MI-ARI

(Computer arithmetics)

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D2. **Division II.**

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D2. **Division II.**

- **Principle of skip over 0's**
- **Principle of skip over 1's**
- **SRT methods**
- **Using fast multipliers**
  - **Fraction extension**
  - **Iteration**
- **Division in decimal system**

Next we will assume **division of unsigned numbers**
and **divisor B satisfy condition**
$\frac{1}{2} \leq B < 1$,
i.e. that it is **normalized.**
We can process analogous in a case, that $1 \leq B < 2$.

Principle of skip over 0's

example of division:

$$0,1000 \; : \; 0,1111 = 0,1000$$

$$
\begin{array}{rrl}
0\,1\,0\,0\,0 & & \\
\underline{1\,0\,0\,0\,1} & & -0{,}1111 \;\triangleright 0 \\
0\;1\,1\,0\,0\,1\,0 & 0\,, & \\
\underline{0\,1\,1\,1\,1} & & +0{,}1111 \;\triangleright 1 \\
1\;0\,0\,0\,0\,1\,0 & 1 & \\
\underline{1\,0\,0\,0\,1} & & -0{,}1111 \;\triangleright 2 \\
0\;1\,0\,0\,1\,1 & 0 & \\
0\,0\,1\,0\,0 \;\text{(restoring)} & & \\
\underline{1\,0\,0\,0\,1} & & -0{,}1111 \;\triangleright 3 \\
0\;1\,0\,1\,0\,1 & 0 & \\
0\,1\,0\,0\,0 \;\text{(restoring)} & & \\
\underline{1\,0\,0\,0\,1} & & -0{,}1111 \;\triangleright 4 \\
0\;1\,1\,0\,0\,1 & 0 & \\
0\,1\,0\,0\,0 \;\text{(restoring)} & &
\end{array}
$$

Principle of skip over 0's ii

0 , 1 0 0 0 : 0 , 1 1 1 1 = 0 , 1 0 0 0

0 1 0 0 0		
1 0 0 0 1		$-0{,}1111 \ \triangleright 0$
0 1 1 0 0 1 0	0 ,	
0 1 1 1 1		$+0{,}1111 \ \triangleright 1$
1 0 0 0 0 1 0 0 0	1 0 0	
1 0 0 0 1		$-0{,}1111 \ \triangleright 4$
0 1 1 0 0 1	0	
0 1 1 1 1 restoring		
0 1 0 0 0		

3 zeroes $\Longrightarrow$ **record 2 zeroes into quotient and shift of partial remainder by 3 positions**

generally:

k **zeroes** $\Longrightarrow$ **record $k-1$ zeroes in quotient and shift partial remainder by k positions**

Principle of skip over 1's

example of division:

0 , 1 1 1 0 : 0 , 1 1 1 1 =	0 , 1 1 1 0	
0 1 1 1 0		
1 0 0 0 1		$-0{,}1111 \triangleright 0$
0 1 1 1 1 0	0 ,	
0 1 1 1 1		$+0{,}1111 \triangleright 1$
1 0 1 1 0 1	1	
1 1 1 0 0 (after „anti-restoring" ?!)		
0 1 1 1 1		$+0{,}1111 \triangleright 2$
1 0 1 0 1 1	1	
1 1 0 0 0 (after „anti-restoring" ?!)		
0 1 1 1 1		$+0{,}1111 \triangleright 3$
1 0 0 1 1 1	1	
1 0 0 0 0 (after „anti-restoring" ?!)		
0 1 1 1 1		$-0{,}1111 \triangleright 4$
0 1 1 1 1 1	0	
0 1 1 1 0 (after restoring)		

$$0,1110 \;:\; 0,1111 = 0,1110$$

$$
\begin{array}{rl}
\begin{array}{r}
0\,1\,1\,1\,0 \\
\underline{1\,0\,0\,0\,1} \\
0\,1\,1\,1\,1\,1\,0\,0\,0\,0 \\
\underline{0\,1\,1\,1\,1} \\
0\,1\,1\,1\,1\,1 \\
\underline{0\,1\,1\,1\,1} \\
0\,1\,1\,1\,0
\end{array}
&
\begin{array}{l}
\\ -0{,}1111 \;\triangleright 0 \\
0\,,\,1\,1\,1 \\
+0{,}1111 \;\triangleright 4 \\
0 \\
\text{restoring} \\
\\
\end{array}
\end{array}
$$

4 ones $\Longrightarrow$ **record 3 ones into quotient and shift partial remainder by 4 positions**

generally:

k ones $\Longrightarrow$ **record $k{-}1$ ones into quotient and shift partial remainder by k positions**

SRT methods

SRT . . . Sweeney - Robertson - Tocher

- Representation of quotient is determined in signed-digit number system and next is converted into standard number system.
- Partial remainder is normalized after each partial operation addition / subtraction by using principle of (skip over 0's and 1's).
- Redundant number of signed-digit number system is used.
  - Only several (few) first bits of partial remainder is sufficient to determine one digit of quotient.
  - The principle of skipping 0's and 1's is used as much as possible.
- The number system with bases $z > 2$are also used.

Robertsons diagram

$A \in \langle 0,5,1)$ ... **dividend**

$B \in \langle 0,5,1)$ ... **divisor**

$Q \in \langle 0,1)$ ... **quotient**

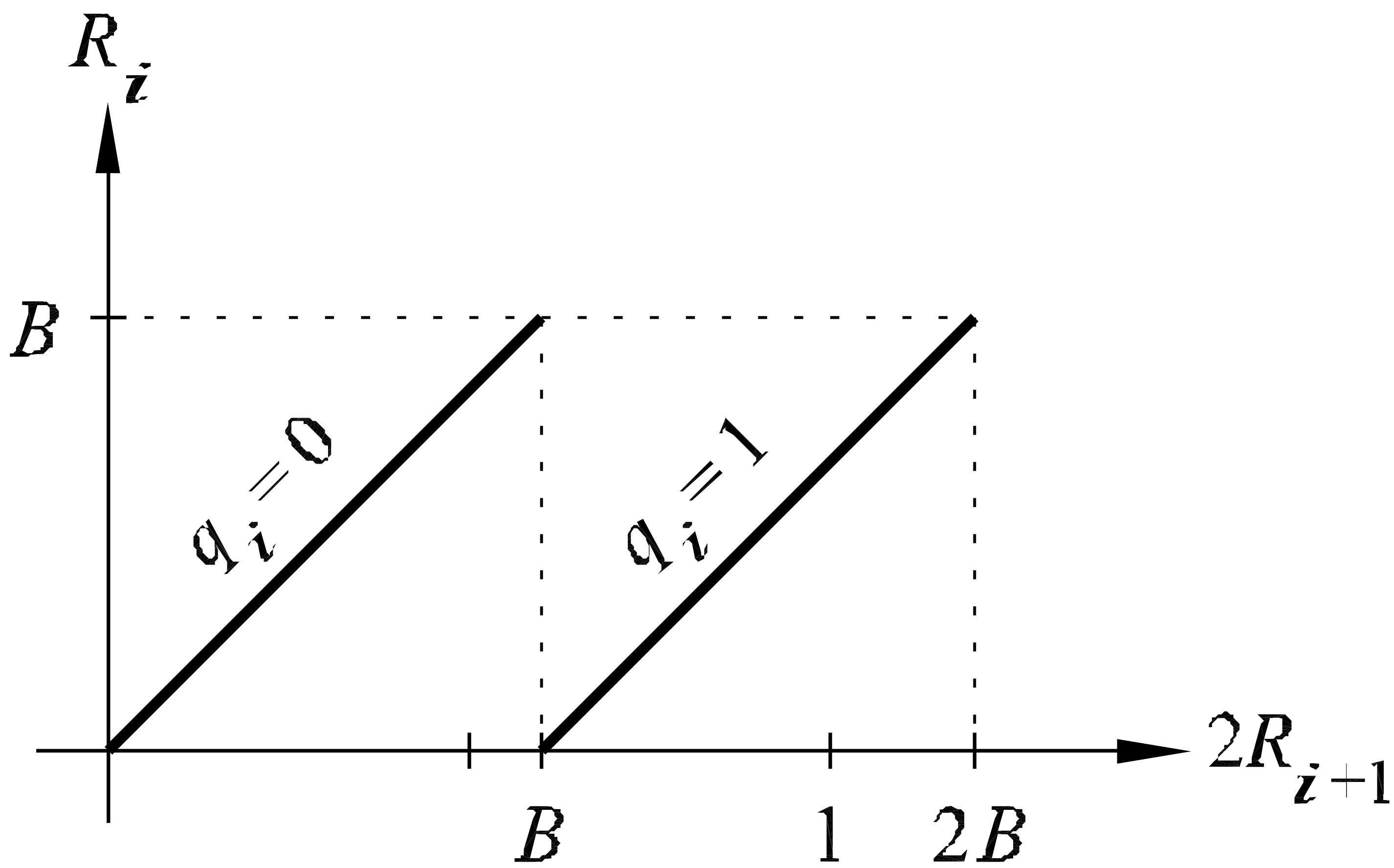
$q_0, q_{-1} q_{-2} \dots q_{-m}$ $\leftarrow$ notation Q

$R \in (-1,1)$ ... **remainder**

$R_i \in (-1,1)$ ... **partial remainder belonging to q_i shifted by m position to the left**

Dividend A is considered as a first partial remainder.

division with restoring:



ex.: division with restoring:

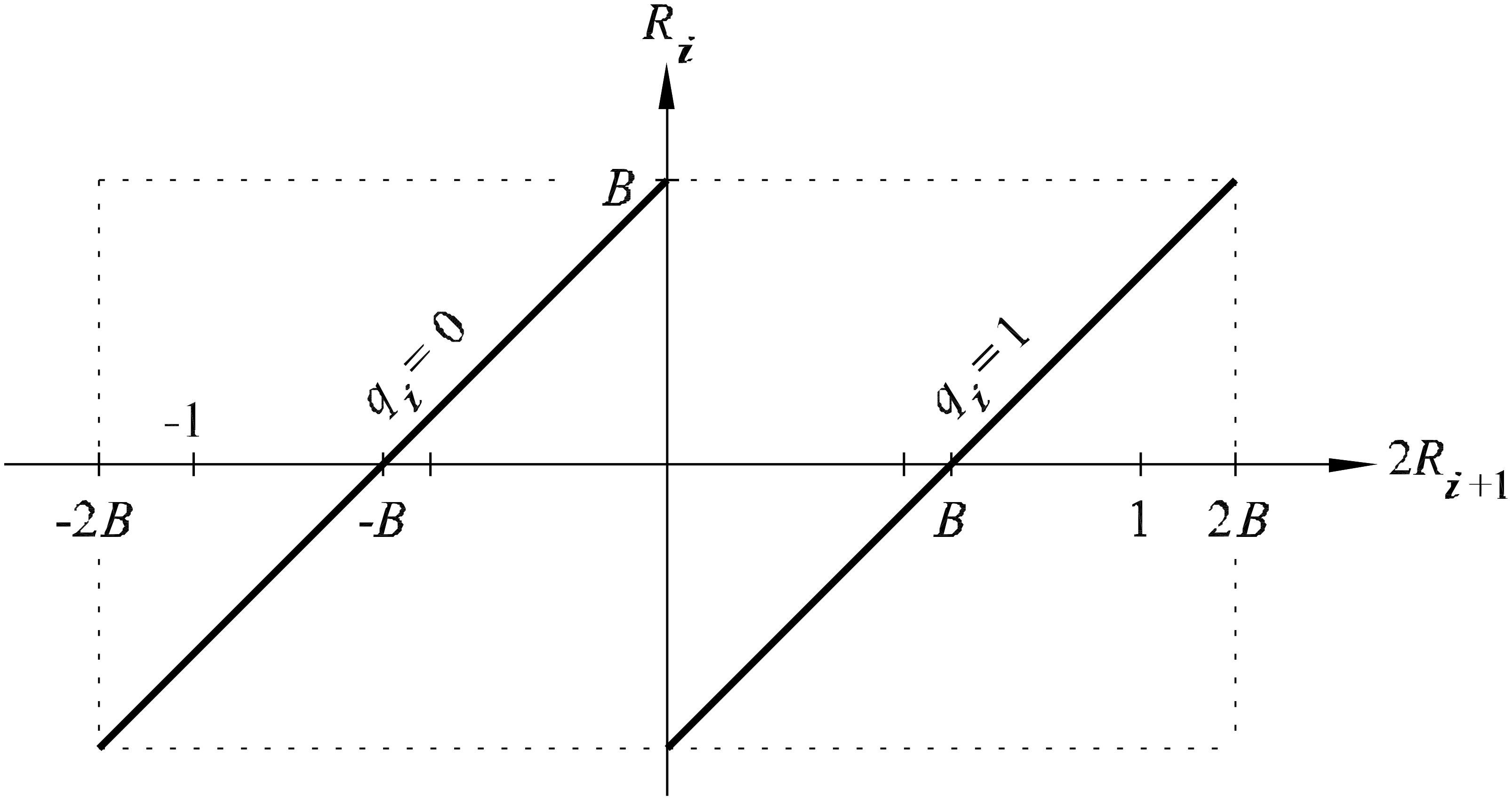
$$A = 0{,}101$$
$$B = 0{,}110$$
$$Q = 0{,}110$$
$$R = 0{,}000\,100$$

$$
\begin{array}{lccccccl}
 & & 0 & 1 & 0 & 1 & & A \sim 2R_1 \\
-B & & 1 & 0 & 1 & 0 & & q_0 = 0 \\
\hline
 & 0 & 1 & 1 & 1 & 1 & & R_0
\end{array}
$$

restoring and shift

$$
\begin{array}{lccccccl}
 & & 1 & 0 & 1 & 0 & & \\
-B & & 1 & 0 & 1 & 0 & & q_{-1} = 1 \\
\hline
 & 1 & 0 & 1 & 0 & 0 & & R_{-1} \\
 & & \swarrow & \swarrow & \swarrow & & & \\
 & & 1 & 0 & 0 & 0 & & \\
-B & & 1 & 0 & 1 & 0 & & q_{-2} = 1 \\
\hline
 & 1 & 0 & 0 & 1 & 0 & & R_{-2} \\
 & & \swarrow & \swarrow & \swarrow & & & \\
 & & 0 & 1 & 0 & 0 & & \\
-B & & 1 & 0 & 1 & 0 & & q_{-3} = 0 \\
\hline
 & 0 & 1 & 1 & 1 & 0 & & R_{-3} \\
 & & 0 & 1 & 0 & 0 & & \text{(after restoring)}
\end{array}
$$

division without restoring:

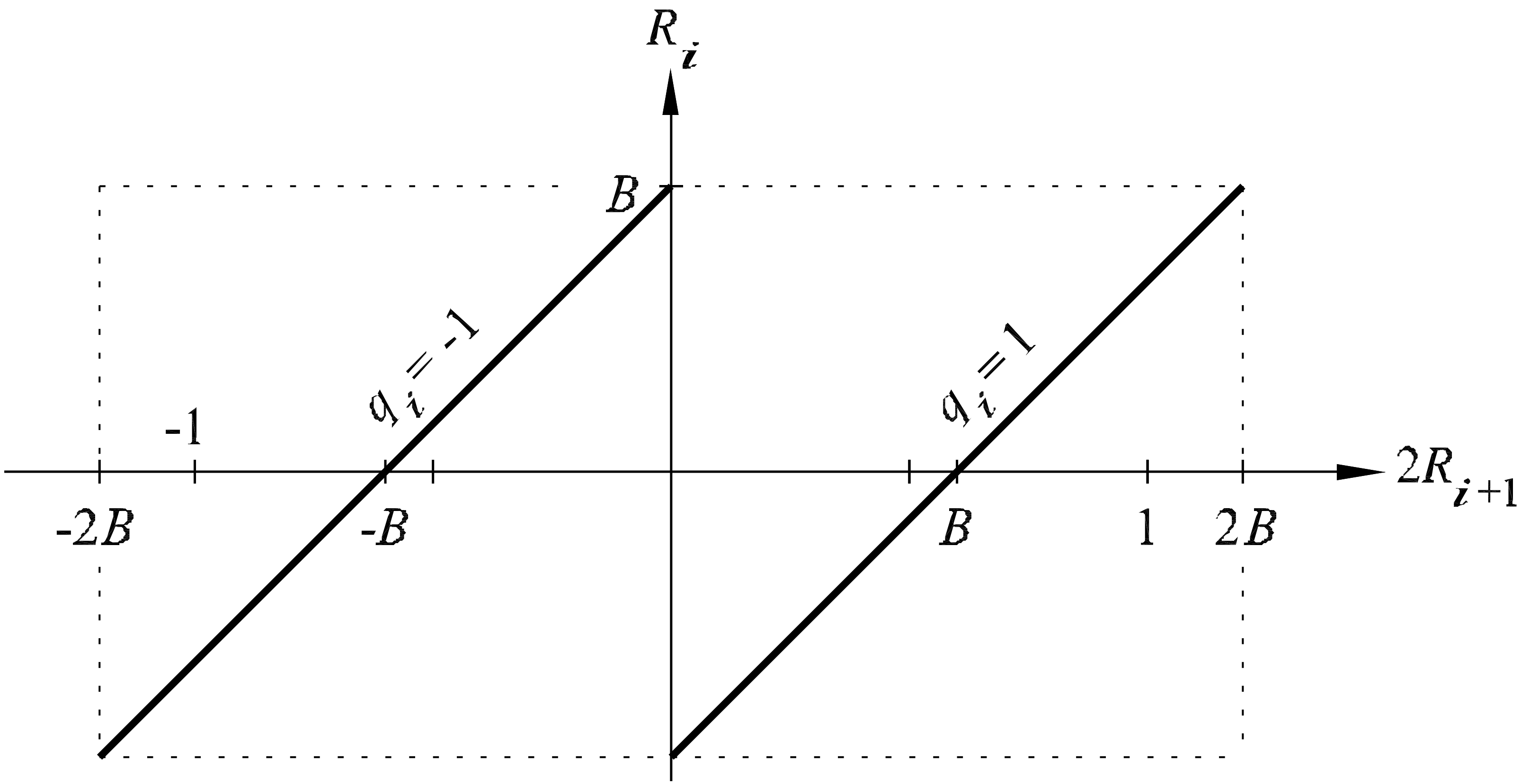


ex.: division without restoring:

$A = 0{,}101$
$B = 0{,}110$
$Q = 0{,}110$
$R = 0{,}000\,100$

$$
\begin{array}{rl|l}
 & \,0\;1\;0\;1 & A \sim 2R_1 \\
-B & \,1\;0\;1\;0 & q_0 = \color{red}{0} \\
\hline
 & \color{red}{0}\,1\;1\;1\;1 & R_0 \\
 & \,\swarrow\;\swarrow\;\swarrow & \\
 & \,1\;1\;1\;0 & \\
+B & \,0\;1\;1\;0 & q_{-1} = \color{red}{1} \\
\hline
 & \color{red}{1}\,0\;1\;0\;0 & R_{-1} \\
 & \,\swarrow\;\swarrow\;\swarrow & \\
 & \,1\;0\;0\;0 & \\
-B & \,1\;0\;1\;0 & q_{-2} = \color{red}{1} \\
\hline
 & \color{red}{1}\,0\;0\;1\;0 & R_{-2} \\
 & \,\swarrow\;\swarrow\;\swarrow & \\
 & \,0\;1\;0\;0 & \\
-B & \,1\;0\;1\;0 & q_{-3} = \color{red}{0} \\
\hline
 & \color{red}{0}\,1\;1\;1\;0 & R_{-3} \\
 & & \\
 & \,0\;1\;0\;0 & \text{(after restoring)}
\end{array}
$$

using of signed-digit numbers (division without restoring):

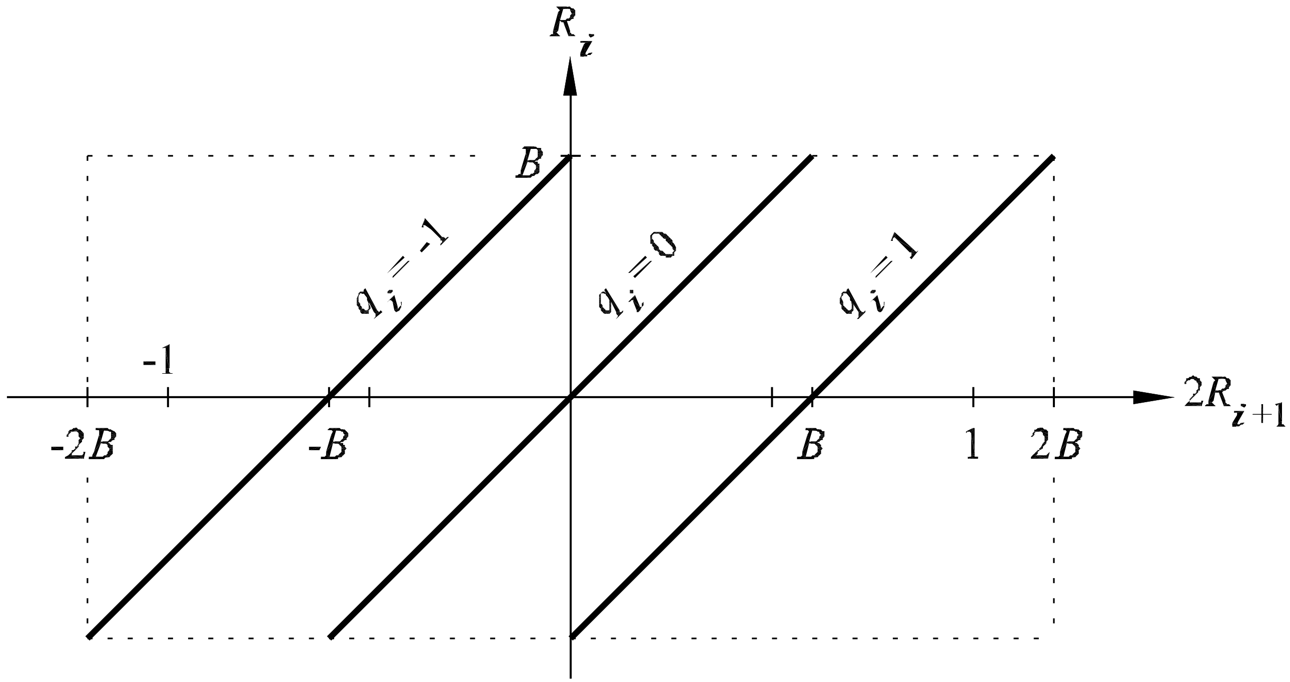


ex.: using of signed-digit numbers
(division without restoring):

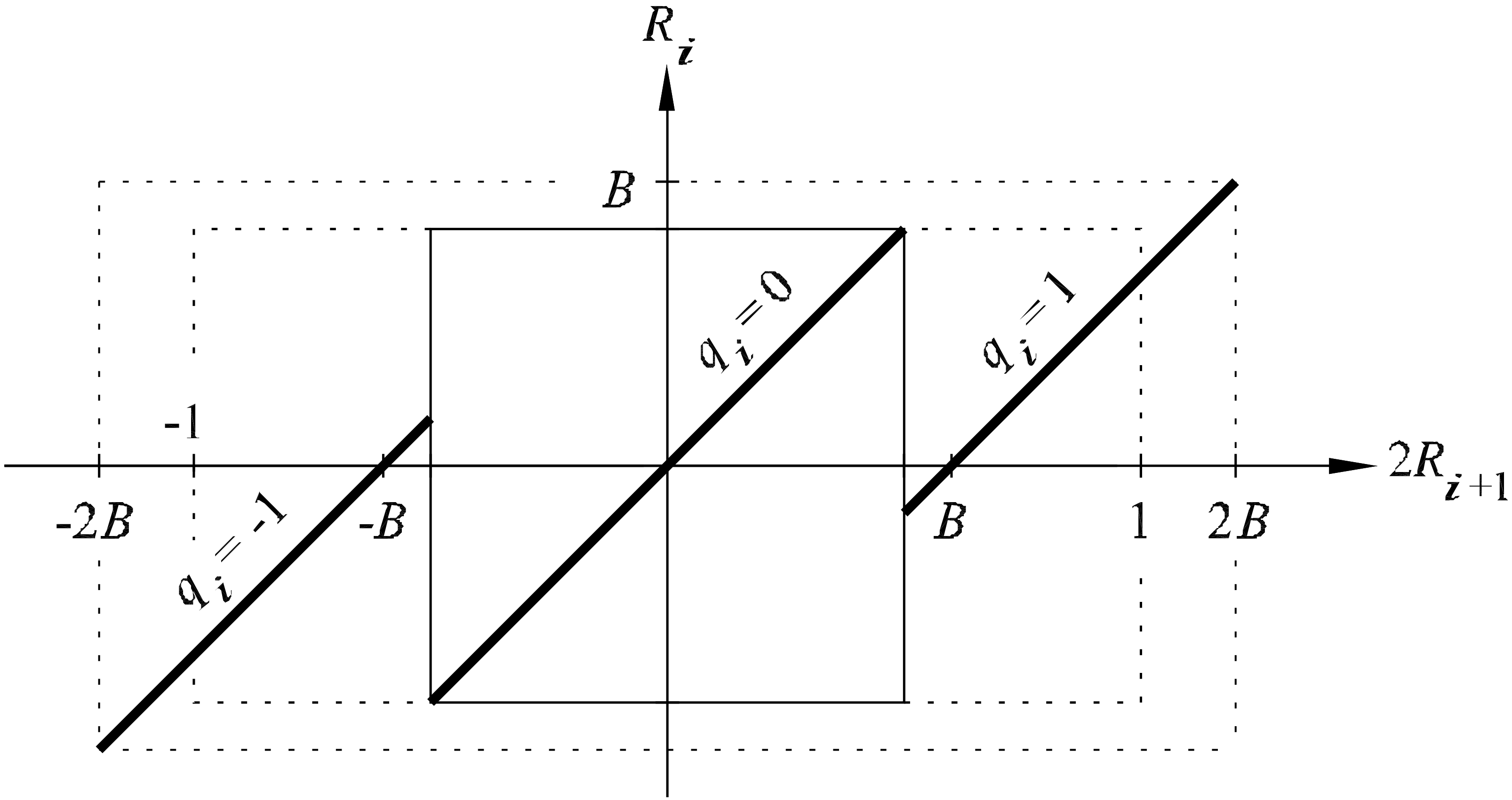
$A = 0{,}101$
$B = 0{,}110$
$Q = 1{,}\widehat{1}10 = 0{,}111$
$R = -\,0{,}000\,010$

$$
\begin{array}{r c c c c c l}
 & & 0 & 1 & 0 & 1 & A \sim 2R_1 \\
\color{red}{-B} & & 1 & 0 & 1 & 0 & q_0 = \color{red}{1} \\ \cline{3-6}
 & 0 & 1 & 1 & 1 & 1 & R_0 \\
 & & \swarrow & \swarrow & \swarrow & & \\
 & & 1 & 1 & 1 & 0 & \\
\color{red}{+B} & & 0 & 1 & 1 & 0 & q_{-1} = \color{red}{-1} \\ \cline{3-6}
 & 1 & 0 & 1 & 0 & 0 & R_{-1} \\
 & & \swarrow & \swarrow & \swarrow & & \\
 & & 1 & 0 & 0 & 0 & \\
\color{red}{-B} & & 1 & 0 & 1 & 0 & q_{-2} = \color{red}{1} \\ \cline{3-6}
 & 1 & 0 & 0 & 1 & 0 & R_{-2} \\
 & & \swarrow & \swarrow & \swarrow & & \\
 & & 0 & 1 & 0 & 0 & \\
\color{red}{-B} & & 1 & 0 & 1 & 0 & q_{-3} = \color{red}{1} \\ \cline{3-6}
 & 0 & 1 & 1 & 1 & 0 & R_{-3}
\end{array}
$$

using of redundant signed-digit numbers
(division without restoring):



$B \in \langle 0,5,1)$ $\Rightarrow$ **condition**
$2R_{i+1} \geq B$, **resp.** $2R_{i+1} \leq -B$,
can be replaced with condition
$2R_{i+1} \geq 0.5$, **resp.** $2R_{i+1} \leq -0.5$.
$\Rightarrow$ **First bits are enough for „correct“ selection.**



R_i
B
$q_i=0$
$q_i=1$
-1
$2R_{i+1}$
$-2B$
$q_i=-1$
$-B$
B
1
$2B$

ex.: $A = 0{,}101$
$B = 0{,}110$
$Q = 1{,}00\hat{1} = 0{,}111$
$R = -0{,}000\,010$

comment $\nearrow$
(Binnary point is
befor 3th bits
from right.)
$A > 1/2$
R_0 = -1/8
$2R_0 \in \langle$-1/2, 1/2$\rangle$
R_{-1} = -1/4
$2R_{-1} \in \langle$-1/2, 1/2$\rangle$
R_{-2} = -1/2
$2R_{-2} < -1/2$

$$
\begin{array}{lccccc l}
 & & 0 & 1 & 0 & 1 & A \sim 2R_1 \\
-B & & 1 & 0 & 1 & 0 & q_0 = 1 \\
\hline
 & 0 & 1 & 1 & 1 & 1 & R_0 \\
 & & \swarrow & \swarrow & \swarrow & & \\
\text{nic} & & 1 & 1 & 1 & 0 & q_{-1} = 0 \\
 & & \swarrow & \swarrow & \swarrow & & \\
\text{nic} & & 1 & 1 & 0 & 0 & q_{-2} = 0 \\
 & & \swarrow & \swarrow & \swarrow & & \\
 & & 1 & 0 & 0 & 0 & \\
+B & & 0 & 1 & 1 & 0 & q_{-3} = -1 \\
\hline
 & 0 & 1 & 1 & 1 & 0 & R_{-3}
\end{array}
$$

Fraction extension

$$\frac{A}{B} = \frac{A_0}{B_0} = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3} = \dots \qquad \begin{aligned} A_{i+1} &= A_i \cdot K_i \\ B_{i+1} &= B_i \cdot K_i \end{aligned}$$

$$A_0 = A \qquad B_0 = B \qquad \boxed{B_i \to 1 \implies A_i \to \frac{A}{B}}$$

$$\boxed{B = 1 - \delta} \qquad \tfrac{1}{2} \leq B < 1 \quad \Rightarrow \quad 0 < \delta \leq \tfrac{1}{2}$$

$$B_i = 1 - \delta_i$$

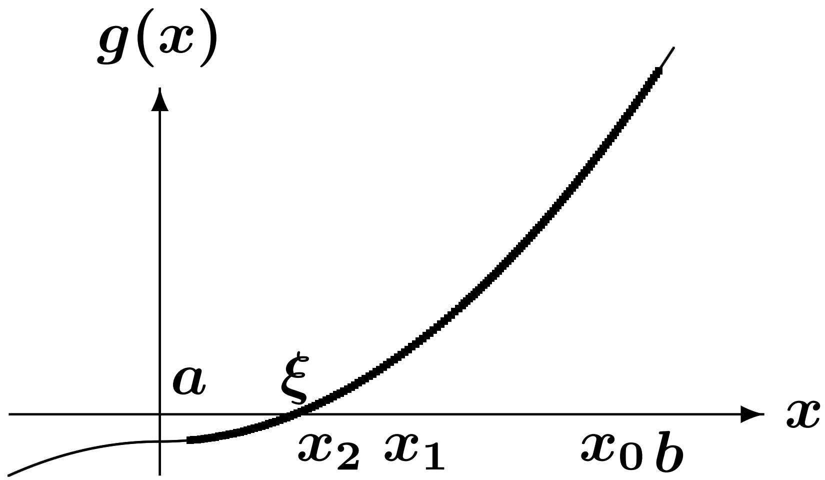
$$K_i = 1 + \delta_i = 2 - B_i \quad \Rightarrow \quad \boxed{K_i = \overline{B_i} + \varepsilon}$$

$$B_{i+1} = B_i \cdot K_i = (1 - \delta_i) \cdot (1 + \delta_i) \;\Rightarrow\; \boxed{B_{i+1} = 1 - \delta_i^2}$$

If $\delta_i \ll 1$ then double valid digits is obtained by each step.

Newtons method (method of tangent) [Newton – Raphson]

Observation:



Assumption: **The function $g(x)$ is gaining for interval $\langle a\ ,\ b\rangle$ value 0**
(pro $x = \xi$) and it is continuous, growing and perfectly convex.

$$g'(x_0) = \frac{g(x_0)}{x_0 - x_1} \qquad \Longrightarrow \qquad x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

$$g'(x_1) = \frac{g(x_1)}{x_1 - x_2} \qquad \Longrightarrow \qquad x_2 = x_1 - \frac{g(x_1)}{g'(x_1)}$$

$$\vdots$$

Therefor

$$\boxed{x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}}$$

for $i = 0, 1, 2, \ldots$

Equation

$$\boxed{g(x) = 0}$$

has root in interval $\langle a \ , \ b \rangle$ **namely, that is**

$$\xi = \lim_{i \to \infty} x_i.$$

exactly — see next slide

If there exist such as a and $b > a$, that

- **function $g(x)$ is continuous in interval $\langle a\ ,\ b\rangle$ and has there continuous first derivation $g'(x)$ and continuous second derivation $g''(x)$,**
- $g(a) \cdot g(b) < 0$,
- $(\forall x \in \langle a\ ,\ b\rangle) \quad g'(x){\cdot}g'(a) > 0$,
- $(\forall x \in \langle a\ ,\ b\rangle) \quad g''(x){\cdot}g''(a) > 0$ **and**
- $(\exists x_0 \in \langle a\ ,\ b\rangle) \quad g(x_0) \cdot g''(x_0) > 0$,

the equation $g(x) = 0$ has only one root in $\langle a\ ,\ b\rangle$, that is

$$\xi = \lim_{i\to\infty} x_i\ ,$$

where

$$\boxed{x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}}$$

for $i = 0,\ 1,\ 2,\ 3,\ \ldots$.

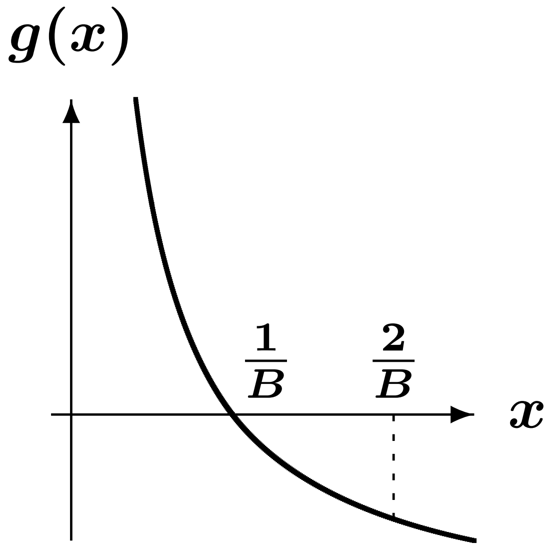
Conditions are sufficient but not necessary. The equation $g(x) = 0$ can have in interval $\langle a\ ,\ b\rangle$ only one root, which can be determined with given procedure, also in a case, when some of conditions are not met.

$$\boxed{\frac{A}{B} = A \cdot \frac{1}{B}}$$

$\xi = \dfrac{1}{B}$ **is a root of equation** $\boxed{\dfrac{1}{x} - B = 0}$.

ξ **is therefor a root of equation** $g(x) = 0$, **where**

$g(x) = \dfrac{1}{x} - B$, $g'(x) = -\dfrac{1}{x^2}$ **a** $g''(x) = \dfrac{2}{x^3}$.



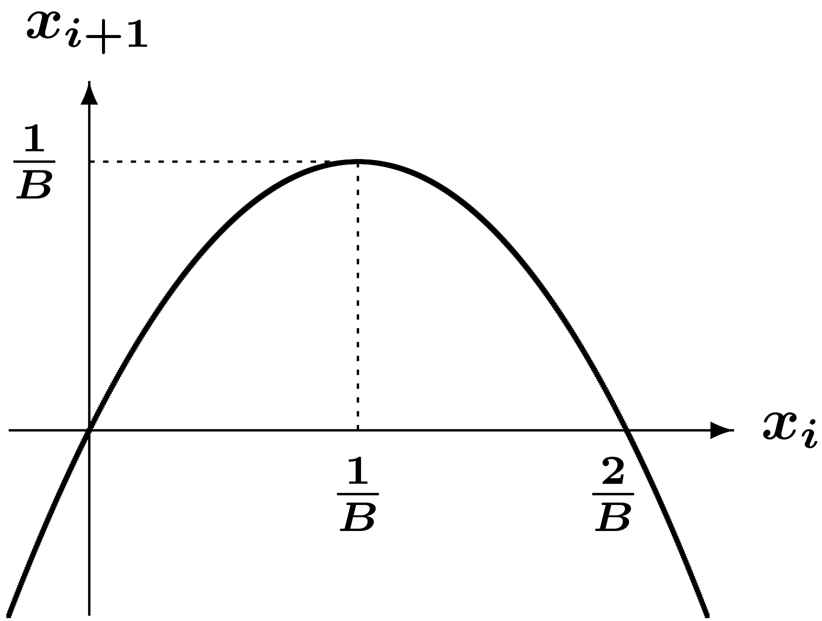
Therefor $x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} = x_i{\cdot}(2 - B{\cdot}x_i)$

and suit any $a \in (0\ ,\ \frac{1}{B})$,

any $b \in (\frac{1}{B}\ ,\ \infty)$ a

any $x_0 \in (0\ ,\ \frac{1}{B})$.

Note.: It can be also used any $x_0 \in \langle \frac{1}{B}\ ,\ \frac{2}{B})$



Calculation value of $\frac{1}{B}$

$$\boxed{x_{i+1} = x_i \cdot (2 - B \cdot x_i)}$$

normalized B: $\frac{1}{2} \leq B < 1 \implies 1 < \frac{1}{B} \leq 2$

$[\forall x_0 \in (0\ ,\ 1)]\ x_0 \in \left(0\ ,\ \frac{1}{B}\right)$

speed of convergence:

$$x_i = \frac{1}{B} \cdot (1 - \delta) \qquad \implies \qquad x_{i+1} = \frac{1}{B} \cdot (1 - \delta^2)$$

$|\delta| \ll 1 \implies$ **double number of valid positions are obtained with each iteration**

It is suitable to choose x_0 so that, $|\delta|$ was as small as possible.

$\rightarrow$ **small table in memory ROM addressed by some of first bits B**

Division in decimal system

Division can be converted on subtraction (or addition) and shifts.

ex.: 9 876 : 312 = ?
quotient = 31
remainder = 204

$$
\begin{array}{rl}
987 & \\
\underline{-312} & \\
675 & 1 \\
\underline{-312} & \\
363 & 2 \\
\underline{-312} & \\
51 & 3 \\
\underline{-312} & \\
-261 & < 0
\end{array}
\qquad\qquad
\begin{array}{rl}
516 & \\
\underline{-312} & \\
204 & 1 \\
\underline{-312} & \\
-108 & < 0
\end{array}
$$

It is also possible use division without restoring but it is not so advantageous such as in binary system (see next slide).

Division in decimal system ii