MI-ARI

(Computer arithmetics)

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N1. **Multiplication I.**

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Department of digital design

Faculty of Information technology

Czech Technical University in Prague

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N1. **Multiplication I.**

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- **Sign and magnitude multiplication**
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Shifts:

- **logical**
- cyclic [rotation]
- **arithmetic**

$$
\begin{array}{llll}
X^* & \sim & X & \\
X^* \lhd i & \sim & X \cdot z^{i} & + \ \text{overflow detection} \\
X^* \rhd i & \sim & X \cdot z^{-i} & + \ \text{? detection of accuracy loss ?}
\end{array}
$$

sign and magnitude code:

- no change of sign bit;
- shift of magnitude;
- left shift & MSB of magnitude is non-zero, $\Rightarrow$ overflow
- right shift of odd magnitude, $\Rightarrow$ loss of accuracy

2's complement code

Arithmetic shift to the left (by 1 position):

$\mathcal{D}(A) \sim a_n^{\mathcal{D}} a_{n-1}^{\mathcal{D}} \dots a_1^{\mathcal{D}} a_0^{\mathcal{D}}$

- $A \geq 0 \;\Rightarrow\; \mathcal{D}(A \cdot z) = A \cdot z$
  log.: $\mathcal{D}(A) \cdot z - a_n^{\mathcal{D}} \cdot \mathcal{Z} = A \cdot z - a_n^{\mathcal{D}} \cdot \mathcal{Z}$
  arith.: $A \cdot z$
  diff: $a_n^{\mathcal{D}} \cdot \mathcal{Z} \quad \Rightarrow \quad a_n^{\mathcal{D}} = 0$ ☛ **ok.**
- $A < 0 \;\Rightarrow\; \mathcal{D}(A) = \mathcal{Z} + A$
  log.: $\mathcal{D}(A) \cdot z - a_n^{\mathcal{D}} \cdot \mathcal{Z} = (\mathcal{Z} + A) \cdot z - a_n^{\mathcal{D}} \cdot \mathcal{Z}$
  arith.: $\mathcal{Z} + A \cdot z$
  diff: $(a_n^{\mathcal{D}} - (z-1)) \cdot \mathcal{Z} \quad \Rightarrow \quad a_n^{\mathcal{D}} = z - 1$ ☛ **ok.**
- **conclusion: arithmetic shift $\rightarrow$ logical shift +**
  + overflow detection:
  - **The sign can not change.**
  - **Only zero or $z-1$ digit can came out.**

(2's complement code — 2)

Arithmetic shift to the right

If there is neither overflow nor loss of accuracy:

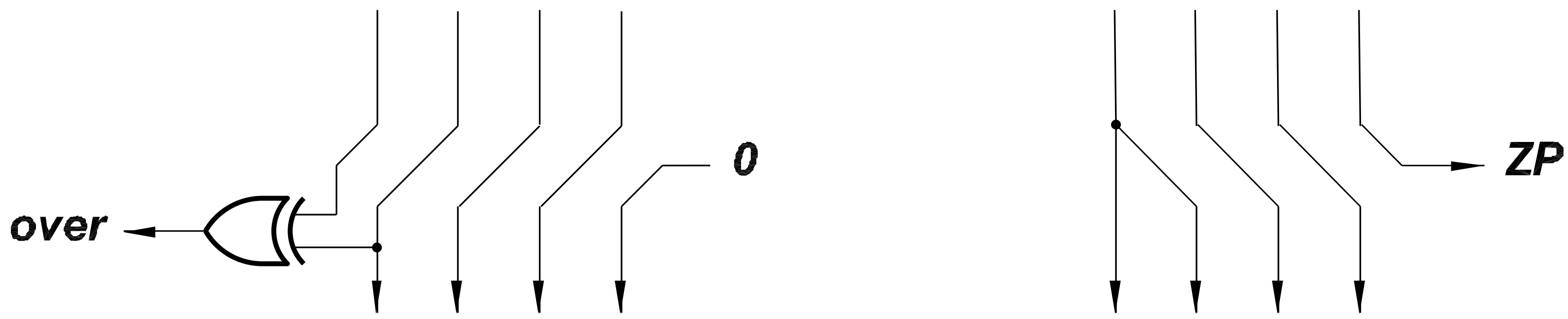
$$(A \triangleright 1) \triangleleft 1 = A = (A \triangleleft 1) \triangleright 1.$$

$\Rightarrow$ **from left the digit is inserted**

$\mathbf{0}$	**for unsigned number and**
$z-1$	**for negative numbers.**

$\Rightarrow$ **If non-zero digit come out from right, there is the loss of accuracy.**

ex.: $z = 2$



1's complement code (integers)

Arithmetic shift to the left (by 1 position):

$\mathcal{I}(A) \sim a_n^{\mathcal{I}} a_{n-1}^{\mathcal{I}} \dots a_1^{\mathcal{I}} a_0^{\mathcal{I}}$

- $A \geq 0 \;\Rightarrow\; \mathcal{I}(A \cdot z) = A \cdot z$
  log.: $\mathcal{I}(A) \cdot z - a_n^{\mathcal{I}} \cdot \mathcal{Z} = A \cdot z - a_n^{\mathcal{I}} \cdot \mathcal{Z}$
  arith.: $A \cdot z$
  diff: $a_n^{\mathcal{I}} \cdot \mathcal{Z} \quad \Rightarrow \quad a_n^{\mathcal{I}} = 0$ ☛ **ok.**

- $A \leq 0 \;\Rightarrow\; \mathcal{I}(A) = (\mathcal{Z} - 1) + A$
  log.: $\mathcal{I}(A) \cdot z - a_n^{\mathcal{I}} \cdot \mathcal{Z} = ((\mathcal{Z} - 1) + A) \cdot z - a_n^{\mathcal{I}} \cdot \mathcal{Z}$
  arith.: $(\mathcal{Z} - 1) + A \cdot z$
  diff: $(a_n^{\mathcal{I}} - (z - 1)) \cdot \mathcal{Z} + (z - 1)$
  $\Rightarrow \quad a_n^{\mathcal{I}} = z - 1 \quad \& \quad a_0^{\mathcal{I}} = z - 1$ ☛ **ok.**

(1's complement code — 2)

Arithmetic shift to the left:

logical shift and

insert digit $\left\{\begin{array}{ll} \mathbf{0} & \textbf{for } A \geq 0 \\ z-1 & \textbf{for } A \leq 0 \end{array}\right\}$ **from right and**

overflow detection:

The sign can not change.

Only zero or $z-1$ digit can came out.

Arithmetic shift to the right:

If there is neither overflow nor loss of accuracy:

$$(A \triangleright 1) \triangleleft 1 = A = (A \triangleleft 1) \triangleright 1.$$

$\Rightarrow$ **From left the digit is inserted**

- **0** **for positive numbers and positive zero**
- $z-1$ **for negative numbers and negative zero.**

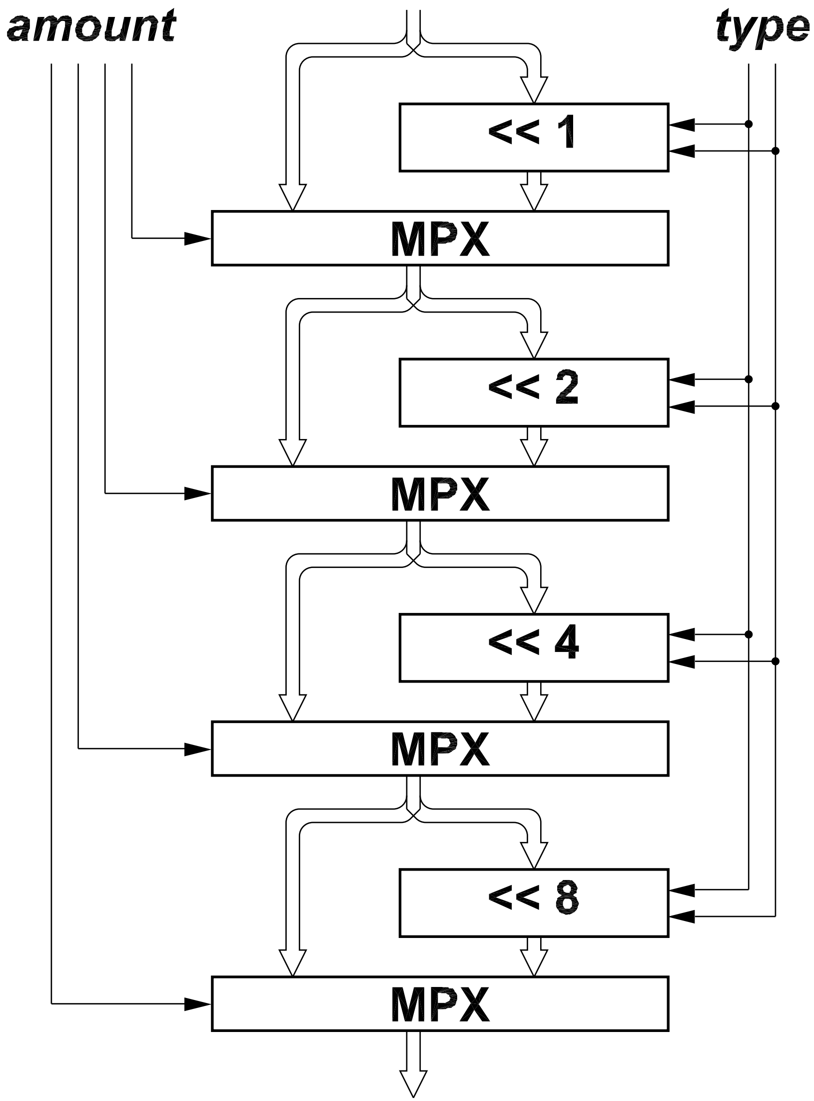
$\Rightarrow$ **If there come out other digit from right**

- **0** **for positive numbers and positive zero**
- $z-1$ **for negative numbers and negative zero**

there is a loss of accuracy.

Barrel Shifter

„barrel shifter scheme“



Format extension

format extension:

$$\left.\begin{array}{l}\textbf{module } \mathcal{Z} = z^{n+1} \\ \textbf{unit } \varepsilon = z^{-m}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\textbf{module } \mathcal{Z}^* = z^{n+2} \\ \textbf{unit } \varepsilon^* = z^{-m-1}\end{array}\right.$$

sign and magnitude — **trivial: zero-extension of magnitude**

no change of sign.

other representations: **zero-extension + correction**

correction: $\mathcal{Z}^* - \mathcal{Z} = (z-1) \cdot z^{n+1}$ **. . . new MSB**

$\frac{1}{2}\mathcal{Z}^* - \frac{1}{2}\mathcal{Z} = (z-1) \cdot z^{n}$ **. . . old MSB**

$\varepsilon - \varepsilon^* = (z-1) \cdot z^{-m-1}$ **. . . new LSB**

(If $z = 2$, then $(z-1) = 1$)

Format extension ii

example — *diminished radix complement:*

$$\mathcal{I}(X) = \begin{cases} X & \textbf{pro } X \geq 0 \quad \text{(positive zero including)} \\ \overline{|X|} = \\ = \mathcal{Z} - \varepsilon + X & \textbf{pro } X \leq 0 \quad \text{(negative zero including)} \end{cases}$$

$$X \geq 0 \quad \Longrightarrow \quad \mathcal{I}^*(X) - \mathcal{I}(X) = \mathbf{0} \quad \Longrightarrow \quad \textbf{no correction}$$

$$X \leq 0 \quad \Longrightarrow \quad \mathcal{I}^*(X) - \mathcal{I}(X) = \mathcal{Z}^* - \mathcal{Z} + \varepsilon - \varepsilon^*$$

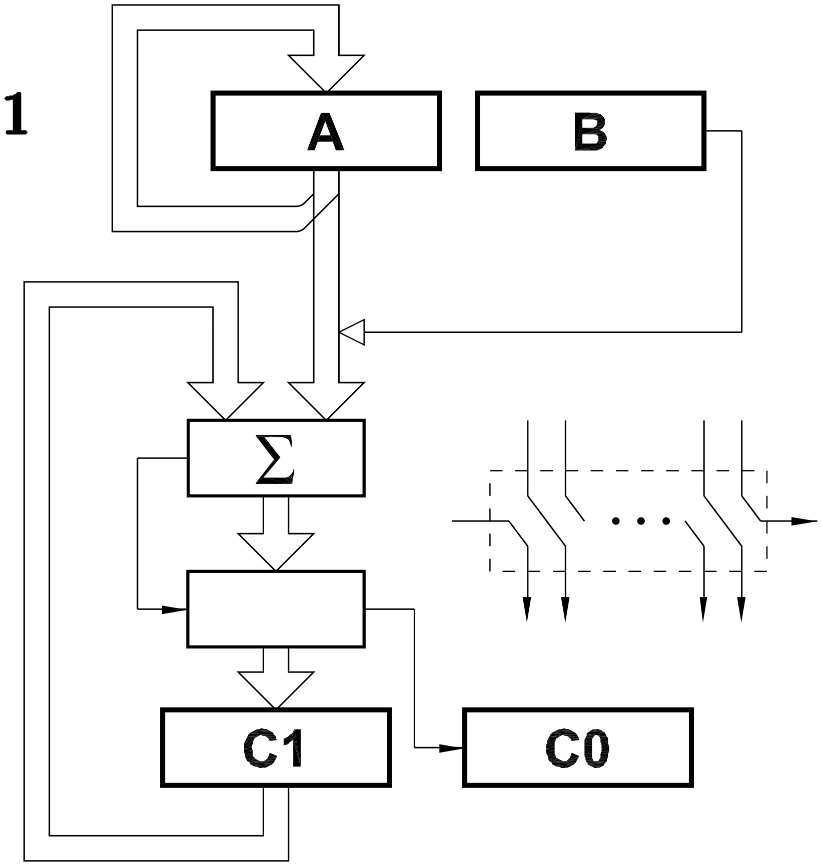
$$\Longrightarrow \quad \mathcal{I}^*(X) - \mathcal{I}(X) = z^{n+1} + z^{-m-1}$$

$$\Longrightarrow \quad \textbf{old MSB} \rightarrow \begin{cases} \textbf{new MSB} \\ \textbf{new LSB} \end{cases}$$

$\mathcal{D}$ $\qquad$ $\mathcal{I}$ $\qquad$ $\mathcal{A}_0$ $\qquad$ $\mathcal{A}_1$

Multiplication of unsigned numbers

```
     A          B            C1        C0
  1 1 0 1 · 1 0 1 1  =  1 0 0 0   1 1 1 1
  ---------------
            ↓ ↓ ↓ ↓
  0 0 0 0
  1 1 0 1           ☜ 1
---------
0 1 1 0 1
1 1 0 1           ☜ 1
---------
1 0 0 1 1
0 0 0 0           ☜ 0
---------
0 1 0 0 1
1 1 0 1           ☜ 1
---------
1 0 0 0 1
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
1 0 0 0 1 1 1 1
\_____/ \_____/
  C1      C0
```



Only one register can be used instead of B as well as C0 !

Signed-digit number systems and their using

Only integers will be considered !

(In essence it is without the loss of generality.)

$$B = \sum_{i=0}^{n} b_i z^i \quad \sim \quad b_n b_{n-1} \ldots b_0 \,{}_z, \text{ where } \qquad 0 \le b_i < z$$

$$B = \sum_{i=0}^{n+1} \beta_i z^i \quad \sim \quad \beta_{n+1} \beta_n \ldots \beta_0 \,{}_{z\pm}, \text{ where } \quad -\tfrac{1}{2} z \le \beta_i \le \tfrac{1}{2} z$$

convention: $\boxed{\widehat{x} = -x}$

Ex.: $z = 10$ $\qquad 6\;7\;2_{\,10} = 1\;\widehat{3}\;\widehat{3}\;2_{\,10\pm}$

conversion:

$$b_i < \tfrac{1}{2} z \quad \Rightarrow \quad \beta_i \leftarrow b_i$$

$$b_i > \tfrac{1}{2} z \quad \Rightarrow \quad \beta_i \leftarrow b_i - z; \quad b_{i+1} \leftarrow b_{i+1} + 1$$

$$b_i = \tfrac{1}{2} z \quad \Rightarrow \quad \beta_i \leftarrow ?$$

conversion (consideration continue):

	$b_i < \frac{1}{2}z$	$b_i = \frac{1}{2}z$	$b_i > \frac{1}{2}z$
Q_0 Q_1	b_i $b_i + 1$	? $b_i + 1 - z$	$b_i - z$ $b_i + 1 - z$
Q_0 Q_1	Q_0 Q_0	? Q_1	Q_1 Q_1

	$b_i < \frac{1}{2}z$	$b_i \geq \frac{1}{2}z$
Q_0 Q_1	b_i $b_i + 1$	$b_i - z$ $b_i + 1 - z$
Q_0 Q_1	Q_0 Q_0	Q_1 Q_1

conversion $(b_i \to \beta_i)$:

	$b_i < \frac{1}{2}z$	$b_i \geq \frac{1}{2}z$
$b_{i-1} < \frac{1}{2}z$	b_i	$b_i - z$
$b_{i-1} \geq \frac{1}{2}z$	$b_i + 1$	$b_i + 1 - z$

$b_{-1} \overset{\text{def}}{=} 0$

$z = 4$

b_{i-1}	b_i			
	00	01	10	11
0x	0	1	-2	-1
1x	1	2	-1	0

$z = 2$

b_{i-1}	b_i	
	0	1
0	0	-1
1	1	0

Modified Booth method (unsigned numbers)

convertion of multiplier to quaternary signed-digit system

ex.: $A = \mathbf{198}_{10} = \mathbf{11\ 00\ 01\ 10}_2$

$B = \mathbf{231}_{10} = \mathbf{11\ 10\ 01\ 11}_2 = \mathbf{3\,2\,1\,3},0_{\,4} =$

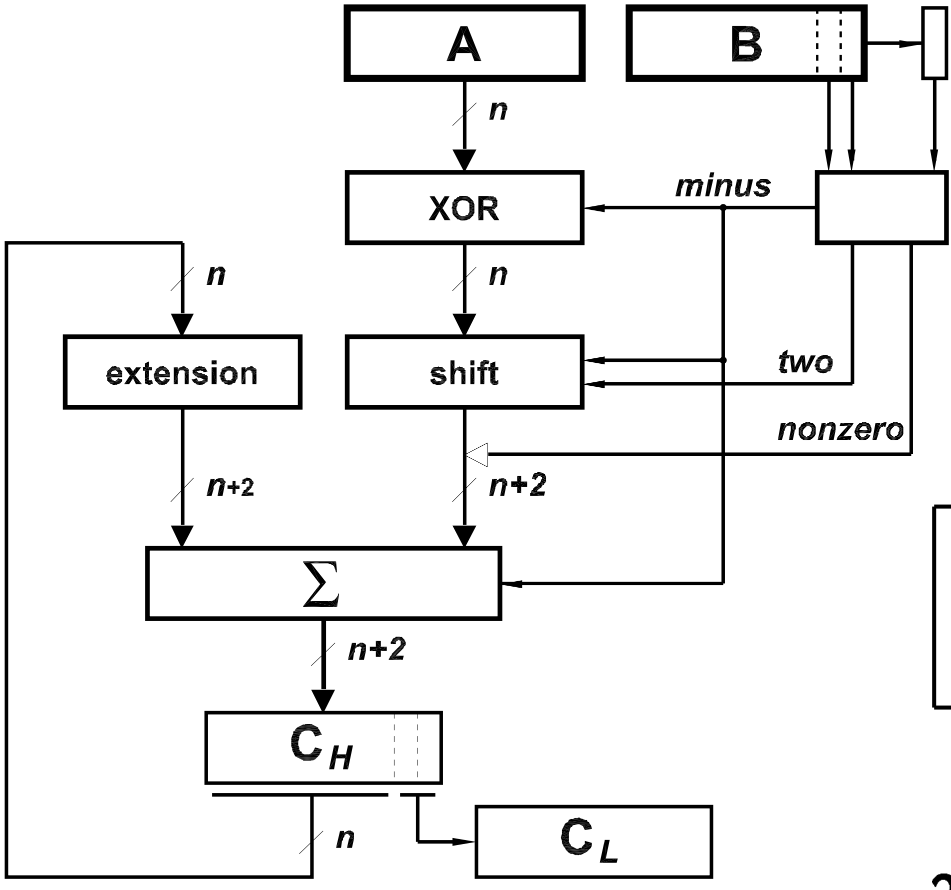
$= \mathbf{1\,0\,\widehat{2}\,2\,\widehat{1}}_{\,4\pm}$

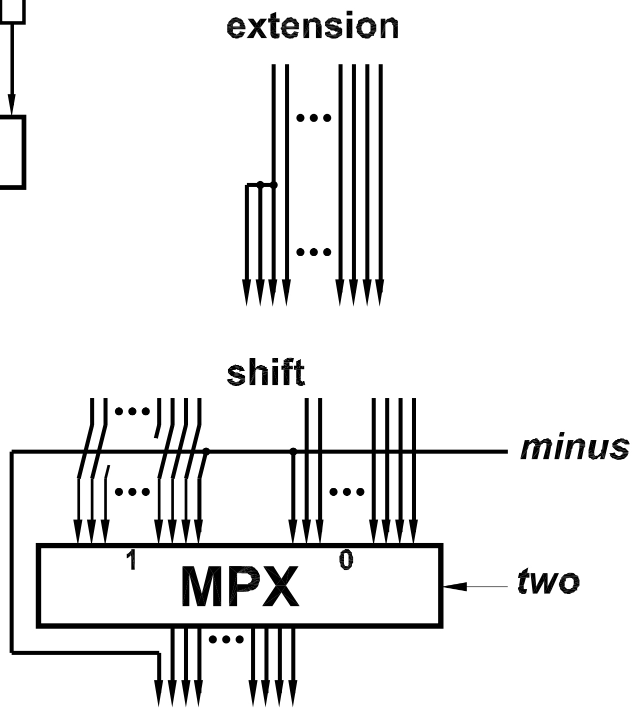
1 1	1 1	1 1	1 1	**0 0**	**1 1**	**1 0**	**0 1**	$\widehat{1}$
							1	
0 0	0 0	0 **1**	**1 0**	**0 0**	**1 1**	**0**		**2**
1 1	1 **0**	**0 1**	**1 1**	**0 0**	**1**			$\widehat{2}$
					1			
0 0	**0 0**	**0 0**	**0 0**	**0 0**				**0**
1 1	**0 0**	**0 1**	**1 0**					**1**
1 0	**1 1**	**0 0**	**1 0**	**1 0**	**1 0**	**1 0**	**1 0**	

$A \cdot B = \mathbf{1011\ 0010\ 1010\ 1010}_{\,2} = \mathbf{45\,738}_{\,10}$

note.: **addition and format extensions can be performed in successive steps**

Modified Booth method (unsigned numbers) *ii*





	minus	*two*	*nonzero*
2	0	1	1
1	0	0	1
0	x	x	0
−1	1	0	1
−2	1	1	1

2's complement multiplication

format: $\mathcal{Z} = z^{n+1}$, $\varepsilon = 1$

b_i	b_{i-1}	β_i
0	0	0
0	1	1
1	0	$\widehat{1}$
1	1	0

$$\underbrace{b_n \dots b_0}_{X} \longrightarrow \underbrace{\beta_n \dots \beta_0}_{Y}$$

$$\boxed{X = \mathcal{D}(B)}$$

1. $b_n = 0 \;\Rightarrow\; B \geq 0 \;\Rightarrow\; \boxed{Y = B}$

2. $b_n = 1 \;\Rightarrow\; B < 0 \;\Rightarrow\; X = \mathcal{Z} + B$
   $\beta_{n+1} = 1$ ... out of format
   is omitted $\Rightarrow Y = X - \mathcal{Z} \;\Rightarrow\; \boxed{Y = B}$

Ex.: $X = \mathcal{D}(B) = 1011_2 \;\Rightarrow\; B = -5_{10}$

$1011 \longrightarrow \widehat{1}\,1\,0\,\widehat{1} \;\Rightarrow\; Y = -5_{10}$

Booth method:

Example: $A = -11_2 \quad B = -010_2 \quad A \cdot B = 110_2$

$\mathcal{D}(A) = 101 \quad \mathcal{D}(B) = 110 \quad \mathcal{D}'(A \cdot B) = 000\,110$

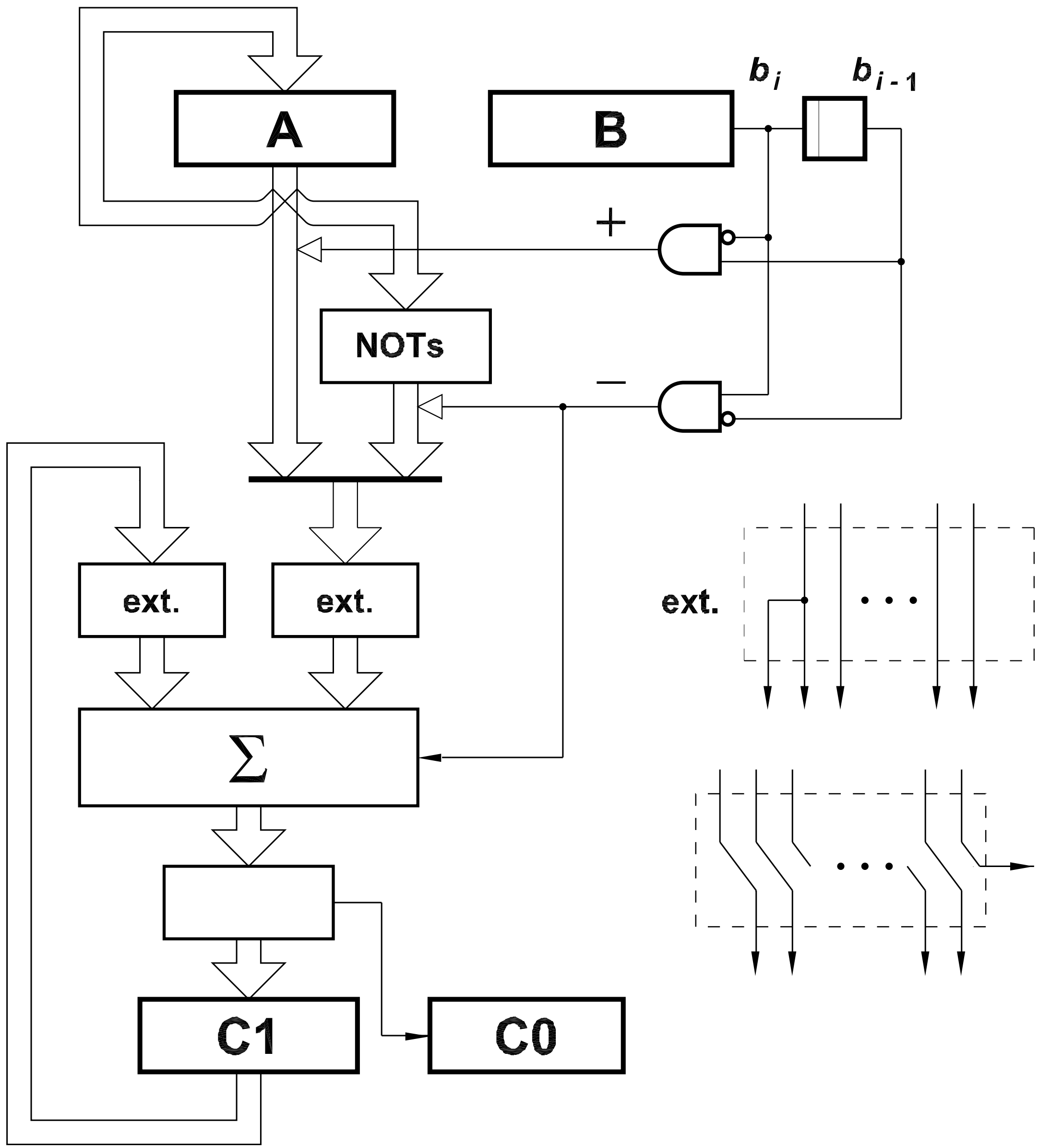
$$B = -010 = 0\hat{1}0$$

$$
\begin{array}{lcl}
1\;1\;1\;\mathbf{1\;0\;1}\cdot\mathbf{0\;\hat{1}\;0} & & \\ \hline
0\;0\;0\;0\;0\;0 & & \\
0\;0\;\mathbf{0\;0\;0\;0} \qquad\quad \mathbf{0} & & \\ \hline
0\;0\;\mathbf{0\;0\;0}\;\boldsymbol{\mathit{0}} & & \\
0\;\mathbf{0\;0\;1\;0} \qquad\quad \hat{1} & & \text{negation} \\
\qquad\quad\;\;\mathbf{1} & & \text{hot 1} \\ \hline
0\;\mathbf{0\;0\;1}\;\boldsymbol{\mathit{1}} & & \\
\mathbf{0\;0\;0\;0} \qquad\quad \mathbf{0} & & \\ \hline
\mathbf{0\;0\;0}\;\boldsymbol{\mathit{1}} & &
\end{array}
$$

4bit adder is used:

$$
\begin{array}{lcl}
1\;\mathbf{1\;0\;1}\cdot\mathbf{0\;\hat{1}\;0} & & \\ \hline
0\;0\;0\;0 & & \\
\mathbf{0\;0\;0\;0} \qquad\quad \mathbf{0} & & \\ \hline
0\;\mathbf{0\;0\;0}\;\boldsymbol{\mathit{0}} & & \\
\mathbf{0\;0\;1\;0} \qquad\quad \hat{1} & & \text{negation} \\
\qquad\quad\;\;\mathbf{1} & & \text{hot 1} \\ \hline
0\;\mathbf{0\;0\;1}\;\boldsymbol{\mathit{1}} & & \\
\mathbf{0\;0\;0\;0} \qquad\quad \mathbf{0} & & \\ \hline
\mathbf{0\;0\;0}\;\boldsymbol{\mathit{1}} & &
\end{array}
$$

2's complement multiplication iii



another way:

$$\mathcal{D}(B) = \sum_{i=0}^{n} b_i^{\mathcal{D}} \cdot 2^i = B' + B'',$$

$$\text{where} \quad B' = b_n^{\mathcal{D}} \cdot 2^n \quad \text{a} \quad B'' = \sum_{i=0}^{n-1} b_i^{\mathcal{D}} \cdot 2^i$$

only the B' is converted $\begin{cases} 10\cdots0 & \longrightarrow \widehat{1}0\cdots0 \\ 00\cdots0 & \longrightarrow 00\cdots0 \end{cases}$

Therefore: The first bit $\mathcal{D}(B)$ from left (it means. $b_n^{\mathcal{D}}$) has oposite weight, it means weight -2^n (not weight $+2^n$).

Conclusion: **everything is same as for unsigned numbers, but in the last step the subtraction operation** (0 or 1 multiple) **is processed instead of addition.**

Ex.: $X = \mathcal{D}(B) = 1011_2 \;\Rightarrow\; B = -5_{10}$

$1\,0\,1\,1 \longrightarrow \widehat{1}\,0\,1\,1 \qquad = -5_{10}$

modified Booth method

— convertion of multiplier to quaternary signed-digit number system

ex.: $A = -\mathbf{58}_{10}$ $\quad \mathcal{D}(A) = \mathbf{11\ 00\ 01\ 10}_2$

$B = -\mathbf{25}_{10}$ $\quad \mathcal{D}(B) = \mathbf{11\ 10\ 01\ 11}_2 = \mathbf{3\,2\,1\,3}\,\mathbf{,}\,0_4$

$B =$ $\quad \mathbf{0}\,\widehat{2}\,\mathbf{2}\,\widehat{1}_{\,4\pm}$

0 0	0 0	0 0	0 0	**0 0**	**1 1**	**1 0**	**0 1**	**-1**
							1	
1 1	1 1	1 **1**	**1 0**	**0 0**	**1 1**	**0**		**2**
0 0	0 **0**	**0 1**	**1 1**	**0 0**	**1**			**-2**
					1			
0 0	**0 0**	**0 0**	**0 0**	**0 0**				**0**
0 0	**0 0**	**0 1**	**0 1**	**1 0**	**1 0**	**1 0**	**1 0**	

$$A \cdot B = \mathbf{1\,450}_{10} = \mathbf{0000\ 0101\ 1010\ 1010}_2$$

$$= \mathcal{D}(A \cdot B)$$

Sign and magnitude multiplication

sign:

$+$	$+$	$+$
$+$	$-$	$-$
$-$	$+$	$-$
$-$	$-$	$+$

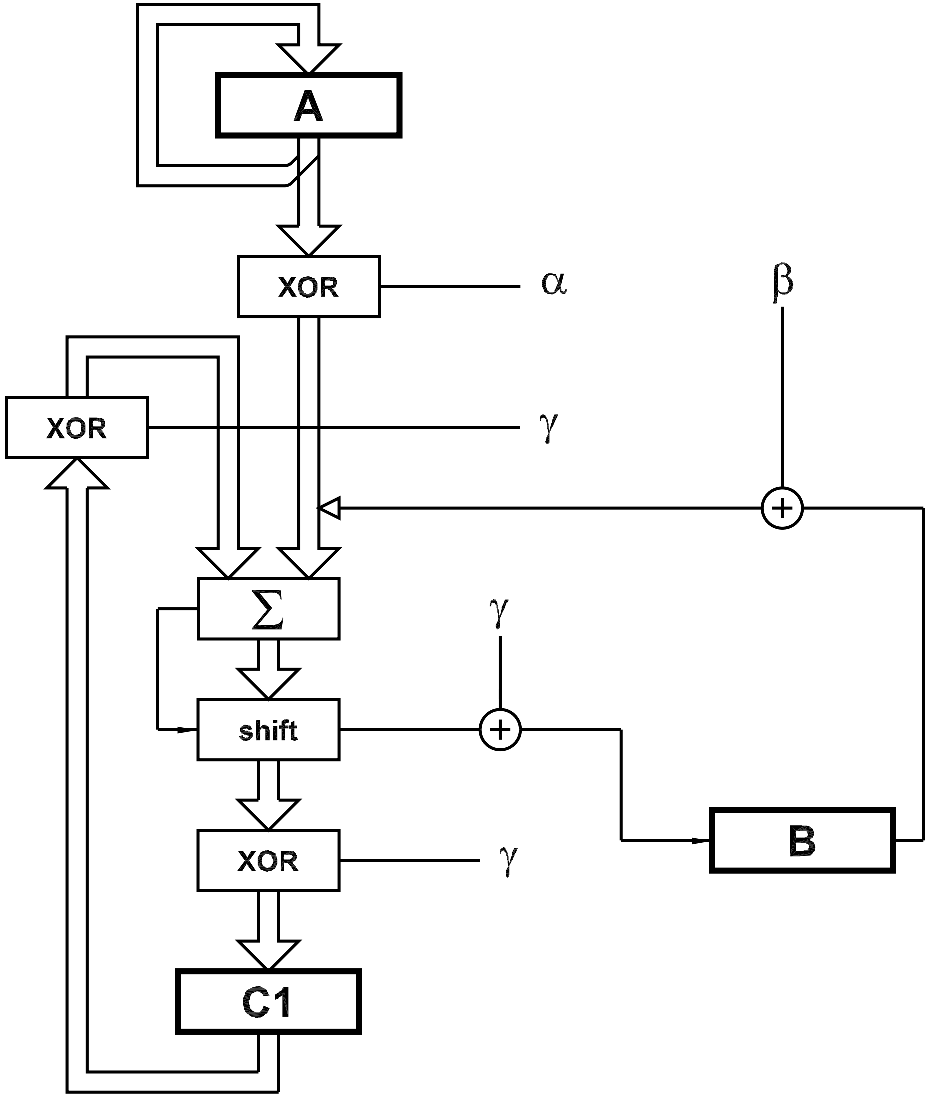
$\Rightarrow$

0	**0**	**0**
0	**1**	**1**
1	**0**	**1**
1	**1**	**0**

$\Rightarrow$ $\boxed{\textbf{XOR}}$

magnitude — **non-negative number** alias
unsigned number
$\Rightarrow$ **multiplication of unsigned numbers**
(and it is known)

1's complement mulitplication



α ... **original MSB of first operand**

β ... **original MSB of second operand**

$$\gamma = \alpha \oplus \beta$$