Lecture 27: Sums of Independent Random Variables: III
Friday, April 13

1 Sums of Independent Gamma Random Variables

Recall that a gamma random variable X with parameters (α, λ) has density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \dfrac{(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $E(X) = \alpha/\lambda$ and $\mathrm{Var}(X) = \alpha/\lambda^2$. The calculations are in the notes for last time, even though we did not go over them in class.

Last time we stated but did not prove the following proposition.

Proposition. *Let X be a gamma random variable with parameters (α, λ), and let Y be a gamma random variable with parameters (β, λ). If X and Y are independent, then $X + Y$ is a gamma random variable with parameters $(\alpha + \beta, \lambda)$.*

We will base the proof on the following lemma.

Lemma. *Let f and g be probability densities. If there is a constant c such that $f(x) = cg(x)$ for almost all $x \in \mathbf{R}$, then*

1. *$c = 1$;*
2. *$f(x) = g(x)$ for almost all $x \in \mathbf{R}$.*

Proof. It suffices to prove (1). We have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\, dx \\ &= c \int_{-\infty}^{\infty} g(x)\, dx \\ &= c \end{aligned}$$

$\square$

Proposition. *Let X be a gamma random variable with parameters (α, λ), and let Y be a gamma random variable with parameters (β, λ). If X and Y are independent, then $X+Y$ is a gamma random variable with parameters $(\alpha+\beta, \lambda)$.*

Proof. Let g be the gamma density with parameters $(\alpha+\beta)$. The strategy of the proof is to prove that there is a constant c such that $f_{X+Y}(x) = cg(x)$ for all $x \in \mathbf{R}$ and then invoke the lemma.

Now $f_{X+Y}(x) = 0 = g(x)$ for all $x < 0$. Let $a > 0$.

$$
\begin{aligned}
f_{X+Y}(a) &= (f_Y * f_X)(a) \\
&= \int_0^a f_Y(y) f_X(a-y)\, dy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^a \lambda e^{-\lambda y} (\lambda y)^{\beta-1} \lambda e^{\lambda(a-y} \left(\lambda(a-y)\right)^{\alpha-1}\, dy \\
&= \frac{\lambda^{\alpha+\beta} e^{-\lambda a}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^a y^{\beta-1}(a-y)^{\alpha-1}\, dy \\
&= \frac{\lambda e^{-\lambda a} \lambda^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (at)^{\beta-1} \left(a(1-t)\right)^{\alpha-1} a\, dt \\
&= \frac{\lambda e^{-\lambda a} (a\lambda)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\, dt \\
&= g(a) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}\, dt
\end{aligned}
$$

$\square$

Remark. The lemma also tells us that the constant factor in the last equation is equal to 1.

Definition. The **beta function** is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

where $x > 0$ and $y > 0$.

Corollary.

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

Remark. The proposition extends to the case of n independent gamma random variables, all with the same second parameter. The proof is a straightforward induction that involves no further calculation. The assertions we made about sums of independent exponential random variables follow from this extension.

2 Sums of Independent Normal Random Variables

$$
\begin{aligned}
f_{X+Y}(a) &= (f_X * f_Y)(a) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{1}{2}\left(\frac{a-x-\nu}{\tau}\right)^2} dx \\
&= \frac{1}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{a-x-\nu}{\tau}\right)^2\right]} dx
\end{aligned}
$$

$$
\begin{aligned}
\left[\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{a-x-\nu}{\tau}\right)^2\right] &= \left(\frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) x^2 - 2\left(\frac{\mu}{\sigma^2} + \frac{a-\nu}{\tau^2}\right) + \left[\frac{\mu^2}{\sigma^2} + \frac{(a-\nu)^2}{\tau^2}\right] \\
&= Ax^2 + -2Bx + C
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{\sigma^2 + \tau^2}{\sigma^2\tau^2} \\
B &= \frac{\tau^2\mu + \sigma^2(a-\nu)}{\sigma^2\tau^2} \\
C &= \frac{\tau^2\mu^2 + \sigma^2(a-\nu)^2}{\sigma^2\tau^2}
\end{aligned}
$$

$$
\begin{aligned}
Ax^2 - 2Bx + C &= A\left(x^2 - 2\frac{B}{A}x\right) + C \\
&= A\left(x - \frac{B}{A}\right)^2 + C - \frac{B^2}{A}
\end{aligned}
$$

$$
\begin{aligned}
\int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left(Ax^2-2Bx+c\right)}\,dx &= \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\left[A\left(x-\frac{B}{A}\right)^2+C-\frac{B^2}{A}\right]}\,dx \\
&= e^{-\frac{1}{2}\left(C-\frac{B^2}{A}\right)} \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}A\left(x-\frac{B}{A}\right)^2}\,dx \\
&= e^{-\frac{1}{2}\left(C-\frac{B^2}{A}\right)} \frac{1}{\sqrt{A}} \int\limits_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\,dt \\
&= e^{-\frac{1}{2}\left(C-\frac{B^2}{A}\right)} \frac{\sqrt{2\pi}}{\sqrt{A}}
\end{aligned}
$$

$$
C-\frac{B^2}{A} \quad =
$$