Supplementary Materials of TAoI for Monitoring Systems

I. Lemmas and Theorem

Lemma 1. *The value function $V(\Delta, F(X))$ is non-decreasing with Δ for any given $F(X)$.*

Proof: See Section II-A. □

Lemma 2. *Given $F(X)$, the value function $V(\Delta, F(X))$ is concave in Δ.*

Proof: See Section II-B. □

Since the value function $V(\Delta, F(X))$ is concave, its slope does not increase monotonically. The lower bound of the slope of $V(\Delta, F(X))$ is given by the following lemma.

Lemma 3. *For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, we have $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)$, where $p_1 = \hat{p}_A$ if $F(X) = 1$ and $p_1 = 1 - \hat{p}_B$ if $F(X) = 0$.*

Proof: See Section II-C. □

Based on the above lemmas, we can derive the structure of the optimal transmission policy as stated in the following theorem.

Theorem 4. *Given $F(X)$, there exists a stationary deterministic optimal policy that is of threshold-type in Δ. Specifically, if $\Delta \geq \Omega$, the $\pi^* = 1$, where Ω denotes the threshold given pair of Δ and $F(X)$.*

Proof: See Section II-D. □

II. Proofs of Lemmas and Theorem

A. Proof of Lemma 1

Based on the value iteration algorithm (VIA) outlined in [1, Ch. 4.3], we utilize mathematical induction to establish the proof of Lemma 1. Initially, we introduce $Q_k(\mathbf{s}, a)$ and $V_k(\mathbf{s})$ to represent the state-action value function and the state value function at the k-th iteration, respectively. Particularly, $Q_k(\mathbf{s}, a)$ is defined as

$$Q_k(\mathbf{s}, a) \triangleq \bar{R}(\mathbf{s}, a) + \sum_{\mathbf{s}' \in \mathcal{S}} \bar{p}(\mathbf{s}'|\mathbf{s}, a) V_k(\mathbf{s}'),\ \forall \mathbf{s} \in \mathcal{S}. \tag{1}$$

where $\mathbf{s}'$ is given by (**??**). For any given state $\mathbf{s}$, the update to the value function can be executed by

$$V_{k+1}(\mathbf{s}) = \min_{a \in \mathcal{A}} Q_k(\mathbf{s}, a),\ \forall \mathbf{s} \in \mathcal{S}. \tag{2}$$

Regardless of how $V_0(s)$ is initially set, the sequence $\{V_k(\mathbf{s})\}$ converges to $V(\mathbf{s})$ that satisfies the Bellman equation, i.e.,

$$\lim_{k \to \infty} V_k(\mathbf{s}) = V(\mathbf{s}),\ \forall \mathbf{s} \in \mathcal{S}. \tag{3}$$

Therefore, the monotonicity of $V(\mathbf{s})$ is validated by showing that, for any two states $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$V_k(\mathbf{s}_1) \leq V_k(\mathbf{s}_2),\ k = 0, 1, \cdots. \tag{4}$$

Next, we prove (4) using mathematical induction. Without loss of generality, we set $V_0(\mathbf{s}) = 0$ for each $s \in \mathcal{S}$, ensuring that (4) is satisfied at $k = 0$. Then, assuming that (4) holds up to $k > 0$, we verify whether it holds for $k + 1$.

For $a = 0$, it follows that

$$Q_k(\mathbf{s}_1, 0) = \Delta_1 + (1-\epsilon)V_k(\mathbf{s}_1) + \epsilon V_k(\mathbf{s}_1'), \tag{5}$$

and

$$Q_k(\mathbf{s}_2, 0) = \Delta_2 + (1-\epsilon)V_k(\mathbf{s}_2) + \epsilon V_k(\mathbf{s}_2'), \tag{6}$$

where $\mathbf{s}_1' = (\Delta_1 + 1, F(X)_+)$ and $\mathbf{s}_2' = (\Delta_2 + 1, F(X)_+)$. Given that $\Delta_1 + 1 \leq \Delta_2 + 1$, $V_k(\Delta_1) \leq V_k(\Delta_2)$ and $V_k(\mathbf{s}_1') \leq V_k(\mathbf{s}_2')$, it can be easily deduced that $Q_k(\mathbf{s}_1, 0) \leq Q_k(\mathbf{s}_2, 0)$.

For $a = 1$, it follows that

$$\begin{aligned} Q_k(\mathbf{s}_1, 1) = &\Delta_1 + \frac{1}{2}(T_u - 1) + \frac{\epsilon}{T_u} p_0 V_k(T_u, F(X)_+) + \\ &\frac{\epsilon}{T_u} p_1 V_k(\Delta_1 + T_u, F(X)_+) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\mathbf{s}_1), \end{aligned} \tag{7}$$

and

$$\begin{aligned} Q_k(\mathbf{s}_2, 1) = &\Delta_2 + \frac{1}{2}(T_u - 1) + \frac{\epsilon}{T_u} p_0 V_k(T_u, F(X)_+) + \\ &\frac{\epsilon}{T_u} p_1 V_k(\Delta_2 + T_u, F(X)_+) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\mathbf{s}_2), \end{aligned} \tag{8}$$

where if $F(X) = 1$, then $p_0 = 1 - \hat{p}_A$ and $p_1 = \hat{p}_A$; if $F(X) = 0$, then $p_0 = \hat{p}_B$ and $p_1 = 1 - \hat{p}_B$. Similar to $a = 0$, we can obtain $Q_k(\mathbf{s}_1, 1) \leq Q_k(\mathbf{s}_2, 1)$ according to $\Delta_1 \leq \Delta_2$ and $V_k(\mathbf{s}_1) \leq V_k(\mathbf{s}_2)$.

By (2), we can get that $V_{k+1}(\mathbf{s}_1) \leq V_{k+1}(\mathbf{s}_2)$ for any k.

This completes the proof of Lemma 1.

B. Proof of Lemma 2

The concavity of $V(\mathbf{s})$ with respect to $\mathbf{s}$ for any given $F(X)$ can be demonstrated by showing that, for any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$ and $w \in N$, whenever $\Delta_1 \leq \Delta_2$, it follows that

$$\begin{aligned} &V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)) \geq \\ &V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X)), k = 0, 1, \ldots. \end{aligned} \tag{9}$$

Without sacrificing generality, we set $V_0(\mathbf{s}) = 0$ for all $\mathbf{s} \in \mathcal{S}$, ensuring that (9) is applicable at $k = 0$. Then, we assume that (9) holds up till $k > 0$ and inspect whether it holds for

$k+1$. Now, let $\mathbf{s}=(\Delta,F(X))$, $\mathbf{s}_1=(\Delta_1,F(X))$, $\mathbf{s}_2=(\Delta_2,F(X))$, $\mathbf{s}'=(\Delta+w,F(X))$, $\mathbf{s}'_1=(\Delta_1+w,F(X))$ and $\mathbf{s}'_2=(\Delta_2+w,F(X))$. For ease of explanation, we introduce $\Delta Q(\mathbf{s}',\mathbf{s},a)=Q(\mathbf{s}',a)-Q(\mathbf{s},a)$.

For $a=0$, it follows that

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}'_1,\mathbf{s}_1,0)-\Delta Q_k(\mathbf{s}'_2,\mathbf{s}_2,0)\\
=&[\Delta_2+(1-\epsilon)V_k(\Delta_2,F(X))+\epsilon V_k(\Delta_2+1,F(X)_+)]\\
&-[\Delta_1+(1-\epsilon)V_k(\Delta_1,F(X))+\epsilon V_k(\Delta_1+1,F(X)_+)]\\
&+[\Delta_1+w+(1-\epsilon)V_k(\Delta_1+w,F(X))\\
&+\epsilon V_k(\Delta_1+w+1,F(X)_+)]\\
&-[\Delta_2+w+(1-\epsilon)V_k(\Delta_2+w,F(X))\\
&+\epsilon V_k(\Delta_2+w+1,F(X)_+)]\\
=&(1-\epsilon)[(V_k(\Delta_1+w,F(X))-V_k(\Delta_1,F(X)))\\
&-(V_k(\Delta_2+w,F(X))-V_k(\Delta_2,F(X)))]\\
&+\epsilon((V_k(\Delta_1+w+1,F(X)_+)-V_k(\Delta_1+1,F(X)_+))\\
&-(V_k(\Delta_2+w+1,F(X)_+)-V_k(\Delta_2+1,F(X)_+))).
\end{aligned}\tag{10}$$

Given that $V_k(\Delta_1+w,F(X))-V_k(\Delta_1,F(X))\geq V_k(\Delta_2+w,F(X))-V_k(\Delta_2,F(X))$ and $V_k(\Delta_1+w+1,F(X)_+)-V_k(\Delta_1+1,F(X)_+)\geq V_k(\Delta_2+w+1,F(X)_+)-V_k(\Delta_2+1,F(X)_+)$, we can easily see that $\Delta Q_k(\mathbf{s}'_1,\mathbf{s}_1,0)-\Delta Q_k(\mathbf{s}'_2,\mathbf{s}_2,0)\geq 0$. Thus, $Q_k(\mathbf{s},0)$ is concave in Δ for any given $F(X)$.

For $a=1$, it follows that

$$\begin{aligned}
&\Delta Q(\mathbf{s}'_1,\mathbf{s}_1,1)-\Delta Q(\mathbf{s}'_2,\mathbf{s}_2,1)\\
=&\Delta_1+w+\frac{1}{2}(T_u-1)+\left(1-\frac{\epsilon}{T_u}\right)V_k(\Delta_1+w,F(X))\\
&+\frac{\epsilon}{T_u}\left(p_0V_k(T_u,F(X)_+)+p_1V_k(\Delta_1+w+T_u,F(X)_+)\right)\\
&-[\Delta_1+\frac{1}{2}(T_u-1)+\left(1-\frac{\epsilon}{T_u}\right)V_k(\Delta_1,F(X))\\
&+\frac{\epsilon}{T_u}\left(V_k(T_u,F(X)_+)+p_1V_k(\Delta_1+T_u,F(X)_+)\right)]\\
&-[\Delta_2+w+\frac{1}{2}(T_u-1)+\left(1-\frac{\epsilon}{T_u}\right)V_k(\Delta_2+w,F(X))\\
&+\frac{\epsilon}{T_u}\left(p_0V_k(T_u,F(X)_+)+p_1V_k(\Delta_2+w+T_u,F(X)_+)\right)]\\
&+[\Delta_2+\frac{1}{2}(T_u-1)+\left(1-\frac{\epsilon}{T_u}\right)V_k(\Delta_2,F(X))\\
&+\frac{\epsilon}{T_u}\left(p_0V_k(T_u,F(X)_+)+p_1V_k(\Delta_2+T_u,F(X)_+)\right)]\\
=&(1-\frac{\epsilon}{T_u})[(V_k(\Delta_1+w,F(X))-V_k(\Delta_1,F(X)))\\
&-(V_k(\Delta_2+w,F(X))-V_k(\Delta_2,F(X)))]\\
&+\frac{\epsilon}{T_u}p_1[(V_k(\Delta_1+w+T_u,F(X)_+)\\
&-V_k(\Delta_1+T_u,F(X)_+))\\
&-(V_k(\Delta_2+w+T_u,F(X)_+)\\
&-V_k(\Delta_2+T_u,F(X)_+))].
\end{aligned}\tag{11}$$

Given that $V_k(\Delta_1+w,F(X))-V_k(\Delta_1,F(X))\geq V_k(\Delta_2+w,F(X))-V_k(\Delta_2,F(X))$ and $V_k(\Delta_1+w+T_u,F(X)_+)-V_k(\Delta_1+T_u,F(X)_+)\geq V_k(\Delta_2+w+T_u,F(X)_+)-V_k(\Delta_2+T_u,F(X)_+)$, we can also get that $\Delta Q_k(\mathbf{s}'_1,\mathbf{s}_1,1)-\Delta Q_k(\mathbf{s}'_2,\mathbf{s}_2,1)\geq 0$. Thus, $Q_k(\mathbf{s},1)$ is concave in Δ for any given $F(X)$.

Since the value function $V_{k+1}(\mathbf{s})$ is the minimum of two concave functions, it is also concave in Δ for any given $F(X)$. Hence, we have $V_k(\Delta_1+w,F(X))-V_k(\Delta_1,F(X))\geq V_k(\Delta_2+w,F(X))-V_k(\Delta_2,F(X))$, i.e., (9) holds for $k+1$. Therefore, we can show that (9) holds for any k by induction.

This completes the proof of Lemma 2.

C. Proof of Lemma 3

The proof follows the same procedure of Lemma 1. The lower bound of $V(\mathbf{s}_2)-V(\mathbf{s}_1)$ can be proved by showing that for any $\mathbf{s}_1=(\Delta_1,F(X))$, $\mathbf{s}_2=(\Delta_2,F(X))\in\mathcal{S}$, such that $\Delta_1\leq\Delta_2$

$$V_k(\Delta_2,F(X))-V_k(\Delta_1,F(X))\geq\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1),\quad k=0,1,\cdots.\tag{12}$$

Without sacrificing generality, we set $V_0(\mathbf{s})=\frac{L(a)}{\epsilon(1-p_1)}\Delta$ for all $\mathbf{s}=(\Delta,F(X))\in\mathcal{S}$, ensuring that (12) is satisfied at $k=0$. Then, we assume that (12) holds up till $k>0$ and hence we have $V_k(\Delta_2,F(X))-V_k(\Delta_1,F(X))\geq\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1)$ and $V_k(\Delta_2+1,F(X))-V_k(\Delta_1+1,F(X))\geq\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1)$.

Then, we inspect whether it holds for $k+1$. We first consider the case when $a=1$ and we have $\frac{L(a)}{\epsilon(1-p_1)}=\frac{T_u}{\epsilon(1-p_1)}$. Since $V_{k+1}(\mathbf{s})=\min_{a\in\mathcal{A}}Q_k(\mathbf{s},a)$, we investigate the two state-action value functions, in the following, respectively.

When $F(X)=1$ and $a=0$, we have

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}_2,\mathbf{s}_1)\\
=&Q_k((\Delta_2,F(X)),0)-Q_k((\Delta_1,F(X)),0)\\
=&\Delta_2-\Delta_1+(1-\epsilon)\left(V_k(\Delta_2,F(X))-V_k(\Delta_1,F(X))\right)\\
&+\epsilon(V_k(\Delta_2+1,F(X)_+)-V_k(\Delta_1+1,F(X)_+))\\
\geq&(\Delta_2-\Delta_1)+\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1)\\
=&\left(1+\frac{L(a)}{\epsilon(1-p_1)}\right)(\Delta_2-\Delta_1)\\
\geq&\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1).
\end{aligned}\tag{13}$$

When $F(X)=1$ and $a=1$, we have

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}_2,\mathbf{s}_1)\\
=&Q_k((\Delta_2,F(X)),1)-Q_k((\Delta_1,F(X)),1)\\
=&\Delta_2-\Delta_1+\left(1-\frac{\epsilon}{T_u}\right)V_k(\Delta_2,F(X))-V_k(\Delta_1,F(X))\\
&\frac{\epsilon}{T_u}p_1\left(V_k(\Delta_2+T_u,F(X)_+)-V_k(\Delta_1+T_u,F(X)_+)\right)\\
\geq&\Delta_2-\Delta_1+\left(1-\frac{\epsilon}{T_u}+\frac{\epsilon}{T_u}p_1\right)\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1)\\
=&\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2-\Delta_1).
\end{aligned}\tag{14}$$

Similarly, the same results can be derived when $F(X) = 0$. When the optimal policy in $\mathbf{s}_1$ and $\mathbf{s}_2$ is two different actions, i.e., a_1 and a_2, we have

$$\begin{aligned}
&V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \\
&= Q_k((\Delta_2, F(X)), a_2) - Q_k((\Delta_1, F(X)), a_1) \\
&\geq Q_k((\Delta_2, F(X)), a_2) - Q_k((\Delta_1, F(X)), a_2) \\
&\geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1).
\end{aligned} \tag{15}$$

This completes the proof of Lemma 3.

D. Proof of Theorem 1

For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
&+ \frac{\epsilon}{L(a)} p_1(V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X))).
\end{aligned} \tag{16}$$

Since the concavity of $V(\mathbf{s})$ have been proved in Lemma 2, we can easily see that $V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X)) \leq V(\Delta_2, F(X)) - V(\Delta_1 + L(a), F(X))$. Therefore, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
\leq&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1), F(X)) \\
&+ \frac{\epsilon}{L(a)} p_1(V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(1 - p_1)(V(\mathbf{s}_2) - V(\mathbf{s}_1)).
\end{aligned} \tag{17}$$

As proved in Lemma 3 that $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq [L(a)/\epsilon(1-p_1)](\Delta_2 - \Delta_1)$, it is easy to see that $Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V(\mathbf{s}_2) - V(\mathbf{s}_1)) \leq 0$.

Now, we can prove the threshold structure of the optimal policy. Suppose $\Delta_2 \geq \Delta_1$ and $\pi^*(\Delta_1, F(X)) = a$, it is easily to see that $V(\Delta_1, F(X)) = Q((\Delta_1, F(X)), a)$, i.e., $V(\mathbf{s}_1) = Q(\mathbf{s}_1, a)$. According to Theorem 1, we know that $V(\mathbf{s}_2) - V(\mathbf{s}_1) \geq Q(\mathbf{s}_2, a) - Q(\mathbf{s}_1, a)$. Therefore, we have $V(\mathbf{s}_2) \geq Q(\mathbf{s}_2, a)$. Since the value function is a minimum of two state-action cost functions, we have $V(\mathbf{s}_2) \leq Q(\mathbf{s}_2, a)$. Altogether, we can assert that $V(\mathbf{s}_2) = Q(\mathbf{s}_2, a)$ and $\pi^*(\Delta_2, F(X)) = a$.

This completes the proof of Theorem 1.

REFERENCES

[1] P. Bertsekas, Dimitri, *Dynamic Programming and Optimal Control-II*, 3rd ed. Belmont, MA, USA: Athena Sci., 2007, vol. 2.