MI-ARI

(Computer arithmetics)

winter semester 2017/18

F1. Elementary functions I.

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https://edux.fit.cvut.cz/courses/MIE-ARI/lectures

F1. **Elementary functions I.**

- **square root**
  - **Pseudo-division**
  - **Extract the root by iterations**
- **The CORDIC method**
  - **Trigonometric functions**
  - **Inverse trigonometric functions**
  - **Other functions**

Square root — pseudo-division

$$N^2 = \sum_{j=1}^{N} (2j-1) = \sum_{i=0}^{N-1} (2i+1)$$

$$1 = 1^2$$
$$1+3 = 2^2$$
$$1+3+5 = 3^2$$
$$1+3+5+7 = 4^2$$

etc.

$$B = \lfloor \sqrt{A} \rfloor$$

```
B = 0;
while (A >= 2*B+1))
   {A -= 2*B+1;
    B++;}
```

(A)

or

```
B = 0;
while (A >= 0)
   {A -= 2*B+1;
    if (A >= 0) B++;}
A += 2*B+1; // restoring
```

(B)

$$\sqrt{5} \qquad\qquad\qquad \sqrt{531}$$

(A)

$$\begin{array}{rr}
5 & 0 \\
\underline{-1} & \hookleftarrow \\
4 & 1 \\
\underline{-3} & \hookleftarrow \\
1 & 2 \\
\multicolumn{2}{l}{1 < 5 \Rightarrow \textbf{end}}
\end{array}$$

(B)

$$\begin{array}{rrl}
5 & 0 & \\
\underline{-1} & \hookleftarrow & \\
4 & 1 & \\
\underline{-3} & \hookleftarrow & \\
1 & 2 & \\
\underline{-5} & \hookleftarrow & \\
-4 & < 0 & \cancel{3} \\
\underline{+5} & \text{restoring} & \\
1 & &
\end{array}$$

check: $5 = 2^2 + 1 =$
$= 4 + 1$

(A)

$$\begin{array}{rr}
531 & 0 \\
\underline{-1} & \hookleftarrow \\
530 & 1 \\
\underline{-3} & \hookleftarrow \\
527 & 2 \\
\underline{-5} & \hookleftarrow \\
\vdots & \vdots \\
\underline{-39} & \hookleftarrow \\
131 & 20 \\
\underline{-41} & \hookleftarrow \\
90 & 21 \\
\underline{-43} & \hookleftarrow \\
47 & 22 \\
\underline{-45} & \hookleftarrow \\
2 & 23
\end{array}$$

check: $531 = 23^2 + 2 =$
$= 529 + 2$

$$\lfloor\sqrt{A}\rfloor = B = K\cdot z + L \qquad B^2 = \sum_{i=0}^{B-1}(2i+1)$$

$$A = U\cdot z^2 + V$$

$$K = \lfloor\sqrt{U}\rfloor \qquad K^2 = \sum_{i=0}^{K-1}(2i+1)$$

$$\begin{aligned} A - B^2 &= A - \sum_{i=0}^{B-1}(2i+1) = \\ &= A - \sum_{i=0}^{K\cdot z+L-1}(2i+1) = \\ &= A - \sum_{i=0}^{K\cdot z-1}(2i+1) - \sum_{i=K\cdot z}^{K\cdot z+L-1}(2i+1) = \\ &= A - (K\cdot z)^2 - \sum_{i=K\cdot z}^{K\cdot z+L-1}(2i+1) \end{aligned}$$

$$A - B^2 = A - \left(\sum_{i=0}^{K-1}(2i+1)\right)\cdot z^2 - \sum_{i=K\cdot z}^{K\cdot z+L-1}(2i+1)$$

$z = 10 \qquad A = 0531 \quad \Longrightarrow \quad U = 05 \quad V = 31$

		A	05 31
i	$2i+1$	U	V
0		05	
	1	−1	
1		04	
		04	
	3	−3	
2		01	

$2 \to K$

		01 31
20		01 31
	41	− 41
21		00 90
	43	− 43
22		00 47
	45	− 45
23		00 02

$23 \to B$

check:

$B = 23 = \lfloor\sqrt{531}\rfloor \quad \Longrightarrow \quad K = 2 \quad L = 3$

$23^2 = 529 = 1 + 3 + \ldots + 45$

$K = 2 = \lfloor\sqrt{5}\rfloor$

$2^2 = 5 = 1 + 3$

$$\sqrt{194_{10}} = \sqrt{11\,00\,00\,10_2}$$

11 00 00 1 0

0	-1	0 0 0 **1 1** **1 1 1 1 1** **1** **0 0 0 1 0**
1	-101	**0 1 0 0 0** **1 1 0 1 1** **1** **0 0 0 1 1**
1 1	-1101	**0 1 1 0 0** **1 0 0 1 1** **0** **1 1 1 1 1**
1 1 0	$+1101$ -11001	**1 1 1 1 1** **restoring** **0 1 1 0 1** **0 1 1 0 0** **1 0 0 1 0** **0 0 1 1 1** **1** **1 1 0 0 1**
1 1 0 1		

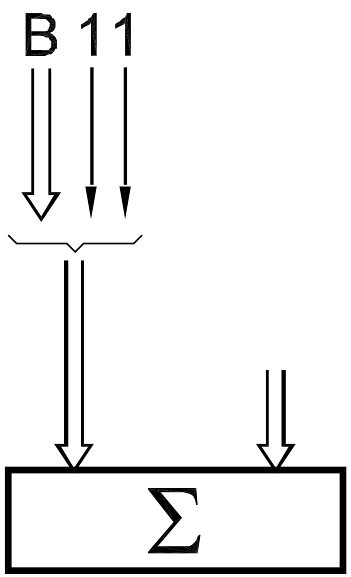
11 00 00 10

0	-1	0 00 11 $\underline{1\ 11\ 11}$ 1 0 00 10
1	-101	0 10 00 $\underline{1\ 10\ 11}$ 1 0 00 11
11	-1101	0 11 00 $\underline{1\ 00\ 11}$ 0 1 11 11
110	$+11011$	1 11 10 $\underline{1\ 10\ 11}$ 1 1 10 01
1101		

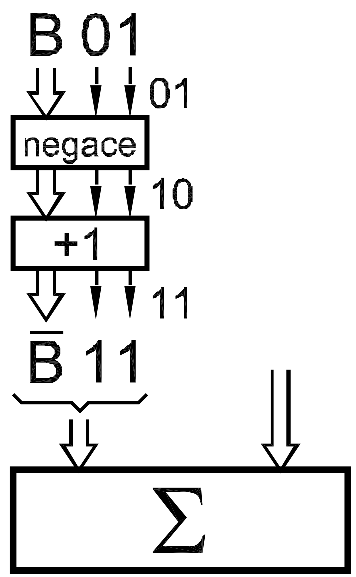
check: $194 = 169 + 25 = 13^2 + 25$

consideration:

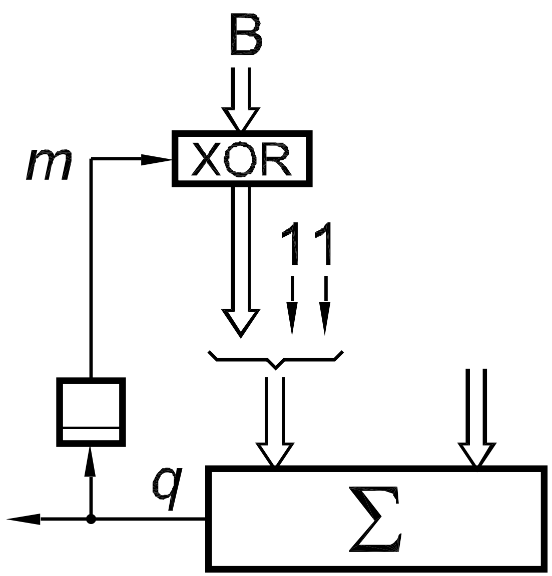
q **. . . carry from higher order**
$q = 0 \implies$ **addition in next step** $\mathbf{4B + 3 = B \triangleleft 2 + 11_2}$
$q = 1 \implies$ **subtraction in next step** $\mathbf{4B + 1 = B \triangleleft 2 + 01_2}$
subtraction: negation + hot one



$\mathbf{+(B \triangleleft 2 + 11_2)}$

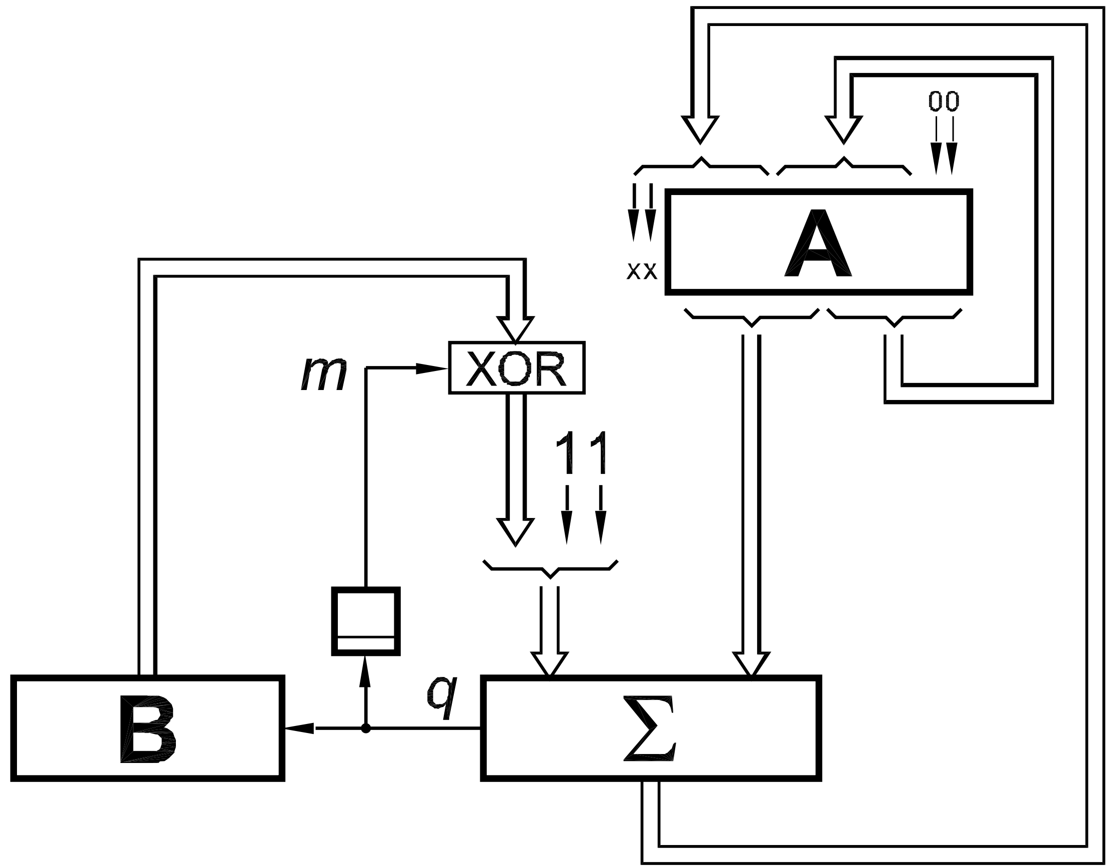


$\mathbf{-(B \triangleleft 2 + 01_2)}$



both

scheme:



at the begin the register A must be filled from left with zeroes

at the begin the register B must be erased

at the begin m must be set to 1

xx . . . is ignored

00 . . . bits inserted during the left shift of the register A

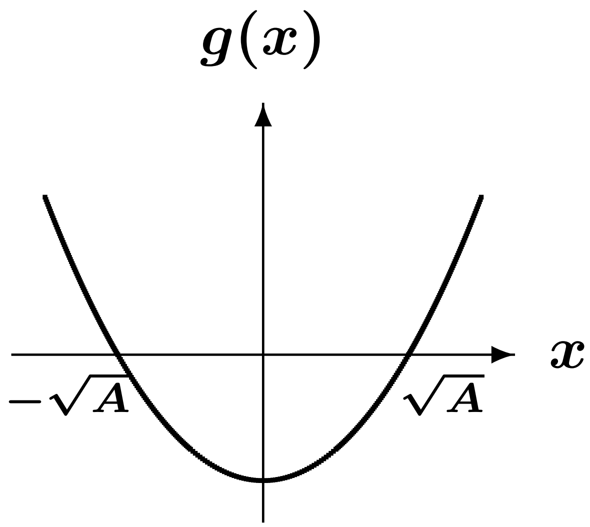
Extract the root by iterations

Observation: $\xi = \sqrt{A}$ **is a root of equation** $\boxed{x^2 - A = 0}$,

i.e. ξ **is a root of equation** $g(x) = 0$**, where** $g(x) = x^2 - A$

,

i.e. $g'(x) = 2x$ **a** $g''(x) = 2$.



Thus:

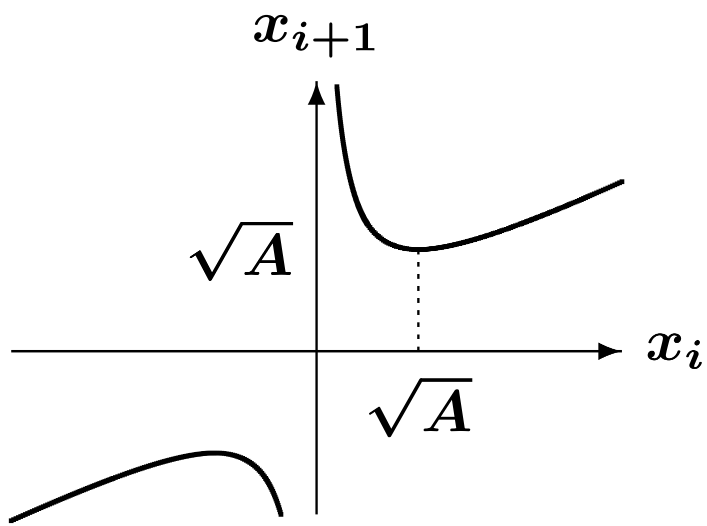
$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} =$$
$$= \tfrac{1}{2} \cdot \left(\frac{A}{x_i} + x_i\right)$$

Satisfy:

any $a \in (0\ ,\ \sqrt{A})$,
any $b \in (\sqrt{A}\ ,\ \infty)$ a

any $\boxed{x_0 > 0}$.

$$\boxed{x_{i+1} = \tfrac{1}{2}\cdot\left(\frac{A}{x_i} + x_i\right)}$$



normalization ☞ $\frac{1}{2} \le A < 2 \implies \frac{1}{\sqrt{2}} \le \sqrt{A} < \sqrt{2}$

speed of convergence:

$$x_i = \sqrt{A}\cdot(1+\delta) \qquad \implies \qquad x_{i+1} = \sqrt{A}\cdot\left(1+\frac{\delta^2}{2(1+\delta^2)}\right)$$

$$|\delta| \ll 1 \implies x_{i+1} \doteq \sqrt{A}\cdot\left(1+\frac{\delta^2}{2}\right)$$

$\implies$ **Almost double valid digits is obtained by each iteration.**

Division operation is required by given solution !!!

$$\boxed{\sqrt{A} = A \cdot \sqrt{\frac{1}{A}}}$$

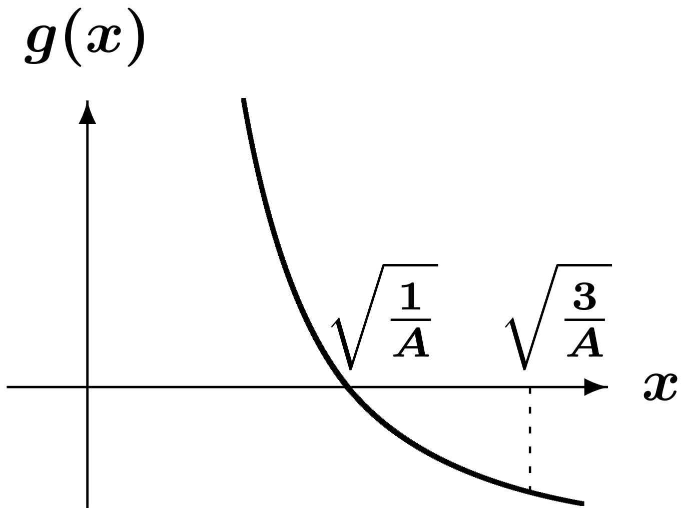
Observation: $\xi = \sqrt{\frac{1}{A}}$ **is a root of equation** $\boxed{\frac{1}{x^2} - A = 0}$,

i.e. ξ **is a root of equation** $g(x) = 0$, **where** $g(x) = \frac{1}{x^2} - A$
,
i.e. $g'(x) = -\frac{2}{x^3}$

a $g''(x) = \frac{6}{x^4}$.

Thus:

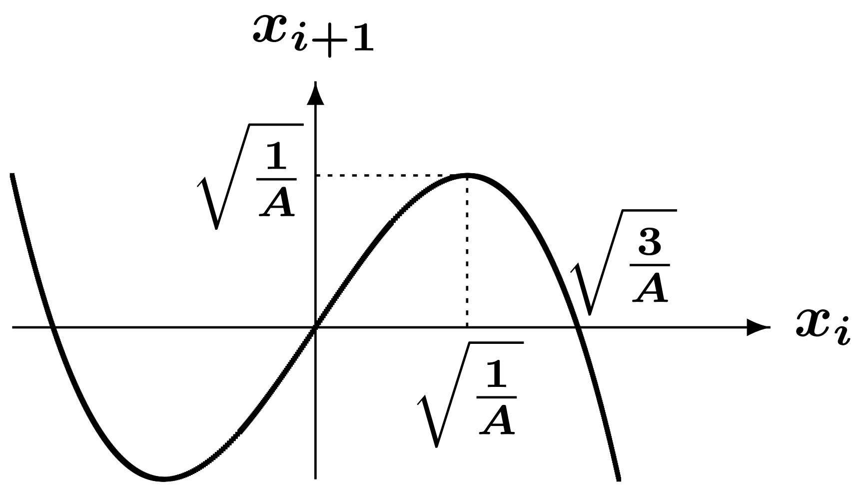
$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}$$
$$= \frac{x_i}{2} \cdot (3 - A \cdot x_i^2)$$



and satisfy: any $a \in (0\ ,\ \sqrt{\frac{1}{A}})$,

any $b \in (\sqrt{\frac{1}{A}}\ ,\ \sqrt{\frac{3}{A}})$ and **any** $\boxed{x_0 \in (0\ ,\ \sqrt{\frac{3}{A}})}$.

$$\boxed{x_{i+1} = \frac{x_i}{2}\cdot(3 - A\cdot x_i^2)}$$



normalization ☞ $\frac{1}{2} \leq A < 2 \implies \frac{1}{\sqrt{2}} \leq \sqrt{A} < \sqrt{2}$

speed of convergence:

$$x_i = \sqrt{\frac{1}{A}}\cdot(1-\delta) \quad \implies \quad x_{i+1} = \sqrt{\frac{1}{A}}\cdot\left(1 - \frac{3}{2}\delta^2 + \frac{1}{2}\delta^3\right)$$

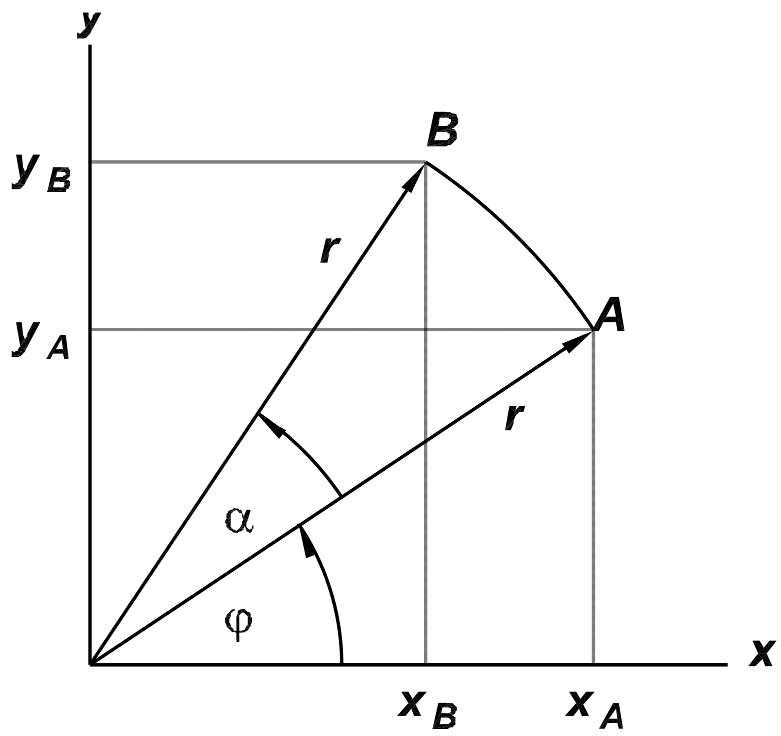
$$|\delta| \ll 1 \implies x_{i+1} \doteq \sqrt{\frac{1}{A}}\cdot\left(1 - \frac{3}{2}\delta^2\right)$$

$\implies$ **Almost double valid digits is obtained by each iteration.**

It is suitable to choose x_0 to reach $|\delta|$ as small as possible.
$\rightarrow$ **small table in memory ROM, addressed with few first bits A**

The CORDIC method

CORDIC ☜ **CO**ordinate **R**otation **DI**gital **C**omputer



$$x_A = r \cdot \cos\varphi \qquad x_B = r \cdot \cos(\varphi + \alpha)$$
$$y_A = r \cdot \sin\varphi \qquad y_B = r \cdot \sin(\varphi + \alpha)$$

$$
\begin{aligned}
x_B &= r \cdot \cos(\varphi + \alpha) \\
y_B &= r \cdot \sin(\varphi + \alpha)
\end{aligned}
$$

$$
\begin{aligned}
x_B &= r \cdot \cos\varphi \cdot \cos\alpha - r \cdot \sin\varphi \cdot \sin\alpha \\
y_B &= r \cdot \sin\varphi \cdot \cos\alpha + r \cdot \cos\varphi \cdot \sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
x_B &= \cos\alpha \cdot (x_A - y_A \cdot \mathrm{tg}\ \alpha) \\
y_B &= \cos\alpha \cdot (y_A + x_A \cdot \mathrm{tg}\ \alpha)
\end{aligned}
$$

$$\alpha = \alpha'_1 + \alpha'_2 + \cdots + \alpha'_n$$

$$
\begin{aligned}
x'_i &= \cos\alpha'_i \cdot (x'_{i-1} - y'_{i-1} \cdot \mathrm{tg}\ \alpha'_i) \\
y'_i &= \cos\alpha'_i \cdot (y'_{i-1} + x'_{i-1} \cdot \mathrm{tg}\ \alpha'_i)
\end{aligned}
$$

$$\mathrm{tg}\ \alpha'_i = \pm 2^{1-i} \qquad \Rightarrow \qquad \alpha_i = |\alpha'_i| = \mathrm{arctg}\ 2^{1-i}$$

Trigonometric functions

$$\begin{aligned} x_i' &= \cos\alpha_i' \cdot (x_{i-1}' - y_{i-1}' \cdot 2^{1-i}) \\ y_i' &= \cos\alpha_i' \cdot (y_{i-1}' + x_{i-1}' \cdot 2^{1-i}) \end{aligned}$$

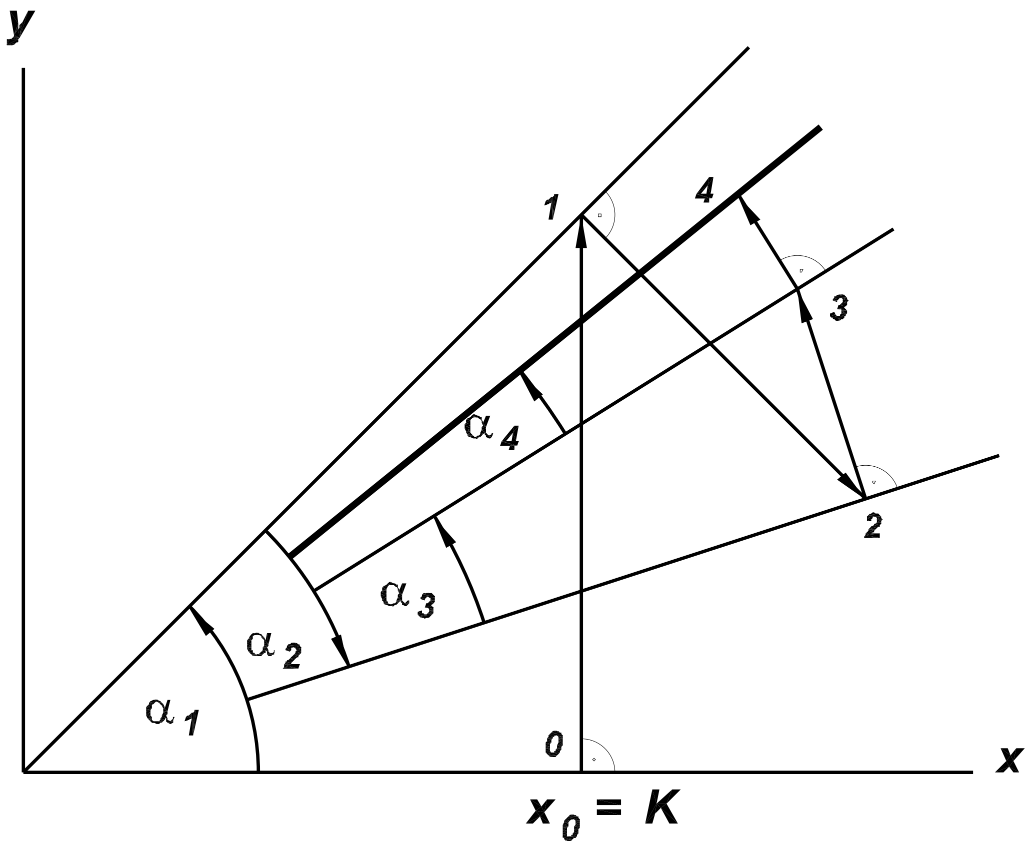
$$\begin{aligned} x_i &= x_{i-1} \mp y_{i-1} \cdot 2^{1-i} \\ y_i &= y_{i-1} \pm x_{i-1} \cdot 2^{1-i} \end{aligned}$$

$$K = \cos\alpha_1 \cdot \cos\alpha_2 \cdot \; \cdots \; \cdot \cos\alpha_n$$

$$\left.\begin{aligned} x_0 &= 1 \\ y_0 &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} \cos\alpha &= K \cdot x_n \\ \sin\alpha &= K \cdot y_n \end{aligned}\right.$$

$$\left.\begin{aligned} x_0 &= K \\ y_0 &= 0 \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} \cos\alpha &= x_n \\ \sin\alpha &= y_n \end{aligned}}$$

$$\mathrm{tg}\ \alpha = \frac{\sin\alpha}{\cos\alpha}\,, \quad \mathrm{cotg}\ \alpha = \frac{\cos\alpha}{\sin\alpha}\,, \quad \ldots$$


y
x
1
4
3
2
0
α_1
α_2
α_3
α_4
$x_0 = K$

Inverse trigonometric functions

$\mathrm{arctg}\,\frac{y}{x} =$?

1. $x_0 := x; \quad y_0 := y; \quad i := 0$
2. If $y_i > 0$ rotate (x_i, y_i) by $-\alpha_{i+1}$, else by α_{i+1};
   it gets so (x_{i+1}, y_{i+1});
   $i := i + 1$.
3. If the required precision is not reached,
   repeat step 2.
4. $-(\alpha_1 + \cdots + \alpha_{i+1}) \doteq \mathrm{arctg}\,\frac{y}{x}$

$\mathrm{arccotg}\,x = \frac{1}{2}\pi - \mathrm{arctg}\,x$

$\arcsin x$ a $\arccos x$ — somewhat more complicated

Other functions

$$
\begin{aligned}
x_B &= r \cdot \cosh(\varphi + \alpha) \\
y_B &= r \cdot \sinh(\varphi + \alpha)
\end{aligned}
$$

-
-
-

$\Rightarrow$ **hyperbolic function, inverse hyperbolic functions**

$\Rightarrow$ **logarithm, exponential functions, ...**

