Control Systems

G V V Sharma*

Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are**

Download python codes using

1 Gain Margin

1.1. (**a**). The open loop transfer function of a feedback control system is

$$G(s) = \frac{1}{s(1+2s)(1+s)} \quad (1.1.1)$$

Find the gain margin of this system and analyse the stability. (**b**). Provide some other example other than this to analyse stability mathematically using the concept of gain margin.

Solution:
(**a**) : **Gain Margin**:The greater the Gain Margin (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.

We can usually read the gain margin directly from the Bode plot. This is done by calculating the vertical distance between the magnitude curve (on the Bode magnitude plot) and the x-axis at the frequency where the Bode phase plot = 180°. This point is known as the phase crossover frequency.

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

Gain Margin is given by,

$$G.M = -20log_{10}|G(j\omega_{pc})| = 20log_{10}k_g \quad (1.1.2)$$

where

$$k_g = \frac{1}{|G(j\omega_{pc})|} \quad (1.1.3)$$

Now let's put s = jω in the equation of G(s) :

$$G(j\omega) = \frac{1}{j\omega(1+2j\omega)(1+j\omega)} \quad (1.1.4)$$

So,

$$G(j\omega) = \frac{1}{j\omega(1+3j\omega-2\omega^2)} = \frac{1}{j\omega-3\omega^2-2j\omega^3} \quad (1.1.5)$$

Hence ,

$$G(j\omega) = \frac{1}{-3\omega^2 + j\omega(1-2\omega^2)} \quad (1.1.6)$$

Now we know that ω_{pc} is the Phase crossover frequency (The frequency at which the phase of open-loop transfer function reaches -180° or +180° depending upon the range of tan inverse function).
Now,

$$\angle G(j\omega) = -tan^{-1}\left(\frac{\omega(1-2\omega^2)}{-3\omega^2}\right) \quad (1.1.7)$$

So,at $\omega = \omega_{pc}$:

$$\omega(1-2\omega^2) = 0 \quad (1.1.8)$$

i.e. the imaginary part of G(jω) = 0.So ,

$$\omega_{pc} = \frac{1}{\sqrt{2}} \quad (1.1.9)$$

as ω_{pc} should be positive and ω_{pc} shuld not be equal to zero.So now G(jω_{pc}) will be :

$$G(j\omega_{pc}) = \frac{1}{-3\omega_{pc}^2} \quad (1.1.10)$$

i.e,

$$|G(j\omega_{pc})| = \frac{1}{(\frac{3}{2})} \tag{1.1.11}$$

$$k_g = \frac{1}{|G(j\omega_{pc})|} = \frac{3}{2} = 1.5 \tag{1.1.12}$$

So , Gain margin in terms of dB is :

$$20log_{10}1.5 = 3.5dB \tag{1.1.13}$$

Plot obtained for verification in python :
(You can download code from codes/ee18btech11016.py)

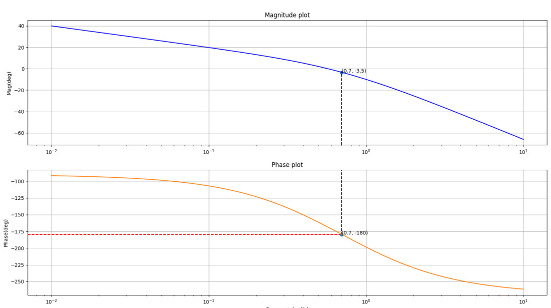

Fig. 1.1

2 Stability

So,in the above figure, since $20log_{10}(G(j\omega_{pc}))$ = -3.5dB at $\omega_{pc} = -180°$ so G.M = +3.5dB And since the gain margin is positive we can say that the system is stable more precisely the system is marginally stable as one of the pole lies on the imaginary axis.(Because for stability , both gain and phase margin should be positive.)

(**b**).

(**1**)Gain of closed loop transfer function would not affect its stability.

(**2**)Gain of open loop transfer function would affect the closed loop stability.

The idea is very simple. When you derive the unit feedback closed loop transfer function of an open loop transfer function, it will be very obvious. Let's analyze it over a simple example;

G = K/(s+1) Unit Feedback Transfer Function = G/(1+G) = K/(s+1+K)

As you can see from UFTF, open loop gain K affects both gain of the closed loop system and the pole location of it, thus its stability. And therefore if we increase the gain above the gain margin then the system will become unstable.The pole location of closed loop system is (-1-K) and as long as this term is not positive, the system will be stable.

Suppose we have G(s) = -0.5/(s+1) i.e K = -0.5. In this case closed loop transfer function is $\frac{-0.5}{s+0.5}$. i.e the pole of this transfer function is lying on the negative side of s-plane . So closed loop transfer function for this value of K is stable.

Now lets find out the gain margin for this case. Since

$$G(j\omega) = \frac{-0.5}{j\omega + 1}. \tag{2.1.1}$$

As discussed previously also , ω_{pc} is the value of ω at which the imaginary part of G(jω) = 0.So, ω_{pc}=0. And hence $|G(j\omega_{pc})|$ = 0.5 . So ,

$$k_g = \frac{1}{|G(j\omega_{pc})|} = \frac{1}{0.5} = 2 \tag{2.1.2}$$

Now,let's see what will happen if we increase the gain above the gain margin, Since K is negative so let now K is increased to -3 . (Gain of the open loop transfer function is now 3).So in this case , the closed loop transfer function will be equal to : K/(s+1+K).

$$\frac{G(s)}{1+G(s)} = \frac{-3}{s-2}. \tag{2.1.3}$$

i.e we can clearly see that the pole of the transfer function is lying on the positive side of s-plane.So,the closed loop system is unstable. i.e. Exceeding the gain margin makes the system unstable