Supplementary Materials of TAoI for Remote Inference with Hybrid Language Models

Shuying Gan*, Xijun Wang*, Chenyuan Feng†, Chao Xu‡, Howard H. Yang§, Xiang Chen*, and Tony Q. S. Quek¶

*School of Electronics and Information Engineering, Sun Yat-sen University, Guangzhou, China

†Department of Communication Systems, EURECOM, Sophia Antipolis, France

‡School of Information Engineering, Northwest A&F University, Yangling, China

§ZJU-UIUC Institute, Zhejiang University, Haining, China

¶Information System and Technology Design Pillar, Singapore University of Technology and Design, Singapore

Email: ganshy7@mail2.sysu.edu.cn, wangxijun@mail.sysu.edu.cn, Chenyuan.Feng@eurecom.fr,

cxu@nwafu.edu.cn, haoyang@intl.zju.edu.cn, chenxiang@mail.sysu.edu.cn, tonyquek@sutd.edu.sg

I. Lemmas and Theorem

Lemma 1. *The value function $V(\Delta)$ is non-decreasing with Δ.*

Proof: See Section II-A. □

Lemma 2. *The value function $V(\Delta)$ is concave in Δ.*

Proof: See Section II-B. □

Since the value function $V(\Delta)$ is non-decreasing and concave, its slope is non-increasing and lower bounded. The lower bound of the slope of $V(\Delta)$ is given by the following lemma. Prior to that, we define an auxiliary variable l_{min} as follows:

$$l_{min} = \min\left(\frac{T_1^u + T_1^c}{p_s}, \frac{T_2^u + T_1^c}{q_s}, \frac{T_1^u + T_2^c}{p_l}, \frac{T_2^u + T_2^c}{q_l}\right). \tag{1}$$

Lemma 3. *For any Δ_1, $\Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, we have $V(\Delta_2) - V(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1)$, where $\hat{\mathbf{a}}$ and $\hat{p}$ are given by*

$$(\hat{\mathbf{a}}, \hat{p}) = \begin{cases} ((0,0), p_s), & \text{if } l_{min} = \frac{T_1^u + T_1^c}{p_s} \\ ((1,0), q_s), & \text{if } l_{min} = \frac{T_2^u + T_1^c}{q_s} \\ ((0,1), p_l), & \text{if } l_{min} = \frac{T_1^u + T_2^c}{p_l} \\ ((1,1), q_l), & \text{if } l_{min} = \frac{T_2^u + T_2^c}{q_l} \end{cases}. \tag{2}$$

Proof: See Section II-C. □

Based on Lemmas 1-3, we can derive the structure of the optimal control policy as stated in the following theorem.

Theorem 4. *For any $\Delta_1, \Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, there exists a stationary deterministic optimal policy with a threshold-based structure, described as follows:*

- *When $l_{min} = \frac{T_1^u + T_1^c}{p_s}$ and $\pi^*(\Delta_1) = (0,0)$, $\pi^*(\Delta_2) = (0,0)$.*
- *When $l_{min} = \frac{T_2^u + T_1^c}{q_s}$ and $\pi^*(\Delta_1) = (1,0)$, $\pi^*(\Delta_2) = (1,0)$.*
- *When $l_{min} = \frac{T_1^u + T_2^c}{p_l}$ and $\pi^*(\Delta_1) = (0,1)$, $\pi^*(\Delta_2) = (0,1)$.*
- *When $l_{min} = \frac{T_2^u + T_2^c}{q_l}$ and $\pi^*(\Delta_1) = (1,1)$, $\pi^*(\Delta_2) = (1,1)$.*

Proof: See Section II-D. □

II. Proofs of Lemmas and Theorem

A. Proof of Lemma 1

Based on the value iteration algorithm (VIA) outlined in [1, Ch. 4.3], we utilize mathematical induction to establish the proof of Lemma 1. Initially, we introduce $Q_k(s, \mathbf{a})$ and $V_k(s)$ to represent the state-action value function and the state value function at the k-th iteration, respectively. Particularly, $Q_k(s, \mathbf{a})$ is defined as

$$Q_k(s, \mathbf{a}) \triangleq \bar{R}(s, \mathbf{a}) + \sum_{s' \in \mathcal{S}} \bar{p}(s'|s, \mathbf{a}) V_k(s'), \ \forall s \in \mathcal{S}. \tag{3}$$

For any given state s, the update to the value function can be executed by

$$V_{k+1}(s) = \min_{\mathbf{a} \in \mathcal{A}} Q_k(s, \mathbf{a}), \ \forall s \in \mathcal{S}. \tag{4}$$

Regardless of how $V_0(s)$ is initially set, the sequence $\{V_k(s)\}$ converges to $V(s)$ that satisfies the Bellman equation, i.e.,

$$\lim_{k \to \infty} V_k(s) = V(s), \ \forall s \in \mathcal{S}. \tag{5}$$

Therefore, the monotonicity of $V(\Delta)$ is validated by showing that, for any two states Δ_1, $\Delta_2 \in \mathcal{S}$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$V_k(\Delta_1) \leq V_k(\Delta_2), \ k = 0, 1, \cdots. \tag{6}$$

Next, we prove (6) using mathematical induction. Without loss of generality, we set $V_0(\Delta) = 0$ for each $\Delta \in \mathcal{S}$, ensuring that (6) is satisfied at $k = 0$. Then, assuming that (6) holds up to $k > 0$, we verify whether it holds for $k + 1$.

For $\mathbf{a} = (0,0)$, it follows that

$$\begin{aligned} Q_k(\Delta_1, (0,0)) =& \Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \\ & \frac{\epsilon}{T_1^u + T_1^c}(1 - p_s) V_k(\Delta_1 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_1), \end{aligned} \tag{7}$$

and

$$\begin{aligned} Q_k(\Delta_2, (0,0)) =& \Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \\ & \frac{\epsilon}{T_1^u + T_1^c}(1 - p_s) V_k(\Delta_2 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_2). \end{aligned} \tag{8}$$

Given that $\Delta_1 + T_1^u + T_1^c \leq \Delta_2 + T_1^u + T_1^c$, $V_k(\Delta_1) \leq V_k(\Delta_2)$ and $V_k(\Delta_1^{'}) \leq V_k(\Delta_2^{'})$, we can obtain $Q_k(\Delta_1, (0,0)) \leq Q_k(\Delta_2, (0,0))$.

Similar to $\mathbf{a} = (0,0)$, it can be easily deduced that $Q_k(\Delta_1, (1,0)) \leq Q_k(\Delta_2, (1,0))$, $Q_k(\Delta_1, (0,1)) \leq Q_k(\Delta_2, (0,1))$, and $Q_k(\Delta_1, (1,1)) \leq Q_k(\Delta_2, (1,1))$ according to $\Delta_1 \leq \Delta_2$ and $V_k(\Delta_1) \leq V_k(\Delta_2)$. By (4), we can get that $V_{k+1}(\Delta_1) \leq V_{k+1}(\Delta_2)$ for any k.

This concludes the proof of Lemma 1.

B. *Proof of Lemma 2*

The concavity of $V(\Delta)$ with respect to Δ can be demonstrated by showing that, for any Δ_1, $\Delta_2 \in \mathcal{S}$ and $w \in N$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$\begin{aligned} &V_k(\Delta_1 + w) - V_k(\Delta_1) \geq \\ &V_k(\Delta_2 + w) - V_k(\Delta_2), k = 0, 1, \ldots. \end{aligned} \tag{9}$$

Without sacrificing generality, we set $V_0(\Delta) = 0$ for all $\Delta \in \mathcal{S}$, ensuring that (9) is applicable at $k = 0$. Then, we assume that (9) holds up till $k > 0$ and inspect whether it holds for $k+1$. Now, let $\Delta' = \Delta + w$, $\Delta_1^{'} = \Delta_1 + w$ and $\Delta_2^{'} = \Delta_2 + w$. For ease of explanation, we introduce $\delta Q(\Delta', \Delta, \mathbf{a}) = Q(\Delta', \mathbf{a}) - Q(\Delta, \mathbf{a})$.

For $\mathbf{a} = (0,0)$, it follows that

$$\begin{aligned}
&\delta Q(\Delta_1^{'}, \Delta_1, (0,0)) - \delta Q(\Delta_2^{'}, \Delta_2, (0,0)) \\
=&[\Delta_1 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_1 + w) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c, I_+) + (1-p_s) V_k(\Delta_1 + w + T_1^u + T_1^c)] \\
&- [\Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_1) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + (1-p_s) V_k(\Delta_1 + T_1^u + T_1^c)] \\
&- [\Delta_2 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_2 + w) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c, I_+) + (1-p_s) V_k(\Delta_2 + w + T_1^u + T_1^c)] \\
&+ [\Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(\Delta_2) \\
&+ \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + (1-p_s) V_k(\Delta_2 + T_1^u + T_1^c)] \\
=&(1 - \frac{\epsilon}{T_1^u + T_1^c})[(V_k(\Delta_1 + w) - V_k(\Delta_1)) \\
&- (V_k(\Delta_2 + w) - V_k(\Delta_2))] \\
&+ \frac{\epsilon}{T_1^u + T_1^c}(1 - p_0)[(V_k(\Delta_1 + w + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)) \\
&- (V_k(\Delta_2 + w + T_1^u + T_1^c) - V_k(\Delta_2 + T_1^u + T_1^c))].
\end{aligned} \tag{10}$$

Given that $V_k(\Delta_1+w)-V_k(\Delta_1) \geq V_k(\Delta_2+w)-V_k(\Delta_2)$ and $V_k(\Delta_1+w+1)-V_k(\Delta_1+1) \geq V_k(\Delta_2+w+1)-V_k(\Delta_2+1)$, we can easily see that $\delta Q_k(\Delta_1^{'}, \Delta_1, (0,0)) - \delta Q_k(\Delta_2^{'}, \Delta_2, (0,0)) \geq 0$. Thus, $Q_k(\Delta, (0,0))$ is concave in Δ.

Similar to $\mathbf{a} = (0,0)$, we can also get that $Q_k(\Delta, (1,0))$, $Q_k(\Delta, (0,1))$,and $Q_k(\Delta, (1,1))$ is concave in Δ. Since the value function $V_{k+1}(\Delta)$ is the minimum of two concave functions, it is also concave in Δ. Hence, we have $V_k(\Delta_1+w)-V_k(\Delta_1) \geq V_k(\Delta_2+w)-V_k(\Delta_2)$, i.e., (9) holds for $k+1$. Therefore, we can show that (9) holds for any k by induction.

This completes the proof of Lemma 2.

C. Proof of Lemma 3

The proof follows the same procedure of Lemma 1. The lower bound of $V(\Delta_2) - V(\Delta_1)$ can be proved by showing that for any Δ_1, $\Delta_2 \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$

$$V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1), k = 0, 1, \cdots . \tag{11}$$

where $p = p_s$ if $I = 0$, and $p = p_l$ if $I = 1$.

Without sacrificing generality, we set $V_0(\Delta) = \frac{L(\mathbf{a})}{\epsilon p}\Delta$ for all $\Delta \in \mathcal{S}$, ensuring that (11) is satisfied at $k = 0$. Then, we assume that (11) holds up till $k > 0$ and hence we have $V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$ and $V_k(\Delta_2+1) - V_k(\Delta_1+1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$.

Then, we inspect whether it holds for $k+1$. Since $V_{k+1}(\Delta) = \min_{\mathbf{a}\in\mathcal{A}} Q_k(\Delta, \mathbf{a})$, we investigate the two state-action value functions, in the following, respectively.

When $\mathbf{a} = (0,0)$, we have

$$\begin{aligned}
&\delta Q_k(\Delta_2, \Delta_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_1^u + T_1^c}(1 - p_s)\left(V_k(\Delta_2 + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)\frac{L(\mathbf{a})}{\epsilon p_s}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1).
\end{aligned} \tag{12}$$

When $\mathbf{a} = (1,0)$, we have

$$\begin{aligned}
&\delta Q_k(\Delta_2, \Delta_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_2^u + T_1^c}(1 - q_s)\left(V_k(\Delta_2 + T_2^u + T_1^c) - V_k(\Delta_1 + T_2^u + T_1^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right)\frac{L(\mathbf{a})}{\epsilon q_s}(\Delta_2 - \Delta_1)
\end{aligned}$$

$$= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1). \tag{13}$$

When $\mathbf{a} = (0, 1)$, we have

$$\begin{aligned}
&\delta Q_k(\Delta_2, \Delta_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_1^u + T_2^c}(1 - p_l)\left(V_k(\Delta_2 + T_1^u + T_2^c) - V_k(\Delta_1 + T_1^u + T_2^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right)\frac{L(\mathbf{a})}{\epsilon p_l}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1).
\end{aligned} \tag{14}$$

When $\mathbf{a} = (1, 1)$, we have

$$\begin{aligned}
&\delta Q_k(\Delta_2, \Delta_1) \\
&= Q_k(\Delta_2, (0,0)) - Q_k(\Delta_1, (0,0)) \\
&= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right)(V_k(\Delta_2) - V_k(\Delta_1)) + \\
&\quad \frac{\epsilon}{T_2^u + T_2^c}(1 - q_l)\left(V_k(\Delta_2 + T_2^u + T_2^c) - V_k(\Delta_1 + T_2^u + T_2^c)\right) \\
&\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right)\frac{L(\mathbf{a})}{\epsilon q_l}(\Delta_2 - \Delta_1) \\
&= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1).
\end{aligned} \tag{15}$$

This concludes the proof of Lemma 3.

D. Proof of Theorem 1

First, we introduce $\delta Q(\Delta_2, \Delta_1, \mathbf{a}) = Q(\Delta_2, \mathbf{a}) - Q(\Delta_1, \mathbf{a})$ for convenience. For any Δ_1, $\Delta_2 \in \mathcal{S}$, and $\Delta_1 \leq \Delta_2$, we have

$$\begin{aligned}
&\delta Q(\Delta_2, \Delta_1, \hat{\mathbf{a}}) - (V(\Delta_2) - V(\Delta_1)) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(\hat{\mathbf{a}})}(V(\Delta_2) - V(\Delta_1)) \\
&+ \frac{\epsilon(1-p)}{L(\hat{\mathbf{a}})}(V(\Delta_2 + L(\hat{\mathbf{a}})) - V(\Delta_1 + L(\hat{\mathbf{a}}))).
\end{aligned} \tag{16}$$

Given that the concavity of $V(s)$ is established in Lemma 2, it follows that $V(\Delta_2 + L(\hat{\mathbf{a}})) - V(\Delta_1 + L(\hat{\mathbf{a}})) \leq V(\Delta_2) - V(\Delta_1))$. Then, we can get that

$$\begin{aligned}
&\delta Q(\Delta_2, \Delta_1, \hat{\mathbf{a}}) - (V(\Delta_2) - V(\Delta_1)) \\
&\leq \Delta_2 - \Delta_1 + \frac{\epsilon(1-p)}{L(\hat{\mathbf{a}})}(V(\Delta_2) - V(\Delta_1)) \\
&\quad - \frac{\epsilon}{L(\hat{\mathbf{a}})}(V(\Delta_2) - V(\Delta_1))
\end{aligned}$$

$$= \Delta_2 - \Delta_1 - \frac{\epsilon p}{L(\hat{\mathbf{a}})}(V(\Delta_2) - V(\Delta_1)). \tag{17}$$

As shown in Lemma 3, we have $V(\Delta_2) - V(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1)$. This implies that $\Delta Q(\Delta_2, \Delta_1) - (V(\Delta_2) - V(\Delta_1)) \leq 0$.

Next, we prove the threshold structure of the optimal policy. Suppose $\Delta_2 \geq \Delta_1$ and $\pi^*(\Delta_1) = \hat{\mathbf{a}}$, we have $V(\Delta_1) = Q(\Delta_1, \hat{\mathbf{a}})$, i.e., $V(\Delta_1) = Q(\Delta_1, \hat{\mathbf{a}})$. It is straightforward to obtain $V(\Delta_2) \geq Q(\Delta_2, \hat{\mathbf{a}})$, since $V(\Delta_2) - V(\Delta_1) \geq Q(\Delta_2, \hat{\mathbf{a}}) - Q(\Delta_1, \hat{\mathbf{a}})$. Moreover, since the value function is a minimum of two state-action value functions, we have $V(\Delta_2) \leq Q(\Delta_2, \hat{\mathbf{a}})$. Therefore, we can conclude that $V(\Delta_2) = Q(\Delta_2, \hat{\mathbf{a}})$ and that $\pi^*(\Delta_2) = \hat{\mathbf{a}}$.

This completes the proof of Theorem 1.

References

[1] P. Bertsekas, Dimitri, *Dynamic Programming and Optimal Control-II*, 3rd ed. Belmont, MA, USA: Athena Sci., 2007, vol. 2.