Assignment-1

Ganraj Borade - EE18BTECH11016

Download all python codes from

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https://github.com/ganrajborade/EE3025/tree/main/A1/codes
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and latex-tikz codes from

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https://github.com/ganrajborade/EE3025/tree/main/A1
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1 Problem

1.1. Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \quad (1.1.1)$$

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \quad (1.1.2)$$

1.2. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \quad (1.2.1)$$

and $H(k)$ using h(n).

1.3. Compute

$$Y(k) = X(k)H(k) \quad (1.3.1)$$

2 Solution

2.1. The Impulse Response of the LTI system is the output of the system when Unit Impulse Signal is given as input to the system. Impulse response h(n) can be found from given difference equation as follows (h(n) is IIR Filter)

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \quad (2.1.1)$$

2.2. DFT of a Input Signal $x(n)$ is

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \quad (2.2.1)$$

2.3. Let $W_N = e^{-j2\pi/N}$
We can express X as Matrix Multiplication of DFT Matrix and x.

$$X = \left[W_N^{ij}\right]_{N\times N} x, \quad i, j = 0, 1, \ldots, N-1 \quad (2.3.1)$$

2.4. Properties :

a)

$$W_N^{k+N/2} = -W_N^k$$

b)

$$W_N^{k+N} = W_N^k$$

c)

$$W_N^2 = W_{N/2}$$

2.5. Using properties to derive FFT from DFT :

$$\mathcal{X}(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \quad (2.5.1)$$

$$= \sum_{n=even} x(n)W_N^{kn} + \sum_{n=odd} x(n)W_N^{kn} \quad (2.5.2)$$

$$= \sum_{m=0}^{2} x(2m)W_N^{2mk} + \sum_{m=0}^{2} x(2m+1)W_N^{(2m+1)k} \quad (2.5.3)$$

using property (c), we get,

$$\mathcal{X}(k) = \sum_{m=0}^{2} x(2m)W_{N/2}^{mk} + W_N^k \sum_{m=0}^{2} x(2m+1)W_{N/2}^{mk} \quad (2.5.4)$$

$$= X_1(k) + W_N^k X_2(k) \quad (2.5.5)$$

- X_1(k) and X_2(k) are 3 point DFTs of x(2m) and x(2m+1) , m=0,1,2.
- X_1(k) and X_2(k) are periodic, Hence X_1(k+3) = X_1(k) and X_2(k+3) = X_1(k).

2.6. Calculating X1 and X2

$$X_1(k) = \sum_{m=0}^{2} x(2m) W_3^{mk} \tag{2.6.1}$$

$$X_2(k) = \sum_{m=0}^{2} x(2m+1) W_3^{mk} \tag{2.6.2}$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 & 0 & 0 & 0 \\ W_3^0 & W_3^1 & W_3^2 & 0 & 0 & 0 \\ W_3^0 & W_3^2 & W_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_3^0 & W_3^0 & W_3^0 \\ 0 & 0 & 0 & W_3^0 & W_3^1 & W_3^2 \\ 0 & 0 & 0 & W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(1) \\ x(3) \\ x(5) \end{bmatrix} \tag{2.6.3}$$

2.7. Calculating X

$$X(k) = X_1(k) + W_N^k X_2(k) \tag{2.7.1}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^2 \\ 1 & 0 & 0 & W_6^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^5 \end{bmatrix} \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_2(0) \\ X_2(1) \\ X_2(2) \end{bmatrix} \tag{2.7.2}$$

$$\implies \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 13 + 0j \\ -4 - \sqrt{3}j \\ 1 + 0j \\ -1 + 0j \\ 1 + 0j \\ -4 + \sqrt{3}j \end{bmatrix} \tag{2.7.3}$$

2.8. DFT of a Impulse Response $h(n)$ is

$$H(k) \triangleq \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{2.8.1}$$

2.9. Now to find $H(k)$ we need to know h(n) first. So we will first calculate h(n). For that we need to first find the H(z) by applying Z-transform on equation (2.1.1) i.e.,

$$H(z) = \frac{2(z^2+1)}{z(2z+1)} \tag{2.9.1}$$

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{2.9.2}$$

From this we can say that h(n) is,

$$h(n) = Z^{-1}\left[\frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}}\right] \tag{2.9.3}$$

$$h(n) = \left[\frac{-1}{2}\right]^n u(n) + \left[\frac{-1}{2}\right]^{n-2} u(n-2) \tag{2.9.4}$$

Similarly, like X(k), we can calculate H(k).

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & W_6^0 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^2 \\ 1 & 0 & 0 & W_6^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & W_6^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & W_6^5 \end{bmatrix} \tag{2.9.5}$$

$$W_2 = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 & 0 & 0 & 0 \\ W_3^0 & W_3^1 & W_3^2 & 0 & 0 & 0 \\ W_3^0 & W_3^2 & W_3^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_3^0 & W_3^0 & W_3^0 \\ 0 & 0 & 0 & W_3^0 & W_3^1 & W_3^2 \\ 0 & 0 & 0 & W_3^0 & W_3^2 & W_3^4 \end{bmatrix} \tag{2.9.6}$$

Same as we done while calculating X(k) using Eq (2.7.2)

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = W_1 W_2 \begin{bmatrix} h(0) \\ h(2) \\ h(4) \\ h(1) \\ h(3) \\ h(5) \end{bmatrix} \tag{2.9.7}$$

$$\implies \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \\ H(4) \\ H(5) \end{bmatrix} = \begin{bmatrix} 1.28125 + 0j \\ 0.51625 - 0.5142j \\ -0.07813 + 1.1096j \\ 3.84375 + 0j \\ -0.07183 - 1.1096j \\ 0.51625 + 0.5142j \end{bmatrix} \tag{2.9.8}$$

2.10. We can now compute $Y(k)$ using Eq (2.10.1)

$$Y(k) = X(k)H(k) \tag{2.10.1}$$

So, $Y(k)$ is obtained element wise multiplication of $X(k)$ and $H(k)$

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} X(0) \cdot H(0) \\ X(1) \cdot H(1) \\ X(2) \cdot H(2) \\ X(3) \cdot H(3) \\ X(4) \cdot H(4) \\ X(5) \cdot H(5) \end{bmatrix} \tag{2.10.2}$$

Computing the above expression we get,

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \end{bmatrix} = \begin{bmatrix} 16.6562 + 0j \\ -2.95312 + 1.16372j \\ -0.07812 + 1.10959j \\ -3.84375 - 9.27556j \\ -0.07812 - 1.10959j \\ -2.95312 - 1.16372j \end{bmatrix} \tag{2.10.3}$$

2.11. The following code computes Y and generates magnitude and phase plots of X, H, Y

https://github.com/ganrajborade/EE3025/blob/main/A1/codes/EE18BTECH11016.py

2.12. The following plots are obtained