EE2010 : Engineering Electromagnetics.

Homework Assignment 3

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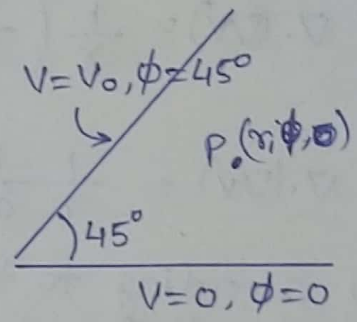
1 It is given that we have a wedge capacitor with two infinite conducting plates at an angle $\phi = 45°$. The structure is invariant in the z-dir^n and there exists an insulating gap b/w the plates.



(a) We have to find the potential between the plates by writing the general solution to Laplace equation, and then applying the boundary conditions.

→ If we confine our attention to places where there is no charge, then the poisson's equation $\left(\nabla^2 V = \frac{-\rho}{\epsilon_0}\right)$ reduces to Laplace's equation :-

$\nabla^2 V = 0$, if we written out in Cartisian coordinates,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In our case of problem, let's solve it in cylindrical coordinate system. Any point P in bet^n those plates can be written as

$P \equiv (r, \phi, 0)$.

→ $\nabla^2 V = 0$ { In between the plates as there is no charge there.)

Now $\nabla^2 V = \nabla \cdot \nabla V$

Now $\nabla V = \frac{1}{h_1}\left(\frac{\partial V}{\partial u}\hat{a}_u\right) + \frac{1}{h_2}\left(\frac{\partial V}{\partial v}\hat{a}_v\right) + \frac{1}{h_3}\left(\left(\frac{\partial V}{\partial w}\right)\hat{a}_w\right)$

In case of cylindrical coordinate system ; $h_1 = 1$, $h_2 = r$, $h_3 = 1$

$\hat{a}_u = \hat{r}$, $\hat{a}_v = \hat{\phi}$, $\hat{a}_w = \hat{z}$

$\therefore \nabla V$ (in our case) $= \left(\frac{\partial V}{\partial r}\right)\hat{r} + \frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi} + \frac{1}{1}\underbrace{\left(\frac{\partial V}{\partial z}\right)}_{=0}\hat{z}$ (as structure is invariant in $\hat{z}$-dirn)

$$\therefore \nabla V = \left(\frac{\partial V}{\partial r}\right)\hat{r} + \frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi}$$

Now $\nabla\cdot(\nabla V) = \frac{1}{h_1h_2h_3}\left(\frac{\partial(\nabla V(h_2h_3))}{\partial u} + \frac{\partial(h_1h_3\nabla V)}{\partial v} + \frac{\partial}{\partial w}(h_1h_2\nabla V)\right)$

$$\therefore \nabla\cdot(\nabla V) = \frac{1}{r}\left(\frac{\partial}{\partial r}(r(\nabla V)) + \frac{\partial}{\partial \phi}(\nabla V) + \underbrace{\frac{\partial(r(\nabla V))}{\partial z}}_{=0}\right)$$

$$\therefore \nabla\cdot(\nabla V) = \frac{1}{r}\frac{\partial(r\nabla V)}{\partial r} + \frac{\partial(\nabla V)}{\partial \phi}$$

Now according to the structure, for a particular direction in which $\vec{r}$ present ($|\vec{r}|$ can be of any value), the voltage along that dirn is constant.

$$\therefore \frac{\partial V}{\partial r} = 0$$

$\therefore \nabla V = \frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi}$ and $\nabla\cdot(\nabla V) = \cancel{\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{r}\frac{\partial V}{\partial \phi}}\, 0 + \frac{\partial(\nabla V)}{\partial \phi}$.

$$\therefore \nabla\cdot\nabla V = \nabla^2 V = \frac{\partial}{\partial \phi}\left(\frac{1}{r}\frac{\partial V}{\partial \phi}\right) = \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2}$$

$\therefore \nabla^2 V = 0$ (In b/w the two plates).

$$\therefore \frac{1}{r}\frac{\partial^2 V}{\partial \phi^2} = 0 \Rightarrow \frac{\partial^2 V}{\partial \phi^2} = 0 \Rightarrow \frac{\partial V}{\partial \phi} = A$$

Hence $\boxed{V = A\phi + B} \longrightarrow$ ~~Sol~~ got by solving Laplace equation

Now If we apply the boundary conditions $\Rightarrow$

At $\phi = 45^\circ$ i.e $\phi = \frac{\pi}{4}$, $V = V_0$

and at $\phi = 0$, $V = 0 \Rightarrow 0 = 0 + B \Rightarrow \boxed{B = 0}$

and At $\phi = \frac{\pi}{4}$, $V = V_0 \Rightarrow V_0 = \left(\frac{\pi}{4}\right)A \Rightarrow \boxed{A = \frac{4V_0}{\pi}}$

Hence, $\boxed{V = \frac{4V_0\phi}{\pi}}$ Volts (In SI system) or ~~$V = \frac{4V_0\phi}{\pi}$ V~~ $\boxed{V = \frac{4V_0}{\pi}\phi\ V}$ In SI system

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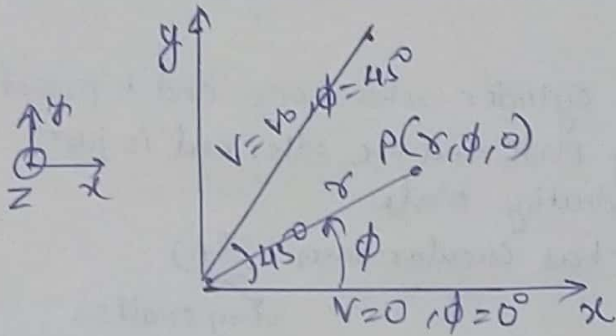
(b)

Now we know that $\vec{E}$ is a conservative field. So, it can be written as $\vec{E} = -\nabla V$

and we have calculated ∇V in part (a) :-

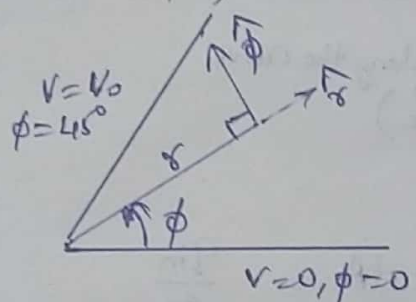
$$\nabla V = \left(\frac{\partial V}{\partial r}\right)\hat{r} + \frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi} = \frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi}$$

Hence $$\boxed{\vec{E} = -\frac{1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi}}$$



I am assuming the shown coordinate system. Here the dirn of $\hat{r}$ is along the radial direction and the dirn of $\hat{\phi}$ is along the tangential dirn to $\hat{r}$ or at 90° (in the dirn of rotation of ϕ) to $\hat{r}$.

Hence the dirn of $\hat{\phi}$ at point P :-



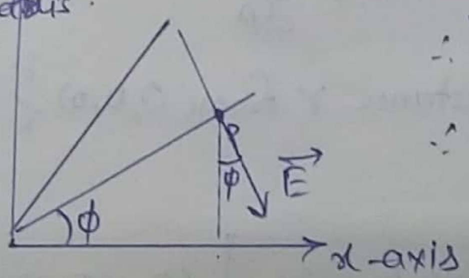
So At angle ϕ, & radial distance r the electric field $\vec{E} = \frac{-1}{r}\left(\frac{\partial V}{\partial \phi}\right)\hat{\phi}$.

$$\Rightarrow \vec{E} = \frac{-1}{r}\frac{\partial}{\partial \phi}\left(\frac{4V_0\phi}{\pi}\right)\hat{\phi}$$

$$= \frac{-1}{r} \times \frac{4V_0}{\pi}\hat{\phi}$$

Hence, $$\boxed{\vec{E} = \frac{-1}{r}\frac{4V_0}{\pi}\hat{\phi}}$$

Let's convert the above $\vec{E}$ into Cartesian coordinate system:-



$$\therefore \vec{E} = |\vec{E}|(+\sin\phi)\hat{i} + |\vec{E}|\cos\phi(-\hat{j})$$

$$\therefore \vec{E} = \left(\frac{4V_0}{\pi r}\sin\phi\right)\hat{i} - \left(\frac{4V_0}{\pi r}\cos\phi\right)\hat{j}$$

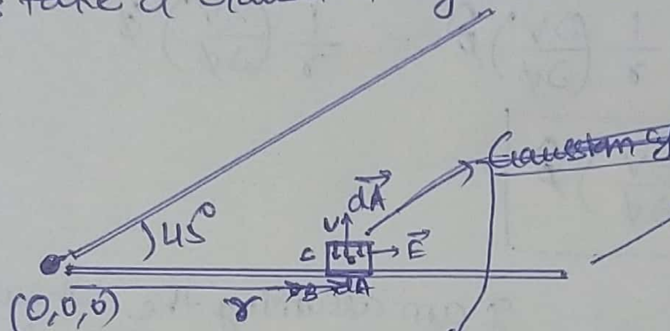
$\therefore \vec{E}$ in Cartesian Coordinate system $= \left(\frac{4V_0}{\pi r}\sin\phi\right)\hat{i} - \left(\frac{4V_0}{\pi r}\cos\phi\right)\hat{j}$

Now we need to find surface charge density of the conductor at $\phi = 0^\circ$

Therefore, $\vec{E}$ at $\phi = 0^\circ \Rightarrow$

$$\vec{E}\Big|_{\phi=0^\circ} = \left(\frac{-4V_0}{\pi r}\right)\hat{j}$$

Now Let's take a Gaussian Cylinder like :-



~~Gaussian cylinder whose one surface is in conducting plate~~

we can assume thickness is very very small.

Gaussian cylinder whose one end is present in the conducting plate and the other end is just outside the conducting plate
(Cylinder has circular area = dA)

very small so we can assume σ to be const.

Hence $\oint \vec{E}\cdot\vec{dA} = \frac{q_{in}}{\epsilon_0}$ → on Gaussian cylinder

Since $\vec{E}$ is $\perp^{er}$ to plate (lower), hence

$\int \vec{E_c}\cdot\vec{dA} = 0$ ($\vec{E_c}$ = Electric field along the curve and $\vec{dA} = \perp^{er}$ to $\vec{E_c}$)

$\int \vec{E_B}\cdot\vec{dA} = 0$ (As $\vec{E_B} = 0$)

$$\oint \vec{E}\cdot dA = \left(\frac{-4V_0}{\pi r}\right)\hat{j}\cdot(dA)\hat{j} = \frac{-4V_0\, dA}{\pi r} = \frac{q_{in}}{\epsilon_0}$$

$$\therefore \boxed{\frac{-4V_0\epsilon_0}{\pi r} = \frac{q_{in}}{dA}}$$

Here, q_{in} is charge on conducting plate (lower) on dA area.

$$\therefore \sigma = \frac{q_{in}}{dA}$$

$$\therefore \boxed{\sigma = \frac{-4V_0\epsilon_0}{\pi r}\ C/m^2}$$ at distance r from (0,0,0) {basically $\sigma \propto \frac{1}{r}$ on lower plate)

Hence at $\phi = 0^\circ$, $\boxed{\sigma = \frac{-4V_0\epsilon_0}{\pi r}\ C/m^2}$ (In SI system)

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a → I have plotted the graphs for both Relative error v/s Iterations and meshgrid to show potential.

b Here we have

→ $V = \left(\frac{4V_0}{\pi}\right)\phi$ V (In SI system)

Now at $\phi = \frac{45^\circ}{2}$ i.e $\phi = \frac{\pi}{8}^c$ (ϕ=22.5°)

then $V = \left(\frac{4V_0}{\pi}\right) \times \frac{\pi}{8} \Rightarrow V_0 = \frac{1}{2}V. \Rightarrow \boxed{V_0 = 0.5V}$

In graph also, I have plotted it.

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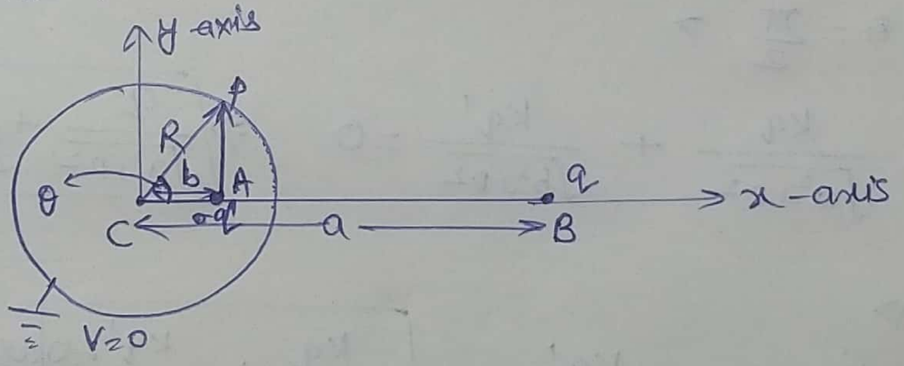
3 It is given that a point charge q is situated at a distance

→ a from the center of a grounded conducting sphere of radius R, & we need to find ~~Find~~ the potential outside the sphere using method of images.



By symmetry, the image charge q' must lie somewhere on the x-axis defined by q and the center of the sphere.

Let's assume that image charge(q') is located at A and CA=b.

Now since the conducting sphere is grounded. So $V_{sphere} = 0$.

Now if we consider only q' and q charges and find the potential on the surface of sphere (at point P) :-

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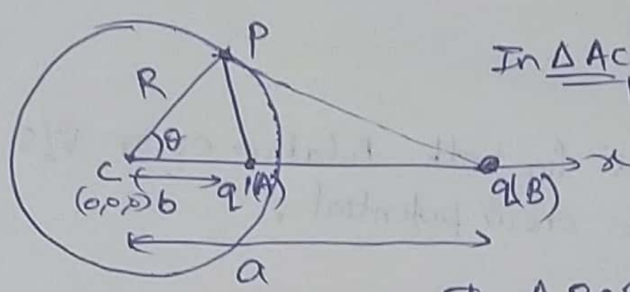


we have to find AP and BP.

In $\Delta ACP \Rightarrow$
Now $\cos\theta = \dfrac{(b)^2+(R)^2-(AP)^2}{2bR}$

$\therefore (AP)^2 = b^2+R^2-2bR\cos\theta$

In $\Delta BCP \Rightarrow$

$$\cos\theta = \frac{a^2+R^2-(BP)^2}{2aR}$$

$$\therefore (BP)^2 = a^2+R^2-2aR\cos\theta$$

$$\therefore V_P = \frac{Kq}{(BP)} + \frac{K(q')}{(AP)} \quad \text{where } K = \frac{1}{4\pi\epsilon_0}$$

$$\therefore \boxed{V_P = \frac{Kq}{\sqrt{a^2+R^2-2aR\cos\theta}} + \frac{Kq'}{\sqrt{b^2+R^2-2bR\cos\theta}}} = 0 \quad \rightarrow \boxed{I}$$

Now above formula is valid for $0 \le \theta < 2\pi$ as it should satisfy on the complete surface of sphere

$\therefore (V_P)$ at $\theta = \frac{\pi}{2} \Rightarrow$

$$V_P = \frac{Kq}{\sqrt{a^2+R^2}} + \frac{Kq'}{\sqrt{b^2+R^2}} = 0 \Rightarrow \boxed{\frac{q}{\sqrt{a^2+R^2}} + \frac{q'}{\sqrt{b^2+R^2}} = 0} \quad \text{— (1)}$$

For $\theta = 0 \Rightarrow$

$$V_P = \frac{Kq}{|a-R|} + \frac{Kq'}{|b-R|} = \boxed{\frac{Kq}{a-R} + \frac{Kq'}{R-b} = 0} \text{ as } b \le R \text{ and } a > R \quad \text{— (2)}$$

$\therefore$ ~~$q(R-b)$~~ from (2)nd eq^n $\Rightarrow$

$$q^2(R-b)^2 = (q')^2(a-R)^2 \quad \text{— (3)}$$

and from (1)st eq^n $\Rightarrow$

$$q^2(b^2+R^2) = (q')^2(a^2+R^2) \quad \text{— (4)}$$

Now Subtracting eqⁿ 3 from 4 $\Rightarrow$

$$q^2(2bR) = (q')^2\,(2aR) \quad \Rightarrow \quad q^2 b = (q')^2 a$$

Now if we put above value of $(q')^2 = (q^2 b/a)$ in ~~eqⁿ~~ eqⁿ 4 $\Rightarrow$

$$q^2(b^2+R^2) = q^2\left(\frac{b}{a}\right)(a^2+R^2)$$

$$\Rightarrow ab^2 + aR^2 = a^2 b + bR^2 \qquad \Rightarrow ab^2 - (a^2+R^2)b + aR^2 = 0$$

$$\Rightarrow ab^2 - a^2 b + R^2(a-b) = 0 \qquad \Rightarrow b = \frac{(a^2+R^2) \pm \sqrt{(a^2+R^2)^2 - 4a(aR^2)}}{2a}$$

$$\Rightarrow \cancel{b^2 - ab + \left(\frac{a-b}{a}\right)R^2 = 0}$$

$$\therefore \cancel{b = a \pm \sqrt{a^2 - 4}}$$

$$\therefore b = \frac{(a^2+R^2) \pm \sqrt{a^4 + R^4 + 2a^2R^2 - 4a^2R^2}}{(2a)}$$

Since $a > R$ $\therefore |a^2 - R^2| = a^2 - R^2$

$$b = \frac{(a^2+R^2) \pm |a^2 - R^2|}{2a}$$

$$\therefore b = \frac{(a^2+R^2) \pm (a^2-R^2)}{2a} \quad \Rightarrow \quad b = a \text{ or } b = R^2/a.$$

at this pt. q charge is present.

Hence $\boxed{b = \frac{R^2}{a}}$

$$\therefore (q')^2 = \frac{q^2 b}{a} = q^2 \times \frac{R^2}{a^2} \quad \Rightarrow \quad q' = \pm \frac{qR}{a}$$

$q' = +\frac{qR}{a}$ is not acceptible as we want $V_P = 0$

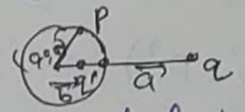
$$\therefore \boxed{q' = -q\,(R/a)} \; C \text{ (in SI system)}$$

Substituting above eqⁿ into $\boxed{I}$ correctly produces $V_P = 0$ for all pts. P.

By uniqueness, we conclude that a point charge q located at any pt. (a) outside the grounded conducting sphere of radius R produces the following electrostatic potential at any other point $\vec{r}$ outside the sphere.

$$\boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{a}|} - \frac{(R/a)}{\left|\vec{r}-\left(\frac{R^2}{a^2}\right)\vec{a}\right|}\right\} \quad a, r > R}$$

By coulomb's law $\Rightarrow$ $\vec{a}$ is the ~~dirn~~ position vector in which q is present.
(Assumed that centre of sphere $\equiv (0,0,0)$ origin.



$\therefore$ The potential outside the sphere using method of images $\Rightarrow$

$$V = \frac{kq}{\cancel{4\pi\epsilon}|\vec{r}-\vec{a}|} + \frac{kq'}{(\vec{r}-\vec{b})}$$

$$= \frac{kq}{|\vec{r}-\vec{a}|} + \frac{k\left(-\frac{qR}{a}\right)}{\left|\vec{r}-\left(\frac{R^2}{a^2}\right)\vec{a}\right|} = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{a}|} - \frac{(R/a)}{\left|\vec{r}-\left(\frac{R^2}{a^2}\right)\vec{a}\right|}\right\}$$

$a, r > R.$

$$\therefore \boxed{V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{|\vec{r}-\vec{a}|} - \frac{(R/a)}{\left|\vec{r}-\left(\frac{R^2}{a^2}\right)\vec{a}\right|}\right\} V \quad a, r > R}$$ (in SI system)

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(4)

(i) $V = x^2+y^2$

$\rightarrow$ Laplace eq^n is given by $\nabla^2 V = 0$
In cartisian Coordinate system, $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$$\therefore \nabla^2 V = \frac{\partial^2 (x^2+y^2)}{\partial x^2} + \frac{\partial^2 (x^2+y^2)}{\partial y^2} + 0$$

$$\therefore \nabla^2 V = \frac{\partial^2 (x^2)}{\partial x^2} + \frac{\partial^2 (y^2)}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial x^2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} y^2\right)$$

$$\therefore \nabla^2 V = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) = 2+2 = 4$$

$$\therefore \nabla^2 V \neq 0$$

Hence $V = x^2+y^2$ does not satisfy Laplace eq^n.

(ii) $V = x^2 - y^2$

→ Laplace equation is given by $\nabla^2 V = 0$

In Cartisian Coordinate system, $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

$$\therefore \nabla^2 V = \frac{\partial^2 (x^2 - y^2)}{\partial x^2} + \frac{\partial^2 (x^2 - y^2)}{\partial y^2} + 0$$

$$\nabla^2 V = \frac{\partial}{\partial x}\left(\frac{\partial x^2}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial y^2}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y)$$

$$\therefore \nabla^2 V = 2 - 2 = 0$$

Hence $V = x^2 - y^2$ satisfy laplace eq^n.

→ General Laplace equation is $\nabla^2 V = 0$ and such a 'V' which satisfy the above condition is the solution for this laplace eq^n.

Now, the solution of Laplace eq^n can not have local maxima or minima. Extreme values must occur at end points (the boundaries). This is true because $V(x)$ is the average of $V(x+R)$ & $V(x-R)$ for any R as long as $(x+R)$ & $(x-R)$ are located in the region b/w the boundary points.

Now If we consider (i) $V = x^2 + y^2$ and plot it as a 3D-surface plot, we observe that there is a local minima at $x = y = 0$ but the Sol^n of Laplace eq^n can't have local maxima or minima. (Extreme values must occur at end pts). Hence $V = x^2 + y^2$ is not a solution of Laplace eq^n.

~~(ii) $V = x^2 - y^2$ ⇒ If we plot a 3D surface plot, we observe that~~

~~① Local minima or local maxima~~ .

$V = x^2 + y^2$

$V_x = \frac{\partial V}{\partial x} = 2x$

$V_y = \frac{\partial V}{\partial y} = 2y$

$V_{xx} = \frac{\partial^2 V}{\partial x^2} = 2$

$V_{yy} = \frac{\partial^2 V}{\partial y^2} = 2$

$V_{xy} = \frac{\partial^2 V}{\partial x \partial y} = 0$

Now For stationary pts, we need $V_x = 0$ and $V_y = 0$

$\therefore 2x = 0$ & $2y = 0 \Rightarrow x = 0, y = 0.$

Now we ~~need~~ need to classify it.

~~first~~ $V_{xx}V_{yy} - V_{xy}^2 = (2)(2) - 0 = 4 > 0$

So it is either maxima or a minima.

But $V_{xx} = 2 > 0$ and $V_{yy} = 2 > 0$. Hence it is minimum: i.e $(0,0)$ pt is a local minima.

Hence $V = x^2 + y^2$ don't satisfy Laplace equation.

(ii) $V = x^2 - y^2$

$V_x = \frac{\partial V}{\partial x} = 2x$

$V_y = \frac{\partial V}{\partial y} = -2y$

$V_{xx} = \frac{\partial^2 V}{\partial x^2} = 2$

$V_{yy} = \frac{\partial^2 V}{\partial y^2} = -2$

$V_{xy} = \frac{\partial^2 V}{\partial x \partial y} = 0$

Now for stationary points, we need $V_x = 0$ & $V_y = 0$

$\therefore 2x = 0$ and $-2y = 0 \Rightarrow x = 0, y = 0$

Now we need to classify it.

$V_{xx}V_{yy} - V_{xy}^2 = 2(-2) - 0 = -4 < 0$

$\therefore (0,0)$ is a Saddle point.

$\therefore$ the function $V = x^2 - y^2$ has no local maxima or minima. And extreme values are occuring at end pts (Boundaries) if we see the 3D-plot

Hence $V = x^2 - y^2$ is a sol^n for Laplace eq^n.