Advanced Cryptology

Algebraic Cryptanalysis

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„Breaking a good cipher should require as much work as solving a system of simultaneous equations in a large number of unknowns of a complex type.“

C.E. Shannon, 1949

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Introduction

- Algebraic cryptanalysis, abbr. AC, is a new area of cryptanalysis which has gained much attention in recent years.
- The principle of AC consists in transferring the problem of breaking the cryptosystem to the problem of solving system of polynomial equations over a finite field.
- AC is mainly used in symmetric cryptanalysis:
  - examples of block ciphers: AES, DES
  - examples of stream ciphers: E0 - Bluetooth, Toyocrypt,

  however AC was used also in assymetric cryptography.

Process

- Process of AC is divided into the following two steps:

  1: Convert the cipher and possibly some supplemental information into a system of polynomial equations.
  2: Apply some algorithm to calculate the solution of the system of polynomial equations and derive a secret key.

- However there is a fundamental problem: solving a system of quadratic equations over any finite field is NP-Complete.
- Why should we convert the problem of breaking a cipher to the problem for which we do not know fast algorithm(i.e. with polynomial complexity)?

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1.step

- The first step of AC consists of using the structure of cipher and supplemental information for creating a system of equations, that describe behavior of ciphers for a specific case.
- We consider the system of equations over a finite field, usually GF(2).
- Our aim is to obtain the smallest possible system of equations, that contains the polynomials with the lowest degree. Method which derives the system of equations depends on the cipher.
- If the system contains only linear equations, then we use e.g. Gaussian elimination, which has a cubic complexity and we receive the result relatively quickly.
- Well proposed ciphers provide a system of polynomial equations that we can not solve in a short time.

2.step

- The main part of AC is the second step, in which we solve the system of the polynomial equations over a finite field.
- Example: AC of AES-128 system includes approximately 8000 equations with 1600 variables and AC of AES-256 system has 22400 equations with 4480 variables.
- There are several methods for calculating these nonlinear systems, however none of them is fast.
- If there is an efficient algorithm for solving this problem (i.e. with polynomial complexity), then P = NP and that would be considered as surprise.

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Methods for solving a system of polynomial equations

- Method of the type „guess and determine“ is simply, however it is not very effective.
- We „guess“(using brute force) values of appropriate variables and then we can easily calculate the rest of the system.
- Other methods are: linearization, that we will introduce later, XL algorithm and the method of Gröbner bases.
- Gröbner bases is very perspective method and has been successfully applied e.g. on AES.

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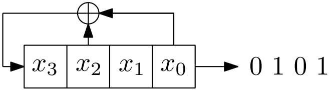
Application AC on some types of stream ciphers

- Examples of using AC - three classes of stream ciphers:
  - LFSR(Linear Feedback Shift Register) - cipher consists of only one LFSR
  - NLCG(Nonlinear Combination Generator) - cipher consists of more LFSRs and nonlinear boolean function which is used to compute output(function uses only output bits from LFSRs)
  - NLFG(Nonlinear Filter Generator) - cipher consists of only one LFSR and nonlinear boolean function which is used to compute output(function uses all bits from LFSR)
- AC is „known-plaintext attack“ since we assume that attacker knows bits of keystream.

Example of application AC on LFSR

- Let's consider the following simple LFSR of length 4:



- Our goal is to gain values $x_0, \ldots, x_3 \in \mathrm{GF}(2)$.
-

clock	LFSR state	output	equations
1.	(x_3, x_2, x_1, x_0)	x_0	$x_0 = 0$
2.	$(x_0 \oplus x_2, x_3, x_2, x_1)$	x_1	$x_1 = 1$
3.	$(x_1 \oplus x_3, x_0 \oplus x_2, x_3, x_2)$	x_2	$x_2 = 0$
4.	$(x_0, x_1 \oplus x_3, x_0 \oplus x_2, x_3)$	x_3	$x_3 = 1$

- In this case the system of equations is solved immediately.

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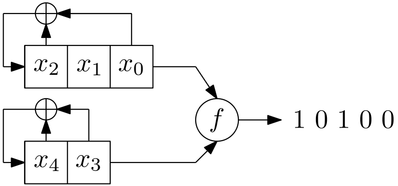
Example of application AC on NLCG

- Let's consider the following example with two LFSRs:



  and nonlinear function $f(v_1, v_2) = v_1 + v_1 v_2$, where v_1 is the output bit of the first LFSR and v_2 of the second one.

- Goal is to compute values $x_0, \ldots, x_4 \in \mathrm{GF}(2)$.

Generating the system of equations

- Let the first LFSR is R_1 and the second LFSR is R_2 and their output bits are v_1 and v_2 respectively

clock	state of R_1	state of R_2	v_1, v_2
1.	(x_2, x_1, x_0)	(x_4, x_3)	x_0, x_3
2.	$(x_0 \oplus x_2, x_2, x_1)$	$(x_3 \oplus x_4, x_4)$	x_1, x_4
3.	$(x_0 \oplus x_1 \oplus x_2, x_0 \oplus x_2, x_2)$	$(x_3, x_3 \oplus x_4)$	$x_2, x_3 \oplus x_4$
4.	$(x_0 \oplus x_1, x_0 \oplus x_1 \oplus x_2, x_0 \oplus x_2)$	(x_4, x_3)	$x_0 \oplus x_2, x_3$
5.	$(x_1 \oplus x_2, x_0 \oplus x_1, x_0 \oplus x_1 \oplus x_2)$	$(x_3 \oplus x_4, x_4)$	$x_0 \oplus x_1 \oplus x_2, x_4$

System of polynomial equations

$$
\begin{aligned}
x_0 + x_0x_3 &= 1 \\
x_1 + x_1x_4 &= 0 \\
x_2 + x_2x_3 + x_2x_4 &= 1 \\
x_0 + x_2 + x_0x_3 + x_2x_3 &= 0 \\
x_0 + x_1 + x_2 + x_0x_4 + x_1x_4 + x_2x_4 &= 0
\end{aligned}
$$

- Can you solve it?
- (e.g. from the first equation is obvious that $x_0 = 1$)

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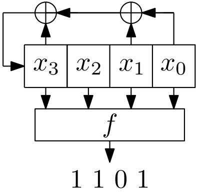
Example of application AC on NLFG

- Let‘s consider the following example of LFSR:



  and nonlinear function $f(x_0, x_1, x_2, x_3) = x_0 + x_0x_1 + x_1x_3$.
- Our goal is to compute values $x_0, \ldots, x_3 \in$ GF(2).

Generating the system of equations

clock	state of LFSR
1.	(x_3, x_2, x_1, x_0)
2.	$(x_0 + x_1 + x_3, x_3, x_2, x_1)$
3.	$(x_0 + x_2 + x_3, x_0 + x_1 + x_3, x_3, x_2)$
4.	$(x_0, x_0 + x_2 + x_3, x_0 + x_1 + x_3, x_3)$

System of polynomial equations

$$\begin{aligned}
x_0 + x_0x_1 + x_1x_3 &= 1 \\
x_1 + x_0x_2 + x_2x_3 &= 1 \\
x_2 + x_0x_3 + x_3^2 &= 0 \\
x_3 + x_1x_3 + x_3^2 + x_0^2 + x_0x_1 &= 1
\end{aligned}$$

- Can you solve it?

Notes

- In NLFG, degree of the generated equations is upper bounded by the degree of nonlinear function f.
- The maximal degree of the equations is important factor and AC is more effective for lower degree.
- The important fact is that generating the equations does not depend on keystream.
- Therefore a generation of the system of polynomial equations can be done as a precomputation step and that's why the second step - solving of equations - is the most complicated.
- *Challenge: Try to create a system of equation for A5/1 stream cipher, which consists of 3 LFSRs and nonlinear clocking mechanism.*

Linearization

- At the end we present the basic technique for solving the system of multivariate polynomial equations called linearization.
- The algorithm is based on the following three steps:

  1: Substitute any product of variables by fresh variable.
  2: Solve the linear system (e.g. using Gaussian elimination).
  3: Plug the solution into the original system and check correctness of the solution.

- During linearization the linearized system can contain linear dependent equations and there have been proposed several improvements.

Linearization - example 1

- Consider the following system of equations over GF(2):

$$\begin{aligned} x + xy &= 1 \\ x + y &= 1 \\ x + y + xy &= 1 \end{aligned}$$

- In the first step we replace term xy with fresh variable z:

$$\begin{aligned} x + z &= 1 \\ x + y &= 1 \\ x + y + z &= 1 \end{aligned}$$

- In step 2 we easily compute the solution: $x = 1$, $y = 0$, $z = 0$
- In step 3 we check the correctness of the solution.

Linearization - example 2

- The following example illustrates the necessity of checking correctness of the solution:

$$\begin{aligned} x + xy &= 1 \\ x + y &= 0 \\ x + y + xy &= 0 \end{aligned}$$

- In step 1 we replace term xy with fresh variable z:

$$\begin{aligned} x + z &= 1 \\ x + y &= 0 \\ x + y + z &= 0 \end{aligned}$$

- By solving the system we obtain solution: $x = 1, y = 1, z = 0$.
- However after plugging it into the first equation we obtain: $1 + 1 * 1 \neq 1$, therefore checking the solution is necessary.