Problem 1: A beam of variable cross-section is stretched by its own weight and compressed by a force P applied at the end of the beam as shown in Figure 1. Determine the magnitude of the force at which the displacement of the section A-A will be zero. The material density is ρ, modulus of elasticity is E, and the cross-sectional area is $F(z)$.

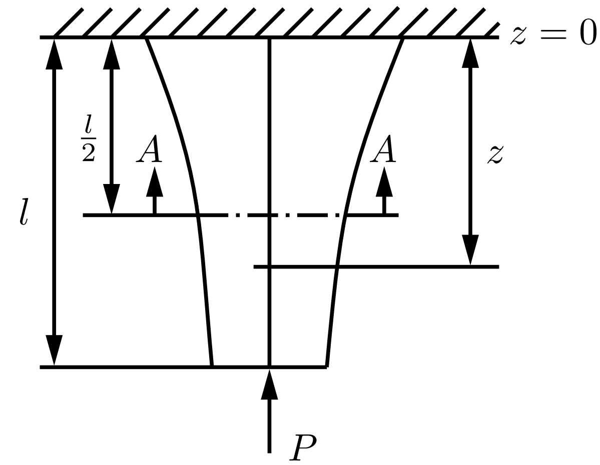

Figure 1: A beam of variable cross-section.

Solution: Let us determine the magnitude of the normal force $N(z)$ in an arbitrary section z.

$$N(z) = \int_z^l p(z)\; dz - P. \tag{1}$$

Since

$$p(z) = \rho F(z) \tag{2}$$

then

$$N(z) = \rho \int_z^l F(z)\; dz - P. \tag{3}$$

and

$$\sigma(z) = \frac{N(z)}{F(z)} = \frac{\rho}{F(z)} \int_z^l F(z_1)\; dz_1 - \frac{P}{F(z)}. \tag{4}$$

We know that the displacement $w(z)$ of an arbitrary section z is equal to

$$w(z) = \int_0^z \varepsilon(z_1)\; dz_1 = \frac{1}{E}\int_0^z \sigma(z_1)\; dz_1 = \frac{1}{E}\int_0^z \frac{\rho \int_z^l F(z_1)\; dz_1 - P}{F(z_1)}\; dz_1 \tag{5}$$

or

$$w(z) = \frac{1}{E}\int_0^z \frac{\rho \int_z^l F(z_1)\; dz_1}{F(z_1)}\; dz - \frac{P}{E}\int_0^z \frac{dz_1}{F(z_1)} \tag{6}$$

Now we can find force P at which the displacement of the section A-A with the coordinate $z^\star$ is zero.

$$w(z^\star) = \frac{1}{E}\int_0^{z^\star} \frac{\rho \int_z^l F(z_1)\; dz_1}{F(z_1)}\; dz - \frac{P}{E}\int_0^{z^\star} \frac{dz_1}{F(z_1)} = 0 \tag{7}$$

$$\frac{1}{E}\int_0^{z^\star} \frac{\rho \int_z^l F(z_1)\; dz_1}{F(z_1)} = \frac{P}{E}\int_0^{z^\star} \frac{dz_1}{F(z_1)} \tag{8}$$

$$P = \rho \frac{\int_0^{z^\star} \frac{\int_z^l F(z_1)\ dz_1}{F(z_1)}\ dz}{\int_0^{z^\star} \frac{dz_1}{F(z_1)}} \tag{9}$$

If the beam has a constant cross section, that is, $F(z) = F$, then at $z^\star = \frac{l}{2}$, we have

$$\begin{aligned}
P =& \rho F \frac{\int_0^{z^\star} \left[\int_z^l 1\ dz_1\right]\ dz}{\int_0^{z^\star} 1\ dz_1} \Rightarrow \\
P =& \rho F \frac{\int_0^{z^\star} (l-z)\ dz}{\int_0^{z^\star} 1\ dz_1} = \rho F \frac{\left[l z^\star - \frac{(z^\star)^2}{2}\right]}{z^\star} = \rho F \left[l - \frac{z^\star}{2}\right] \Rightarrow \\
P =& \frac{3}{4} \rho F l
\end{aligned} \tag{10}$$

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