MI-ARI

(Computer arithmetics)

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CS. Number systems and basic operations

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CS. Number systems and basic operations

- Number systems
- Number formats
  - Basic operation in general format
- Binary adders
- Format respecting addition
- Subtractor
- Subtractor using adder
- Signed number representation
  - Sign and magnitude
  - Radix complement
  - Diminished radix complement
  - Biased code
    - Type 0 of biased code
    - Type 1 of biased code

Number systems

number systems

- **position number systems**
  - **standard**
    - **decimal**
    - **binary (or dyadic)**
    - **octal**
    - **hexadecimal**
    - **ternary (or triadic)**

    etc.
  - **non-standard number systems**
    - **negative radix number systems** (Polish system)
    - **signed digit number systems**
    - **multiple radix number systems**

    and others.
- **non-position number systems**
  - **so caller Roman numerals**
  - **residue number systems (RNS),** (Czech system)

  etc.

Standard number system –position number system

$$A \sim a_n a_{n-1} \ldots a_0, a_{-1} \ldots a_{-m}$$

$$\boxed{A = \sum_{i=-m}^{n} a_i z^i}$$

$$\boxed{z \geq 2}$$

radix (or **base) of system — natural number**

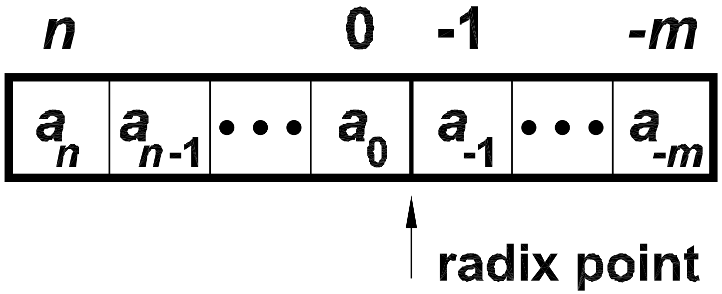
$$\boxed{a_i \in \langle 0;\ z)}$$

digits — non-negative integers

$$\boxed{0 \leq A \leq z^{n+1} - z^{-m}}$$

!!! Negative numbers can not be represented !!!

Number formats



n ... **highest position**
$-m$ ... **lowest position**

$A \sim \mathbf{a}_n \mathbf{a}_{n-1} \ldots \mathbf{a}_0, \mathbf{a}_{-1} \ldots \mathbf{a}_{-m}$
$A = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 + a_{-1} z^{-1} \ldots a_{-m} z^{-m}$
z ... **radix** (or base) of the number system

$\mathcal{Z} = z^{n+1}$ **module of the format** — it is out of format
$\varepsilon = z^{-m}$ **unit of the format** — smallest positive number in the format

representable numbers A: $\boxed{0 \leq A = k \cdot \varepsilon < \mathcal{Z}}$,
k **is an integer**

$k = A/\varepsilon = A^* \implies \boxed{A = A^* \text{ of units } \varepsilon}$

A^* ☞ **digit number at notation of** A
ε ☞ **radix point position**

Addition and subtraction

the same format of both operands and the result:

$$\left.\begin{aligned} A &= A^*\cdot\varepsilon \\ B &= B^*\cdot\varepsilon \end{aligned}\right\} \Rightarrow\ A \pm B = A^*\cdot\varepsilon \pm B^*\cdot\varepsilon = (A^* \pm B^*)\cdot\varepsilon$$

Ex.: $z = 10,\ \mathcal{Z} = 10 = 10^{n+1},\ \varepsilon = 0,01 = 10^{-m}$

$n = 0,\ m = 2$ (or $-m = -2$)

$A = 1,23 \quad \Rightarrow \quad A^* = 1,23/0,01 = 123$

$B = 4,56 \quad \Rightarrow \quad B^* = 4,56/0,01 = 456$

$A = 1,23 + 4,56 = (123 + 456)\cdot 0,01 =$

$= 579 \cdot 0,01 = 5,79$

different formats:

transformation numbers into suitable format — zeroes adding

Ex.: **1,234+56,7 = 01,234+56,700 = 57,934**

Conclusion: Addition a subtraction (in general format) **can be easy transformed on addition a subtraction of integers.**

Multiplication:

$$\left.\begin{array}{l} A = A^{*} \cdot \varepsilon_A \\ B = B^{*} \cdot \varepsilon_B \end{array}\right\} \Rightarrow \qquad \begin{aligned} A \cdot B &= A^{*} \cdot \varepsilon_A \cdot B^{*} \cdot \varepsilon_B = \\ &= (A^{*} \cdot B^{*}) \cdot \varepsilon_A \cdot \varepsilon_B \end{aligned}$$

Ex.: $z = 10$

$$\mathcal{Z}_A = 10, \quad \varepsilon_A = 0,01, \; n_A = 0, \; m_A = 2$$
$$\mathcal{Z}_B = 100, \; \varepsilon_B = 0,1, \quad n_B = 1, \; m_B = 1$$
$$\begin{aligned} 7,01 \cdot 80,3 &= (701 \cdot 803) \cdot 0,001 = \\ &= 562\,903 \cdot 0,001 = 562,903 \end{aligned}$$

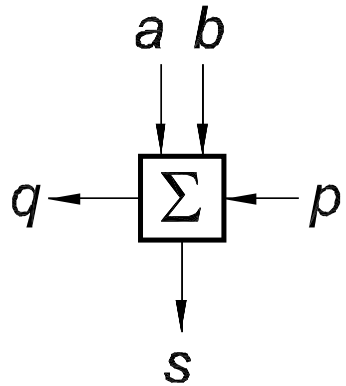
Conclusion: Multiplication(in general format) **can be easy transformed on addition a subtraction of integers.**

Full-adder

a	b	p	q	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$\begin{aligned} s &= a \oplus b \oplus p = \\ &= \overline{a}\,\overline{b}p + \overline{a}b\overline{p} + a\overline{b}\,\overline{p} + abp \end{aligned}$$

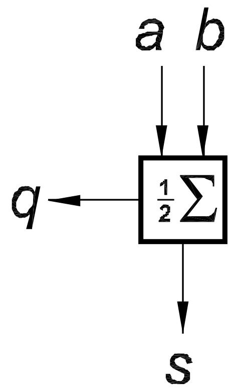
$$\begin{aligned} q &= \mathsf{M}_3(a, b, p) = \\ &= ab + ap + bp = \\ &= ab \oplus ap \oplus bp = \\ &= ab + (ap \oplus bp) \end{aligned}$$

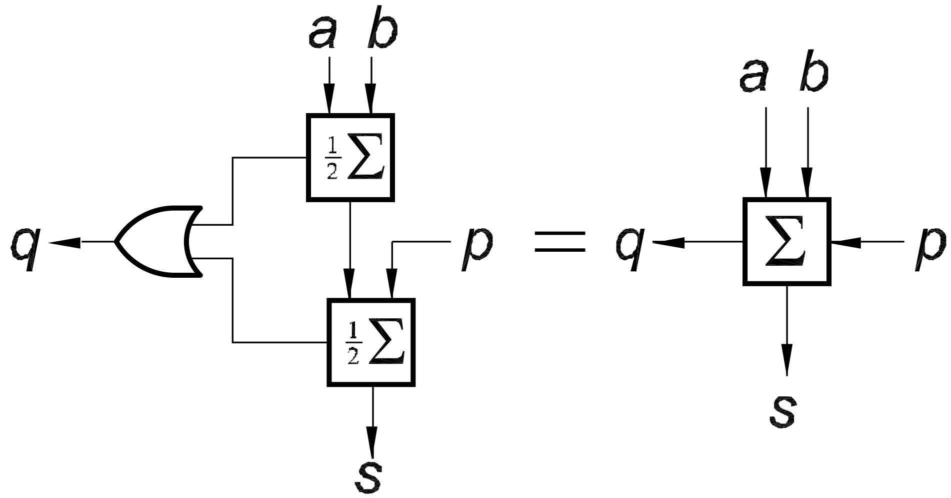


Half-adder

a	b	q	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

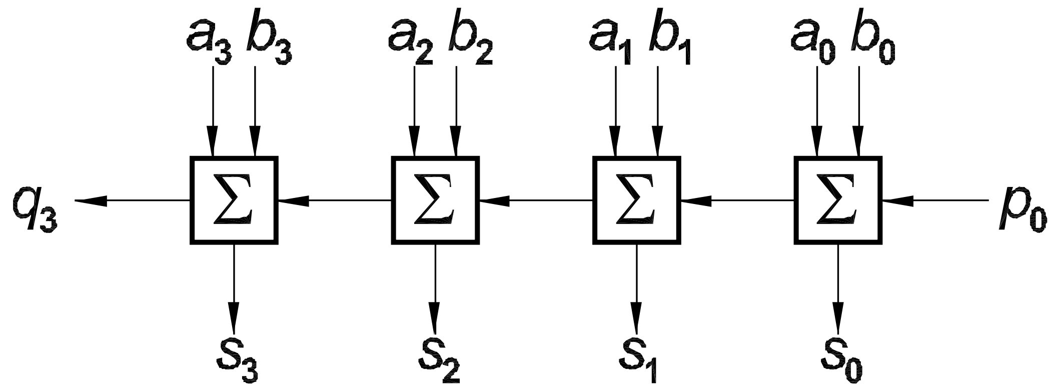
$$\begin{aligned} s &= a \oplus b \\ &= \overline{a}b + a\overline{b} \\ q &= a \cdot b \end{aligned}$$





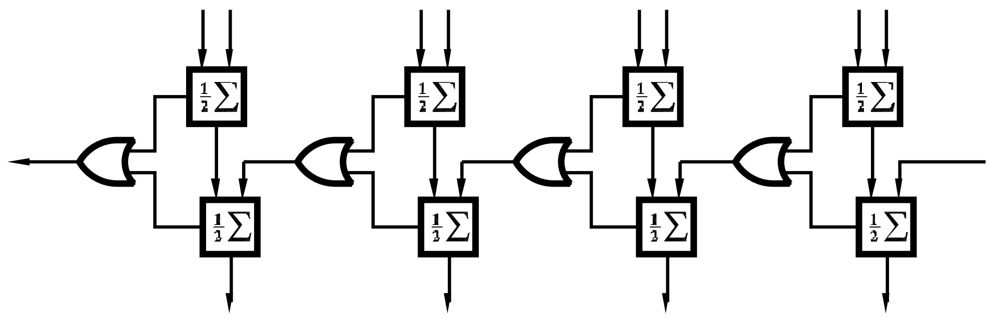
Ripple-carry adder
(also $\boldsymbol{parallel\ adder}$**)**

$\boldsymbol{n = 3}$
$\boldsymbol{\mathcal{Z} = 16}$
$\boldsymbol{A \sim a_3 a_2 a_1 a_0}$
$\boldsymbol{B \sim b_3 b_2 b_1 b_0}$
$\boldsymbol{S \sim s_3 s_2 s_1 s_0}$
$\boldsymbol{p_{i+1} = q_i}$



$$\boldsymbol{S = A + B + p_0 - q_n \cdot \mathcal{Z}}$$

the same using half-adders



Format respecting addition

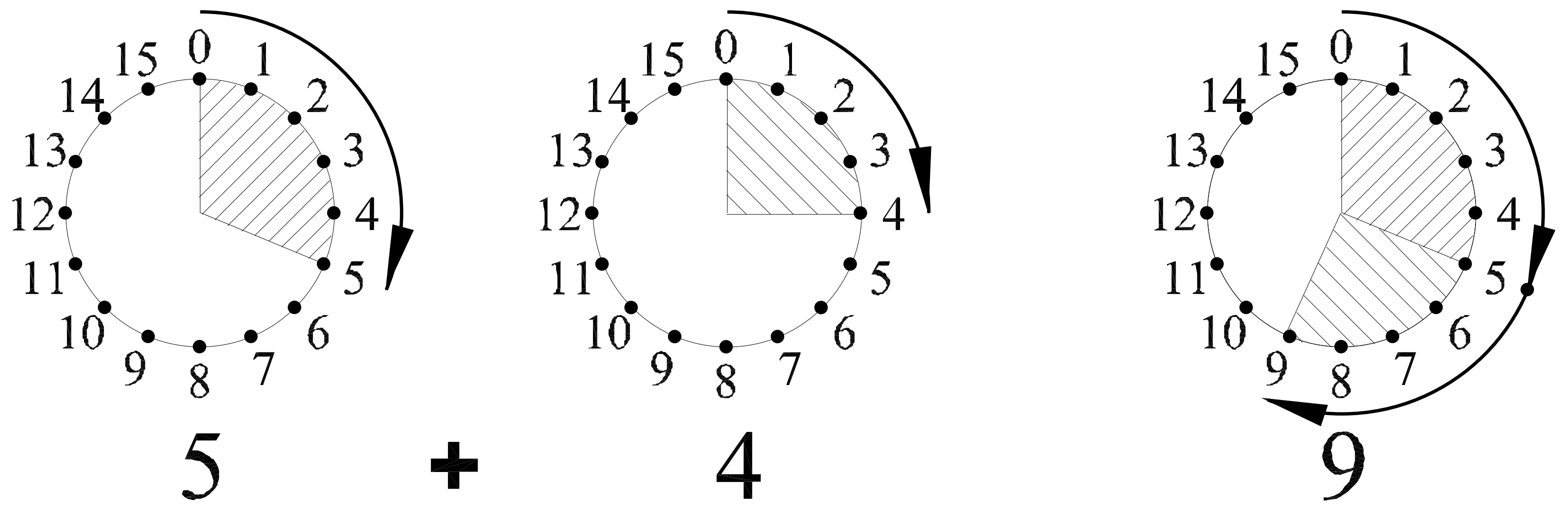
Output of adder: $S = A + B + p_0 - q_n \cdot \mathcal{Z}$
Let $p_0 = 0$
(or half adder is used in zero order of adder):

$$\boxed{S = A + B - q_n \cdot \mathcal{Z}}$$

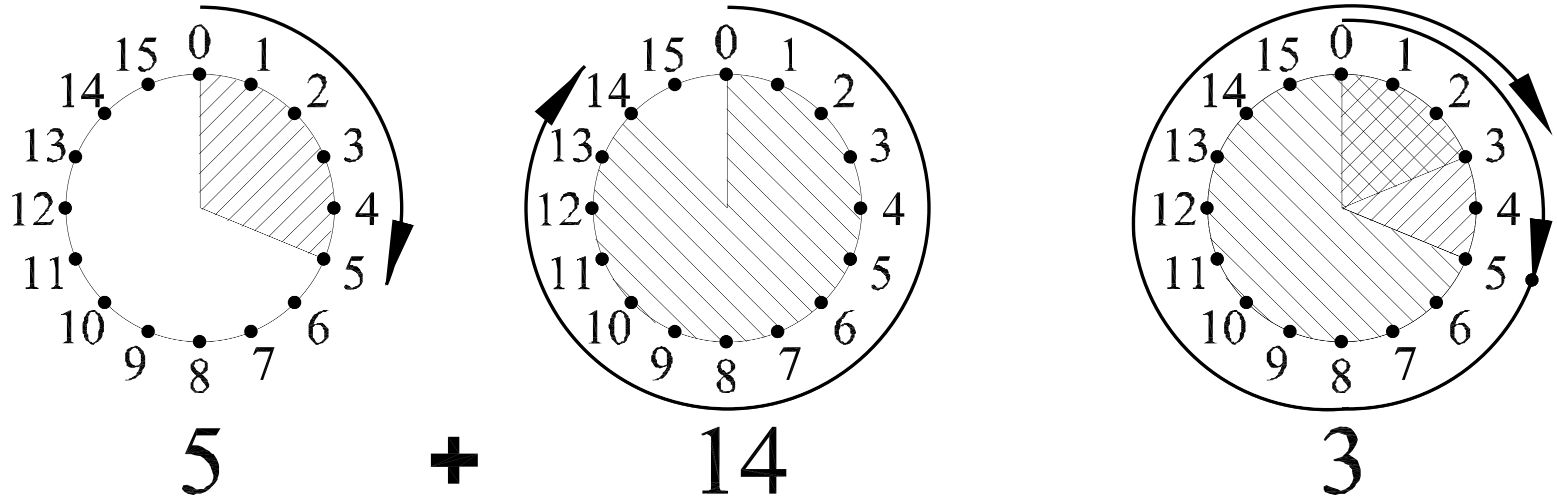
S **differ from** $A + B$ **by multiple of** $\mathcal{Z}$ **so that** $\mapsto$
$S \equiv A + B \pmod{\mathcal{Z}}$

graphic view (analogy of clock face):

$$0101 + 0100 = {}_0\,1001 \rightarrow 1001$$



$$\mathbf{0101 + 1110 = {}_1 0011 \rightarrow 0011}$$



$\boxed{\textbf{passing thru} \quad \Leftrightarrow \quad \text{carry } \textbf{from the highest order } q_n = 1}$

in this case (additon using unsigned numbers):

$q_n = 1 \quad \Rightarrow \quad A + B \geq \mathcal{Z} \qquad (\mathcal{Z} = 1\,0000_2 = 16_{10})$

$q_n = 1 \quad \Rightarrow \quad$ overflow $\sim$ **the result is out of format**

however generally: $\boxed{\text{!!! carry} \neq \text{overflow !!!}}$

Subtractor

Full adder (! *How to do it in wrong way.* !)

a	b	v	u	r
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	1	0
1	0	0	0	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

$$r = a \oplus b \oplus v =$$
$$= \overline{a}\overline{b}v + \overline{a}b\overline{v} + a\overline{b}\overline{v} + abv$$

$$u = = \overline{a}b + \overline{a}v + bv$$

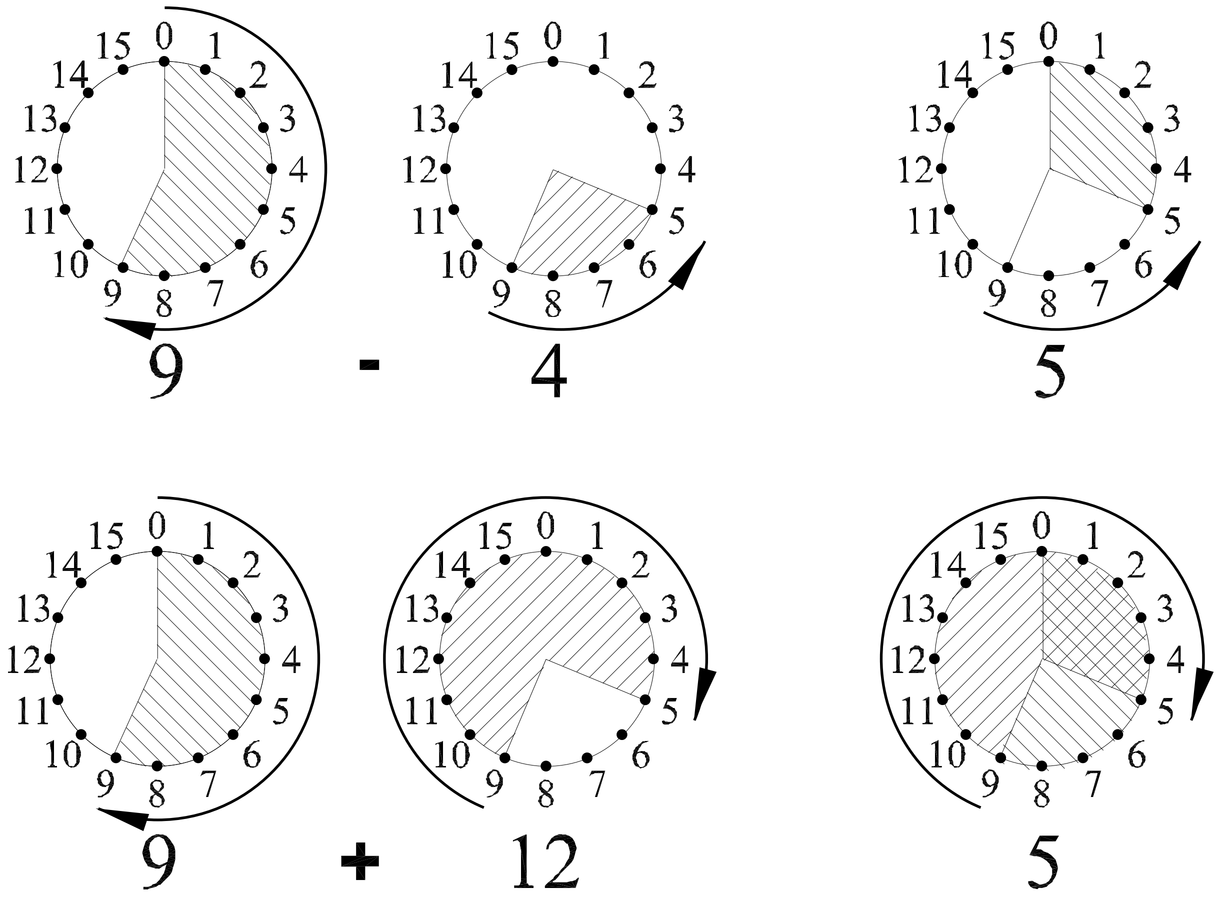
v_i borrow for order i
u_i borrof from order i

etc. - similarly as for addition

Is it possible modify an adder for subtraction?

Subtractor using adder

$$\boxed{A - B \equiv A + (\mathcal{Z} - B) \pmod{\mathcal{Z}}}$$



How to find $\mathcal{Z}-B$?

$$X = \sum_{i=0}^{n} x_i z^i \qquad x_i \in \langle 0, z-1 \rangle$$

$$X_{max} = \sum_{i=0}^{n} (z-1) z^i = \sum_{j=1}^{n+1} z^j - \sum_{i=0}^{n} z^i = z^{n+1} - 1 =$$

$$= \mathcal{Z} - 1$$

$z = 2$: $\quad X_{max} = \boxed{11\ldots11} = \mathcal{Z} - 1$ $\qquad \boxed{\mathrm{xx}\ldots\mathrm{xx}}$ — module format

$$\mathcal{Z} = \boxed{11\ldots11} + 1$$

$$\mathcal{Z} - B = \boxed{11\ldots11} - B + 1$$

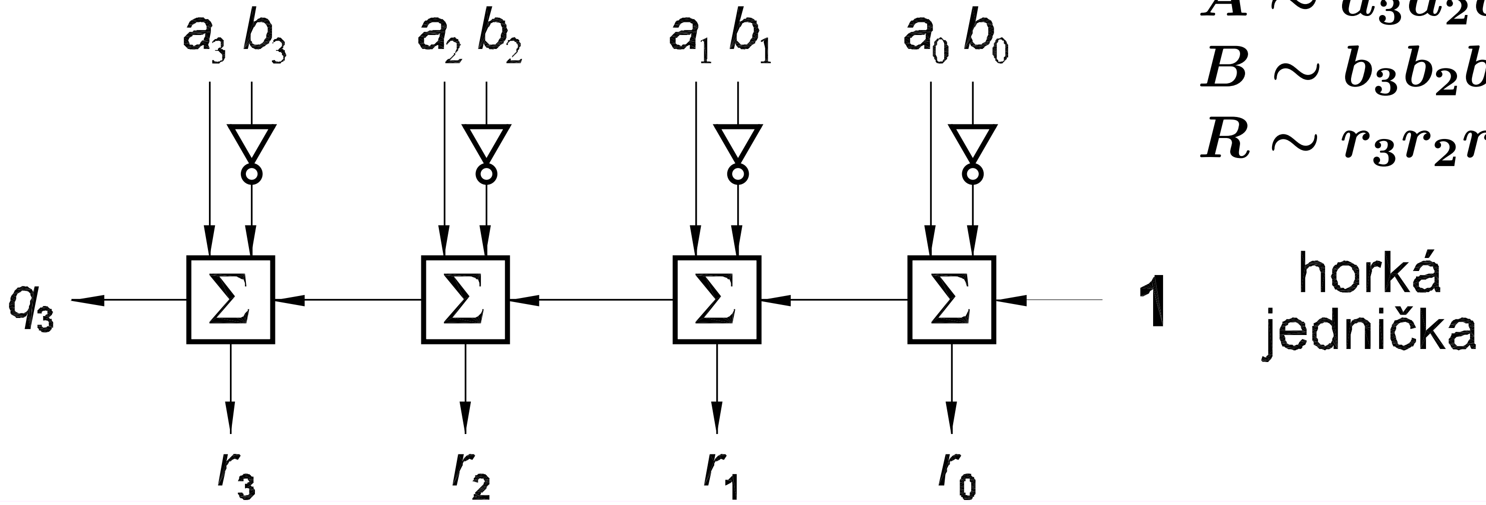
$$\boxed{\boxed{\mathcal{Z} - B = \overline{B} + 1}}$$

$\overline{B}$ **. . . negation of all bits**

$\ldots + 1$ — **so called** hot one

note.: $B = 0 \Rightarrow \mathcal{Z} - B = \mathcal{Z} \equiv 0 \pmod{\mathcal{Z}}$

$$\overline{B} + 1 = \boxed{11\ldots11} + 1 = 1\boxed{00\ldots00}$$



$$A \sim a_3a_2a_1a_0$$
$$B \sim b_3b_2b_1b_0$$
$$R \sim r_3r_2r_1r_0$$

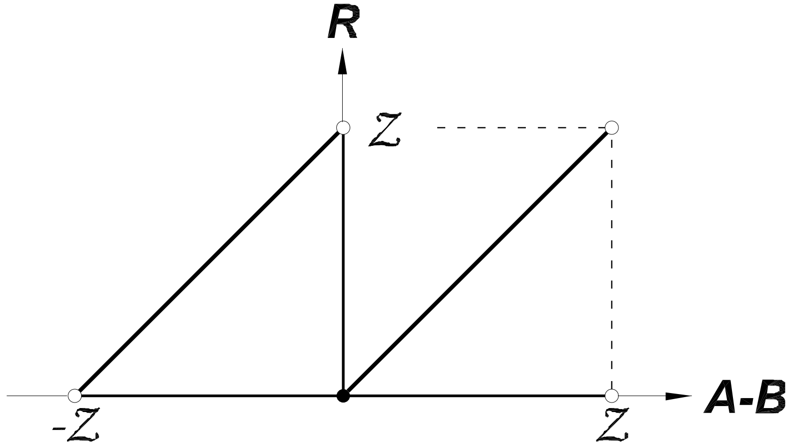
$$R = A + (\mathcal{Z} - B) - q_n \cdot \mathcal{Z} = A - B + (1 - q_n) \cdot \mathcal{Z}$$

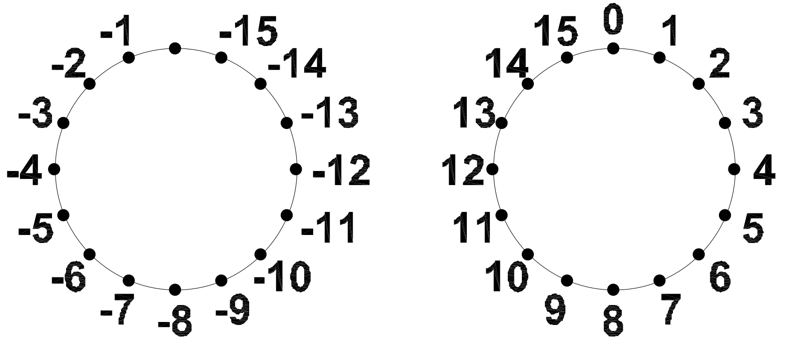
$$R = A - B + \overline{q_n} \cdot \mathcal{Z} \qquad 0 \leq R < \mathcal{Z}$$

$$q_n = 1 \quad \Rightarrow \quad R = A - B \geq 0$$
$$q_n = 0 \quad \Rightarrow \quad R = A - B + \mathcal{Z} < \mathcal{Z} \quad \Rightarrow A - B < 0$$

$$\boxed{\begin{array}{llll} q_n = 1 & \Rightarrow & A \geq B & R = A - B \\ q_n = 0 & \Rightarrow & A < B & R = \mathcal{Z} - (B - A) \end{array}}$$

complementary pseudocode





$q_n = 0$ $\qquad$ $q_n = 1$

If it is $q_n = 0$, **it means** $B > A$, **then**

$$R = \mathcal{Z} - (B - A) \quad \Rightarrow \quad B - A = \mathcal{Z} - R$$

$$\Rightarrow \quad \boxed{B - A = \overline{R} + 1}$$

Signed number representation

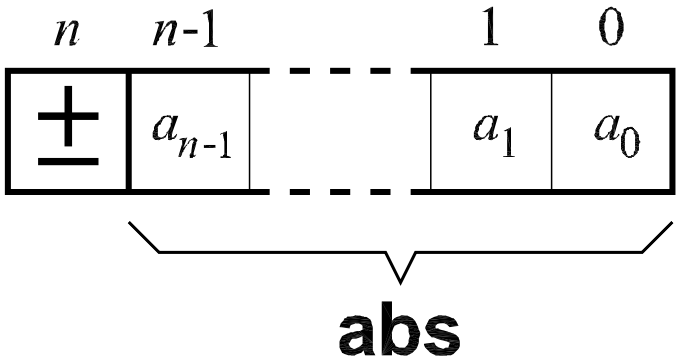
5 ways to represent
negative (as well as non-negative) number X:

1. **sign and magnitude** ... $\mathcal{P}(X)$
2. **radix complement** ... $\mathcal{D}(X)$
   2's complement - $z = 2$
   10's complement - $z = 10$
3. **diminished radix complement** ... $\mathcal{I}(X)$
   1's complement - $z = 2$
   9's complement - $z = 10$
4. **biased code (**or **excess K)** ... $\mathcal{A}(X)$
   a. **type 0** ... $\mathcal{A}_0(X)$
   b. **type 1** ... $\mathcal{A}_1(X)$

Sign and magnitude

sign & abs value (magnitude):

z=2:

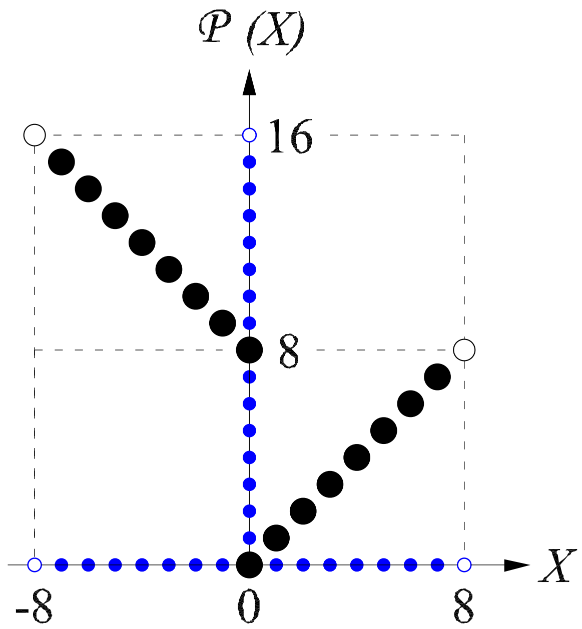


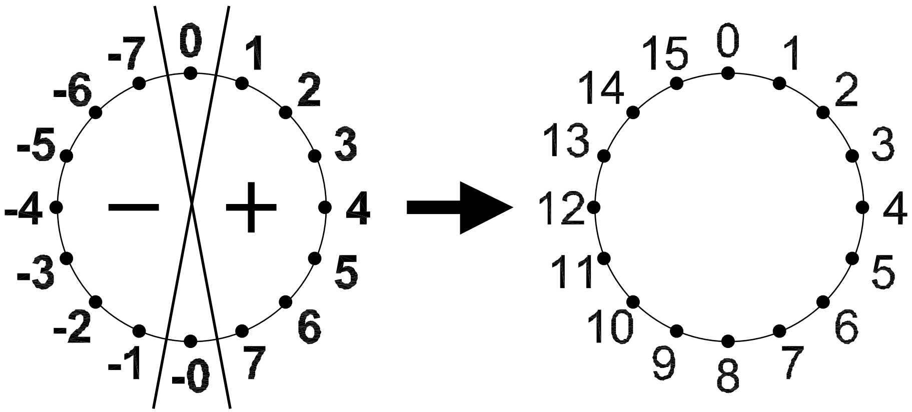
$$\text{sign bit} = \begin{cases} \mathbf{0} & \text{for } X \geq 0 \\ \mathbf{1} & \text{for } X \leq 0 \end{cases}$$

$$\boxed{\mathcal{P}(X) = \begin{cases} X & \text{for } X \geq 0 \\ 2^n + |X| & \text{for } X \leq 0 \end{cases}}$$

$-\frac{1}{2}\mathcal{Z} < X < \frac{1}{2}\mathcal{Z}$ ☞ symmetric range

2 zero representation: $\begin{cases} \text{so called „positive zero“} & 0\ 0\ldots0 \\ \text{so called „negative zero“} & 1\ 0\ldots0 \end{cases}$





X	$\mathcal{P}(X)$
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-0	1000
-1	1001
-2	1010
-3	1011
-4	1100
-5	1101
-6	1110
-7	1111

addition :

$$\boxed{(zA, aA) + (zB, aB) \rightarrow (zC, aC)} ,$$

where zA, zB and zC are the sign bits and aA, aB and aC are the magnitudes

$$
\boxed{
\begin{array}{lll}
\textbf{if (zA=zB)} & \{\textbf{aA+aB} \rightarrow \textbf{aC;} & \\
 & \ \textbf{zA} \rightarrow \textbf{zC;} & \\
 & \ \textbf{if (q = 1) overflow; }\} & \\
\textbf{else} & \{\textbf{aA+}\overline{\textbf{aB}}\textbf{+1} \rightarrow \textbf{aC;} & \\
 & \ \textbf{zA} \rightarrow \textbf{zC;} & \\
 & \ \textbf{if (q = 0)} & \{\overline{\textbf{aC}}\textbf{+1} \rightarrow \textbf{aC;} \\
 & & \ \overline{\textbf{zC}} \rightarrow \textbf{zC; }\}\ \}
\end{array}
}
$$

q carry-out from higher order

sign change:

$$\boxed{\mathcal{P}(-X) \to \mathcal{P}(X) \quad \Longleftrightarrow \quad \overline{\mathsf{MSB}} \to \mathsf{MSB}}$$

MSB - bit in higher order (first from left)
- Most Significant Bit

subtraction:

$$\boxed{A - B = A + (-B)}$$

to swap $\overline{zB}$ by zB

else alike the addition

absolute value:

$$\boxed{\mathcal{P}(X) \to \mathcal{P}(\,|X|\,) \quad \Longleftrightarrow \quad 0 \to \mathsf{MSB}}$$

Radix complement

radix complement $\begin{cases} \text{2's complement for } z = 2 \\ \text{10's complement for } z = 10 \end{cases}$

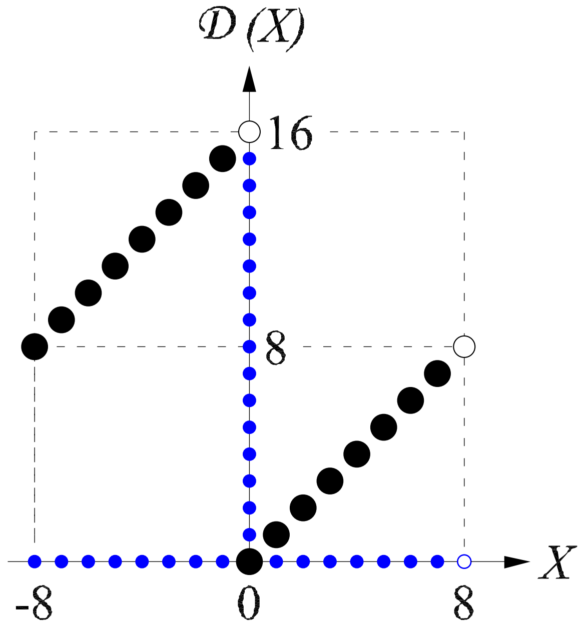
$$\boxed{\mathcal{D}(X) = \begin{cases} X & \text{pro } X \geq 0 \\ \mathcal{Z} + X = \mathcal{Z} - |X| & \text{pro } X < 0 \end{cases}}$$

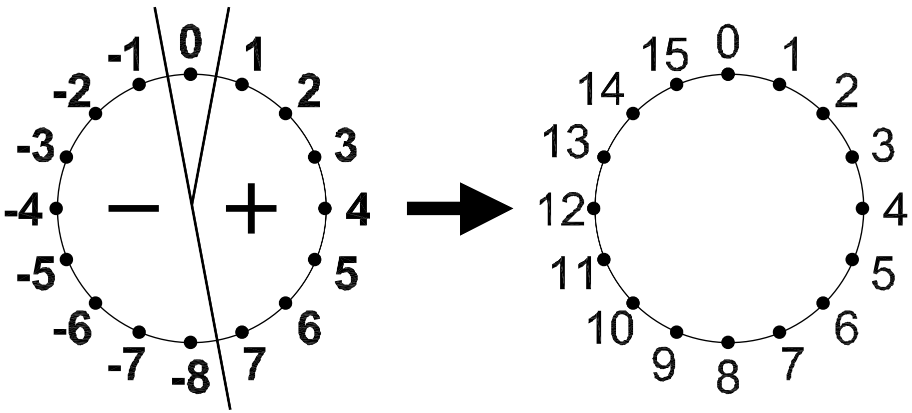
$-\frac{1}{2}\mathcal{Z} \leq X < \frac{1}{2}\mathcal{Z}$ ☞ asymmetric range

$$\boxed{\begin{array}{lcr} \text{MSB} = 0 & \Longleftrightarrow & X \geq 0 \\ \text{MSB} = 1 & \Longleftrightarrow & X < 0 \end{array}}$$

$$\mathcal{D}(X) \equiv X \pmod{\mathcal{Z}}$$

z=2:





X	$\mathcal{D}(X)$
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-8	1000
-7	1001
-6	1010
-5	1011
-4	1100
-3	1101
-2	1110
-1	1111

addition: $\mathcal{D}(A+B) \equiv A+B \equiv \mathcal{D}(A) + \mathcal{D}(B) \pmod{\mathcal{Z}}$

☞ **add up** $\mathcal{D}(A) + \mathcal{D}(B)$ **and ignore carry** q_n **from higher order**

subtraction: $\mathcal{D}(A-B) \equiv A - B \equiv \mathcal{D}(A) - \mathcal{D}(B) \pmod{\mathcal{Z}}$

☞ **subtract** $\mathcal{D}(B)$ **from** $\mathcal{D}(A)$ **and ignore borrow** v_n **from higher order**

or

☞ **convert subtraction on addition, that is add up** $\mathcal{D}(A) + \overline{\mathcal{D}(B)} + 1$ **and ignore carry** q_n **from higher order**

Carry (eventually borrow) **is ignored.**

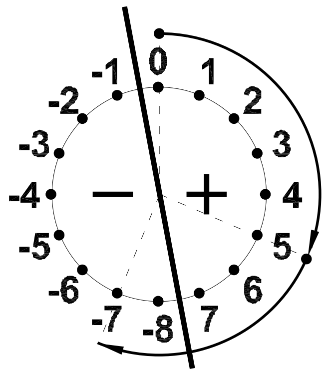
? **How to detect overflow?** **(that is** $A+B \geq \frac{1}{2}\mathcal{Z}$ **or** $A+B < -\frac{1}{2}\mathcal{Z}$**)** **?**

overflow during addition in 2's complement code:

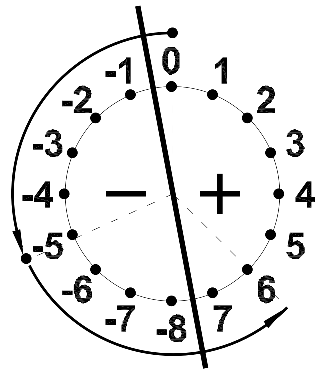
1. $A < 0$ a $B \geq 0 \;\Rightarrow\; A \;\leq A + B < B$
   $B < 0$ a $A \geq 0 \;\Rightarrow\; B \leq A + B < A$

   ! <u>in this case overflow can not occur</u> **!**

2. $A \geq 0$ a $B \geq 0$
   overflow: $A \; + B \geq \frac{1}{2}\mathcal{Z}$
   result has opposite sign
   see example $5 + 4 \rightarrow -7$



3. $A < 0$ a $B < 0$
   overflow: $A + B < -\frac{1}{2}\mathcal{Z}$
   result has oposite sign
   see example $(-5) + (-5) \rightarrow 6$



overflow during addition: **same sign of both operands and opposite sign of result**

$$\boxed{+}\ \boxed{+} \rightarrow \boxed{-}$$
$$\boxed{-}\ \boxed{-} \rightarrow \boxed{+}$$

overflow during addition:

subtraction converted on addition

$\Rightarrow$ **There is nothing to solve.**

will be referred to: $\mathcal{D}(A) + \mathcal{D}(B) - \, q_n \cdot \mathcal{Z} = S$

$\mathcal{D}(A) \sim a_n^{\mathcal{D}} a_{n-1}^{\mathcal{D}} \ldots a_1^{\mathcal{D}} a_0^{\mathcal{D}}$

$\mathcal{D}(B) \sim b_n^{\mathcal{D}} b_{n-1}^{\mathcal{D}} \ldots b_1^{\mathcal{D}} b_0^{\mathcal{D}}$

$S \ \sim s_n^{\mathcal{D}} s_{n-1}^{\mathcal{D}} \ldots s_1^{\mathcal{D}} s_0^{\mathcal{D}}$

$over$ — **overflow**

detection of overflow:

$\boxed{1}$ $a_n^{\mathcal{D}} = b_n^{\mathcal{D}} = 0$ **a** $s_n^{\mathcal{D}} = 1$ **or**

$a_n^{\mathcal{D}} = b_n^{\mathcal{D}} = 1$ **a** $s_n^{\mathcal{D}} = 0$

$$\boxed{over = \overline{a_n^{\mathcal{D}}} \cdot \overline{b_n^{\mathcal{D}}} \cdot s_n^{\mathcal{D}} + a_n^{\mathcal{D}} \cdot b_n^{\mathcal{D}} \cdot \overline{s_n^{\mathcal{D}}}}$$

$\boxed{2}$

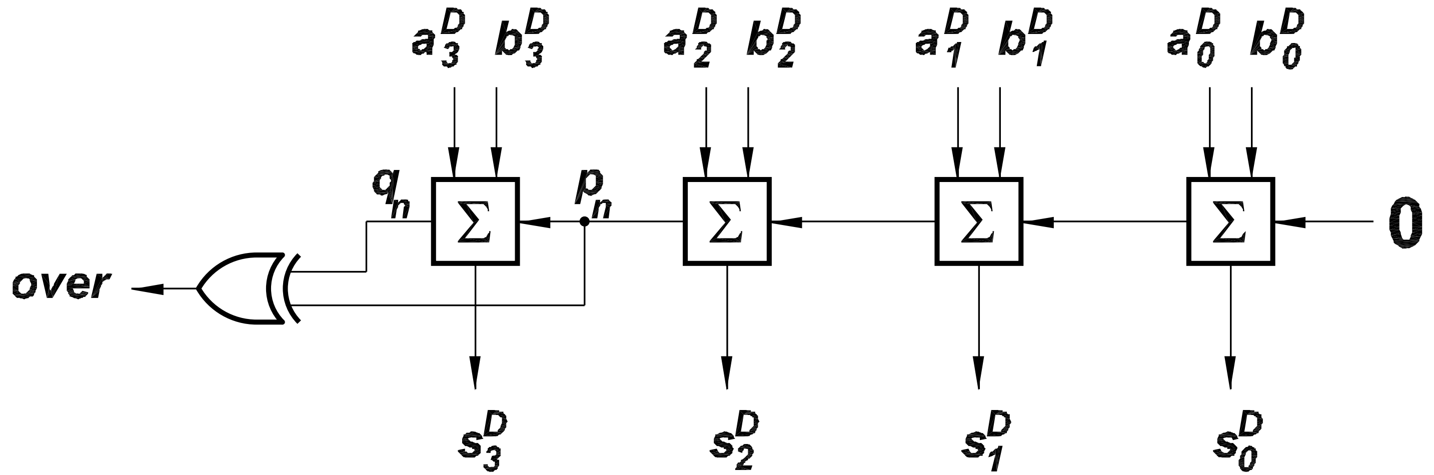
$a_n^{\mathcal{D}}$	$b_n^{\mathcal{D}}$	$p_n^{\mathcal{D}}$	$q_n^{\mathcal{D}}$	$s_n^{\mathcal{D}}$		
0	0	0	0	0		$q_n = p_n$
$\boxed{0}$	$\boxed{0}$	1	0	$\boxed{1}$	$q_n \neq p_n$	
0	1	0	0	1		$q_n = p_n$
0	1	1	1	0		$q_n = p_n$
1	0	0	0	1		$q_n = p_n$
1	0	1	1	0		$q_n = p_n$
$\boxed{1}$	$\boxed{1}$	0	1	$\boxed{0}$	$q_n \neq p_n$	
1	1	1	1	1		$q_n = p_n$

$$\boxed{over = q_n \oplus p_n}$$

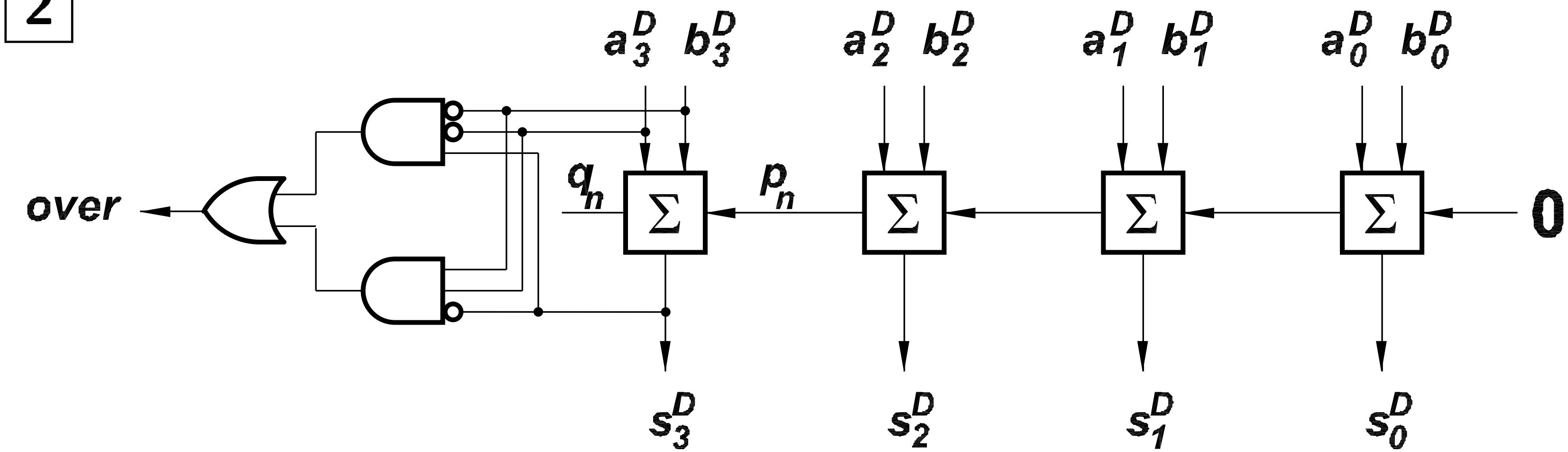
z=2:

addition : $\mathcal{D}(A+B) \equiv \mathcal{D}(A) + \mathcal{D}(B)$ **(mod $\mathcal{Z}$)**

1



2

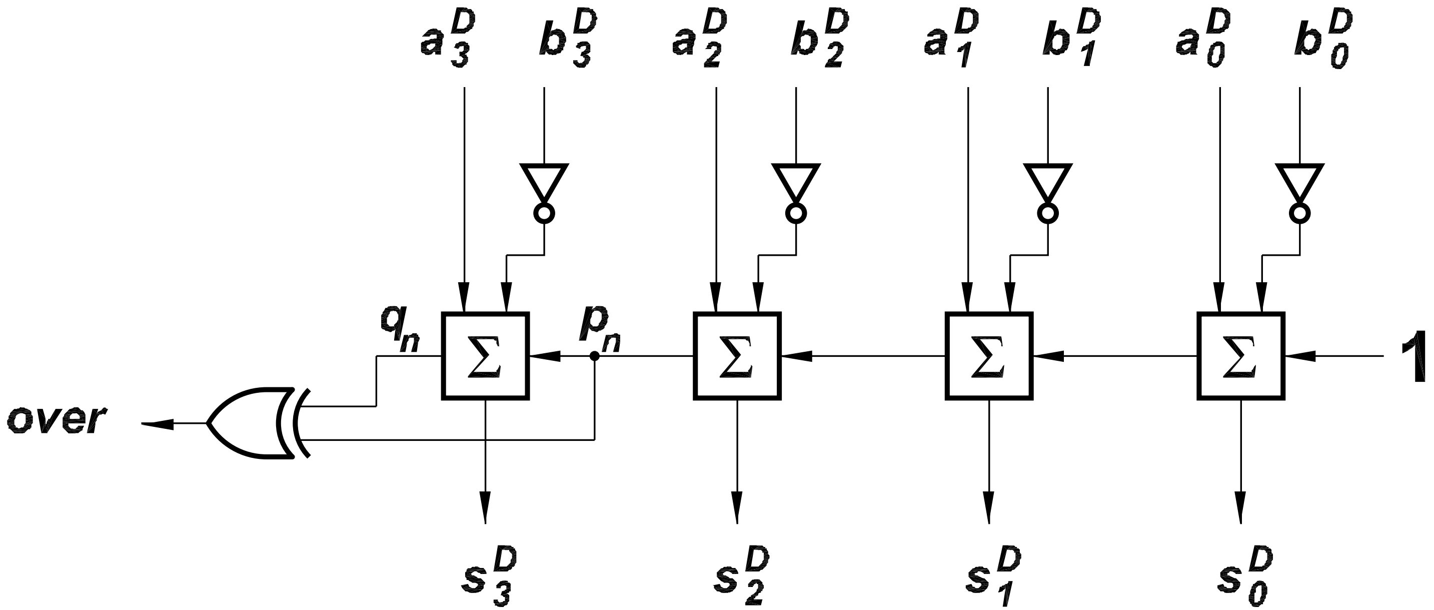


z=2:

sign change:

$$\boxed{\mathcal{D}(-X) \equiv \overline{\mathcal{D}(X)} + 1 \ (\text{mod } \mathcal{Z})}$$

subtraction: $A - B = A + (-B)$



Diminished radix complement

$$\text{radix complement}\begin{cases}\text{1's complement for } z = 2\\ \text{9's complement for } z = 10\end{cases}$$

$$\boxed{\mathcal{I}(X) = \begin{cases} X & \text{for } X \geq 0 \\ \overline{|X|} & \text{for } X \leq 0 \end{cases}}$$

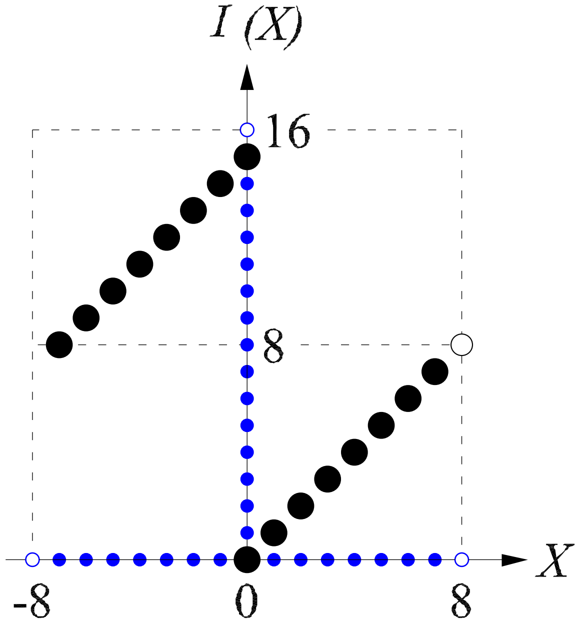
$-\frac{1}{2}\mathcal{Z} < X < \frac{1}{2}\mathcal{Z}$ ☞ symmetric range

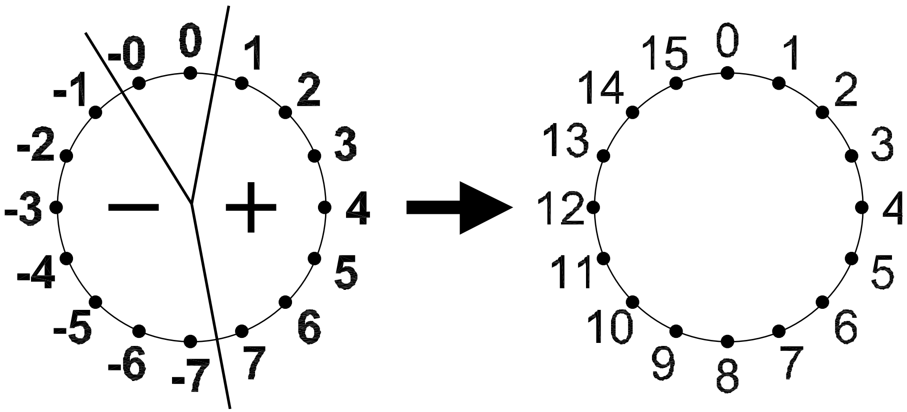
$$\text{two zero representation:}\begin{cases}\text{so called „positive zero“} & 0\,0\ldots0\\ \text{so called „negative zero“} & 1\,1\ldots1\end{cases}$$

$$\boxed{\begin{array}{lcl} \mathsf{MSB} = 0 & \Longrightarrow & X \geq 0 \\ \mathsf{MSB} = 1 & \Longrightarrow & X \leq 0 \end{array}}$$

$T + \overline{T} = 11\ldots11 = \mathcal{Z}{-}1$ $\boxed{\mathcal{I}(X) \equiv X \pmod{\mathcal{Z}{-}1}}$

z=2:



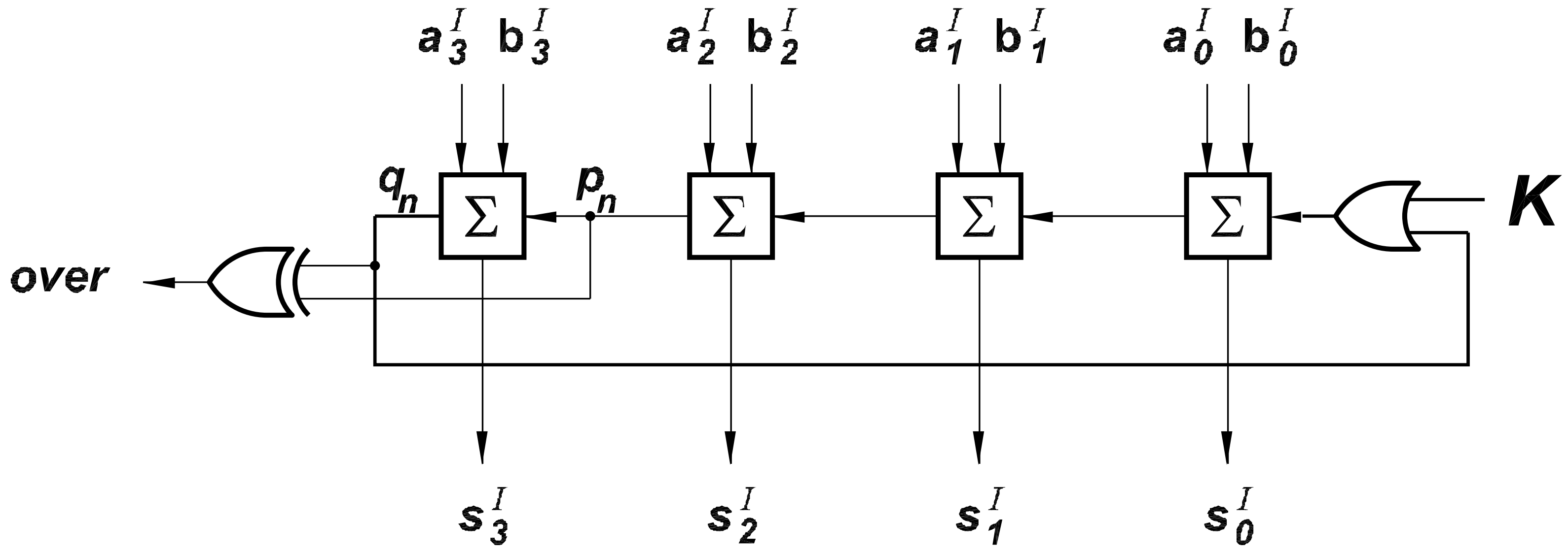


X	$\mathcal{I}(X)$
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
-7	1000
-6	1001
-5	1010
-4	1011
-3	1100
-2	1101
-1	1110
- 0	1111

addition: $\mathcal{I}(A+B) \equiv \mathcal{I}(A) + \mathcal{I}(B) \pmod{\mathcal{Z}-1}$

$\mathcal{I}(A) + \mathcal{I}(B) \geq \mathcal{Z} \quad \Rightarrow \quad q=1$

necessary to subtract $\mathcal{Z}$ and add 1 $\Longrightarrow$ circular carry



feedback ☞ sequential circuit

solution: correction K on input carry into lower order

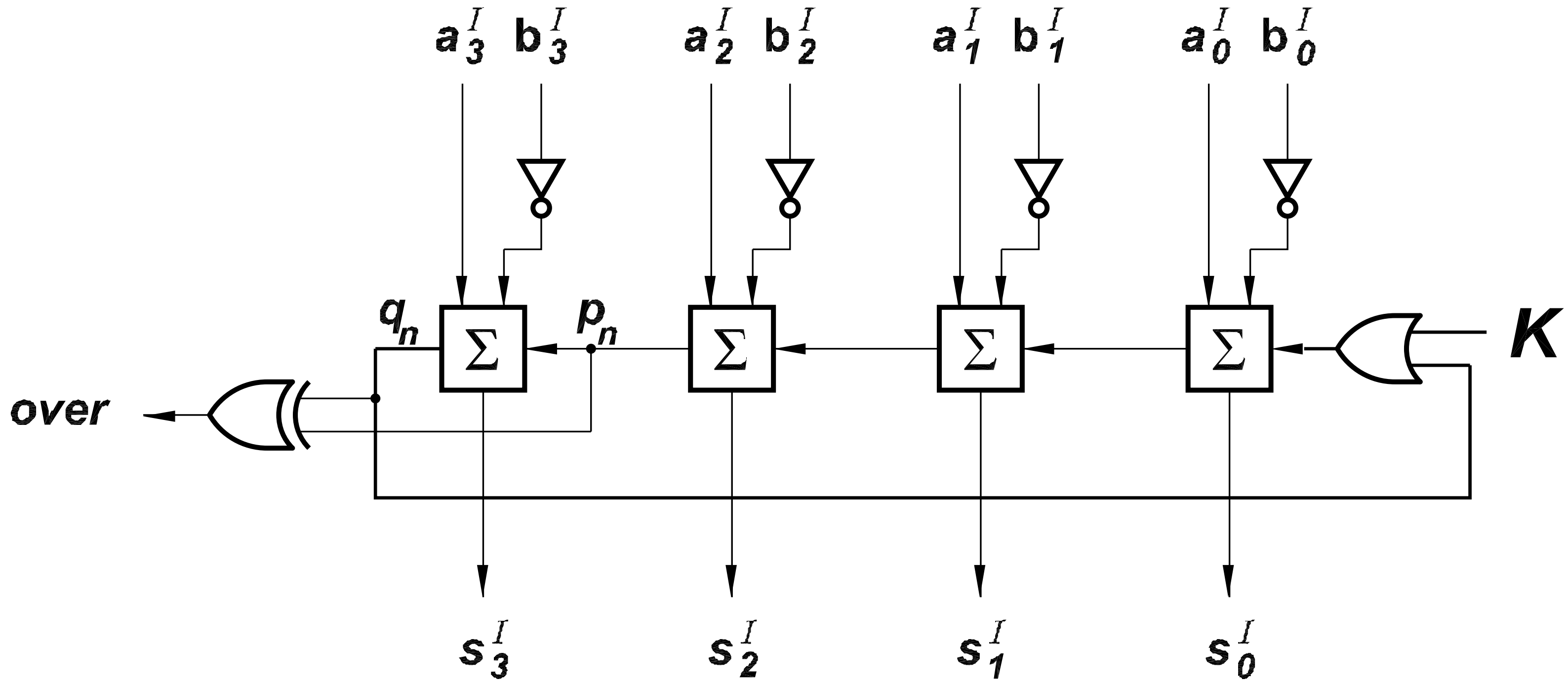
$K=1$, in a case, when carry go over all orders

z=2:

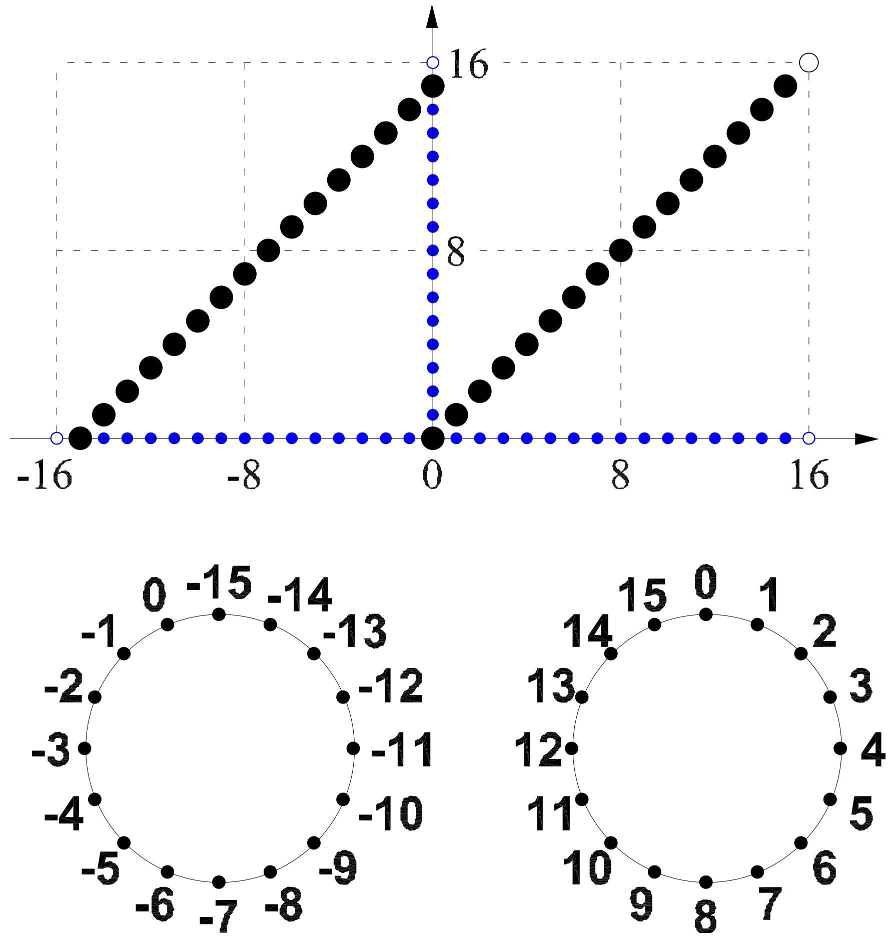
sign change:

$$\boxed{\mathcal{I}(-X) \equiv \overline{\mathcal{I}(X)}}$$

subtraction: $A - B = A + (-B)$



unsigned numbers + adder for radix complement code ⇨
⇨ **radix complement pseudo-code** (comparison with 2's complement pseudo-code)



Biased code

$$\boxed{\mathcal{A}(X) = X + K},$$

where K is „suitable“ constant

„suitable“ constants:

$\frac{1}{2}\mathcal{Z}$ ☞ $\mathcal{A}_0(X)$ ... biased code type 0

$\frac{1}{2}\mathcal{Z}-1$ ☞ $\mathcal{A}_1(X)$ ... biased code type 1

Code is monotonous — growing.

addition: $\boxed{\mathcal{A}(A+B) = \mathcal{A}(A) + \mathcal{A}(B) - K}$

subtraction: $\boxed{\mathcal{A}(A-B) = \mathcal{A}(A) - \mathcal{A}(B) + K}$

It is true (of course) *iff overflow doesn't occurs!*

Biased code - type 0

$$\boxed{\mathcal{A}_{\mathbf{0}}(X) = X + \tfrac{1}{2}\mathcal{Z}}$$

$-\frac{1}{2}\mathcal{Z} \leq X < \frac{1}{2}\mathcal{Z}$ ☞ **asymmetric range**

$$\boxed{\begin{array}{lcl} \mathsf{MSB} = 1 & \iff & X \geq 0 \\ \mathsf{MSB} = 0 & \iff & X < 0 \end{array}}$$

$$\mathcal{A}_{\mathbf{0}}(X) \equiv \mathcal{D}(X) \ (\mathbf{mod}\ \tfrac{1}{2}\mathcal{Z})$$

! $\mathcal{A}_{\mathbf{0}}(X)$ and $\mathcal{D}(X)$ differ in the MSB only !

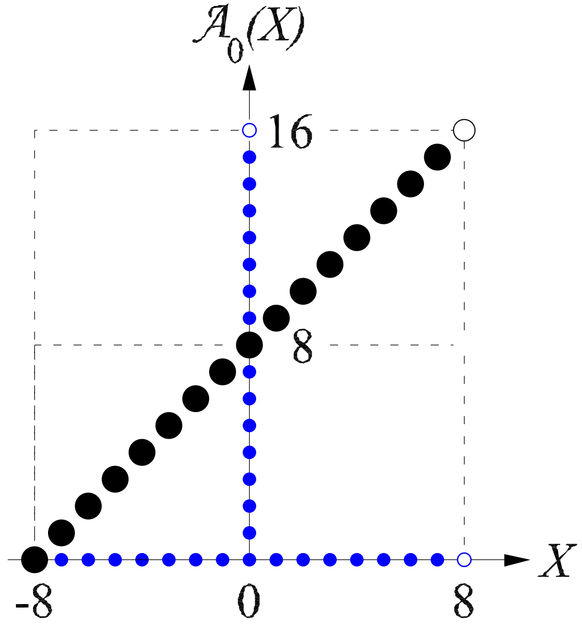
$$\mathsf{MSB}(\mathcal{A}_{\mathbf{0}}(X)) = \overline{\mathsf{MSB}(\mathcal{D}(X))}$$

$\Longrightarrow$ **operations** (addition, subtraction, sign change etc.):

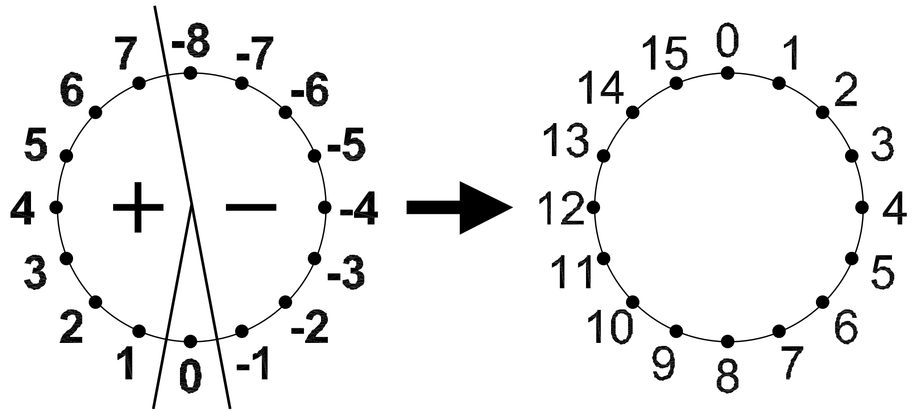
$$\mathcal{A}_{\mathbf{0}}(operands) \rightarrow \mathcal{D}(operands)$$

$$\mathcal{D}(result) \rightarrow \mathcal{A}_{\mathbf{0}}(result)$$

z=2:



X	$\mathcal{A}_0(X)$
-8	0000
-7	0001
-6	0010
-5	0011
-4	0100
-3	0101
-2	0110
-1	0111
0	1000
1	1001
2	1010
3	1011
4	1100
5	1101
6	1110
7	1111



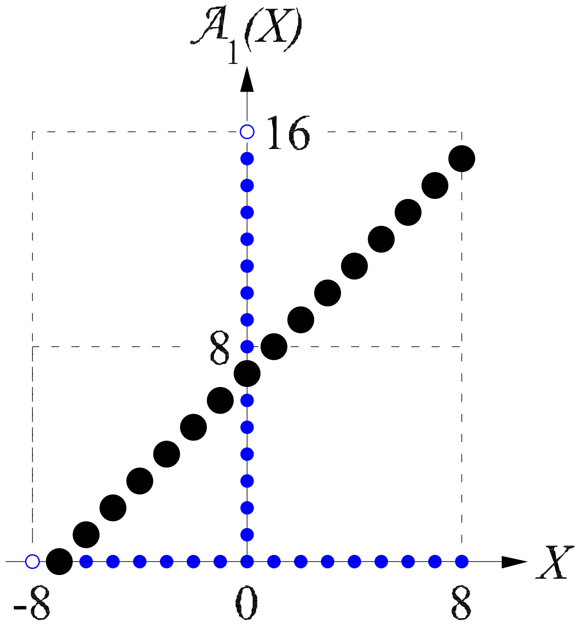
Biased code - type 1

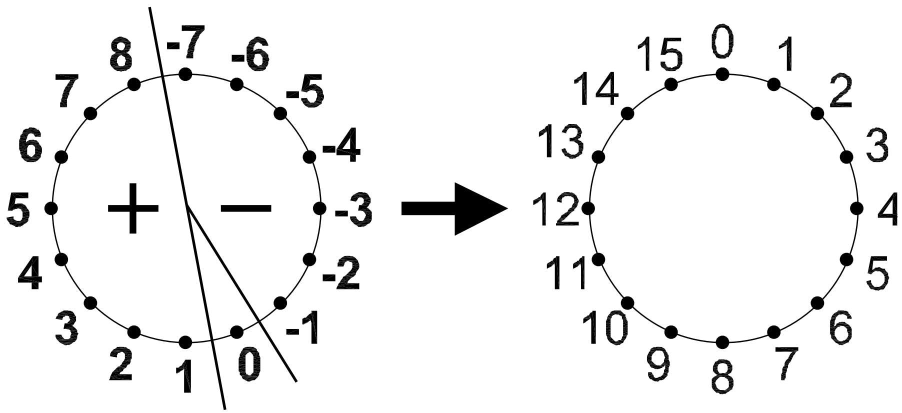
$$\boxed{\mathcal{A}_1(X) = X + \tfrac{1}{2}\mathcal{Z} - 1}$$

$-\frac{1}{2}\mathcal{Z} < X \leq \frac{1}{2}\mathcal{Z}$ ☞ **asymmetric range**

$$\boxed{\begin{array}{lcr} \text{MSB} = 1 & \iff & X > 0 \\ \text{MSB} = 0 & \iff & X \leq 0 \end{array}}$$

z=2:





X	$\mathcal{A}(1)$**X**
-7	0000
-6	0001
-5	0010
-4	0011
-3	0100
-2	0101
-1	0110
0	0111
1	1000
2	1001
3	1010
4	1011
5	1100
6	1101
7	1110
8	1111

addition and subtraction:

$$\mathcal{A}(A+B) = \mathcal{A}(A) + \mathcal{A}(B) - K$$
$$\mathcal{A}(A-B) = \mathcal{A}(A) - \mathcal{A}(B) + K$$

$$\mathcal{A}(A+B) = \mathcal{A}(A) + \mathcal{A}(B) - \tfrac{1}{2}\mathcal{Z} + 1$$
$$\mathcal{A}(A-B) = \mathcal{A}(A) - \mathcal{A}(B) + \tfrac{1}{2}\mathcal{Z} - 1$$

$-\frac{1}{2}\mathcal{Z} \equiv +\frac{1}{2}\mathcal{Z} \pmod{\mathcal{Z}}$ $\qquad \frac{1}{2}\mathcal{Z} \sim \boxed{100\ldots00}$

$-\frac{1}{2}\mathcal{Z} \equiv +\frac{1}{2}\mathcal{Z} \pmod{\mathcal{Z}} \rightsquigarrow$ **negation of bit in higher order**

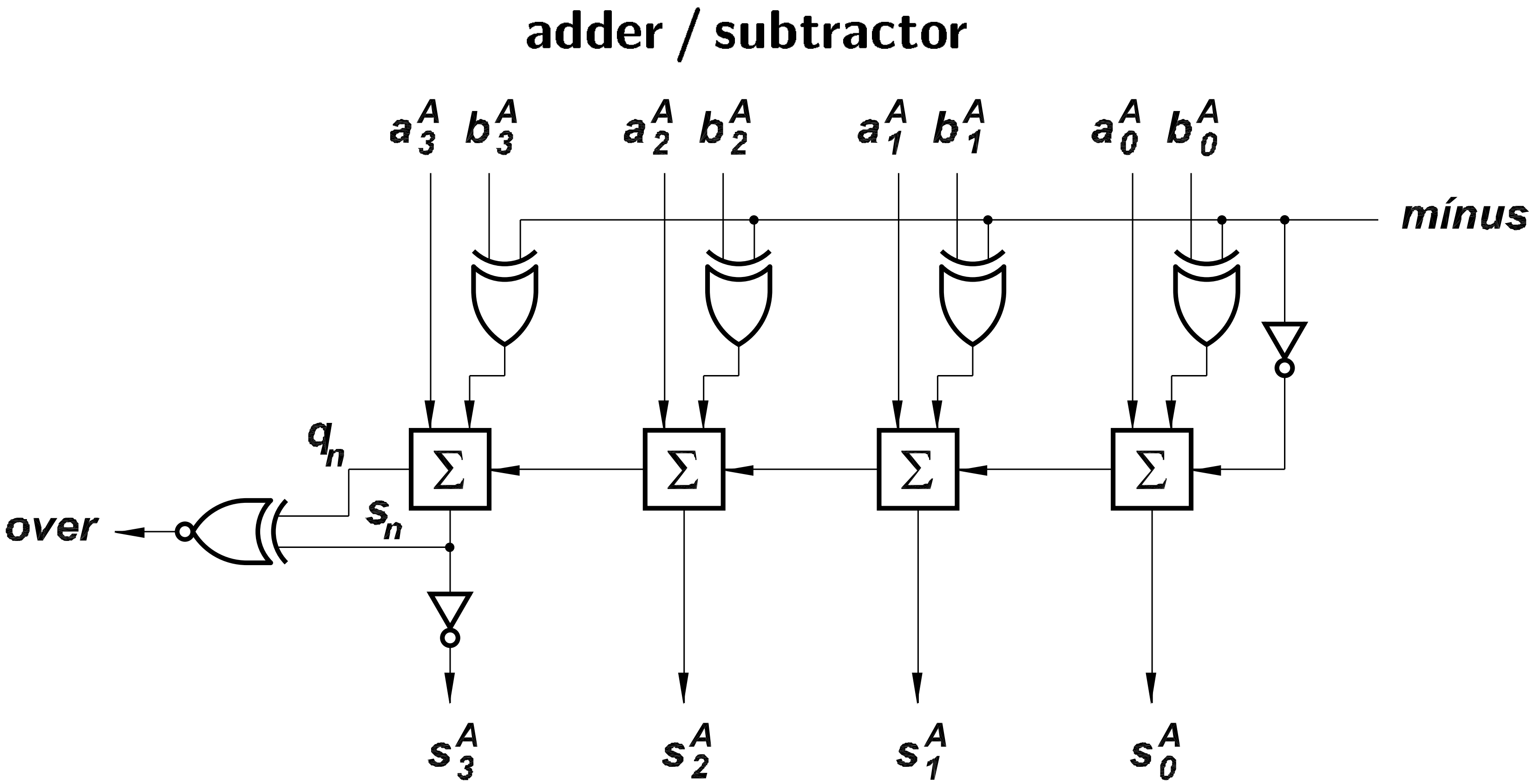
overflow:

addition: **same sign of addend**
and different sign of result

subtraction: **sign of minuend and subtrahend are different**
signs of minuend and result are different

(derivation is analogous as in 2's complement code)

z=2:



sign change: $\quad -X \quad = \quad \mathbf{0} - X$

overflow detection

a_n	b_n	p_n	q_n	s_n		
$\boxed{0}$	$\boxed{0}$	0	0	$\boxed{0}$	$q_n = s_n$	
0	**0**	**1**	**0**	**1**		$q_n \neq s_n$
0	**1**	**0**	**0**	**1**		$q_n \neq s_n$
0	**1**	**1**	**1**	**0**		$q_n \neq s_n$
1	**0**	**0**	**0**	**1**		$q_n \neq s_n$
1	**0**	**1**	**1**	**0**		$q_n \neq s_n$
1	**1**	**0**	**1**	**0**		$q_n \neq s_n$
$\boxed{1}$	$\boxed{1}$	1	1	$\boxed{1}$	$q_n = s_n$	

$$\boxed{over = \overline{q_n \oplus s_n}}$$