

Supplementary Materials of TAoI for Remote Inference with Hybrid Language Models

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I. LEMMAS AND THEOREM

Lemma 1. *The value function $V(\Delta)$ is non-decreasing with Δ .*

Proof: See Section II-A. □

Lemma 2. *The value function $V(\Delta)$ is concave in Δ .*

Proof: See Section II-B. □

Since the value function $V(\Delta)$ is non-decreasing and concave, its slope is non-increasing and lower bounded. The lower bound of the slope of $V(\Delta)$ is given by the following lemma. Prior to that, we define an auxiliary variable l_{min} as follows:

$$l_{min} = \min \left(\frac{T_1^u + T_1^c}{p_s}, \frac{T_2^u + T_1^c}{q_s}, \frac{T_1^u + T_2^c}{p_l}, \frac{T_2^u + T_2^c}{q_l} \right). \quad (1)$$

Lemma 3. *For any $\Delta_1, \Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, we have $V(\Delta_2) - V(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1)$, where $\hat{\mathbf{a}}$ and \hat{p} are given by*

$$(\hat{\mathbf{a}}, \hat{p}) = \begin{cases} ((0, 0), p_s), & \text{if } l_{min} = \frac{T_1^u + T_1^c}{p_s} \\ ((1, 0), q_s), & \text{if } l_{min} = \frac{T_2^u + T_1^c}{q_s} \\ ((0, 1), p_l), & \text{if } l_{min} = \frac{T_1^u + T_2^c}{p_l} \\ ((1, 1), q_l), & \text{if } l_{min} = \frac{T_2^u + T_2^c}{q_l} \end{cases}. \quad (2)$$

Proof: See Section II-C. □

Based on Lemmas 1-3, we can derive the structure of the optimal control policy as stated in the following theorem.

Theorem 4. For any $\Delta_1, \Delta_2 \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, there exists a stationary deterministic optimal policy with a threshold-based structure, described as follows:

- 1) When $l_{min} = \frac{T_1^u + T_1^c}{p_s}$, if $\pi^*(\Delta_1) = (0, 0)$, then $\pi^*(\Delta_2) = (0, 0)$.
- 2) When $l_{min} = \frac{T_2^u + T_1^c}{q_s}$, if $\pi^*(\Delta_1) = (1, 0)$, then $\pi^*(\Delta_2) = (1, 0)$.
- 3) When $l_{min} = \frac{T_1^u + T_2^c}{p_l}$, if $\pi^*(\Delta_1) = (0, 1)$, then $\pi^*(\Delta_2) = (0, 1)$.
- 4) When $l_{min} = \frac{T_2^u + T_2^c}{q_l}$, if $\pi^*(\Delta_1) = (1, 1)$, then $\pi^*(\Delta_2) = (1, 1)$.

Proof: See Section II-D. □

II. PROOFS OF LEMMAS AND THEOREM

A. Proof of Lemma 1

Based on the value iteration algorithm (VIA) outlined in [1, Ch. 4.3], we utilize mathematical induction to establish the proof of Lemma 1. Initially, we introduce $Q_k(s, \mathbf{a})$ and $V_k(s)$ to represent the state-action value function and the state value function at the k -th iteration, respectively. Particularly, $Q_k(s, \mathbf{a})$ is defined as

$$Q_k(s, \mathbf{a}) \triangleq \bar{R}(s, \mathbf{a}) + \sum_{s' \in \mathcal{S}} \bar{p}(s'|s, \mathbf{a}) V_k(s'), \quad \forall s \in \mathcal{S}. \quad (3)$$

where s' is given by (??). For any given state s , the update to the value function can be executed by

$$V_{k+1}(s) = \min_{\mathbf{a} \in \mathcal{A}} Q_k(s, \mathbf{a}), \quad \forall s \in \mathcal{S}. \quad (4)$$

Regardless of how $V_0(s)$ is initially set, the sequence $\{V_k(s)\}$ converges to $V(s)$ that satisfies the Bellman equation (??), i.e.,

$$\lim_{k \rightarrow \infty} V_k(s) = V(s), \quad \forall s \in \mathcal{S}. \quad (5)$$

Therefore, the monotonicity of $V(s)$ is validated by showing that, for any two states $s_1 = \Delta_1, s_2 = \Delta_2 \in \mathcal{S}$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$V_k(s_1) \leq V_k(s_2), \quad k = 0, 1, \dots \quad (6)$$

Next, we prove (6) using mathematical induction. Without loss of generality, we set $V_0(s) = 0$ for each $s \in \mathcal{S}$, ensuring that (6) is satisfied at $k = 0$. Then, assuming that (6) holds up to $k > 0$, we verify whether it holds for $k + 1$.

For $\mathbf{a} = (0, 0)$, it follows that

$$\begin{aligned} Q_k(s_1, (0, 0)) = & \Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \\ & \frac{\epsilon}{T_1^u + T_1^c} (1 - p_s) V_k(\Delta_1 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) V_k(s_1), \end{aligned} \quad (7)$$

and

$$Q_k(s_2, (0, 0)) = \Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \frac{\epsilon}{T_1^u + T_1^c} p_s V_k(T_1^u + T_1^c) + \quad (8)$$

$$\frac{\epsilon}{T_1^u + T_1^c}(1 - p_s)V_k(\Delta_2 + T_1^u + T_1^c) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)V_k(s_2),$$

where $s_1' = \Delta_1 + T_1^u + T_1^c$ and $s_2' = \Delta_2 + T_1^u + T_1^c$. Given that $\Delta_1 + T_1^u + T_1^c \leq \Delta_2 + T_1^u + T_1^c$, $V_k(\Delta_1) \leq V_k(\Delta_2)$ and $V_k(s_1') \leq V_k(s_2')$, we can obtain $Q_k(s_1, (0, 0)) \leq Q_k(s_2, (0, 0))$.

Similar to $\mathbf{a} = (0, 0)$, it can be easily deduced that $Q_k(s_1, (1, 0)) \leq Q_k(s_2, (1, 0))$, $Q_k(s_1, (0, 1)) \leq Q_k(s_2, (0, 1))$, and $Q_k(s_1, (1, 1)) \leq Q_k(s_2, (1, 1))$ according to $\Delta_1 \leq \Delta_2$ and $V_k(s_1) \leq V_k(s_2)$. By (4), we can get that $V_{k+1}(s_1) \leq V_{k+1}(s_2)$ for any k .

This concludes the proof of Lemma 1.

B. Proof of Lemma 2

The concavity of $V(s)$ with respect to s can be demonstrated by showing that, for any $s_1 = \Delta_1$, $s_2 = \Delta_2 \in \mathcal{S}$ and $w \in N$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$V_k(\Delta_1 + w) - V_k(\Delta_1) \geq \tag{9}$$

$$V_k(\Delta_2 + w) - V_k(\Delta_2), k = 0, 1, \dots$$

Without sacrificing generality, we set $V_0(s) = 0$ for all $s \in \mathcal{S}$, ensuring that (9) is applicable at $k = 0$. Then, we assume that (9) holds up till $k > 0$ and inspect whether it holds for $k + 1$. Now, let $s = \Delta$, $s_1 = \Delta_1$, $s_2 = \Delta_2$, $s' = \Delta + w$, $s_1' = \Delta_1 + w$ and $s_2' = \Delta_2 + w$. For ease of explanation, we introduce $\Delta Q(s', s, \mathbf{a}) = Q(s', \mathbf{a}) - Q(s, \mathbf{a})$.

For $\mathbf{a} = (0, 0)$, it follows that

$$\begin{aligned} & \Delta Q(s_1', s_1, (0, 0)) - \Delta Q(s_2', s_2, (0, 0)) \\ &= [\Delta_1 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)V_k(\Delta_1 + w) \\ & \quad + \frac{\epsilon}{T_1^u + T_1^c}p_s V_k(T_1^u + T_1^c, I_+) + (1 - p_s)V_k(\Delta_1 + w + T_1^u + T_1^c)] \\ & \quad - [\Delta_1 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)V_k(\Delta_1) \\ & \quad + \frac{\epsilon}{T_1^u + T_1^c}p_s V_k(T_1^u + T_1^c) + (1 - p_s)V_k(\Delta_1 + T_1^u + T_1^c)] \\ &= [\Delta_2 + w + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)V_k(\Delta_2 + w) \\ & \quad + \frac{\epsilon}{T_1^u + T_1^c}p_s V_k(T_1^u + T_1^c, I_+) + (1 - p_s)V_k(\Delta_2 + w + T_1^u + T_1^c)] \\ & \quad - [\Delta_2 + \frac{1}{2}(T_1^u + T_1^c - 1) + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right)V_k(\Delta_2) \\ & \quad + \frac{\epsilon}{T_1^u + T_1^c}p_s V_k(T_1^u + T_1^c) + (1 - p_s)V_k(\Delta_2 + T_1^u + T_1^c)] \\ &= (1 - \frac{\epsilon}{T_1^u + T_1^c})[(V_k(\Delta_1 + w) - V_k(\Delta_1)) \\ & \quad - (V_k(\Delta_2 + w) - V_k(\Delta_2))] \\ & \quad + \frac{\epsilon}{T_1^u + T_1^c}(1 - p_0)[(V_k(\Delta_1 + w + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)) \\ & \quad - (V_k(\Delta_2 + w + T_1^u + T_1^c) - V_k(\Delta_2 + T_1^u + T_1^c))] \end{aligned}$$

$$- (V_k(\Delta_2 + w + T_1^u + T_1^c) - V_k(\Delta_2 + T_1^u + T_1^c))]. \quad (10)$$

Given that $V_k(\Delta_1 + w) - V_k(\Delta_1) \geq V_k(\Delta_2 + w) - V_k(\Delta_2)$ and $V_k(\Delta_1 + w + 1) - V_k(\Delta_1 + 1) \geq V_k(\Delta_2 + w + 1) - V_k(\Delta_2 + 1)$, we can easily see that $\Delta Q_k(s'_1, s_1, (0, 0)) - \Delta Q_k(s'_2, s_2, (0, 0)) \geq 0$. Thus, $Q_k(s, (0, 0))$ is concave in Δ .

Similar to $\mathbf{a} = (0, 0)$, we can also get that $Q_k(s, (1, 0))$, $Q_k(s, (0, 1))$, and $Q_k(s, (1, 1))$ is concave in Δ . Since the value function $V_{k+1}(s)$ is the minimum of two concave functions, it is also concave in Δ . Hence, we have $V_k(\Delta_1 + w) - V_k(\Delta_1) \geq V_k(\Delta_2 + w) - V_k(\Delta_2)$, i.e., (9) holds for $k + 1$. Therefore, we can show that (9) holds for any k by induction.

This completes the proof of Lemma 2.

C. Proof of Lemma 3

The proof follows the same procedure of Lemma 1. The lower bound of $V(s_2) - V(s_1)$ can be proved by showing that for any $s_1 = \Delta_1, s_2 = \Delta_2 \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$

$$V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\hat{\mathbf{a}})}{\epsilon p}(\Delta_2 - \Delta_1), k = 0, 1, \dots. \quad (11)$$

where $p = p_s$ if $I = 0$, and $p = p_l$ if $I = 1$.

Without sacrificing generality, we set $V_0(s) = \frac{L(\mathbf{a})}{\epsilon p}\Delta$ for all $s = \Delta \in \mathcal{S}$, ensuring that (11) is satisfied at $k = 0$. Then, we assume that (11) holds up till $k > 0$ and hence we have $V_k(\Delta_2) - V_k(\Delta_1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$ and $V_k(\Delta_2 + 1) - V_k(\Delta_1 + 1) \geq \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1)$.

Then, we inspect whether it holds for $k + 1$. Since $V_{k+1}(s) = \min_{\mathbf{a} \in \mathcal{A}} Q_k(s, \mathbf{a})$, we investigate the two state-action value functions, in the following, respectively.

When $\mathbf{a} = (0, 0)$, we have

$$\begin{aligned} \Delta Q_k(s_2, s_1) &= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\ &= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) (V_k(\Delta_2) - V_k(\Delta_1)) + \\ &\quad \frac{\epsilon}{T_1^u + T_1^c} (1 - p_s) (V_k(\Delta_2 + T_1^u + T_1^c) - V_k(\Delta_1 + T_1^u + T_1^c)) \\ &\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_1^c}\right) \frac{L(\mathbf{a})}{\epsilon p_s} (\Delta_2 - \Delta_1) \\ &= \frac{L(\mathbf{a})}{\epsilon p} (\Delta_2 - \Delta_1). \end{aligned} \quad (12)$$

When $\mathbf{a} = (1, 0)$, we have

$$\begin{aligned} \Delta Q_k(s_2, s_1) &= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\ &= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right) (V_k(\Delta_2) - V_k(\Delta_1)) + \\ &\quad \frac{\epsilon}{T_2^u + T_1^c} (1 - q_s) (V_k(\Delta_2 + T_2^u + T_1^c) - V_k(\Delta_1 + T_2^u + T_1^c)) \\ &\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_1^c}\right) \frac{L(\mathbf{a})}{\epsilon q_s} (\Delta_2 - \Delta_1) \end{aligned}$$

$$= \frac{L(\mathbf{a})}{\epsilon p}(\Delta_2 - \Delta_1). \quad (13)$$

When $\mathbf{a} = (0, 1)$, we have

$$\begin{aligned} & \Delta Q_k(s_2, s_1) \\ &= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\ &= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right) (V_k(\Delta_2) - V_k(\Delta_1)) + \\ & \quad \frac{\epsilon}{T_1^u + T_2^c} (1 - p_l) (V_k(\Delta_2 + T_1^u + T_2^c) - V_k(\Delta_1 + T_1^u + T_2^c)) \\ &\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_1^u + T_2^c}\right) \frac{L(\mathbf{a})}{\epsilon p_l} (\Delta_2 - \Delta_1) \\ &= \frac{L(\mathbf{a})}{\epsilon p} (\Delta_2 - \Delta_1). \end{aligned} \quad (14)$$

When $\mathbf{a} = (1, 1)$, we have

$$\begin{aligned} & \Delta Q_k(s_2, s_1) \\ &= Q_k(\Delta_2, (0, 0)) - Q_k(\Delta_1, (0, 0)) \\ &= \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right) (V_k(\Delta_2) - V_k(\Delta_1)) + \\ & \quad \frac{\epsilon}{T_2^u + T_2^c} (1 - q_l) (V_k(\Delta_2 + T_2^u + T_2^c) - V_k(\Delta_1 + T_2^u + T_2^c)) \\ &\geq \Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_2^u + T_2^c}\right) \frac{L(\mathbf{a})}{\epsilon q_l} (\Delta_2 - \Delta_1) \\ &= \frac{L(\mathbf{a})}{\epsilon p} (\Delta_2 - \Delta_1). \end{aligned} \quad (15)$$

This concludes the proof of Lemma 3.

D. Proof of Theorem 1

For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$, we have

$$\begin{aligned} & Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\ &= \Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)} (V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\ & \quad + \frac{\epsilon}{L(a)} p_1 (V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X))). \end{aligned} \quad (16)$$

Since the concavity of $V(\mathbf{s})$ have been proved in Lemma 2, we can easily see that $V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X)) \leq V(\Delta_2, F(X)) - V(\Delta_1 + L(a), F(X))$. Therefore, we have

$$\begin{aligned} & Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\ &\leq \Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)} (V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\ & \quad + \frac{\epsilon}{L(a)} p_1 (V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \end{aligned}$$

$$= \Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(1 - p_1)(V(\mathbf{s}_2) - V(\mathbf{s}_1)). \quad (17)$$

As proved in Lemma 3 that $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq [L(a)/\epsilon(1 - p_1)](\Delta_2 - \Delta_1)$, it is easy to see that $Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V(\mathbf{s}_2) - V(\mathbf{s}_1)) \leq 0$.

Now, we can prove the threshold structure of the optimal policy. Suppose $\Delta_2 \geq \Delta_1$ and $\pi^*(\Delta_1, F(X)) = a$, it is easily to see that $V(\Delta_1, F(X)) = Q((\Delta_1, F(X)), a)$, i.e., $V(\mathbf{s}_1) = Q(\mathbf{s}_1, a)$. According to Theorem 1, we know that $V(\mathbf{s}_2) - V(\mathbf{s}_1) \geq Q(\mathbf{s}_2, a) - Q(\mathbf{s}_1, a)$. Therefore, we have $V(\mathbf{s}_2) \geq Q(\mathbf{s}_2, a)$. Since the value function is a minimum of two state-action cost functions, we have $V(\mathbf{s}_2) \leq Q(\mathbf{s}_2, a)$. Altogether, we can assert that $V(\mathbf{s}_2) = Q(\mathbf{s}_2, a)$ and $\pi^*(\Delta_2, F(X)) = a$.

This completes the proof of Theorem 1.

REFERENCES

- [1] P. Bertsekas, Dimitri, *Dynamic Programming and Optimal Control-II*, 3rd ed. Belmont, MA, USA: Athena Sci., 2007, vol. 2.