Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Mason's Gain Formula

1.1. The Block diagram of a system is illustrated in the figure shown, where $X(s)$ is the input and $Y(s)$ is the output. Draw the equivalent signal flow graph.

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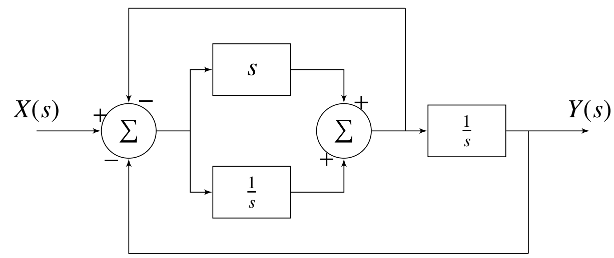

Fig. 1.1.1: Block Diagram

Solution: The signal flow graph of the block diagram in Fig. 1.1.1 is available in Fig. 1.1.2

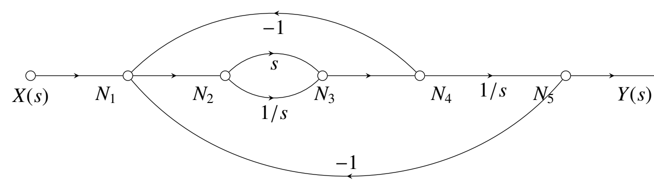

Fig. 1.1.2: Signal Flow Graph

1.2. Draw all the forward paths in Fig. 1.1.2 and compute the respective gains.
Solution: The forward paths are available in Figs. 1.2.3 and 1.2.4. The respective gains are

$$P_1 = s\left(\frac{1}{s}\right) = 1 \tag{1.2.1}$$

$$P_2 = (1/s)(1/s) = 1/s^2 \tag{1.2.2}$$

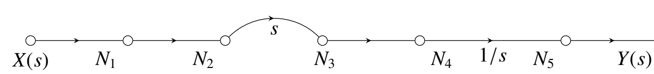

Fig. 1.2.3: P_1

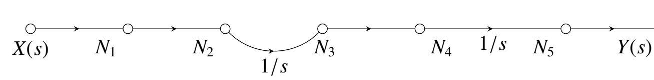

Fig. 1.2.4: P_2

1.3. Draw all the loops in Fig. 1.1.2 and calculate the respective gains.

Solution: The loops are available in Figs. 1.3.5-1.3.8 and the corresponding gains are

$$L_1 = (-1)(s) = -s \tag{1.3.1}$$

$$L_2 = s\left(\frac{1}{s}\right)(-1) = -1 \tag{1.3.2}$$

$$L_3 = \left(\frac{1}{s}\right)(-1) = -\frac{1}{s} \tag{1.3.3}$$

$$L_4 = \left(\frac{1}{s}\right)\left(\frac{1}{s}\right)(-1) = -\frac{1}{s^2} \tag{1.3.4}$$

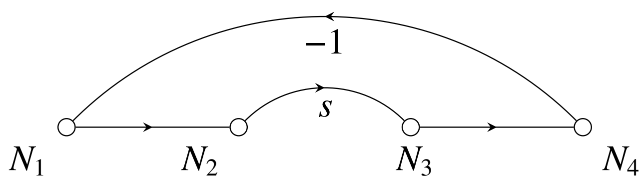

Fig. 1.3.5: L_1

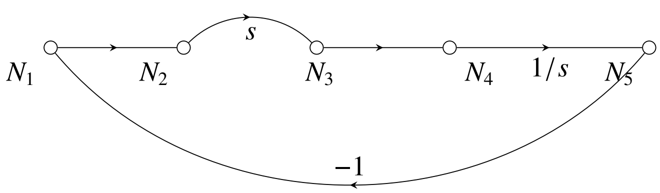

Fig. 1.3.6: L_2

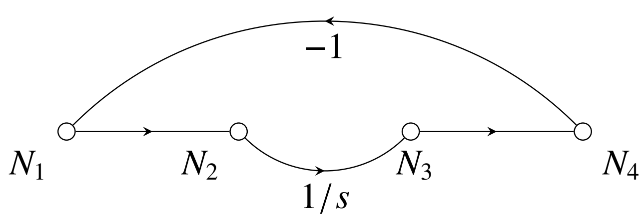

Fig. 1.3.7: L_3

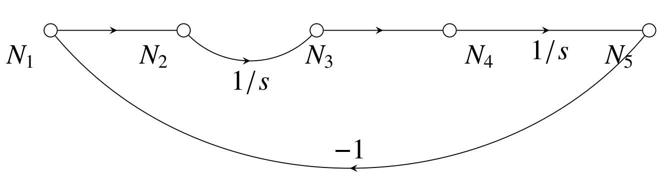

Fig. 1.3.8: L_4

1.4. State Mason's Gain formula and explain the parameters through a table.
Solution: According to Mason's Gain Formula,

$$T = \frac{Y(s)}{X(s)} \tag{1.4.1}$$

$$= \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \tag{1.4.2}$$

where the parameters are described in Table 1.4

Variable	Description
P_i	ith forward path
L_j	jth loop
Δ	$1 - \sum_{L_i} + \sum_{L_i \cap L_j = \phi} L_i L_j - \sum_{L_i \cap L_j \cap L_k = \phi} L_i L_j L_k + \ldots$
Δ_i	$1 - \sum_{L_k \cap P_i = \phi} L_k + \sum_{L_k \cap L_j \cap P_i = \phi} L_k L_j - \ldots$

TABLE 1.4

1.5. List the parameters in Table 1.4 for Fig. 1.1.2.
Solution: The parameters are available in Table 1.5

Path	Value	Parameter	Value	Remarks
P_1	1	Δ_1	1	All loops intersect with P_1
P_2	$\frac{1}{s^2}$	Δ_2	1	All loops intersect with P_2
L_1	$-s$	Δ	$1 - \sum_i L_i$	All loops intersect
L_2	-1			
L_3	-$\frac{1}{s}$			
L_4	-$\frac{1}{s^2}$			

TABLE 1.5

1.6. Find the transfer function using Mason's Gain Formula.
Solution: From (1.4.2) and 1.5,

$$T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{1.6.1}$$

$$= \frac{1 + \frac{1}{s^2}}{1 - (-s - 1 - \frac{1}{s} - \frac{1}{s^2})} \tag{1.6.2}$$

$$= \frac{s^2 + 1}{s^3 + 2s^2 + s + 1} \tag{1.6.3}$$

after simplification.

1.7. Write a program to compute Mason's gain formula, given the branch nodes and gains for each path.

2 Bode Plot

2.1 Introduction

2.1. For an LTI system, the Bode plot for its gain defined as

$$G(s) = 20\log|H(s)| \tag{2.1.1}$$

is as illustrated in the Fig. 2.1. Express $G(f)$ in terms of f.

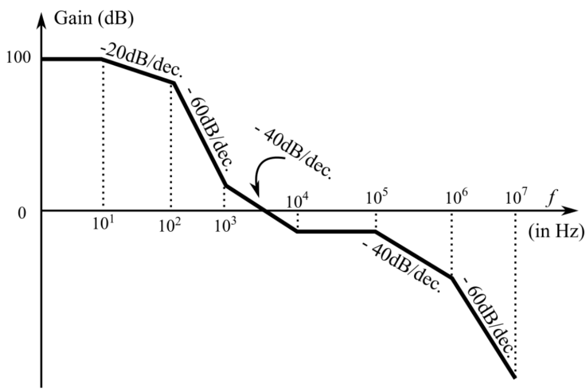

Fig. 2.1

Solution:

$$G(f) = \begin{cases} 100 & 0 < f < 10^1 \\ 120 - 20\log(f) & 10 < f < 10^2 \\ 200 - 60\log(f) & 10^2 < f < 10^3 \\ 140 - 40\log(f) & 10^3 < f < 10^4 \\ -20 & 10^4 < f < 10^5 \\ 180 - 40\log(f) & 10^5 < f < 10^6 \\ 300 - 60\log(f) & 10^6 < f < 10^7 \end{cases} \tag{2.1.2}$$

2.2. Express the slope of $G(f)$ in terms of f.
Solution: The desired slope is

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} \tag{2.2.1}$$

$$\nabla G(f) = \begin{cases} 0 & 0 < f < 10^1 \\ -20 & 10 < f < 10^2 \\ -60 & 10^2 < f < 10^3 \\ -40 & 10^3 < f < 10^4 \\ 0 & 10^4 < f < 10^5 \\ -40 & 10^5 < f < 10^6 \\ -60 & 10^6 < f < 10^7 \end{cases} \tag{2.2.2}$$

2.3. Express the change of slope of $G(f)$ in terms of f.
Solution:
$\Delta(\nabla G(f))$ = Change of slope G(f) at f

$$\Delta(\nabla G(f)) = \begin{cases} -20 & f = 10^1 \\ -40 & f = 10^2 \\ +20 & f = 10^3 \\ +40 & f = 10^4 \\ -40 & f = 10^5 \\ -20 & f = 10^6 \end{cases} \tag{2.3.1}$$

2.4. Tabulate the poles and zeros of $H(s)$ using (2.3.1).
Solution: Table 2.4 provides the details.

f **(Hz)**	$\Delta(\nabla G(f))$	**Pole**	**Zero**
10^1	-20	1	0
10^2	-40	2	0
10^3	20	0	1
10^4	40	0	2
10^5	-40	2	0
10^6	-20	1	0
Total		**6**	**3**

TABLE 2.4

2.5. Obtain the transfer function of $H(s)$.
Solution: From Table 2.4,

$$H(s) = \frac{K(s + j2\pi 10^3)(s + j2\pi 10^4)^2}{(s + j2\pi 10^1)(s + j2\pi 10^2)^2(s + j2\pi 10^5)^2(s + j2\pi 10^6)} \tag{2.5.1}$$

2.6. Justify the above results.
Solution: Let us consider a generalized transfer gain

$$H(s) = k\frac{(s - z_1)(s - z_2)...(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)....(s - p_{n-1})(s - p_n)} \tag{2.6.1}$$

The gain

$$\begin{aligned}G(f) &= 20\log|H(s)| \\ &= 20\log|k| + 20\log|s-z_1| \\ &+ 20\log|s-z_2| + \cdots + 20\log|s-z_m| \\ &- 20\log|s-p_1| - 20\log|s-p_2| \\ &- \cdots - 20\log|s-z_n| \end{aligned} \quad (2.6.2)$$

Substituting $s = \jmath\omega$, for real z_1

$$20\log|s-z_1| = 20\log\left|\sqrt{\omega^2+z_1^2}\right| \quad (2.6.3)$$

$$= \begin{cases}20\log|z_1|, & \omega \ll z_1 \\ 20\log|\omega|, & \omega \gg z_1\end{cases} \quad (2.6.4)$$

Taking the derivative,

$$\frac{d\,(20\log|s-z_1|)}{d\,(\log|\omega|)} = \begin{cases}0, & \omega \ll z_1 \\ 20, & \omega \gg z_1\end{cases} \quad (2.6.5)$$

Thus, when a zero is encountered, the gradient of $H(\jmath\omega)$ jumps by +20 in the log scale. When a pole is encountered, the gradient falls by -20. Note that this is a very loose justification, but works well in practice.

2.7. Obtain the Bode plot and the slope plot for $H(s)$ and verify with Fig. 2.1
Solution: Bode Plot of obtained Transfer Function is Fig. **??**, obtained from (2.5.1), is a close

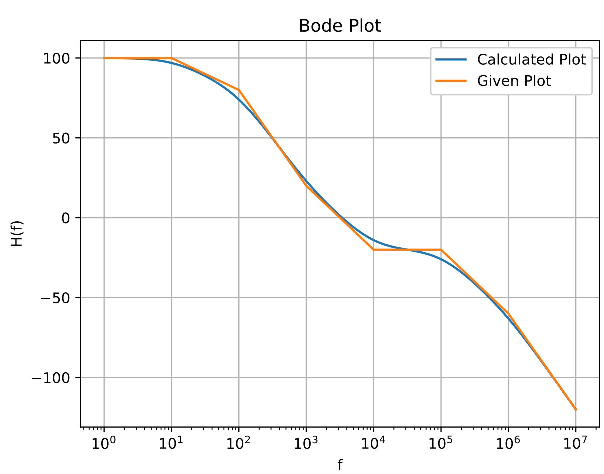

Fig. 2.7

reconstruction of Fig. **??**.

2.2 Example

2.2.1. The asymptotic Bode magnitude plot of minimum phase transfer function G(s) is show below. Express $20\log\left|G(\jmath\omega)\right|$ as a function of

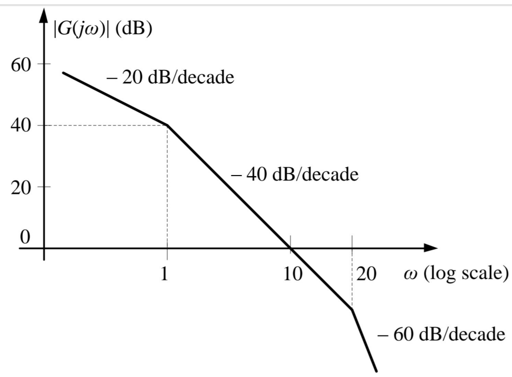

Fig. 2.2.1

ω using Fig. 2.2.1.

2.2.2. Express the slope of $20\log\left|G(\jmath\omega)\right|$ as a function of ω.

2.2.3. Express the change of slope of $20\log\left|G(\jmath\omega)\right|$ as a function of ω.

2.2.4. Find the poles and zeros of $G(s)$.

2.2.5. Find $G(s)$

2.2.6. Obtain the Bode plot of $G(s)$ through a python code and compare with the line plot of the expression that you obtained in Problem 2.2.1

2.2.7. Verify if at very high frequency ($\omega \to \infty$), the phase angle $\angle G(j\omega) = -3\pi/2$ **Solution:**
Since, each pole corresponds to -20 dB/decade and each zero corresponds to +20 dB/decade. Therefore, from the given Bode plot we can get the Transfer equation,

$$G(s) = \frac{k}{s(1+s)(20+s)} \quad (2.2.7.1)$$

Now, from the Transfer equation we can conclude that, there are three poles (0, -1 and -20) and no zeros.

$\therefore$ Statement 1 is false ....(1)

Calculating phase:

Since we know that,
phase ϕ is the sum of all the phases corresponding to each pole and zero.

phase corresponding to pole is =

$$-tan^{-1}(\frac{imaginary}{real}) \quad (2.2.7.2)$$

phase corresponding to zero is =

$$tan^{-1}(\frac{imaginary}{real}) \quad (2.2.7.3)$$

Now take,

$$s = j\omega \quad (2.2.7.4)$$

$$\Rightarrow G(j\omega) = \frac{k}{j\omega(1 + j\omega)(20 + j\omega)} \quad (2.2.7.5)$$

Therefore,

$$\phi = -tan^{-1}(\frac{\omega}{0}) - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20}) \quad (2.2.7.6)$$

$$\phi = -90^\circ - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20}) \quad (2.2.7.7)$$

$$\because \omega \to \infty \quad (2.2.7.8)$$

$$\phi = -90^\circ - 90^\circ - 90^\circ \quad (2.2.7.9)$$

$$\phi = -270^\circ \quad (2.2.7.10)$$

$$\phi = -3\pi/2 \quad (2.2.7.11)$$

$\therefore$ Statement 2 is true ....(2)
thus, from (1) and (2) option (B) is correct.

2.2.8.

3 Second order System

3.1 Damping

3.1.1. List the different kinds of damping for a second order system defined by

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2} \quad (3.1.1.1)$$

where ω is the natural frequency and ζ is the damping factor.
Solution: The details are available in Table 3.1.1

Damping Ratio	**Damping Type**
$\zeta > 1$	Overdamped
$\zeta = 1$	Critically Damped
$0 < \zeta < 1$	Underdamped
$\zeta = 0$	Undamped

TABLE 3.1.1

3.1.2. Classify the following second-order systems according to damping.

a) $H(s) = \frac{15}{s^2+5s+15}$
b) $H(s) = \frac{25}{s^2+10s+25}$
c) $H(s) = \frac{35}{s^2+18s+35}$

Solution: For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \quad (3.1.2.1)$$

$$\omega^2 = 25, 2\zeta\omega = 10 \quad (3.1.2.2)$$

$$\implies \omega = 1, \zeta = 1 \quad (3.1.2.3)$$

and the system is critically damped. Similarly, the damping factors for other systems in Problem 3.1.2 are calculated and listed in Table 3.1.2

$H(s)$	ω	ζ	**Damping Type**
$\frac{35}{s^2+18s+35}$	$\sqrt{35}$	$\sqrt{\frac{81}{35}} > 1$	Overdamped
$\frac{25}{s^2+10s+25}$	5	1	Critically Damped
$\frac{15}{s^2+5s+15}$	$\sqrt{15}$	$\sqrt{\frac{5}{12}} < 1$	Underdamped

TABLE 3.1.2

3.1.3. By choosing an appropriate input, illustrate the effect of damping using a Python code to sketch the response.

3.2 Example

3.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \quad (3.1.1)$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \quad (3.1.2)$$

are at

$$s = -1 \quad (3.1.3)$$

i.e., the left half of s-plane. Hence the system is stable.

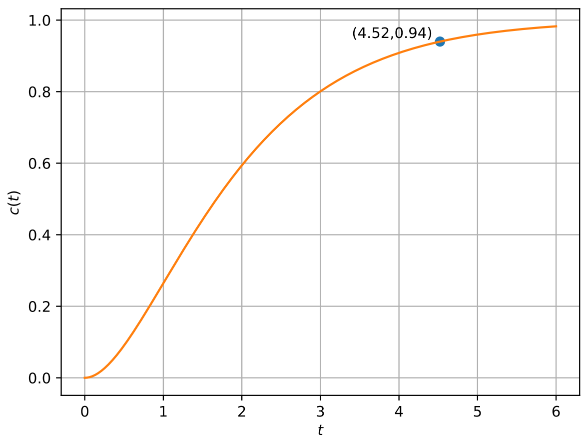

Fig. 3.2

3.2. Find and sketch the step response $c(t)$ of the system.
Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1+2s+s^2}\right] \tag{3.2.1}$$

$$= \frac{1}{s(1+s)^2} \tag{3.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \tag{3.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \tag{3.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right)u(t) \tag{3.2.5}$$

The following code plots $c(t)$ in Fig. 3.2

```
codes/ee18btech11002/plot.py
```

3.3. Find the steady state response of the system using the final value theorem. Verify using 3.2.5
Solution: To know the steady response value of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{3.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1+s+s^2}\right) = \frac{1}{1+0+0} = 1 \tag{3.3.2}$$

Using 3.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right)u(t) \tag{3.3.3}$$

$$= (1 - 0 - 0) = 1 \tag{3.3.4}$$

3.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{3.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{3.4.2}$$

4 Routh Hurwitz Criterion

4.1 Routh Array

4.1.1. Generate the Routh array for the polynomial,

$$f(s) = s^7 + s^6 + 7s^5 + 14s^4 + 31s^3 + 73s^2 + 25s + 200 \tag{4.1.1.1}$$

Solution:

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \end{vmatrix} \tag{4.1.1.2}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \end{vmatrix} \tag{4.1.1.3}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 0 & 0 & 0 & \end{vmatrix} \tag{4.1.1.4}$$

When such a case is encountered, we take the derivative of the expression formed the the coefficients above it i.e derivative of $8s^4+48s^2+200$.

$$\frac{d}{dx}(8s^4 + 48s^2 + 200) = 32s^3 + 96s$$

The coefficients of obtained expression are placed in the table.

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \end{vmatrix} \tag{4.1.1.5}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \end{vmatrix} \tag{4.1.1.6}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \end{vmatrix} \tag{4.1.1.7}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \\ 200 & & & \end{vmatrix} \tag{4.1.1.8}$$

So, the above one is the Routh-Hurwitz Table.

4.1.2. Find the number of roots of the polynomial in the right half of the s-plane.
Solution: The number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column. From 4.1.1.8, the polynomial in (4.1.1.1) has 4 roots lie on right-side of Imaginary Axis.

4.1.3. Write a Python code for generating each stage of the Routh Table.
Solution: The following code

```
codes/ee18btech11014/ee18btech11014.py
```

generates the various stages.

4.1.4. Find the roots of the polynomial in in (4.1.1.1) and verify that 4 roots are in the right half s-plane.
Solution: The following code generates the necessary roots.

```
codes/ee18btech11014/Roots.py
```

4.2 Marginal Stability

4.2.1. Consider a unity feedback system as shown in Fig. 4.2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{4.2.1.1}$$

where k greater than 0. Find its closed loop transfer function.

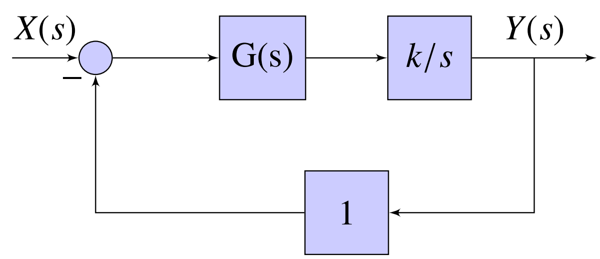

Fig. 4.2.1

Solution: $\because H(s) = 1$ in Fig. 4.2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{4.2.1.2}$$

$$\implies T(s) = \frac{k}{s^3 + 3s^2 + 2s} \tag{4.2.1.3}$$

4.2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{4.2.2.1}$$

$$\implies 1 + \left[\frac{k}{s^3 + 3s^2 + 2s}\right] = 0 \tag{4.2.2.2}$$

$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{4.2.2.3}$$

4.2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two

poles of unity feedback system on jω axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0 \tag{4.2.3.1}$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix}\begin{vmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{vmatrix} \tag{4.2.3.2}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \tag{4.2.3.3}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \tag{4.2.3.4}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \tag{4.2.3.5}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \tag{4.2.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 4.2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \tag{4.2.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{4.2.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{4.2.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{4.2.3.10}$$

4.2.4. Repeat the above using the determinant method.
Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 \cdots & & \\ 0 & a_1 & a_3 \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{pmatrix} \tag{4.2.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{4.2.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{4.2.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{4.2.4.4}$$

$$\ldots \tag{4.2.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (4.2.2.1),

$$D_1 = 1 \neq 0 \tag{4.2.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \tag{4.2.4.7}$$

4.2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.
Solution: The following code verifies the stability using the tabular method

```
codes/ee18btech11005_2.py
```

and the following one verifies using the determinant method.

```
codes/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infin-

ity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.

- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the right side of s-plane.That means it has sign changes.

4.3 Stability

4.3.1. The characteristic equation of linear time invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.1)$$

Find the condition for the system to be BIBO stable using the Routh Array.
solution

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.2)$$

The Routh hurwitz criterion:-

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 3 & k \\ 3 & 1 & 0 \\ \frac{8}{3} & k & 0 \\ \frac{\frac{8}{3}-3k}{\frac{8}{3}} & 0 & 0 \\ k & 0 & 0 \end{vmatrix} \quad (4.3.1.3)$$

From the above array, the given system is stable if

$$k > 0$$
$$\frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0 \quad (4.3.1.4)$$
$$\implies 0 < k < \frac{8}{9} \quad (4.3.1.5)$$

4.3.2. Modify the Python code in Problem 4.2.5 to verify your solution by choosing two different values of k.
Solution: The following code

```
codes/ee18btech11008.py
```

provides the necessary soution for $k = 0.5, 3$.

- $k = 0.5 < \frac{8}{9}$ has no sign changes in first column of its routh array.So the system is stable.
- $k = 3 > \frac{8}{9}$ has 2 sign changes in first column of its routh array.So the system is unstable.

5 State-Space Model

5.1 Controllability and Observabiity

5.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5.1.1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (5.1.2)$$

with parameters listed in Table 5.1.

Variable	Size	Description
$\mathbf{u}$	$p \times 1$	input(control) vector
$\mathbf{y}$	$q \times 1$	output vector
$\mathbf{x}$	$n \times 1$	state vector
$\mathbf{A}$	$n \times n$	state or system matrix
$\mathbf{B}$	$n \times p$	input matrix
$\mathbf{C}$	$q \times n$	output matrix
$\mathbf{D}$	$q \times p$	feedthrough matrix

TABLE 5.1

5.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (5.2.1)$$
$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \quad (5.2.2)$$
$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (5.2.3)$$
$$+ (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \quad (5.2.4)$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + D\mathbf{I}\mathbf{U}(s) \quad (5.2.5)$$

Substituting from (5.2.4) in the above,

$$\mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})\mathbf{U}(s)$$
$$+ \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \quad (5.2.6)$$

5.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I} \quad (5.3.1)$$

5.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.4.1}$$
$$D = 0 \tag{5.4.2}$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.3}$$

find **A** and **C** such that the state-space realization is in *controllable canonical form*.
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{5.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{5.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{5.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{5.4.7}$$

so the above equation can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.4.8}$$

Letting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.4.9}$$
$$\mathbf{X}_1 = \begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} \tag{5.4.10}$$
$$\mathbf{X} = \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix}, \tag{5.4.11}$$

$$\mathbf{X}_1(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{5.4.12}$$
$$Y = \mathbf{C}\mathbf{X}_1(s) \tag{5.4.13}$$

where

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{5.4.14}$$

5.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form*.
Solution: Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}, \tag{5.5.1}$$
$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{5.5.2}$$
$$\implies U(s) = Y(s)(s^3 + 3s^2 + 2s + 1) \tag{5.5.3}$$
$$\text{or, } Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.4}$$

Let

$$X_1(s) = Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.5}$$
$$X_2(s) = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{5.5.6}$$
$$X_3(s) = s^{-1}(U(s) - Y(s)) \tag{5.5.7}$$

$$\implies \begin{aligned} sX_1(s) &= -3Y(s) + X_2(s) \\ sX_2(s) &= -2Y(s) + X_3(s) \\ sX_3(s) &= U(s) - Y(s) \end{aligned} \tag{5.5.8}$$

Substituting $Y = X_1(s)$ the above,

$$sX_1(s) = -3X_1(s) + X_2(s) \tag{5.5.9}$$
$$sX_2(s) = -2X_1(s) + X_3(s) \tag{5.5.10}$$
$$sX_3(s) = U(s) - X_1(s) \tag{5.5.11}$$

which can be expressed as

$$\begin{pmatrix} sX_1(s) \\ sX_2(s) \\ sX_3(s) \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.5.12}$$

$$\text{or, } \begin{aligned} s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \\ Y(s) &= \mathbf{B}\mathbf{X}(s) \end{aligned} \tag{5.5.13}$$

where

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{5.5.14}$$
$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{5.5.15}$$

5.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

5.7. Theoretically, show that eigenvaues of $\mathbf{A}$ are the poles of $H(s)$.
Solution: As we know that the characteristic equation is det(sI-A)

$$s\mathbf{I}-\mathbf{A}=\begin{pmatrix}s&0&0\\0&s&0\\0&0&s\end{pmatrix}-\begin{pmatrix}0&1&0\\0&0&1\\-1&-2&-3\end{pmatrix} \quad (5.7.1)$$

$$=\begin{pmatrix}s&-1&0\\0&s&-1\\1&2&s+3\end{pmatrix} \quad (5.7.2)$$

$$\implies \left|s\mathbf{I}-\mathbf{A}\right| = s(s^2+3s+2)+1(1) \quad (5.7.3)$$

$$= s^3+3s^2+2s+1 \quad (5.7.4)$$

which is the denominator of $H(s)$ in (5.4.1)

5.2 Second Order System

5.2.1. Consider a state-variable model of a system

$$\begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix}=\begin{pmatrix}0&1\\-\alpha&-2\beta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}b_1\\b_2\end{pmatrix}r \quad (5.2.1.1)$$

$$y=\begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} \quad (5.2.1.2)$$

where y is the output, and r is the input.

5.2.2. List the various state matrices in (5.2.1.1)

5.2.3. Find the the system transfer function $H(s)$.
Solution: From (**??**) and , (5.3.1), the transfer function for the state space model is

$$H(s)=C(sI-A)^{-1}B+D \quad (5.2.3.1)$$

$$=\frac{\begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}s+2\beta&1\\-\alpha&s\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}}{s(s+2\beta)+\alpha} \quad (5.2.3.2)$$

$$=\frac{b_1(s+2\beta)+b_2}{s^2+2s\beta+\alpha} \quad (5.2.3.3)$$

$$\implies H(s)=\frac{b_1s}{s^2+2s\beta+\alpha}+\frac{2b_1\beta+b_2}{s^2+2s\beta+\alpha} \quad (5.2.3.4)$$

5.2.4. Find the Damping ratio ζ and the Undamped natural frequency ω_n of the system.
Solution: Generally for a second order system the transfer function is given by 3.1.1.1

$$H(s)=\frac{\omega_n^2}{s^2+2s\zeta\omega_n+\omega_n^2} \quad (5.2.4.1)$$

Comparing the denominator of the above with (5.2.3.4),

$$2\zeta\omega_n=2\beta, \quad (5.2.4.2)$$

$$\omega_n^2=\alpha \quad (5.2.4.3)$$

$$\implies \zeta=\frac{\beta}{\sqrt{\alpha}},\omega_n=\sqrt{\alpha} \quad (5.2.4.4)$$

5.2.5. Using Table 3.1.1, explain how the damping conditions depend upon α and β.

6 Nyquist Plot

6.1. The open loop transfer function of a unity feedback system is given by

$$G(s)=\frac{\pi e^{-0.25s}}{s} \quad (6.1.1)$$

6.2. Find $\mathrm{Re}\{G(\jmath\omega)\}$ and $\mathrm{Im}\{G(\jmath\omega)\}$.
Solution: From (6.1.1),

$$G(j\omega)=\frac{\pi}{\omega}(-\sin 0.25\omega-j\cos 0.25\omega) \quad (6.2.1)$$

$$\implies \mathrm{Re}\{G(\jmath\omega)\}=\frac{\pi}{\omega}(-\sin 0.25\omega) \quad (6.2.2)$$

$$\mathrm{Im}\{G(\jmath\omega)\}=\frac{\pi}{\omega}(-j\cos 0.25\omega) \quad (6.2.3)$$

6.3. Sketch the Nyquist plot.
Solution: The Nyquist plot is a graph of $\mathrm{Re}\{G(\jmath\omega)\}$ vs $\mathrm{Im}\{G(\jmath\omega)\}$. The following python code generates the Nyquist plot in Fig. 6.3

6.4. Find the point at which the Nyquist plot of G(s) passes through the negative real axis
Solution: Nyquist plot cuts the negative real axis at ω for which

$$\angle G(\jmath\omega)=-\pi \quad (6.4.1)$$

From (6.1.1),

$$G(\jmath\omega)=\frac{\pi e^{-\frac{\jmath\omega}{4}}}{\jmath\omega}=\frac{\pi e^{-\jmath\left(\frac{\omega}{4}+\frac{\pi}{2}\right)}}{\omega} \quad (6.4.2)$$

$$\implies \angle G(\jmath\omega)=-\left(\frac{\omega}{4}+\frac{\pi}{2}\right) \quad (6.4.3)$$

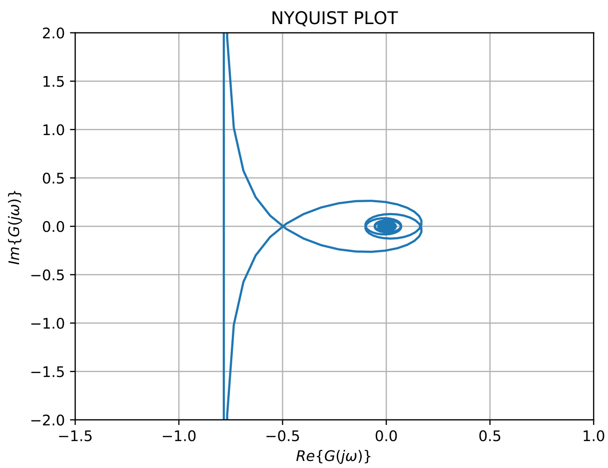

Fig. 6.3

From (6.4.3) and (6.4.1),

$$\frac{\omega}{4} + \frac{\pi}{2} = \pi \tag{6.4.4}$$

$$\implies \omega = 2\pi \tag{6.4.5}$$

Also, from (6.1.1),

$$\left|G(\jmath\omega)\right| = \frac{\pi}{|\omega|} \tag{6.4.6}$$

$$\implies \left|G(\jmath 2\pi)\right| = \frac{1}{2} \tag{6.4.7}$$

6.5. Use the Nyquist Stability criterion to determine if the system in (6.4.3) is stable.

Variable	**Value**	**Description**
Z	0	Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane
P	0	Poles of $G(s)H(s)$ in right half of s plane
N	0	No of clockwise encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot

TABLE 6.5

Solution: Consider Table 6.5. According to the Nyquist stability criterion,

a) If the open-loop transfer function $G(s)$ has a zero pole of multiplicity l, then the Nyquist plot has a discontinuity at ω =0. During further analysis it should be assumed that the phasor travels l times clock-wise along a semicircle of infinite radius. After applying this rule, the zero poles should be neglected, i.e. if there are no other unstable poles, then the open-loop transfer function $G(s)$ should be considered stable.
b) If the open-loop transfer function $G(s)$ is stable, then the closed-loop system is unstable for any encirclement of the point -1. If the open-loop transfer function $G(s)$ is unstable, then there must be one counter clock-wise encirclement of -1 for each pole of $G(s)$ in the right-half of the complex plane.
c) The number of surplus encirclements (N + P greater than 0) is exactly the number of unstable poles of the closed-loop system.
d) However, if the graph happens to pass through the point $-1+\jmath 0$, then deciding upon even the marginal stability of the system becomes difficult and the only conclusion that can be drawn from the graph is that there exist zeros on the $\jmath\omega$ axis.

From (6.1.1), $G(s)$ is stable since it has a single pole at $s = 0$. Further, from Fig. 6.3, the Nyquist plot doesnot encircle $s = -1$. From Theorem 6.5b, we may conclude that the system is stable.

7 Compensators

7.1. The Transfer function of Phase Lead Compensator is given by

$$D(s) = \frac{3(s + \frac{1}{3T})}{(s + \frac{1}{T})} \tag{7.1.1}$$

Find out the frequency (in rad/sec), at which $\angle D(j\omega)$ is maximum?

Solution: The basic requirement of the phase lead network is that all poles and zeros of the transfer function of the network must lie on negative real axis interlacing each other with a zero located as the nearest point to origin.
Substituting $s = j\omega$ in D(s), we get

$$D(j\omega) = \frac{3(j\omega + \frac{1}{3T})}{(j\omega + \frac{1}{T})} \tag{7.1.2}$$

The phase of this transfer function $\phi(\omega)$ is given by,

$$\phi(\omega) = \tan^{-1}(3\omega T) - \tan^{-1}(\omega T) \tag{7.1.3}$$

$\phi(\omega)$ has its maximum at ω_c Where $\phi'(\omega_c) = 0$,

$$\phi'(\omega_c) = 0 = \frac{3T}{1+(3\omega_c T)^2} - \frac{T}{1+(\omega_c T)^2} \tag{7.1.4}$$

After solving and Simplification , we have

$$\omega_c^2 T^2 = \frac{1}{3} \tag{7.1.5}$$

$$\omega_c = \sqrt{\frac{1}{3T^2}} \tag{7.1.6}$$

7.2. Verify your result through a plot.
Solution: The following plots the Phase value of the transfer function,

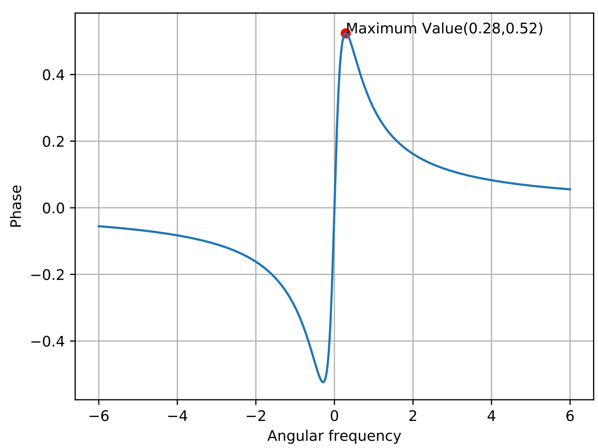

Fig. 7.2

Applications:

a) Phase lead Compensators can be used as High pass filters,Differentiators.
b) They are used to reduce steady state errors.
c) Increases Phase Margin , relative stability.

7.3. What is purpose of of a Phase Lead Compensator?

7.4. Through an example, show how the compensator in Problem 7.1 can be used in a control system.

8 Phase Margin

8.1. The open loop transfer function of a system is

$$G(s) = \frac{2}{(s+1)(s+2)} \tag{8.1.1}$$

Find its magnitude and phase response.
Solution: Substituting $s = \jmath\omega$ in (8.1.1),

$$G(\jmath\omega) = \frac{1}{(j\omega+1)(j\omega+2)} \tag{8.1.2}$$

$$\implies \left|G(\jmath\omega)\right| = \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} \tag{8.1.3}$$

$$\angle G(\jmath\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) \tag{8.1.4}$$

8.2. Find ω for which the gain of (8.1.1) first becomes 1.
Solution: From (8.1.3)

$$\left|G(\jmath\omega)\right| = 1 \tag{8.2.1}$$

$$\implies \frac{2}{\left(\sqrt{\omega^2+1}\right)\left(\sqrt{\omega^2+4}\right)} = 1 \tag{8.2.2}$$

$$\implies \omega_{gc} = 0 \tag{8.2.3}$$

which is the desired frequency.

8.3. Find $\angle G(\jmath\omega_{gc}) + 180°$. This is known as the *phase margin*(PM)
Solution: From (8.1.4),

$$\angle G(\jmath\omega) = 0° \implies PM = 180° \tag{8.3.1}$$

8.4. Verify your result by plotting the gain and phase plots of $G(\jmath\omega)$.
Solution: The following code plots Fig. 8.4

```
codes/ee18btech11017.py
```

The Phase plot is as shown,

8.5. A positive phase margin for the open loop system indicates a stable closed loop system. (8.3.1) indicates that $G(s)$ with unity feedback is stable. Show that the roots of $1+G(s)$ lie in the left half plane proving closed loop stability.
Solution: Let the closed loop transfer function

$$T(s) = \frac{G(s)}{1+G(s)} \tag{8.5.1}$$

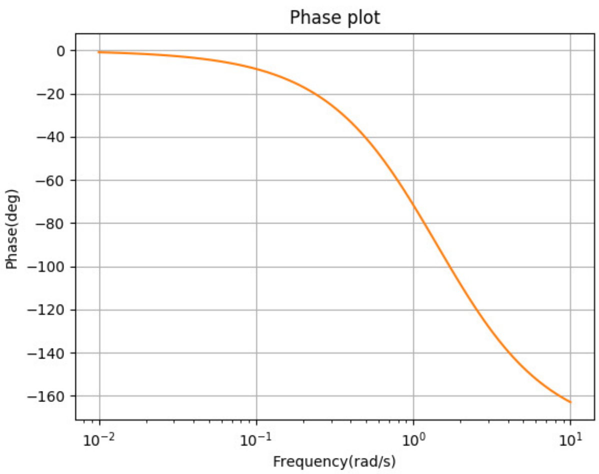

Fig. 8.4

Then

$$1 + G(s) = 0 \tag{8.5.2}$$
$$\implies s^2 + 3s + 4 = 0 \tag{8.5.3}$$
$$\text{or } s = -1.5 + 1.3j, -1.5 - 1.3j \tag{8.5.4}$$

Since the roots are in the left half plane, the system is stable.

8.6. Instead of unity feedback, consider a system with

$$H(s) = \frac{50}{s+1} \tag{8.6.1}$$

Compute the open loop phase margin for this system.

Solution:

$$\because G(s)H(s) = \frac{100}{(s+1)^2(s+2)}, \tag{8.6.2}$$

the magnitude and phase are

$$\left|G(\jmath\omega)H(\jmath\omega)\right| = \frac{10^2}{\sqrt{(\omega^2+1)^2}\sqrt{\omega^2+4}} \tag{8.6.3}$$
$$\angle G(\jmath\omega)H(\jmath\omega) = -\tan^{-1}\frac{\omega}{2} - 2\tan^{-1}(\omega) \tag{8.6.4}$$

The gain crossover frequency is given by

$$\frac{10^2}{\sqrt{\omega_{gc}^2+4}\sqrt{(\omega_{gc}^2+10^2)^2}} = 1 \tag{8.6.5}$$
$$\tag{8.6.6}$$
$$\omega_{gc}^6 + 6\omega_{gc}^4 + 9\omega_{gc}^2 - 9996 = 0 \tag{8.6.7}$$
$$\implies \omega_{gc} = 4.42 \tag{8.6.8}$$

From (8.6.4) and (8.6.8), the phase margin is

$$PM = 180^\circ - 2\tan^{-1}(\omega_{gc}) - \tan^{-1}\left(\frac{\omega_{gc}}{2}\right) \tag{8.6.9}$$
$$\implies P.M = -40.15^\circ \tag{8.6.10}$$

8.7. Verify your result through the magnitude and phase plot.
Solution: The following code plots Fig. **??**

```
codes/ee18btech11017_2.py
```

8.8. Since the PM in (8.6.10) is negative, the closed loop system is unstable . Verify this using the Routh-Hurwitz criterion.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{8.8.1}$$
$$\implies s^3 + 4s^2 + 5s + 102 = 0 \tag{8.8.2}$$

Constructing the routh array for (8.8.2),

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \end{vmatrix} \tag{8.8.3}$$
$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \\ 102 & 0 & 0 \end{vmatrix} \tag{8.8.4}$$

$\because$ there are two sign changes in the first column of the routh array, two poles lie on right half of s-plane. Therefore,the system is unstable.

9 Gain Margin

9.1. The open loop transfer function of a feedback control system is

$$G(s) = \frac{1}{s(1+2s)(1+s)} \tag{9.1.1}$$

Find the magnitude and phase of $\left|G(\jmath\omega)\right|$.
Solution:

$$G(\jmath\omega) = \frac{1}{\jmath\omega(1+2\jmath\omega)(1+\jmath\omega)} \tag{9.1.2}$$

$$= \frac{1}{\jmath\omega(1+3\jmath\omega-2\omega^2)} = \frac{1}{\jmath\omega-3\omega^2} \tag{9.1.3}$$

$$= \frac{1}{-3\omega^2+\jmath\omega(1-2\omega^2)} \tag{9.1.4}$$

$$\implies \angle G(\jmath\omega) = -tan^{-1}\left(\frac{\omega(1-2\omega^2)}{-3\omega^2}\right) \tag{9.1.5}$$

9.2. The frequency at which the phase of open-loop transfer function reaches -180° or +180° depending upon the range of tan inverse function) is defined to be the phase crossover frequency. Find the phase crossover frequency for (9.1.1).
Solution: From (9.1.5), at $\omega = \omega_{pc}$

$$\omega(1-2\omega^2) = 0 \tag{9.2.1}$$

$$\implies \omega_{pc} = \frac{1}{\sqrt{2}} \tag{9.2.2}$$

9.3. The gain Margin is given by,

$$GM = -20log_{10}\left|G\left(\jmath\omega_{pc}\right)\right| = 20log_{10}k_g \tag{9.3.1}$$

where

$$k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} \tag{9.3.2}$$

Find the GM for (9.1.5).
Solution:

$$\left|G\left(\jmath\omega_{pc}\right)\right| = \frac{1}{\left(\frac{3}{2}\right)} \implies k_g = \frac{1}{\left|G\left(\jmath\omega_{pc}\right)\right|} = \frac{3}{2} \tag{9.3.3}$$

i.e G.M = 3.5dB

The greater the Gain Margin (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.

9.4. Obtain the GM from the Bode plot. **Solution:** The following code

```
codes/ee18btech11016.py
```

plots the amplitude and phase of (9.1.1) in Fig. . So,in the above figure, since $20log_{10}(G(\jmath\omega_{pc}))$

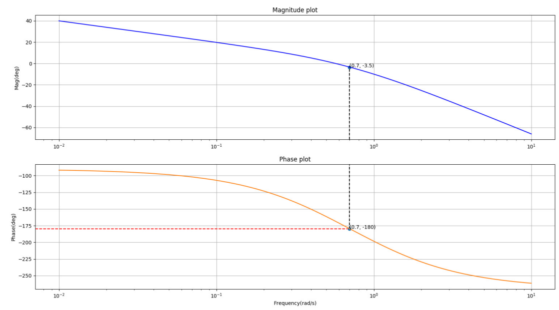

Fig. 9.4

= -3.5dB at $\omega_{pc} = -180°$ so GM = +3.5dB.

9.5. A positive GM indicates closed loop stability with unity feedback. Verify this for (9.1.1).
Solution: The characteristic equation is

$$1+G(s) = 0 \implies 2s^3+3s^2+s+1 = 0 \tag{9.5.1}$$

Constructing the routh array

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \end{vmatrix} \tag{9.5.2}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (1/3) & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{9.5.3}$$

There are no sign changes in the first column of the routh array. $\therefore$ the system is stable.

9.6. Instead of unity feedback, consider a system with

$$H(s) = \frac{1}{s+1} \tag{9.6.1}$$

Compute the open loop gain margin for this system.
Solution:

$$\because G(s)H(s) = \frac{1}{s(1+2s)(s+1)^2}, \tag{9.6.2}$$

Find the magnitude and phase of $\left|G\left(\jmath\omega\right)H\left(\jmath\omega\right)\right|$

$$G(j\omega)H(j\omega) = \frac{1}{(2\omega^4 - 4\omega^2) + j(\omega - 5\omega^3)} \tag{9.6.3}$$

$$=> \angle G(j\omega)H(j\omega) = -tan^{-1}(\frac{\omega - 5\omega^3}{2\omega^4 - 4\omega^2}) \tag{9.6.4}$$

Now , for ω=ω_{pc} , the imaginary part of $G(j\omega)H(j\omega)$=0. i.e,

$$\omega(1 - 5\omega^2) = 0 => \omega_{pc} = \frac{1}{\sqrt{5}} \tag{9.6.5}$$

$$G.M = -20log_{10}|G(j\omega_{pc})H(j\omega_{pc})| = 20log_{10}k_g \tag{9.6.6}$$

where

$$k_g = \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|} \tag{9.6.7}$$

So, $k_g = \frac{1}{\frac{1}{\frac{18}{25}}} = 18/25$.

And hence G.M=-2.853dB.i.e the gain margin is negative and hence we can say that the system is unstable.

9.7. Let's verify this using the RH array : So,the closed loop transfer function is given by

$$T(s) = \frac{1}{1 + (s(1 + 2s)(1 + s)^2)} \tag{9.7.1}$$

$$=> D(s) = 1 + s(1 + 2s)(1 + s)^2 = 2s^4 + 5s^3 + 4s^3 + s + 1 \tag{9.7.2}$$

So,the characteristics equation is given by D(s) = 0.i.e,

$$=> 2s^4 + 5s^3 + 4s^3 + s + 1 = 0 \tag{9.7.3}$$

Constructing routh array for above equation of D(s),

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 2 & 4 & 1 \\ 5 & 1 & 0 \\ (18/5) & 1 & 0 \end{vmatrix} \tag{9.7.4}$$

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (18/5) & 1 & 0 \\ (-7/18) & 0 & 0 \end{vmatrix} \tag{9.7.5}$$

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix}\begin{vmatrix} 2 & 1 & 0 \\ 3 & 1 & 0 \\ (18/5) & 1 & 0 \\ (-7/18) & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \tag{9.7.6}$$

There are 2 sign changes in the first column of the routh array. So, 2 poles lie on right half of s-plane.
Therefore,the system is unstable.

Hence,we can say that from both Routh-hurwitz criterion and from the gain margin concept we are getting the same answers.