On the solvability of 8-puzzle

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1 Introduction

Putting integers 1-8 randomly into a matrix of order 3, leaving a blank, gives us a valid "**pattern**". We define the **target pattern** T as follows:

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \square \end{bmatrix}$$

If by moving the blank to its neighbors, one pattern can be transformed to another in finite steps, we say the two patterns are connected.

Define the "**sequential ordering**" of a pattern $I = (a_{ij})_{3\times 3}$ as $(a_{11}a_{12}a_{13}a_{21}a_{22}a_{23}a_{31}a_{32}a_{33})$, which is a permutation of 1-8(omitting the blank symbol).

Define the "**snake ordering**"(because the path looks like a snake !) of a pattern $I = (a_{ij})_{3\times 3}$ as $S(I) = (a_{11}a_{12}a_{13}a_{23}a_{22}a_{21}a_{31}a_{32}a_{33})$, which is also a permutation of 1-8 (omitting the blank symbol).

We will prove that a pattern is connected to the target pattern if and only if its sequential ordering has even number of inversions.[1]

2 Necessity

It's obvious that every move doesn't change the parity of the number of inversions in its sequential ordering. And the sequential ordering of T is $(1, 2, 3, 4, 5, 6, 7, 8)$.

Thus the proof is trival.

And it's easy to see by bijection that there are $\dfrac{9!}{2}$ patterns which have odd number of inversions.

3 Sufficiency

We consider the groups of patterns with the same snake ordering.

It's trival that:

1. Every snake ordering has 9 different patterns correspondent to it.
2. In a given pattern, the blank can move to any place without changing the snake ordering. (Just walk along the "snake path")

Every move can be regarded as a permutation applied to the snake ordering. For example:

$$P_1 = \begin{bmatrix} a & \square & b \\ c & d & e \\ f & g & h \end{bmatrix} \to P_2 = \begin{bmatrix} a & d & b \\ c & \square & e \\ f & g & h \end{bmatrix}$$

[1] http://en.wikipedia.org/wiki/inversions

The correspondent snake ordering: $(a, b, e, d, c, f, g, h) \to (a, d, b, e, c, f, g, h)$
The permutation is: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 4 & 2 & 5 & 6 & 7 & 8 \end{pmatrix}$
Or to write in the form of cycle: $\sigma = (2, 3, 4)$

Label the places in the matrix as followed: $L = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}$, and denote σ_{ij} as the permutation of moving the blank from the place i to the place j. The above example shows that $\sigma_{25} = (2, 3, 4)$.

We have the following results by calculation:

1. $\sigma_{ij} = \sigma_{ji}^{-1}$
2. $\sigma_{i(i+1)}$ is an identical permutation.(Since the matrix is labelled along the "snake path")
3. $\sigma_{16} = (1, 2, 3, 4, 5)$
4. $\sigma_{25} = (2, 3, 4)$
5. $\sigma_{49} = (4, 5, 6, 7, 8)$
6. $\sigma_{58} = (5, 6, 7)$

We first try to find the subgroup G generating by all the σ_{ij}.

It's obvious that G is the subgroup of the symmetric group with 8 symbols S_8. We claim that: G is the group of all **even permutations** with 8 symbols E_8.[2]

The proof requires the following lemmas in Group Theory:

Lemma 1. *All the 3-cycles can generate E_n.*

Proof. By definition, every element in E_n is the product of even number of transpositions. But every product of two transpositions is equal to the product of some 3-cycles by the following rules:

1. $(a, b)(a, c) = (a, b, c)$
2. $(a, b)(c, b) = (a, c, b)$
3. $(a, b)(a, b) = ()$
4. $(a, b)(c, d) = (a, b, c)(a, d, c)$

Therefore the proof is done. □

Lemma 2. *All the 3-cycles of the form $(1, i, j)$ can generate E_n.*

Proof. Use $(a, b, c) = (1, a, b)(1, b, c)$ and Lemma 1. □

Lemma 3. *All the 3-cycles of the form $(1, 2, k)$ can generate E_n.*

Proof. Use $(1, 2, k) = (1, 2, k), (1, k, 2) = (1, 2, k)^{-1}, (1, i, j) = (1, 2, j)(1, 2, j)(1, 2, i)(1, 2, i)$ and Lemma 2. □

Lemma 4. *All the consecutive 3-cycles (cycles of the form $(k, k+1, k+2), k \le n-2$) can generate $E_n (n \ge 5)$.*

[2] http://en.wikipedia.org/wiki/Parity_of_a_permutation

Proof. Since $(1,2,3) = (1,2,3), (1,2,4) = (2,3,4)(2,3,4)(1,2,3)$, we can apply induction on the formula

$$(1,2,i) = (1,2,i-2)(1,2,i-1)(i-2,i-1,i)(1,2,i-2)(1,2,i-1), (i \geq 5)$$

It follows that all the consecutive 3-cycles generate all the $(1,2,k)$. Then by Lemma 3, the proof is done.

□

Theorem 1. $G = E_8$

Proof. We have:

$$\begin{aligned}
(1,2,3) &= \sigma_{16}\sigma_{25}\sigma_{61} \\
(2,3,4) &= \sigma_{25} \\
(3,4,5) &= \sigma_{61}\sigma_{25}\sigma_{16} \\
(4,5,6) &= \sigma_{49}\sigma_{58}\sigma_{94} \\
(5,6,7) &= \sigma_{58} \\
(6,7,8) &= \sigma_{94}\sigma_{58}\sigma_{49}
\end{aligned}$$

Then by Lemma 4, E_8 is the subgroup of G.

Moreover, since $\sigma_{16}, \sigma_{25}, \sigma_{49}, \sigma_{58} \in E_8$, therefore $E_8 = G$. □

Now we go back our original problem:

Theorem 2. *Every pattern whose sequential ordering has even number of inversions is connected to the target pattern.*

Proof. As pointed out before, the blank can move freely without changing the snake ordering. Then to transform a snake ordering to another, all the permutations σ_{ij} can be combined sequentially in any possible order.

Since $|G| = |E_8| = \frac{8!}{2}$, therefore from $S(T)$, $\frac{8!}{2}$ different snake ordering can be achieved. So $\frac{8!}{2} \times 9 = \frac{9!}{2}$ different patterns are connected with T.

From the proof of necessity, $\frac{9!}{2}$ patterns are proved disconnected with T. Thus the proof is finished. □

Writing a simple program is enough to prove our proposition, saving the time to read all these...