# MI-ARI

## (Computer arithmetics)

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# NS. Non-standard number systems

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# NS. Non-standard number systems

- **Standard number systems**
- **Non-standard positional number systems**
  - **Negative radix number system (Polish system)**
  - **Signed digit number system (with symmetrical base)**
  - **Multiple radix digit system**
- **Non-positional number system**
  - **Residue number system (Czech system)**

## *Standard number systems*

$$a_n a_{n-1} \ldots a_0, a_{-1} \ldots a_{-m} \quad \sim \quad A$$

$$\boxed{A = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + a_{-1} z^{-1} \ldots a_{-m} z^{-m}}$$

or $$A = \sum_{i=-m}^{n} a_i z^i$$

$$\boxed{z \geq 2}$$

**radix (**or **base) system — natural number**

$$\boxed{0 \leq a_i \leq z-1 < z}$$

**digits — positive natural integers**

$$\boxed{0 \leq A \leq z^{n+1} - z^{-m}}$$

## *Negative radix number system*

$\boxed{z \leq -2}$ $z$ is an integer $z = -2$ ... Polish system

$0 \leq a_i < |z|$

$$a_n a_{n-1} \dots a_0, a_{-1} \dots a_{-m} \quad \sim \quad A = \sum_{i=-m}^{n} a_i z^i$$

ex.: $z = -2$

| | | | | | |
|---|---|---|---|---|---|
| $0_{-2}$ | $=$ | $0_{10}$ | | | |
| $1_{-2}$ | $=$ | $1_{10}$ | $0,01_{-2}$ | $=$ | $0,25_{10}$ |
| $10_{-2}$ | $=$ | $-2_{10}$ | $0,10_{-2}$ | $=$ | $-0,50_{10}$ |
| $11_{-2}$ | $=$ | $-1_{10}$ | $0,11_{-2}$ | $=$ | $-0,25_{10}$ |
| $100_{-2}$ | $=$ | $4_{10}$ | $1,00_{-2}$ | $=$ | $1,00_{10}$ |
| $101_{-2}$ | $=$ | $5_{10}$ | $1,01_{-2}$ | $=$ | $1,25_{10}$ |
| $110_{-2}$ | $=$ | $2_{10}$ | $1,10_{-2}$ | $=$ | $0,25_{10}$ |
| $111_{-2}$ | $=$ | $3_{10}$ | $1,11_{-2}$ | $=$ | $0,75_{10}$ |
| $1000_{-2}$ | $=$ | $-8_{10}$ | $10,00_{-2}$ | $=$ | $-2,00_{10}$ |
| $1001_{-2}$ | $=$ | $-7_{10}$ | $10,01_{-2}$ | $=$ | $-1,75_{10}$ |
| $1010_{-2}$ | $=$ | $-10_{10}$ | $10,10_{-2}$ | $=$ | $-2,50_{10}$ |
| | $\vdots$ | | | $\vdots$ | |

It is possible **to represent** not only positive numbers but **also negative numbers.**

$$\left.\begin{array}{l} a_j \neq 0 \\ (\forall\, i > j)\ a_i = 0 \end{array}\right\} \;☞\; \begin{cases} j \text{ is even} \Leftrightarrow A > 0 \\ j \text{ is odd} \Leftrightarrow A < 0 \end{cases}$$

☛ **There are not used special codes (e.g. 1's complement and etc.) to represent negative numbers.**

**significantly non-symmetrical range:**

**ex.:** $z = -10,\ n = 3,\ m = 0 \Longrightarrow -9090 \leq A \leq 909$

$z = -2,\ n = 6,\ m = 0 \Longrightarrow -42 \leq A \leq 85$

**addition:**

**ex.:** $z = -10$ $\quad 5+8 = 13 = 3 - z \quad \Rightarrow \text{carry} = -1$

$0+(-1) = -1 = 9 + z \quad \Rightarrow \text{carry} = +1$

$z = -2$ $\quad 1+1 = 2 = 1 - z \quad \Rightarrow \text{carry} = -1$

$0+(-1) = -1 = 1 + z \quad \Rightarrow \text{carry} = +1$

 

## full adder for base $z = -2$

| $a$ | $b$ | $p$ | $q$ | $s$ |
|---|---|---|---|---|
| 0 | 0 | $-1$ | $+1$ | 1 |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | $+1$ | 0 | 1 |
| 0 | 1 | $-1$ | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | $+1$ | $-1$ | 0 |
| 1 | 0 | $-1$ | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | $+1$ | $-1$ | 0 |
| 1 | 1 | $-1$ | 0 | 1 |
| 1 | 1 | 0 | $-1$ | 0 |
| 1 | 1 | $+1$ | $-1$ | 1 |

$$\text{carry} = \begin{cases} -1 \\ \phantom{+}0 \\ +1 \end{cases}$$

 

**subtraction:** $A - B = A + (-B)$

$-B = ?$

$z = -2$ ☛ $-B = B \cdot (-1) = B \cdot 11_{-2}$

$-B = B + B \triangleleft 1$

**multiplication:** similarly to the standard number system (shifts and addition)

**division:** a little bit complicated

 

## *Signed digit number system*

$z \geq 2$ **$z$ is an integer** $\boxed{-\frac{1}{2}z \leq a_i \leq \frac{1}{2}z}$

$$a_n a_{n-1} \ldots a_0, a_{-1} \ldots a_{-m} \quad \sim \quad A = \sum_{i=-m}^{n} a_i z^i$$

(our) **notation:** $\boxed{\widehat{a} = -a}$

**ex.:** $z = 10$ $\quad 166_{10} = 2\widehat{3}\widehat{4}_{10\pm}$

$155_{10} = 155_{10\pm} = 2\widehat{4}\widehat{5}_{10\pm} = 2\widehat{5}5_{10\pm}$

**even $z$ $\Rightarrow$ One number can be represented by many images,**

because $\boxed{\frac{1}{2}z = z - \frac{1}{2}z}$.

**This ambiguous can be excluded by some restrictions:**

$-\frac{1}{2}z < a_i \leq \frac{1}{2}z$ **or** $-\frac{1}{2}z \leq a_i < \frac{1}{2}z$

**Restrictions brings other problems to solve** (e.g. for multiplication)**.**
**It is better to do not use any restrictions.**

## *Signed digit number system*

It is possible **to represent** not only positive numbers, but **also negative numbers.**

$$\left.\begin{array}{l} a_j \neq \mathbf{0} \\ (\forall\ i > j)\ a_i = \mathbf{0} \end{array}\right\} ☞ \begin{cases} a_j > \mathbf{0} \ \Leftrightarrow \ A > 0 \\ a_j < \mathbf{0} \ \Leftrightarrow \ A < 0 \end{cases}$$

☛ **There are not used special codes (e.g. 1's complement and etc.) to represent negative numbers.**

### addition:

**one-positional (full) adder:**

$$\begin{array}{lll} a+b+p \geq \frac{1}{2}z & \Rightarrow & q=+1,\ s = a+b+p-z \\ a+b+p \leq -\frac{1}{2}z & \Rightarrow & q=-1,\ s = a+b+p+z \\ \text{other:} & & q=0, \quad s = a+b+p \end{array}$$

### subtraction: $A - B = A + (-B)$

$$\boxed{\begin{array}{l} A \quad \sim \quad a_n \ \ldots \ a_0, a_{-1} \ \ldots \ a_{-m} \quad \Longrightarrow \\ \Longrightarrow \quad -A \quad \sim \quad \widehat{a_n} \ \ldots \ \widehat{a_0}, \widehat{a_{-1}} \ \ldots \ \widehat{a_{-m}} \end{array}}$$

**multiplication:** **similar to the standard number system (shifts and addition)**

**division:** **a little bit complicated**

## significant applications:

- **„temporarysystem“ for multiplication (e.g. Booth method)**
- **„temporarysystem“ for division (SRT methods)**

 

## *Multiple radix digit system*

*The integers are take into account for the next consideration.*

**bases** $z_n, \dots, z_1, z_0$ — **natural numbers** $> 1$

$0 \le a_i \le z_i - 1 < z_i$

$a_n \dots a_2 a_1 a_0 \quad \sim \quad A$

$$\boxed{\begin{aligned} A &= \\ &= a_n \cdot z_{n-1} \cdots z_2 \cdot z_1 \cdot z_0 + \cdots + a_2 \cdot z_1 \cdot z_0 + a_1 \cdot z_0 + a_0 = \\ &= (\cdots(a_n \cdot z_{n-1} + a_{n-1}) \cdot z_{n-2} + \cdots + a_1) \cdot z_0 + a_0 \end{aligned}}$$

**note:** 1. **base** $z_n$ **is significant only for limit** $a_n$.

2. $\forall i \; z_i = z \;\Rightarrow$ **standard number system with base** $z$

**ex.:** $A = 5^{\text{weeks}}\ 3^{\text{days}}\ 13^{\text{hours}}\ 50^{\text{minutes}}\ 11^{\text{seconds}}$

**notation of time information**
**in system with bases ?, 7, 24, 60, 60**

**range of representable numbers $A$:**

$$\boxed{0 \leq A < z_n \cdot z_{n-1} \cdots z_1 \cdot z_0}$$

**conversion of integer $A$ into the system with bases $z_n$, . . . , $z_0$:**

$A_0 := A$;
**for** $i := 0$ **to** $n$ **do**
    **begin**
      $a_i := A_i \,\%\, z_i$;
      $A_{i+1} := A_i \div z_i$;
    **end**;

**practical application — see residual number system**

 

## *Residue number system*

*The integers are take into account for the next consideration.*
*(Scale can be used to represent rational numbers.)*

**bases** $z_n, \ldots, z_1, z_0$ — **different prime numbers**

$$\boxed{A \quad \sim \quad a_n \ldots a_2 a_1 a_0, \quad \text{where} \;\; a_i = A \;\%\; z_i}$$

**ex.:** $z_2 = 5$, $z_1 = 3$, $z_0 = 2$

| | | | | |
|---|---|---|---|---|
| 0 ∼ 000 | 6 ∼ 100 | 12 ∼ 200 | 18 ∼ 300 | 24 ∼ 400 |
| 1 ∼ 111 | 7 ∼ 211 | 13 ∼ 311 | 19 ∼ 411 | 25 ∼ 011 |
| 2 ∼ 220 | 8 ∼ 320 | 14 ∼ 420 | 20 ∼ 020 | 26 ∼ 120 |
| 3 ∼ 301 | 9 ∼ 401 | 15 ∼ 001 | 21 ∼ 101 | 27 ∼ 201 |
| 4 ∼ 410 | 10 ∼ 010 | 16 ∼ 110 | 22 ∼ 210 | 28 ∼ 310 |
| 5 ∼ 021 | 11 ∼ 121 | 17 ∼ 221 | 23 ∼ 321 | 29 ∼ 421 |

**range of representable numbers $A$:**

$$\boxed{0 \leq A < z_n \cdot z_{n-1} \cdots z_1 \cdot z_0}$$

## operation addition, subtraction and multiplication:

$a_i$ ... 1. digit of operand in position $i$

$b_i$ ... 2. digit of operand in position $i$

$c_i$ ... digit of result in position $i$

$\odot$ ... symbol of operation (is replacing $+$, $-$ or $\times$)

$$\boxed{\forall i \quad c_i \equiv a_i \odot b_i \mod z_i}$$

ex.: $211 + 301 = 010 \quad \sim \quad 7+3 = 10$

$211 - 301 = 410 \quad \sim \quad 7-3 = 4$

$211 \times 301 = 101 \quad \sim \quad 7\times 3 = 21$

 

## problems:

- division
- overflow detection
- detection of negative number result
- comparison of two numbers ☜ !!!

without problems: division if the reminder is=0
it is converted on the multiplication with inverted va

ex.: $101\ /\ 211 = 101 \times 311 = 301 \quad \sim \quad 21 \times 7 = 3$

## comparison of two operands

- convert both numbers to the standard number system with same base
  - required operations: subtraction and division without reminder
- Greater number is detected by the bigger digit in first positions , where the numbers are different.

 

ex.: $z_2 = 5$, $z_1 = 3$, $z_0 = 2$
$2^{-1} \sim 32?$ $\quad 3^{-1} \sim 2?1$

$\alpha_2$, $\alpha_1$, $\alpha_0$ — digit in positional number system with bases 5, 3, 2

$$
\begin{array}{ll}
A & \sim 211 = (a_2, a_1, a_0) \\
A_0 = A & \sim 211 \\
\alpha_0 = A_0 \,\%\, z_0 = a_0 = 1 & \sim 111 \\
A_1 = \dfrac{A_0 - \alpha_0}{z_0} & \sim 211 - 111 = 100 \\
 & \phantom{\sim\ } 100 \times 32? = 30? \\
\alpha_1 = A_1 \,\%\, z_1 = a'_1 = 0 & \sim 000 \\
A_2 = \dfrac{A_1 - \alpha_1}{z_1} & \sim 30? \times 2?1 = 1?? \\
\alpha_2 = A_2 \,\%\, z_2 = a'_2 = 1 & \sim 111
\end{array}
$$

$(\alpha_2, \alpha_1, \alpha_0) = 101$

 

**conversion to the different number system — e.g. to the standard decimal number system (required operations are processed in this number system)**

**orthogonal base**

$$\boxed{\begin{array}{lcl} B_i & \equiv & 1 \ \text{mod}\ z_i \\ B_i & \equiv & 0 \ \text{mod}\ \frac{z_n \cdots z_1 \cdot z_0}{z_i} \end{array}}$$

$$\boxed{A \equiv a_n \cdot B_n + \cdots + a_1 \cdot B_1 + a_0 \cdot B_0 \quad \text{mod}\ z_n \cdots z_1 \cdot z_0}$$

**ex.:** $z_2 = 5$, $z_1 = 3$, $z_0 = 2$

$$\begin{array}{lll} B_0 = 15 & \sim & 001 \\ B_1 = 10 & \sim & 010 \\ B_2 = 6 & \sim & 100 \end{array}$$

$$\begin{aligned} 211 &\sim 2 \cdot 6 + 1 \cdot 10 + 1 \cdot 15 = \\ &= 12 + 10 + 15 = 37 \equiv 7 \quad \text{mod}\ 30 \end{aligned}$$