# Supplementary Materials of TAoI for Monitoring Systems

## I. Lemmas and Theorem

**Lemma 1.** *The value function $V(\Delta, F(X))$ is non-decreasing with $\Delta$ for any given $F(X)$.*

*Proof:* See Section II-A. □

**Lemma 2.** *Given $F(X)$, the value function $V(\Delta, F(X))$ is concave in $\Delta$.*

*Proof:* See Section II-B. □

Since the value function $V(\Delta, F(X))$ is concave, its slope does not increase monotonically. The lower bound of the slope of $V(\Delta, F(X))$ is given by the following lemma.

**Lemma 3.** *For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$ with $\Delta_1 \leq \Delta_2$, we have $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)$, where $p_1 = \hat{p}_A$ if $F(X) = 1$ and $p_1 = 1 - \hat{p}_B$ if $F(X) = 0$.*

*Proof:* See Section II-C. □

Based on the above lemmas, we can derive the structure of the optimal transmission policy as stated in the following theorem.

**Theorem 4.** *Given $F(X)$, there exists a stationary deterministic optimal policy that is of threshold-type in $\Delta$. Specifically, if $\Delta \geq \Omega$, the $\pi^* = 1$, where $\Omega$ denotes the threshold given pair of $\Delta$ and $F(X)$.*

*Proof:* See Section II-D. □

## II. Proofs of Lemmas and Theorem

### A. Proof of Lemma 1

Based on the value iteration algorithm (VIA) outlined in [1, Ch. 4.3], we utilize mathematical induction to establish the proof of Lemma 1. Initially, we introduce $Q_k(\mathbf{s}, a)$ and $V_k(\mathbf{s})$ to represent the state-action value function and the state value function at the $k$-th iteration, respectively. Particularly, $Q_k(\mathbf{s}, a)$ is defined as

$$Q_k(\mathbf{s}, a) \triangleq \bar{R}(\mathbf{s}, a) + \sum_{\mathbf{s}' \in \mathcal{S}} \bar{p}(\mathbf{s}'|\mathbf{s}, a) V_k(\mathbf{s}'),\ \forall \mathbf{s} \in \mathcal{S}. \tag{1}$$

where $\mathbf{s}'$ is given by (**??**). For any given state $\mathbf{s}$, the update to the value function can be executed by

$$V_{k+1}(\mathbf{s}) = \min_{a \in \mathcal{A}} Q_k(\mathbf{s}, a),\ \forall \mathbf{s} \in \mathcal{S}. \tag{2}$$

Regardless of how $V_0(s)$ is initially set, the sequence $\{V_k(\mathbf{s})\}$ converges to $V(\mathbf{s})$ that satisfies the Bellman equation, i.e.,

$$\lim_{k \to \infty} V_k(\mathbf{s}) = V(\mathbf{s}),\ \forall \mathbf{s} \in \mathcal{S}. \tag{3}$$

Therefore, the monotonicity of $V(\mathbf{s})$ is validated by showing that, for any two states $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, whenever $\Delta_1 \leq \Delta_2$ , it follows that

$$V_k(\mathbf{s}_1) \leq V_k(\mathbf{s}_2),\ k = 0, 1, \cdots. \tag{4}$$

Next, we prove (4) using mathematical induction. Without loss of generality, we set $V_0(\mathbf{s}) = 0$ for each $s \in \mathcal{S}$, ensuring that (4) is satisfied at $k = 0$. Then, assuming that (4) holds up to $k > 0$, we verify whether it holds for $k + 1$.

For $a = 0$, it follows that

$$Q_k(\mathbf{s}_1, 0) = \Delta_1 + (1-\epsilon)V_k(\mathbf{s}_1) + \epsilon V_k(\mathbf{s}_1'), \tag{5}$$

and

$$Q_k(\mathbf{s}_2, 0) = \Delta_2 + (1-\epsilon)V_k(\mathbf{s}_2) + \epsilon V_k(\mathbf{s}_2'), \tag{6}$$

where $\mathbf{s}_1' = (\Delta_1 + 1, F(X)_+)$ and $\mathbf{s}_2' = (\Delta_2 + 1, F(X)_+)$. Given that $\Delta_1 + 1 \leq \Delta_2 + 1$, $V_k(\Delta_1) \leq V_k(\Delta_2)$ and $V_k(\mathbf{s}_1') \leq V_k(\mathbf{s}_2')$, it can be easily deduced that $Q_k(\mathbf{s}_1, 0) \leq Q_k(\mathbf{s}_2, 0)$.

For $a = 1$, it follows that

$$\begin{aligned} Q_k(\mathbf{s}_1, 1) =& \Delta_1 + \frac{1}{2}(T_u - 1) + \frac{\epsilon}{T_u} p_0 V_k(T_u, F(X)_+) + \\ & \frac{\epsilon}{T_u} p_1 V_k(\Delta_1 + T_u, F(X)_+) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\mathbf{s}_1), \end{aligned} \tag{7}$$

and

$$\begin{aligned} Q_k(\mathbf{s}_2, 1) =& \Delta_2 + \frac{1}{2}(T_u - 1) + \frac{\epsilon}{T_u} p_0 V_k(T_u, F(X)_+) + \\ & \frac{\epsilon}{T_u} p_1 V_k(\Delta_2 + T_u, F(X)_+) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\mathbf{s}_2), \end{aligned} \tag{8}$$

where if $F(X) = 1$, then $p_0 = 1 - \hat{p}_A$ and $p_1 = \hat{p}_A$; if $F(X) = 0$, then $p_0 = \hat{p}_B$ and $p_1 = 1 - \hat{p}_B$. Similar to $a = 0$, we can obtain $Q_k(\mathbf{s}_1, 1) \leq Q_k(\mathbf{s}_2, 1)$ according to $\Delta_1 \leq \Delta_2$ and $V_k(\mathbf{s}_1) \leq V_k(\mathbf{s}_2)$.

By (2), we can get that $V_{k+1}(\mathbf{s}_1) \leq V_{k+1}(\mathbf{s}_2)$ for any $k$.

This completes the proof of Lemma 1.

### B. Proof of Lemma 2

The concavity of $V(\mathbf{s})$ with respect to $\mathbf{s}$ for any given $F(X)$ can be demonstrated by showing that, for any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$ and $w \in N$, whenever $\Delta_1 \leq \Delta_2$ , it follows that

$$\begin{aligned} & V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)) \geq \\ & V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X)), k = 0, 1, \ldots. \end{aligned} \tag{9}$$

Without sacrificing generality, we set $V_0(\mathbf{s}) = 0$ for all $\mathbf{s} \in \mathcal{S}$, ensuring that (9) is applicable at $k = 0$. Then, we assume that (9) holds up till $k > 0$ and inspect whether it holds for

$k+1$. Now, let $\mathbf{s} = (\Delta, F(X))$, $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X))$, $\mathbf{s}' = (\Delta + w, F(X))$, $\mathbf{s}_1' = (\Delta_1 + w, F(X))$ and $\mathbf{s}_2' = (\Delta_2 + w, F(X))$. For ease of explanation, we introduce $\Delta Q(\mathbf{s}', \mathbf{s}, a) = Q(\mathbf{s}', a) - Q(\mathbf{s}, a)$.

For $a = 0$, it follows that

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}_1', \mathbf{s}_1, 0) - \Delta Q_k(\mathbf{s}_2', \mathbf{s}_2, 0)\\
=&[\Delta_2 + (1-\epsilon)V_k(\Delta_2, F(X)) + \epsilon V_k(\Delta_2 + 1, F(X)_+)]\\
&- [\Delta_1 + (1-\epsilon)V_k(\Delta_1, F(X)) + \epsilon V_k(\Delta_1 + 1, F(X)_+)]\\
&+ [\Delta_1 + w + (1-\epsilon)V_k(\Delta_1 + w, F(X))\\
&+ \epsilon V_k(\Delta_1 + w + 1, F(X)_+)]\\
&- [\Delta_2 + w + (1-\epsilon)V_k(\Delta_2 + w, F(X))\\
&+ \epsilon V_k(\Delta_2 + w + 1, F(X)_+)]\\
=&(1-\epsilon)[(V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)))\\
&- (V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X)))]\\
&+ \epsilon((V_k(\Delta_1 + w + 1, F(X)_+) - V_k(\Delta_1 + 1, F(X)_+))\\
&- (V_k(\Delta_2 + w + 1, F(X)_+) - V_k(\Delta_2 + 1, F(X)_+))).
\end{aligned} \tag{10}$$

Given that $V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)) \geq V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X))$ and $V_k(\Delta_1 + w + 1, F(X)_+) - V_k(\Delta_1 + 1, F(X)_+) \geq V_k(\Delta_2 + w + 1, F(X)_+) - V_k(\Delta_2 + 1, F(X)_+)$, we can easily see that $\Delta Q_k(\mathbf{s}_1', \mathbf{s}_1, 0) - \Delta Q_k(\mathbf{s}_2', \mathbf{s}_2, 0) \geq 0$. Thus, $Q_k(\mathbf{s}, 0)$ is concave in $\Delta$ for any given $F(X)$.

For $a = 1$, it follows that

$$\begin{aligned}
&\Delta Q(\mathbf{s}_1', \mathbf{s}_1, 1) - \Delta Q(\mathbf{s}_2', \mathbf{s}_2, 1)\\
=&\Delta_1 + w + \frac{1}{2}(T_u - 1) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\Delta_1 + w, F(X))\\
&+ \frac{\epsilon}{T_u}\left(p_0 V_k(T_u, F(X)_+) + p_1 V_k(\Delta_1 + w + T_u, F(X)_+)\right)\\
&- [\Delta_1 + \frac{1}{2}(T_u - 1) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\Delta_1, F(X))\\
&+ \frac{\epsilon}{T_u}\left(V_k(T_u, F(X)_+) + p_1 V_k(\Delta_1 + T_u, F(X)_+)\right)]\\
&- [\Delta_2 + w + \frac{1}{2}(T_u - 1) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\Delta_2 + w, F(X))\\
&+ \frac{\epsilon}{T_u}\left(p_0 V_k(T_u, F(X)_+) + p_1 V_k(\Delta_2 + w + T_u, F(X)_+)\right)]\\
&+ [\Delta_2 + \frac{1}{2}(T_u - 1) + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\Delta_2, F(X))\\
&+ \frac{\epsilon}{T_u}\left(p_0 V_k(T_u, F(X)_+) + p_1 V_k(\Delta_2 + T_u, F(X)_+)\right)]\\
=&(1 - \frac{\epsilon}{T_u})[(V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)))\\
&- (V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X)))]\\
&+ \frac{\epsilon}{T_u} p_1 [(V_k(\Delta_1 + w + T_u, F(X)_+)\\
&- V_k(\Delta_1 + T_u, F(X)_+))\\
&- (V_k(\Delta_2 + w + T_u, F(X)_+)\\
&- V_k(\Delta_2 + T_u, F(X)_+))].
\end{aligned} \tag{11}$$

Given that $V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)) \geq V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X))$ and $V_k(\Delta_1 + w + T_u, F(X)_+) - V_k(\Delta_1 + T_u, F(X)_+) \geq V_k(\Delta_2 + w + T_u, F(X)_+) - V_k(\Delta_2 + T_u, F(X)_+)$, we can also get that $\Delta Q_k(\mathbf{s}_1', \mathbf{s}_1, 1) - \Delta Q_k(\mathbf{s}_2', \mathbf{s}_2, 1) \geq 0$. Thus, $Q_k(\mathbf{s}, 1)$ is concave in $\Delta$ for any given $F(X)$.

Since the value function $V_{k+1}(\mathbf{s})$ is the minimum of two concave functions, it is also concave in $\Delta$ for any given $F(X)$. Hence, we have $V_k(\Delta_1 + w, F(X)) - V_k(\Delta_1, F(X)) \geq V_k(\Delta_2 + w, F(X)) - V_k(\Delta_2, F(X))$, i.e., (9) holds for $k+1$. Therefore, we can show that (9) holds for any $k$ by induction.

This completes the proof of Lemma 2.

### C. Proof of Lemma 3

The proof follows the same procedure of Lemma 1. The lower bound of $V(\mathbf{s}_2) - V(\mathbf{s}_1)$ can be proved by showing that for any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$

$$V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1), \quad k = 0, 1, \cdots. \tag{12}$$

Without sacrificing generality, we set $V_0(\mathbf{s}) = \frac{L(a)}{\epsilon(1-p_1)}\Delta$ for all $\mathbf{s} = (\Delta, F(X)) \in \mathcal{S}$, ensuring that (12) is satisfied at $k = 0$. Then, we assume that (12) holds up till $k > 0$ and hence we have $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)$ and $V_k(\Delta_2 + 1, F(X)) - V_k(\Delta_1 + 1, F(X)) \geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)$.

Then, we inspect whether it holds for $k+1$. We first consider the case when $a = 1$ and we have $\frac{L(a)}{\epsilon(1-p_1)} = \frac{T_u}{\epsilon(1-p_1)}$. Since $V_{k+1}(\mathbf{s}) = \min_{a \in \mathcal{A}} Q_k(\mathbf{s}, a)$, we investigate the two state-action value functions, in the following, respectively.

When $F(X) = 1$ and $a = 0$, we have

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}_2, \mathbf{s}_1)\\
=&Q_k((\Delta_2, F(X)), 0) - Q_k((\Delta_1, F(X)), 0)\\
=&\Delta_2 - \Delta_1 + (1-\epsilon)\left(V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X))\right)\\
&+ \epsilon(V_k(\Delta_2 + 1, F(X)_+) - V_k(\Delta_1 + 1, F(X)_+))\\
\geq&(\Delta_2 - \Delta_1) + \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)\\
=&\left(1 + \frac{L(a)}{\epsilon(1-p_1)}\right)(\Delta_2 - \Delta_1)\\
\geq&\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1).
\end{aligned} \tag{13}$$

When $F(X) = 1$ and $a = 1$, we have

$$\begin{aligned}
&\Delta Q_k(\mathbf{s}_2, \mathbf{s}_1)\\
=&Q_k((\Delta_2, F(X)), 1) - Q_k((\Delta_1, F(X)), 1)\\
=&\Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_u}\right) V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X))\\
&\frac{\epsilon}{T_u} p_1 \left(V_k(\Delta_2 + T_u, F(X)_+) - V_k(\Delta_1 + T_u, F(X)_+)\right)\\
\geq&\Delta_2 - \Delta_1 + \left(1 - \frac{\epsilon}{T_u} + \frac{\epsilon}{T_u} p_1\right) \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1)\\
=&\frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1).
\end{aligned} \tag{14}$$

Similarly, the same results can be derived when $F(X) = 0$. When the optimal policy in $\mathbf{s}_1$ and $\mathbf{s}_2$ is two different actions, i.e., $a_1$ and $a_2$, we have

$$\begin{aligned}
&V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \\
&= Q_k((\Delta_2, F(X)), a_2) - Q_k((\Delta_1, F(X)), a_1) \\
&\geq Q_k((\Delta_2, F(X)), a_2) - Q_k((\Delta_1, F(X)), a_2) \\
&\geq \frac{L(a)}{\epsilon(1-p_1)}(\Delta_2 - \Delta_1).
\end{aligned} \tag{15}$$

This completes the proof of Lemma 3.

### D. Proof of Theorem 1

For any $\mathbf{s}_1 = (\Delta_1, F(X))$, $\mathbf{s}_2 = (\Delta_2, F(X)) \in \mathcal{S}$, such that $\Delta_1 \leq \Delta_2$, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
&+ \frac{\epsilon}{L(a)} p_1 (V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X))).
\end{aligned} \tag{16}$$

Since the concavity of $V(\mathbf{s})$ have been proved in Lemma 2, we can easily see that $V(\Delta_2 + L(a), F(X)) - V(\Delta_1 + L(a), F(X)) \leq V(\Delta_2, F(X)) - V(\Delta_1 + L(a), F(X))$. Therefore, we have

$$\begin{aligned}
&Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V_k(\mathbf{s}_2) - V_k(\mathbf{s}_1)) \\
\leq&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(V(\Delta_2, F(X)) - V(\Delta_1), F(X)) \\
&+ \frac{\epsilon}{L(a)} p_1 (V(\Delta_2, F(X)) - V(\Delta_1, F(X))) \\
=&\Delta_2 - \Delta_1 - \frac{\epsilon}{L(a)}(1-p_1)(V(\mathbf{s}_2) - V(\mathbf{s}_1)).
\end{aligned} \tag{17}$$

As proved in Lemma 3 that $V_k(\Delta_2, F(X)) - V_k(\Delta_1, F(X)) \geq [L(a)/\epsilon(1-p_1)](\Delta_2 - \Delta_1)$, it is easy to see that $Q_k(\mathbf{s}_2, a) - Q_k(\mathbf{s}_1, a) - (V(\mathbf{s}_2) - V(\mathbf{s}_1)) \leq 0$.

Now, we can prove the threshold structure of the optimal policy. Suppose $\Delta_2 \geq \Delta_1$ and $\pi^*(\Delta_1, F(X)) = a$, it is easily to see that $V(\Delta_1, F(X)) = Q((\Delta_1, F(X)), a)$, i.e., $V(\mathbf{s}_1) = Q(\mathbf{s}_1, a)$. According to Theorem 1, we know that $V(\mathbf{s}_2) - V(\mathbf{s}_1) \geq Q(\mathbf{s}_2, a) - Q(\mathbf{s}_1, a)$. Therefore, we have $V(\mathbf{s}_2) \geq Q(\mathbf{s}_2, a)$. Since the value function is a minimum of two state-action cost functions, we have $V(\mathbf{s}_2) \leq Q(\mathbf{s}_2, a)$. Altogether, we can assert that $V(\mathbf{s}_2) = Q(\mathbf{s}_2, a)$ and $\pi^*(\Delta_2, F(X)) = a$.

This completes the proof of Theorem 1.

## REFERENCES

[1] P. Bertsekas, Dimitri, *Dynamic Programming and Optimal Control-II*, 3rd ed. Belmont, MA, USA: Athena Sci., 2007, vol. 2.