Number of Fixed Points in Random Permutations

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Dec. 2012

Abstract

In this paper, we will focus on various features of the random variable X defined by the number of fixed points in random permutations. The distribution of X will be given in several forms, an approximate distribution will be discussed. The moments as well as the generating functions of X will also be calculated.

Keywords: random permutations, dearrangements, fixed points, rencontres numbers

Contents

Notations

Throughout this paper, we would use the following notations:

- $\mathrm{P}\left\{S\right\}$ is the probability of S (S is usually a set or a proposition).
- $\mathrm{E}\left[X\right]$ is the expectation of random variable X, which is defined as the following for a discrete random variable:
$$\mathrm{E}\left[X\right] = \sum_k k\mathrm{P}\left(k\right)$$
- $\mathrm{Var}\left[X\right]$ is the variance of random variable X, defined as
$$\mathrm{Var}\left[X\right] = \mathrm{E}\left[(X - \mathrm{E}\left[X\right])^2\right]$$
- $\mathrm{Cov}[X, Y]$ is the covariance of random variables X, Y, defined as
$$\mathrm{Cov}[X, Y] = \mathrm{E}\left[XY\right] - \mathrm{E}\left[X\right]\mathrm{E}\left[Y\right]$$

1 Introduction

Let $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ be a permutation of $(1, 2, \cdots, n)$. The *Fixed Points* of the permutation π , denoted as $\mathcal{F}(\pi)$, is defined as followed:

$$\mathcal{F}(\pi) = \{k \in [1, n] \cap \mathbb{N} \mid \pi(k) = k\}$$

Let Π_n be the set of all possible permutations of $(1, 2, \cdots, n)$. The *Rencontres Numbers*[1] $D_{n,k}$ is defined as followed:

$$D_{n,k} = \left\|\{\pi \in \Pi_n \mid \|\mathcal{F}(\pi)\| = k\}\right\|, k = 0, 1, \cdots n$$

where $\|A\|$ is the cardinality of a set A.

In particular, we denote $D_{n,0}$ as D_n for short.

Let π be a random permutation of $(1, 2, \cdots, n)$, where every possible permutation has the same possibility $\dfrac{1}{n!}$. The random variable $X = \|\mathcal{F}(\pi)\|$ is what we will focus on in this paper.

2 Calculation of $D_{n,k}$

We are about to see three totally different approaches of the calculation of D_n.

Then we can easily caculate $D_{n,k}$ by

$$D_{n,k} = \binom{n}{k} D_{n-k}$$

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2.1 Using the Inclusion-Exclusion Principle

Define A_j as:

$$A_j = \{\pi \in \Pi | j \in \mathcal{F}(\pi)\}, j = 1, 2, \cdots, n$$

Then it is obvious to see that, for any $t = 1, 2, \cdots, n$, we have

$$\left\|\bigcap_{i=1}^{t} A_{r_i}\right\| = (n - t)!$$

where r_i is any permutation of $(1, 2, \cdots, n)$.

Therefore, D_n can be calculated by the Inclusion-Exclusion Principle :

$$\begin{aligned} D_n =& n! - \sum_{i=1}^{n} \|A_i\| + \sum_{1 \le i < j \le n} \|A_i \cap A_j\| - \cdots + (-1)^n \left\| \bigcap_{i=1}^{n} A_i \right\| \\ =& n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)! \\ =& n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \end{aligned}$$

2.2 Using Recurrence Relation

For any permutation π of $(1, 2, \cdots n)$, such that $\|\mathcal{F}(\pi)\| = 0$, it is obvious that $\pi(1)$ has $n-1$ possible values. Now consider the value of $\pi(\pi(1))$:

If $\pi(\pi(1)) = 1$, then $\pi' = (\pi(2), \cdots, \pi(\pi(1)-1), \pi(\pi(1)+1), \cdots, \pi(n))$ is a permutation of $(2, \cdots, \pi(1)-1, \pi(1)+1, \cdots, n)$, such that $\|\mathcal{F}(\pi')\| = 0$. It is easy to show that this correspondence from the given π to π' is bijective.

If $\pi(\pi(1)) \neq 1$, assume $\pi(j) = 1$, then $\pi' = (\pi(2), \cdots, \pi(j-1), \pi(1), \pi(j+1), \cdots, \pi(n))$ is a permutation of $(2, \cdots, n)$, such that $\|\mathcal{F}(\pi')\| = 0$. It is easy to show that this correspondence from the given π to π' is also bijective.

Therefore, we have the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \tag{1}$$
$$\Leftrightarrow D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] \tag{2}$$
$$D_2 = 1, D_1 = 0$$

Continue applying (2), we obtain

$$\begin{aligned} & D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] = \cdots = (-1)^{n-2}(D_2 - 2D_1) = (-1)^n \\ \Rightarrow & \frac{D_n}{n!} - \frac{D_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!} \\ \Rightarrow & D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \end{aligned}$$

2.3 Using the Inversion Formula

Lemma 1. ***(Inversion Formula)*** *Given two sequences $\{a_n\}, \{b_n\}$. If the formula*

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k$$

holds for all $n = 1, 2, \cdots$, then

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k, n = 1, 2, \cdots$$

Proof. Note that in series $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+x)^k$, the coefficient of the term x^p is given by $\sum_{k=p}^{n} (-1)^k \binom{n}{k} \binom{k}{p}$.

On the other hand, by *Binomial Theorem*,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+x)^k = [1 - (1+x)]^n = (-1)^n x^n,$$

therefore the coefficient of the term x^p is $(-1)^n\delta_{pn}$, where δ_{ij} is the *Kronecker delta function*:

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

Thus,

$$\sum_{k=p}^{n}(-1)^k\binom{n}{k}\binom{k}{p} = (-1)^n\delta_{pn}$$

It follows that

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k}b_k &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\sum_{p=0}^{k}(-1)^i\binom{k}{i}a_i\right] \\
&= \sum_{p=0}^{n}(-1)^p\left[\sum_{k=p}^{n}(-1)^k\binom{n}{k}\binom{k}{p}a_p\right] \\
&= \sum_{p=0}^{n}(-1)^p(-1)^n\delta_{pn}a_p \\
&= a_n
\end{aligned}$$

□

By definition, $\sum_{k=0}^{n}D_{n,k} = \|\Pi_n\| = n!$, which is equivalent to

$$n! = \sum_{k=0}^{n}\binom{n}{k}D_k = \sum_{k=0}^{n}\left[(-1)^k\binom{n}{k}\right]\left[(-1)^kD_k\right] \tag{3}$$

According to Lemma 1, we immediately obtain:

$$\begin{aligned}
(-1)^nD_n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}k! \\
\Rightarrow D_n &= \sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{(n-k)!} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}
\end{aligned}$$

3 Characteristics of X

3.1 Expectation and Variance

By definition, the probability mass function of X is given by

$$\mathrm{P}\{X = k\} = \frac{D_{n,k}}{n!} = \frac{\binom{n}{k}D_{n-k}}{n!}, k = 0, 1, \cdots, n \tag{4}$$

Recall that $\sum_{k=0}^{n}\binom{n-1}{n-k}D_{n-k} = (n-1)!$ according to (3), the expectation of X can be calculated as followed:

$$\mathrm{E}[X] = \sum_{k=0}^{n}\frac{k\binom{n}{k}D_{n-k}}{n!} = \sum_{k=0}^{n}\frac{n\binom{n-1}{n-k}D_{n-k}}{n!} = \sum_{k=0}^{n}\frac{\binom{n-1}{n-k}D_{n-k}}{(n-1)!} = 1 \tag{5}$$

By similiar approach, the variance can also be derived:

$$\begin{aligned}\mathrm{E}\left[X(X-1)\right] &= \sum_{k=0}^{n} \frac{k(k-1)\binom{n}{k}D_{n-k}}{n!} = \sum_{k=0}^{n} \frac{n(n-1)\binom{n-2}{n-k}D_{n-k}}{n!} = 1 \\ \mathrm{Var}\left[X\right] &= \mathrm{E}\left[X(X-1)\right] + \mathrm{E}\left[X\right] - \mathrm{E}\left[X\right]^2 = 1\end{aligned} \tag{6}$$

Another way of calculating expectation and variance is by treating X as a sum of n identical random variables S_i, where S_i is defined by:

$$S_i = \begin{cases} 1, \pi(i) = i \\ 0, \pi(i) \neq i \end{cases}$$

Obviously $X = \displaystyle\sum_{i=1}^{n} S_i$ and $S_i \sim b(1, \dfrac{1}{n})$, where $b(n, p)$ denotes the *binomial distribution* with sample size n and success probability p.

It can be easily shown that for $i \neq j, \mathrm{P}\left\{S_iS_j = 1\right\} = \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)}$, therefore $S_iS_j \sim b(1, \dfrac{1}{n(n-1)})$. Then we can obtain:

$$\mathrm{E}\left[X\right] = \sum_{i=1}^{n} \mathrm{E}\left[S_i\right] = 1$$

$$\mathrm{Cov}[S_i, S_j] = \mathrm{E}\left[S_iS_j\right] - \mathrm{E}\left[S_i\right]\mathrm{E}\left[S_j\right] = \frac{1}{n^2(n-1)}$$

$$\mathrm{Var}\left[X\right] = \sum_{i=1}^{n} \mathrm{Var}\left[S_i\right] + \sum_{i\neq j} \mathrm{Cov}[S_i, S_j] = 1$$

3.2 Moments

Next, we calculate the moments of X.

Introducing the *Stirling number of the second kind*[2] $\left\{ m \atop k \right\}$, it is well-known that:

$$\sum_{k=0}^{m} \left\{ m \atop k \right\} x^{\underline{k}} = x^m$$

where $x^{\underline{k}}$ is the kth *falling factorial* of x defined by:

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1). \quad x^{\underline{0}} = 1, \text{in particular}$$

Therefore, we have

$$\sum_{k=0}^{m} \left\{ m \atop k \right\} \mathrm{E}\left[X^{\underline{k}}\right] = \mathrm{E}\left[X^m\right]$$

To calculate the moments, we only have to calculate $\mathrm{E}\left[X^{\underline{k}}\right]$. Continue the derivation in (5) and (6), it is easy to show that

$$\mathrm{E}\left[X^{\underline{k}}\right] = \begin{cases} 1, 0 \leq k \leq n \\ 0, k > n \end{cases}$$

(Another proof of this will be shown in Section 6.3).

As a result,

$$\mathrm{E}\left[X^m\right] = \begin{cases} \displaystyle\sum_{k=0}^{m} \left\{ {m \atop k} \right\} = B_m, m = 1, 2, \cdots n \\ \\ \displaystyle\sum_{k=0}^{n} \left\{ {m \atop k} \right\}, m = n+1, n+2, \cdots \end{cases}$$

where B_m is the mth *Bell number*[3].

Note that $\left\{ {m \atop k} \right\} = 0$ for $m < k$. It is reasonable to rewrite the above formula as follows:

$$\mathrm{E}\left[X^m\right] = \sum_{k=0}^{n} \left\{ {m \atop k} \right\}, m = 1, 2, \cdots \tag{7}$$

4 Connections to Gamma Function

Continue working on (4), we can obtain:

$$\mathrm{P}\left\{X = k\right\} = \frac{\binom{n}{k}}{n!}(n-k)! \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} = \frac{1}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \tag{8}$$

Introducing the *incomplete gamma function*[4] defined as:

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} \mathrm{d}t$$

By *integration by parts*, a recurrence relation can be found:

$$\Gamma(s, x) = (s-1)\Gamma(s-1, x) + x^{s-1} e^{-x}$$

Thus,

$$\Gamma(n, x) = (n-1)! e^{-x} \sum_{i=0}^{n-1} \frac{x^i}{i!}, \forall n \in \mathbb{N} \tag{9}$$

Then we can rewrite the previous formula in a more elegant way:

$$D_n = \frac{\Gamma(n+1, -1)}{e}$$

$$\mathrm{P}\left\{X = k\right\} = \frac{\binom{n}{k}\Gamma(n-k+1, -1)}{en!} = \frac{\Gamma(n-k+1, -1)}{ek!(n-k)!}$$

Another way of rewriting D_n is:

$$D_n = \mathrm{E}\left[(Y-1)^n\right]$$

where Y is a random variable such that $Y \sim \mathit{Exp}(1)$, here $\mathit{Exp}(\lambda)$ denotes *exponential distribution*.

To prove this, it is sufficient to show that

$$\mathrm{E}\left[Y^k\right] = \int_{\mathbb{R}^+} x^k e^{-x} \mathrm{d}x = \Gamma(k+1) = k!$$

where $\Gamma(x)$ is the *complete gamma function*.

Thus,

$$\mathrm{E}\left[(Y-1)^n\right] = \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\left[Y^{n-k}\right](-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = D_n \quad \square$$

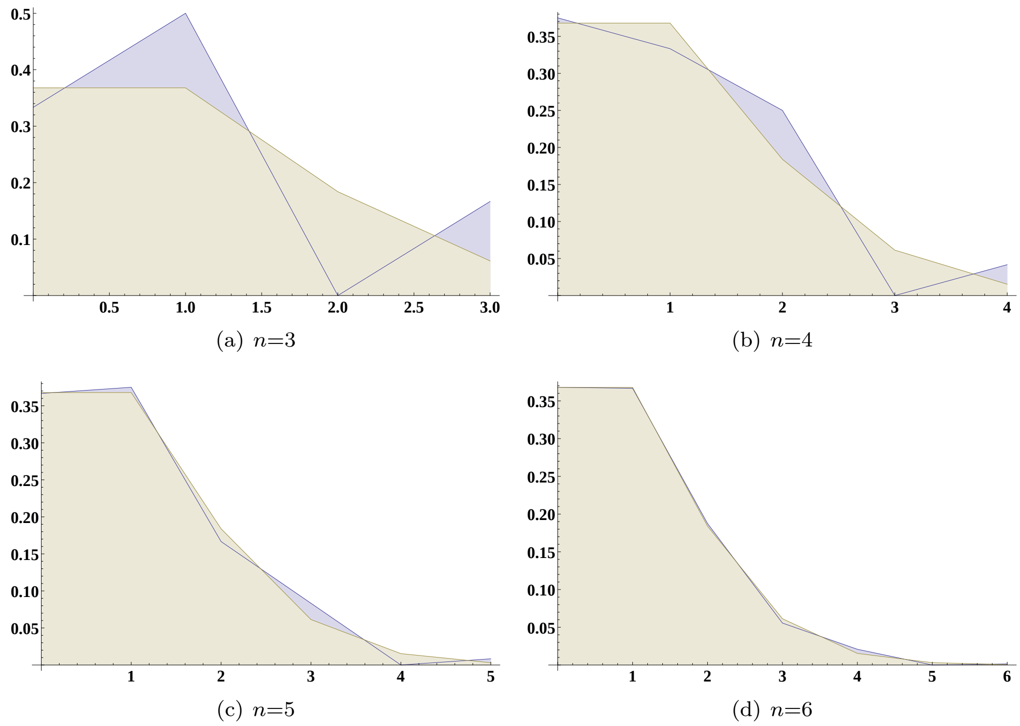
5 Approximation

Note that $e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$, which indicates that $\mathrm{P}\{X = k\} \approx \frac{e^{-1}}{k!}$, $D_n \approx e^{-1}n!$. Now we focus on this two approximations.

5.1 Approximation of $\mathrm{P}\{X = k\}$

Let Y be a random variable such that $Y \sim \text{Poisson}(1)$, then we have $\mathrm{P}\{Y = k\} = \frac{1}{ek!}$. Therefore when n is large enough, X approximately obeys *Poisson distribution* with the parameter 1.

Actually, even when n is small, there is very little difference between $\mathrm{P}\{X = k\}$ and $\mathrm{P}\{Y = k\} = \frac{1}{ek!}$, as shown in Figure 1.



(a) n=3 (b) n=4 (c) n=5 (d) n=6

Figure 1: Values of $\mathrm{P}\{X = k\}$ and $\mathrm{P}\{Y = k\}$

We can then conclude that Poisson distribution is a very good approximation to X.

It is worth mentioning that since we have

$$\frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!} = B_n$$

by the famous *Dobinski's formula*[5], therefore $\mathrm{E}[Y^n] = B_n$. This indicates that X and Y also have the same first n moments.

5.2 Approximation of D_n

Inspired by the previous discussion, we now consider the difference between D_n and $e^{-1}n!$. It is obvious that

$$\begin{aligned}|n!e^{-1} - D_n| &\leq \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\ &< \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots \\ &= \frac{1}{n} \\ &\leq \frac{1}{2}(n \geq 2)\end{aligned}$$

And for $n = 1$, we still have $|n!e^{-1} - D_n| < \frac{1}{2}$.

But knowing that D_n is an integer, we immediately get a neat form of D_n:

$$D_n = [n!e^{-1}]$$

where $[x]$ denote the nearest integer of x, and consequently,

$$\mathrm{P}\{X = k\} = \frac{\binom{n}{k}[(n-k)!e^{-1}]}{n!} = \frac{[(n-k)!e^{-1}]}{(n-k)!k!}$$

We then clearly see a beautiful result (which is also shown by (8)):

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}.$$

6 Generating Function

6.1 Ordinary Generating Function

Let the *ordinary generating function* of D_n be

$$y(x) = D_0 + D_1x + D_2x^2 + \cdots$$

here, we let $D_0 = 1$ to be consistent with (1). The following formulae are obvious:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= D_1 + 2D_2x + 3D_3x^2 + \cdots \\ \frac{\mathrm{d}(xy)}{\mathrm{d}x} &= D_0 + 2D_1x + 3D_2x^2 + \cdots \\ \frac{\mathrm{d}(y+xy)}{\mathrm{d}x} &= (D_0 + D_1) + 2(D_1 + D_2)x + 3(D_2 + D_3)x^2 + \cdots \\ &= \sum_{k=0}^{\infty}(k+1)(D_k + D_{k+1})x^k \\ &= \sum_{k=2}^{\infty} D_k x^{k-2} \quad \text{(applying (1))} \\ &= \frac{y - D_1x - D_0}{x^2} \\ &= \frac{y-1}{x^2}\end{aligned}$$

We obtain an ODE:

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\frac{y-1}{x^2}\\ \Leftrightarrow&\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-x}{x^2}y-\frac{1}{x^2+x^3}\end{aligned}$$

By the *method of variation of parameters*, we can get its general solution of the following form:

$$\begin{aligned}y&=e^{\int\frac{1-x}{x^2}\mathrm{d}x}\left[\int\frac{-1}{x^2+x^3}e^{-\int\frac{1-x}{x^2}\mathrm{d}x}\mathrm{d}x+C\right]\\&=\frac{1}{xe^{\frac{1}{x}}}\left[-\int\frac{e^{\frac{1}{x}}}{x(x+1)}\mathrm{d}x+C\right]\\&=\frac{1}{xe^{\frac{1}{x}}}\left[\frac{1}{e}\int\frac{e^{1+\frac{1}{x}}}{1+\frac{1}{x}}\mathrm{d}(1+\frac{1}{x})+C\right]\\&=\frac{1}{xe^{\frac{1}{x}+1}}\left[\int_{-\infty}^{1+\frac{1}{x}}\frac{e^t}{t}\mathrm{d}t+C\right]\\&=\frac{-1}{xe^{\frac{1}{x}+1}}\left[\int_{-(1+\frac{1}{x})}^{\infty}\frac{e^{-t}}{t}\mathrm{d}t+C\right]\\&=-\frac{1}{xe^{\frac{1}{x}+1}}\left[\Gamma(0,-\frac{x+1}{x})+C\right]\end{aligned}$$

$\Gamma(0,-\dfrac{x+1}{x})$ have the series form $-e^{1+\frac{1}{x}}(x+x^3+2x^4\cdots)$ (this can be verified by software), which indicates that

$$\lim_{x\to 0}-\frac{1}{xe^{1+\frac{1}{x}}}\Gamma(0,-\frac{x+1}{x})=1=D_0=y(0)$$

Therefore $C=0$, and $y(x)=-\dfrac{1}{xe^{1+\frac{1}{x}}}\Gamma(0,-\dfrac{x+1}{x})$

6.2 Exponential Generating Function

Let the *exponential generating function* of D_n be

$$y(x)=D_0+D_1\frac{x}{1!}+D_2\frac{x^2}{2!}+\cdots$$

Note that we still take $D_0=1$, and it follows that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\sum_{k=1}^{\infty}kD_k\frac{x^{k-1}}{k!}=\sum_{k=0}^{\infty}D_{k+1}\frac{x^k}{k!}\\ y+\frac{\mathrm{d}y}{\mathrm{d}x}&=\sum_{k=0}^{\infty}(D_k+D_{k+1})\frac{x^k}{k!}\\&=\frac{1}{x}\sum_{k=0}^{\infty}D_{k+2}\frac{x^{k+1}}{(k+1)!}\quad\text{(applying(1))}\\&=\frac{1}{x}\sum_{k=1}^{\infty}D_{k+1}\frac{x^k}{k!}\\&=\frac{1}{x}(\frac{\mathrm{d}y}{\mathrm{d}x}-D_1)\\&=\frac{\mathrm{d}y}{x\mathrm{d}x}\end{aligned}$$

We obtain an ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\frac{x-1}{x}+y=0 \Leftrightarrow \frac{\mathrm{d}y}{y}=\frac{x\mathrm{d}x}{1-x}$$

This can be easily solved:

$$\ln y=-x-\ln(1-x)+C \Rightarrow y=C'\frac{e^{-x}}{1-x}$$

and $C'=1$ since $y(0)=D_0=1$. Then we can conclude

$$y(x)=\sum_{k=0}^{\infty} D_k\frac{x^k}{k!}=\frac{e^{-x}}{1-x} \tag{10}$$

6.3 Probability Generating Function

Let the *probability generating function* of X be

$$y(n,x)=\sum_{k=0}^{\infty}\mathrm{P}\left\{X=k\right\}x^k$$

Using (4), we have

$$y(n,x)=\sum_{k=0}^{\infty}\frac{D_{n-k}}{(n-k)!}\frac{x^k}{k!}$$

Therefore, for a given x, the sequence $\{y_n\}, y_n=y(n,x)$ is the *convolution* of $\{\frac{D_n}{n!}\}$ and $\{\frac{x^n}{n!}\}$.

By the *convolution formula*, we have

$$\begin{aligned}\sum_{n=0}^{\infty}y(n,x)t^n &= \left(\sum_{k=0}^{\infty}\frac{D_k}{k!}t^k\right)\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}t^k\right)\\ &= \frac{e^{-t}}{1-t}e^{xt} \quad \text{(applying (10))}\\ &= \frac{e^{xt-t}}{1-t}\end{aligned}$$

Denote $[x^k]g(x)$ as the coefficient of the term x^k in $g(x)$, then we have

$$y(n,x)=[t^n]\frac{e^{xt-t}}{1-t}$$

Using the probability generating function, we can calculate the moments of X again by:

$$\begin{aligned}\mathrm{E}\left[X^{\underline{k}}\right] &= \left(\frac{\mathrm{d}^k y(n,x)}{\mathrm{d}x^k}\right)\bigg|_{x=1}\\ &= [t^n]\left(\frac{\mathrm{d}^k(\frac{e^{xt-t}}{1-t})}{\mathrm{d}x^k}\right)\bigg|_{x=1}\\ &= [t^n]\left(\frac{t^k e^{xt-t}}{1-t}\right)\bigg|_{x=1}\\ &= [t^n]\frac{t^k}{1-t}=[t^n]t^k(1+t+t^2+\cdots)\\ &= \begin{cases}1 & ,1\le k\le n\\ 0 & ,k>n\end{cases}\end{aligned}$$

Then, following the arguments in Section 3.2, we still get (7).

6.4 Moment Generating Function & Characteristic Function

Let the *moment generating function* of X be

$$y(n,t) = \mathrm{E}\left[e^{tX}\right] = \sum_{k=0}^{n} e^{tk}\mathrm{P}\left\{X = k\right\}$$

It can be calculated as followed:

$$\begin{aligned}
\mathrm{E}\left[e^{tX}\right] &= \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} e^{tk} \\
&= \sum_{s=0}^{n} \sum_{k=0}^{s} \frac{1}{k!} \frac{(-1)^{s-k}}{(s-k)!} e^{tk} \quad \text{(let } s = k+i\text{)} \\
&= \sum_{s=0}^{n} \frac{1}{s!} \sum_{k=0}^{s} \binom{s}{k} e^{tk} (-1)^{s-k} \\
&= \sum_{s=0}^{n} \frac{(e^t-1)^s}{s!} \\
&= \frac{e^{e^t-1}}{n!} \Gamma(n+1, e^t-1) \quad \text{(applying (9))}
\end{aligned}$$

Another way of calculating $y(n,t)$ is by using $\mathrm{E}\left[e^{tX}\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathrm{E}\left[X^k\right]$ and (7), we can get

$$\mathrm{E}\left[e^{tX}\right] = \sum_{i=0}^{n} \sum_{k=0}^{\infty} \left\{ {k \atop i} \right\} \frac{t^k}{k!} \overset{def}{=\!=} \sum_{i=0}^{n} f_i(t).$$

Since Stirling numbers of the second kind satisfy the following recurrence relation:

$$i \left\{ {k \atop i} \right\} = \left\{ {k+1 \atop i} \right\} - \left\{ {k \atop i-1} \right\}$$

we have an equation $i f_i(t) = \frac{\mathrm{d}f_i(t)}{\mathrm{d}t} - f_{i-1}(t)$. The solution of this equation is $f_i(t) = \frac{(e^t-1)^i}{i!}$, which brings with the desired result. Limited by the length of the paper, the details are left out.

It follows easily that the *characteristic function* of X is:

$$\varphi_n(t) = \mathrm{E}\left[e^{itX}\right] = \sum_{k=0}^{n} \frac{(e^{it}-1)^k}{k!} = \frac{e^{e^{it}-1}}{n!}\Gamma(n+1, e^{it}-1)$$

Note that the incomplete gamma function can be generalized to be defined on complex numbers.

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