Supplementary Materials of TAoI for Inference Systems

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I. Lemmas and Theorem

Lemma 1. *The value function $V(\Delta)$ is non-decreasing with Δ.*

Proof: See Section II-A. □

Lemma 2. *The value function $V(\Delta, F(X))$ is concave in Δ.*

Proof: See Section II-B. □

Since the value function $V(\Delta)$ is concave, its slope does not increase monotonically. The lower bound of the slope of $V(\Delta)$ is given by the following lemma. Before that, we define an intermediate variable l_{min} as follows:

$$l_{min} = \min\left(\frac{T_1^u + T_1^c}{p_s}, \frac{T_2^u + T_1^c}{q_s}, \frac{T_1^u + T_2^c}{p_l}, \frac{T_2^u + T_2^c}{q_l}\right) \quad (1)$$

Lemma 3. *For any $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, we have $V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1)$, where $\hat{\mathbf{a}}$ and p are given by*

$$(\hat{\mathbf{a}}, p) = \begin{cases} ((0,0), p_s) & \text{if } l_{min} = \frac{T_1^u+T_1^c}{p_s} \\ ((1,0), q_s) & \text{if } l_{min} = \frac{T_2^u+T_1^c}{q_s} \\ ((0,1), p_l) & \text{if } l_{min} = \frac{T_1^u+T_2^c}{p_l} \\ ((1,1), q_l) & \text{if } l_{min} = \frac{T_2^u+T_2^c}{q_l} \end{cases}. \quad (2)$$

Proof: See Section II-C. □

Based on Lemma 1-3, we can derive the structure of the optimal transmission policy as stated in the following theorem.

Theorem 4. *For any $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, there exists a stationary deterministic optimal policy that is of threshold-type, as follows.*

1) When $l_{min} = \frac{T_1^u+T_1^c}{p_s}$ and $\pi^*(s_1) = (0,0)$, $\pi^*(s_2) = (0,0)$.
2) When $l_{min} = \frac{T_2^u+T_1^c}{q_s}$ and $\pi^*(s_1) = (1,0)$, $\pi^*(s_2) = (1,0)$.
3) When $l_{min} = \frac{T_1^u+T_2^c}{p_l}$ and $\pi^*(s_1) = (0,1)$, $\pi^*(s_2) = (0,1)$.
4) When $l_{min} = \frac{T_2^u+T_2^c}{q_l}$ and $\pi^*(s_1) = (1,1)$, $\pi^*(s_2) = (1,1)$.

Proof: See Section II-D. □

II. Proofs of Lemmas and Theorem

A. Proof of Lemma 1

Based on the value iteration algorithm (VIA) outlined in [1, Ch. 4.3], we utilize mathematical induction to establish the proof of Lemma 1. Initially, we introduce $Q_k(s, \mathbf{a})$ and $V_k(s)$ to represent the state-action value function and the state value function at the k-th iteration, respectively. Particularly, $Q_k(s, \mathbf{a})$ is defined as

$$Q_k(s, \mathbf{a}) \triangleq \bar{R}(s, \mathbf{a}) + \sum_{s' \in \mathcal{S}} \bar{p}(s'|s, \mathbf{a}) V_k(s'),\ \forall s \in \mathcal{S}. \quad (3)$$

where s' is given by (**??**). For any given state s, the update to the value function can be executed by

$$V_{k+1}(s) = \min_{\mathbf{a} \in \mathcal{A}} Q_k(s, \mathbf{a}),\ \forall s \in \mathcal{S}. \quad (4)$$

Regardless of how $V_0(s)$ is initially set, the sequence $\{V_k(s)\}$ converges to $V(s)$ that satisfies the Bellman equation (**??**) , i.e.,

$$\lim_{k \to \infty} V_k(s) = V(s),\ \forall s \in \mathcal{S}. \quad (5)$$

Therefore, the monotonicity of $V(s)$ is validated by showing that, for any two states $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$V_k(s_1) \leq V_k(s_2),\ k = 0, 1, \cdots. \quad (6)$$

Next, we prove (6) using mathematical induction. Without loss of generality, we set $V_0(s) = 0$ for each $s \in \mathcal{S}$, ensuring that (6) is satisfied at $k = 0$. Then, assuming that (6) holds up to $k > 0$, we verify whether it holds for $k+1$.

For $\mathbf{a} = (0,0)$, it follows that

$$Q_k(s_1, (0,0)) = \Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \quad (7)$$
$$\frac{\epsilon}{T_1^u + T_1^c}(1 - p_s) V_k(\Delta_1 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)$$

and

$$
Q_k(s_2,(0,0)) = \Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \frac{\epsilon}{T_1^u + T_1^c}(1-p_s)V_k(\Delta_2 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(s_2) \tag{8}
$$

where $s_1^{'} = \Delta_1 + T_1^u + T_1^c$ and $s_2^{'} = \Delta_2 + T_1^u + T_1^c$. Given that $\Delta_1 + T_1^u + T_1^c \leq \Delta_2 + T_1^u + T_1^c$, $V_k(\Delta_1) \leq V_k(\Delta_2)$ and $V_k(s_1^{'}) \leq V_k(s_2^{'})$, we can obtain $Q_k(s_1,(0,0)) \leq Q_k(s_2,(0,0))$.

Similar to $\mathbf{a} = (0,0)$, it can be easily deduced that $Q_k(s_1,(1,0)) \leq Q_k(s_2,(1,0))$, $Q_k(s_1,(0,1)) \leq Q_k(s_2,(0,1))$, and $Q_k(s_1,(1,1)) \leq Q_k(s_2,(1,1))$ according to $\Delta_1 \leq \Delta_2$ and $V_k(s_1) \leq V_k(s_2)$.

By (4), we can get that $V_{k+1}(s_1) \leq V_{k+1}(s_2)$ for any k.

This concludes the proof of Lemma 1.

B. *Proof of Lemma 2*

The concavity of $V(s)$ with respect to s can be demonstrated by showing that, for any $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$ and $w \in N$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$
\begin{aligned}
&V_k(\Delta_1 + w) - V_k(\Delta_1) \geq \\
&V_k(\Delta_2 + w) - V_k(\Delta_2), k = 0, 1, \ldots.
\end{aligned} \tag{9}
$$

Without sacrificing generality, we set $V_0(s) = 0$ for all $s \in \mathcal{S}$, ensuring that (9) is applicable at $k = 0$. Then, we assume that (9) holds up till $k > 0$ and inspect whether it holds for $k+1$. Now, let $s = \Delta$, $s_1 = \Delta_1$, $s_2 = \Delta_2$, $s' = \Delta + w$, $s_1^{'} = \Delta_1 + w$ and $s_2^{'} = \Delta_2 + w$. For ease of explanation, we introduce $\Delta Q(s', s, \mathbf{a}) = Q(s', \mathbf{a}) - Q(s, \mathbf{a})$.

For $\mathbf{a} = (0,0)$, it follows that

$$
\begin{aligned}
&\Delta Q(s_1^{'}, s_1, (0,0)) - \Delta Q(s_2^{'}, s_2, (0,0)) \\
=&[\Delta_1 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_1 + w) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c, I_+) + (1-p_s)V_k(\Delta_1 + w + T_1^u + T_1^c)] \\
&- [\Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_1) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + (1-p_s)V_k(\Delta_1 + T_1^u + T_1^c)] \\
&- [\Delta_2 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_2 + w) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c, I_+) + (1-p_s)V_k(\Delta_2 + w + T_1^u + T_1^c)] \\
&+ [\Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_2) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + (1-p_s)V_k(\Delta_2 + T_1^u + T_1^c)] \\
=&(1 - \frac{\epsilon}{T_1^u + T_1^c})[(V_k(\Delta_1 + w) - V_k(\Delta_1)) \\
&- (V_k(\Delta_2 + w) - V_k(\Delta_2))] \\
&+ \frac{\epsilon}{T_1^u + T_1^c}(1-p_0)[(V_k(\Delta_1 + w + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)) \\
&- (V_k(\Delta_2 + w + T_1^u + T_1^c) - V_k(\Delta_2 + T_1^u + T_1^c))].
\end{aligned} \tag{10}
$$

Given that $V_k(\Delta_1 + w) - V_k(\Delta_1) \geq V_k(\Delta_2 + w) - V_k(\Delta_2)$ and $V_k(\Delta_1 + w + 1) - V_k(\Delta_1 + 1) \geq V_k(\Delta_2 + w + 1) - V_k(\Delta_2 + 1)$, we can easily see that $\Delta Q_k(s_1^{'}, s_1, (0,0)) - \Delta Q_k(s_2^{'}, s_2, (0,0)) \geq 0$. Thus, $Q_k(s,(0,0))$ is concave in Δ.

Similar to $\mathbf{a} = (0,0)$, we can also get that $Q_k(s,(1,0))$, $Q_k(s,(0,1))$,and $Q_k(s,(1,1))$ is concave in Δ.

Since the value function $V_{k+1}(s)$ is the minimum of two concave functions, it is also concave in Δ. Hence, we have $V_k(\Delta_1 + w) - V_k(\Delta_1) \geq V_k(\Delta_2 + w) - V_k(\Delta_2)$, i.e., (9) holds for $k+1$. Therefore, we can show that (9) holds for any k by induction.

This completes the proof of Lemma 2.

C. *Proof of Lemma 3*

The proof follows the same procedure of Lemma 1. The lower bound of $V(s_2) - V(s_1)$ can be proved by showing that for any $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$

$$
V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1), k = 0, 1, \cdots. \tag{11}
$$

where $p = p_s$ if $I = 0$, and $p = p_l$ if $I = 1$.

Without sacrificing generality, we set $V_0(\mathbf{s}) = \frac{L(\mathbf{a})}{\epsilon p}\Delta$ for all $s = \Delta \in \mathcal{S}$, ensuring that (11) is satisfied at $k = 0$. Then, we assume that (11) holds up till $k > 0$ and hence we have $V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$ and $V_k(\Delta_2 + 1) - V_k(\Delta_1 + 1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$.

Then, we inspect whether it holds for $k+1$. Since $V_{k+1}(s) = \min_{\mathbf{a} \in \mathcal{A}} Q_k(s, \mathbf{a})$, we investigate the two state-action value functions, in the following, respectively.

When $\mathbf{a} = (0,0)$, we have

$$
\begin{aligned}
&\Delta Q_k(s_2, s_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_1^u + T_1^c}(1-p_s)\left(V_k(\Delta_2 + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)\frac{L(\mathbf{a})}{\epsilon p_s}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1).
\end{aligned} \tag{12}
$$

When $\mathbf{a} = (1,0)$, we have

$$
\begin{aligned}
&\Delta Q_k(s_2, s_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_2^u + T_1^c}(1-q_s)\left(V_k(\Delta_2 + T_2^u + T_1^c) - V_k(\Delta_1 + T_2^u + T_1^c)\right)
\end{aligned}
$$

$$\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right) \frac{L(\mathbf{a})}{\epsilon q_s}(\Delta_2 - \Delta_1)$$
$$= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1). \tag{13}$$

When $\mathbf{a} = (0, 1)$, we have

$$\begin{aligned}
&\Delta Q_k(s_2, s_1) \\
&= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_1^u + T_2^c}(1 - p_l)\left(V_k(\Delta_2 + T_1^u + T_2^c) - V_k(\Delta_1 + T_1^u + T_2^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right) \frac{L(\mathbf{a})}{\epsilon p_l}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1). \qquad (14)
\end{aligned}$$

When $\mathbf{a} = (1, 1)$, we have

$$\begin{aligned}
&\Delta Q_k(s_2, s_1) \\
&= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_2^u + T_2^c}(1 - q_l)\left(V_k(\Delta_2 + T_2^u + T_2^c) - V_k(\Delta_1 + T_2^u + T_2^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right) \frac{L(\mathbf{a})}{\epsilon q_l}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1). \qquad (15)
\end{aligned}$$

This concludes the proof of Lemma 3.

D. Proof of Theorem 1

For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
&+ \frac{\epsilon}{L(a)} p_1 (V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X))).
\end{aligned} \tag{16}$$

Since the concavity of $V(\mathbf{s})$ have been proved in Lemma 2, we can easily see that $V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X)) \leq V(\Delta_2, F(X)) - V(\Delta_1 + L(a), F(X))$. Therefore, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
\leq&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1), F(X)) \\
&+ \frac{\epsilon}{L(a)} p_1 (V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(1 - p_1)(V(\mathbf{s}_2) - V(\mathbf{s}_1)). \qquad (17)
\end{aligned}$$

As proved in Lemma 3 that $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq [L(a)/\epsilon(1 - p_1)](\Delta_2 - \Delta_1)$, it is easy to see that $Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V(\mathbf{s}_2) - V(\mathbf{s}_1)) \leq 0$.

Now, we can prove the threshold structure of the optimal policy. Suppose $\Delta_2 \geq \Delta_1$ and $\pi^*(\Delta_1, F(X)) = a$, it is easily to see that $V(\Delta_1, F(X)) = Q((\Delta_1, F(X)), a)$, i.e., $V(\mathbf{s}_1) = Q(\mathbf{s}_1, a)$. According to Theorem 1, we know that $V(\mathbf{s}_2) - V(\mathbf{s}_1) \geq Q(\mathbf{s}_2, a) - Q(\mathbf{s}_1, a)$. Therefore, we have $V(\mathbf{s}_2) \geq Q(\mathbf{s}_2, a)$. Since the value function is a minimum of two state-action cost functions, we have $V(\mathbf{s}_2) \leq Q(\mathbf{s}_2, a)$. Altogether, we can assert that $V(\mathbf{s}_2) = Q(\mathbf{s}_2, a)$ and $\pi^*(\Delta_2, F(X)) = a$.

This completes the proof of Theorem 1.

REFERENCES

[1] P. Bertsekas, Dimitri, *Dynamic Programming and Optimal Control-II*, 3rd ed. Belmont, MA, USA: Athena Sci., 2007, vol. 2.