MI-ARI

(Computer arithmetics)

winter semester 2017/18

D1. **Division I.**

© **Alois Pluháček, Pavel Kubalík**, 2017
Department of digital design
Faculty of Information technology
Czech Technical University in Prague

https://edux.fit.cvut.cz/courses/MIE-ARI/lectures

D1. **Division I.**

- Introduction
- Division of non-negative numbers less than 1
  - Restoring division
  - Non-restoring division
- Division of non-negative integers
- Division of signed numbers
  - Sign and magnitude
  - 1's complement
  - 2's complement

Introduction

division — complicated operation

- **one operation** (divisor) **must not be 0** $\quad (!\ !\ !)$
- **result „i.e.“ may not be represented exactly**

$$\boxed{A/B = ?} \qquad \boxed{B \neq 0} \text{ !!!}$$

non-negative binary numbers will be assumed first

❶ $\quad \mathcal{Z} = 1 \implies A < 1,\ B < 1$ **a** $A/B < 1,$

popř. $\quad \mathcal{Z} = 2 \implies A < 2,\ B < 2$ **a** $A/B < 2$

❷ $\quad \varepsilon = 1 \implies$ **integers**

- $A \div B = ?$ — **integer quotient**
- $A \,\%\, B = ?$ — **remainder**

❶ **example:** $\quad A = 0{,}101_2$

$\quad B = 0{,}110_2$

$0{,}101 \,/\, 0{,}110 \doteq 0{,}110 \qquad$ **remainder:** $0{,}000\,100$

Restoring division

$$0,101 \;:\; 0,110$$

$$\Downarrow \quad \Downarrow \quad \Downarrow$$

$$101 : 110 = 0,110 \qquad \textbf{quotient}$$

$$\begin{array}{lrl}
- & 110 & \\
\hline
 & -1 & \rightarrow \; 0, \\
+ & 110 & \qquad \textbf{restoring} \\
\hline
 & 101\,_0 & \\
- & 110 & \\
\hline
 & 100\,_0 & \rightarrow \; 1 \\
- & 110 & \\
\hline
 & 10\,_0 & \rightarrow \; 1 \\
- & 110 & \\
\hline
 & -10 & \rightarrow \; 0 \\
+ & 110 & \qquad \textbf{restoring} \\
\hline
 & 100 & \qquad \textbf{remainder}
\end{array}$$

restoring (of remainder) — auxiliary operation re-creating of last partial remainder for next subtraction

Non-restoring division

restorin = addition of some number X $\rightarrow + \quad X$

next partial operation: subtraction of $X \triangleright 1$ $\rightarrow \underline{-X/2}$

$+X/2$

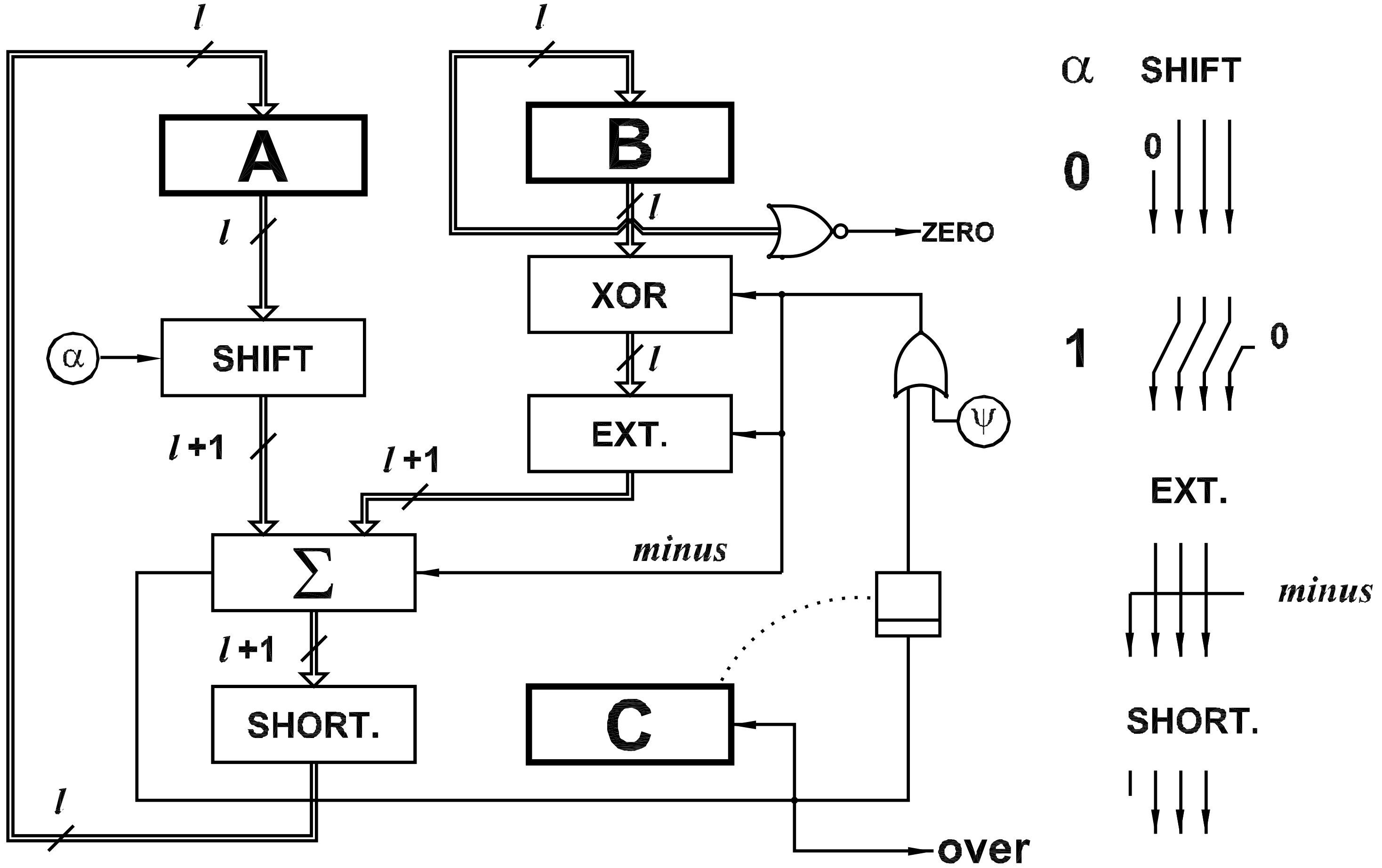
$$
\begin{array}{rl}
 & \mathbf{101} \ : \ \mathbf{110} = \mathbf{0,110} \quad \text{quotient} \\
- & \mathbf{110} \\
\hline
 & -\mathbf{1}\ _0 \quad \rightarrow \quad \mathbf{0,} \\
+ & \mathbf{110} \\
\hline
 & \mathbf{100}\ _0 \quad \rightarrow \quad \mathbf{1} \\
- & \mathbf{110} \\
\hline
 & \mathbf{10}\ _0 \quad \rightarrow \quad \mathbf{1} \\
- & \mathbf{110} \\
\hline
 & -\mathbf{10} \quad \rightarrow \quad \mathbf{0} \\
+ & \mathbf{110} \quad \text{restoring} \\
\hline
 & \mathbf{100} \quad \text{remainder}
\end{array}
$$

restoring: ***at the end*** **only**

if the partial remainder is negative **only**

if the proper remainder is needed **only**

Non-restoring division ii



	α	ψ
1. clk	0	1
next clk	1	0
restoring	0	0

l+1 (or l+2) clock cycles
quotient . . . C
remainder . . . A

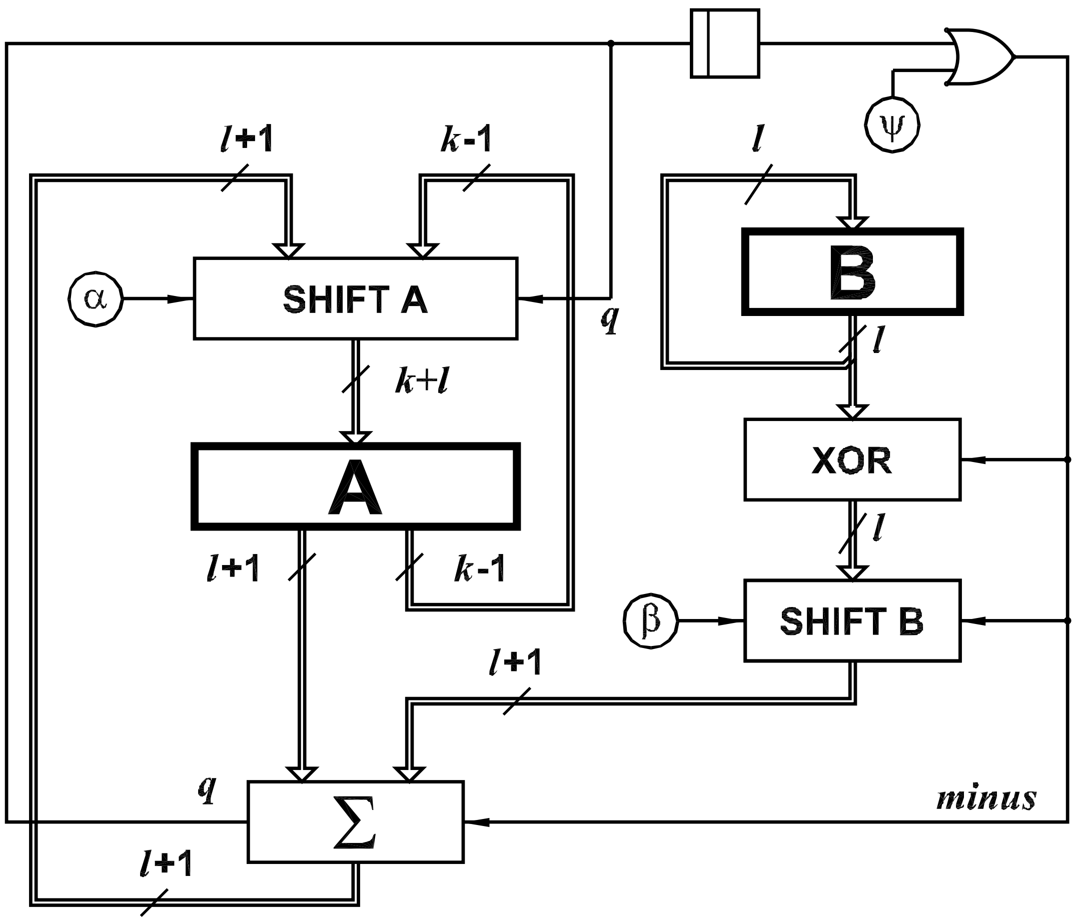
Division of non-negative integers

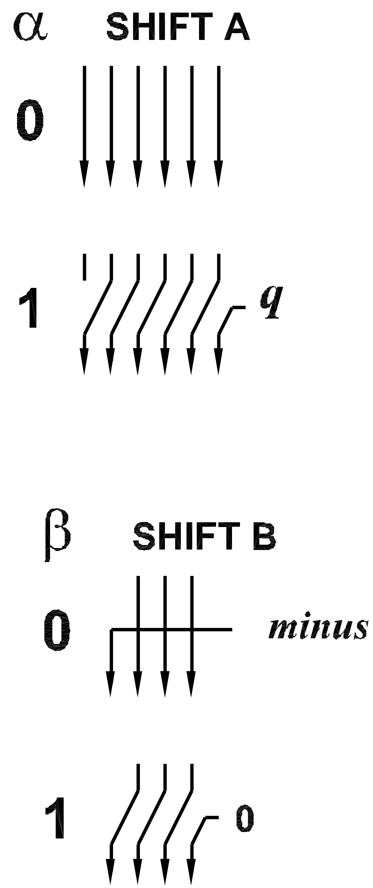
$$\boxed{\mathbf{1\,1\,1 : 0\,1\,1}}$$

$$
\begin{array}{clccccccl}
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \quad \mathbf{:} \quad {\scriptstyle 0} \quad \mathbf{0\ 1\ 1} \\
- & & {\scriptstyle 1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & : & : & \text{negation} \\
 & & & & & \mathbf{1} & : & : & \text{hot 1} \\
\cline{3-6}
 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & : & : & - \quad \Rightarrow \quad \mathbf{0} \\
 & & & \downarrow & \downarrow & \downarrow & \downarrow & : & \\
 & & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & : & \\
+ & & & {\scriptstyle 0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & : & \\
\cline{4-7}
 & & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \quad + \quad \Rightarrow \quad \mathbf{1} \\
 & & & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & & & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \\
- & & & & {\scriptstyle 1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \\
 & & & & & & & \mathbf{1} & \\
\cline{5-8}
 & & & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \qquad - \qquad \Rightarrow \mathbf{0} \\
\mathbb{N} & & & & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \text{restoring} \\
\cline{5-8}
 & & & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \\
 & & & & & \mathbf{0} & \mathbf{0} & \mathbf{1} & \text{— } \textbf{remainder} \qquad \mathbf{0\ 1\ 0} \text{ — } \textbf{quotient}
\end{array}
$$

$$
\begin{array}{cccccccccl}
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & & \mathbf{:}\quad {\scriptstyle 0}\ \mathbf{0}\ \mathbf{1}\ \mathbf{1} \\
- & & {\scriptstyle 1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & : & : & & \text{negation} \\
 & & & & & \mathbf{1} & : & : & & \text{hot 1} \\
\hline
 & {\scriptstyle 0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & : & : & & - \ \Rightarrow\ \mathbf{0} \\
 & & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\
 & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{?} & & \\
+ & & {\scriptstyle 0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & : & : & & \\
\hline
 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & : & & + \ \Rightarrow\ \quad \mathbf{1} \\
 & & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{?} & \mathbf{?} & & \\
- & & {\scriptstyle 1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & : & : & & \\
 & & & & & \mathbf{1} & : & : & & \\
\hline
 & {\scriptstyle 0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & : & : & & - \ \Rightarrow\ \qquad \mathbf{0} \\
 & & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & \swarrow & & \\
 & & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{?} & \mathbf{?} & \mathbf{?} & & \\
\mathbb{N} & & \mathbf{0} & \mathbf{1} & \mathbf{1} & {\scriptstyle 0} & : & : & & \text{restoring} \\
\hline
 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{?} & : & : & & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{?} & \mathbf{?} & \mathbf{?} & & \\
\end{array}
$$

0 0 1 — remainder **0 1 0 — quotient**





	α	β	ψ
1. clk	1	0	1
next clk	1	0	0
restoring	0	1	0

l (or $l+1$) clock cycles
quotient . . . lower order A
remainder . . . higher order A

Division of signed numbers

❶ Sign and magnitude

sign:

$$\begin{array}{|cc|c|} + & + & + \\ + & - & - \\ - & + & - \\ - & - & + \end{array} \Rightarrow \begin{array}{|cc|c|} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{array} \Rightarrow \boxed{\text{XOR}}$$

magnitude — unsigned number

$\Rightarrow$ division of unsigned number

❷ '1s complement — possible way to design:

For different signs combination, the dividend A and the divisor B are modified by known algorithm for division of unsigned numbers $|A|$ and $|B|$ so that:

- **Absolute values of partial remainders are the same.**
- **Bits of quotient are inverted** (or are the same) **depending on $A/B = |A|/|B|$ (or $A/B = -|A|/|B|$).**

A ... **divident or numerator**
B ... **divisor or denominator**
Y ... **partial result - partial remainder or residue**

$A \geq 0$	$B > 0$		$B < 0$	
	$Y \geq 0$	$Y < 0$	$Y \geq 0$	$Y < 0$
bit of quotient	1	0	0	1
next operation	$-$	$+$	$+$	$-$

$A < 0$	$B > 0$		$B < 0$	
	$Y \leq 0$	$Y > 0$	$Y \leq 0$	$Y > 0$
bit of quotient	0	1	1	0
next operation	$+$	$-$	$-$	$+$

summary:

sign(Y) = **sign**(A) $\Rightarrow$ **bit of quotient = 1**

$\Rightarrow$ **next operation = subtraction**

Special care is needed for the sign of zero.

❸ 2's complement

The procedure for the 1's complement can be used for any representation of operands and intermediate results. However the main result (i.e. quotient) **is in 1's complement.**

2's complement division	☜	**modification of 1's complement division**

- **Determine 1's complement representation $\mathcal{I}(C)$ of quotient C using 2's complement code to execute all partial operations.**
- **Convert the 1's complement representation $\mathcal{I}(C)$ to 2's complement representation $\mathcal{D}(C)$:**

  $$\mathcal{D}(C) = \begin{cases} \mathcal{I}(C), & \textbf{pro } C > 0, \\ \mathcal{I}(C) + \varepsilon, & \textbf{pro } C < 0. \end{cases}$$

- **The remainder will be in 2's complement code.**

Method of Z. Pokorný:

- $[\mathcal{D}(A/B)]_{-m} = [\mathcal{I}(A/B)]_{-m}$
  $\lfloor V \rfloor_{-m}$ ... V „cut“ on m places behind point
  $[V]_{-m}$ ... V rounded on m places behind point
- $[W]_{-m} = \lfloor W + \frac{1}{2}\,\varepsilon \rfloor_{-m}$

$$\boxed{[\mathcal{D}(A/B)]_{-m} = \left\lfloor \frac{A + B \cdot \frac{1}{2}\,\varepsilon}{B} \right\rfloor_{-m}}$$

1. To the divident A is added divisor B shifted by$m+1$ positions to the right, where m is a number of positions behind point used in quotient.
2. All possible bits of quotient used for format are determined by the standard procedure valid for 1's complement code.

The rounded quotient representation in 2's complement code and correct remainder is obtained by this procedure in interval $\left\langle -\frac{1}{2} \cdot |B| \triangleright m,\ \frac{1}{2} \cdot |B| \triangleright m \right\rangle$.

ex.: $8 : (-3) = -3$ $\quad \mathcal{D}(8) = $ **0000 1000**

-1 $\quad \mathcal{D}(-3) = $ **1101**

$$
\begin{array}{rl}
\texttt{0 0 0 0 1 0 0 0} & \mathcal{D}(8) \\
\underline{\texttt{1 1 1 1 1 1 1 0 , 1}} & \mathcal{D}((-3)\cdot\frac{1}{2}) \text{ — basic correction} \\
\texttt{0 0 0 0 0 1 1 0 , 1} & \\
\underline{\texttt{1 1 1 0 1}} & +\ (-3)\cdot 2^3 \\
\texttt{1 1 1 0 1 1} & \\
\downarrow \underline{\texttt{0 0 0 1 1}} & -\ (-3)\cdot 2^2 \\
\texttt{1 1 1 1 0 1} & \\
\downarrow \underline{\texttt{0 0 0 1 1}} & -\ (-3)\cdot 2^1 \\
\texttt{0 0 0 0 0 0} & \\
\downarrow \underline{\texttt{1 1 1 0 1}} & +\ (-3)\cdot 2^0 \\
\texttt{1 1 1 0 1 , 1} & \\
\downarrow \underline{\texttt{0 0 0 1 , 1}} & -\ (-3)\cdot 2^{-1} \\
\texttt{1 1 1 1 , 0} & ☛\ \mathcal{D}(-1) \text{ — remainder} \\
\texttt{1 1 0 1} & ☛\ \mathcal{D}(-3) \text{ — quotient}
\end{array}
$$