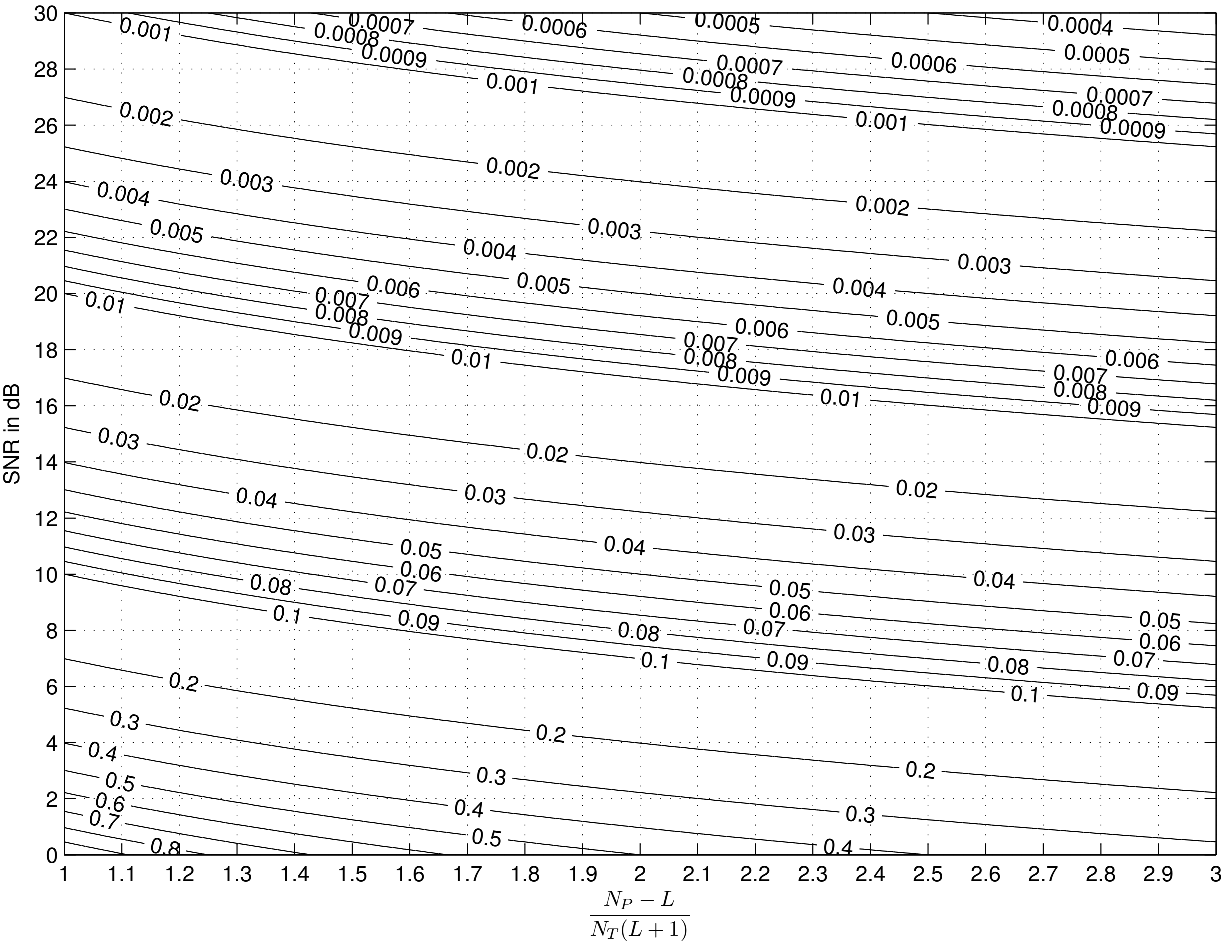

Fig. 1. Required number of training symbols, lines with the same $\text{MMSE}_{\text{SISO,LS}}$

C. Design of Optimal Training Sequences

The matrix $\mathbf{S}$ is cyclic. A matrix which holds (34) has orthogonal columns. A cyclic matrix $\mathbf{S}$ with orthogonal columns can be constructed by writing a perfect root-of-unity sequence (PRUS) into the first row of $\mathbf{S}$ and then filling any next row with the one element right shifted version of the previous row [6].

A root-of-unity sequence $s(k)$ of length N has complex root-of-unity elements with absolut value one and may have P different phases. The elements of a root-of-unity sequence $s(k)$ are of the form

$$s(k) = e^{j\frac{2\pi}{P}f(k)} \text{ with } k = 0, ..., N-1 \text{ and } 0 \leq f(k) \leq P, \tag{37}$$

with $f(k)$ as a whole-number sequence. A root-of-unity sequence is said to be perfect if all of its out-of-phase periodic autocorrelation terms are equal to zero [6].

A perfect root-of-unity sequence $s(k)$ can be constructed for any length N by the Frank-Zadoff-Chu-sequences [6] [7]

$$s(k) = \begin{cases} e^{j\pi M k^2/N} & \text{for } N \text{ even} \\ e^{j\pi M k(k+1)/N} & \text{for } N \text{ odd} \end{cases} \text{ with } k = 0, ..., N-1. \tag{38}$$

M is a natural number greater than zero and needs to be coprime to N.

The length $N = N_T(L+1)$ of the required perfect root-of-unity sequence $s(k)$ is equal to the number of columns of $\mathbf{S}$. The sequence $s(k)$ is written into the first row of $\mathbf{S}$. Any next row is the one element right shifted version of the previous row. The training sequence for each transmit antenna can be extracted from the matrix $\mathbf{S}$, taking into account (13) and (9).

D. Required Number of Training Symbols

The MSE related to one of the $N_T \cdot N_R$ SISO channels of the MIMO channel is given by

$$\text{MSE}_{\text{SISO}} = \frac{1}{N_T} \text{MSE}_{\text{MISO}}. \tag{39}$$

To preserve the comparability with a single-input single-output (SISO) system in terms of equal total transmit energy P, the transmit symbols are multiplied by the factor $\frac{1}{\sqrt{N_T}}$ before transmission. The energy of a training symbol is given by

$$s_m^2(k) = a^2 = P_S = \frac{P}{N_T} = \text{konstant} \quad \forall k \tag{40}$$

with the transmit power P_S of each transmit antenna and the total transmit power $P = 1$.

With (36), (39) and (40) the Minimum Mean Square Error (MMSE) of the Least Square channel estimation follows to

$$\mathrm{MMSE}_{\mathrm{SISO,LS}} = \frac{\sigma_n^2}{P}\,\frac{N_T(L+1)}{(N_P - L)}. \tag{41}$$

If the channel matrices fulfill the normalization condition (4), the term $\frac{P}{\sigma_n^2}$ in (41) can be replaced by the average signal to noise ratio ρ. The Minimum Mean Square Error (MMSE) of the Least Square channel estimation follows to

$$\mathrm{MMSE}_{\mathrm{SISO,LS}} = \frac{1}{\rho}\,\frac{N_T(L+1)}{(N_P - L)} \tag{42}$$

with the signal to noise ratio ρ as a non-logarithmic value.

The required number of training symbols for the Least Square channel estimation depending on the signal to noise ratio (SNR) and the required accuracy of the channel estimate is shown in Figure 1. The required accuracy of the channel estimate is specified by the MMSE. Figure 1 shows lines with the same $\mathrm{MMSE}_{\mathrm{SISO,LS}}$. For an accuracy of the channel estimate of $\mathrm{MMSE}_{\mathrm{SISO,LS}} = 5\cdot 10^{-3}$ for each of the $N_T \cdot N_R$ SISO channels of the MIMO channel, at least $N_P = 2\cdot N_T(L+1)+L$ training symbols per transmit antenna are required for a SNR of 20dB.

V. Normalizing Measured MIMO Channels

Simulating MIMO systems it is possible to create channel matrices $\mathbf{H}(k)$ with random entries $h_{n,m}(k)$ for which (4) holds. Measured systems have to be normalized to fulfill the requirements of the capacity calculation. Normalizing a measured MIMO channel to fulfill (4), one has to multiply every contained SISO channel with another normalization factor and by doing so one would rate the path loss differently for every SISO channel. To avoid this, (4) is relaxed to

$$\sum_{m=1}^{N_T}\sum_{n=1}^{N_R}\sum_{k=0}^{L} |h_{n,m}(k)|^2 = N_T \cdot N_R. \tag{43}$$

By this normalization only one normalization factor is introduced for all SISO channels while the mean energy of the channel still equals one.

Comparing different MIMO channels by their capacity, the same problem as above occurs. Applying (43) every single MIMO channel is normalized by one single factor, but this factor differs from system to system. By introducing the normalization

$$\sum_{i=1}^{N_M}\sum_{m=1}^{N_T}\sum_{n=1}^{N_R}\sum_{k=0}^{L} |h_{n,m}(k,i)|^2 = N_M \cdot N_T \cdot N_R \tag{44}$$

one normalization factor for all N_M different MIMO systems is obtained. However the capacity is calculated for every single MIMO channel and those not necessary fulfill (43) when being normalized to (44). The error causes extra noise for each MIMO channel resulting the MIMO channels to differ in their signal to noise ratio (SNR).

For a comparability based on the same SNR one has to choose (43), for a comparability based on the same path loss select (44). It is not possible to have both. One has to decide which one is more important.

VI. Conclusions

Efficient MIMO channel estimation with optimal training sequences was presented. With optimal training sequences the minimum mean square error of the channel estimate is achieved. On the basis of the required accuracy of the estimate, optimal training sequences of minimum length are determined.

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