E-Companion

EC.1. More Related Work

In addition to the aforementioned works, we briefly review other recent works on market equilibrium computation and fair division.

Market equilibrium computation. For the classical Fisher market setting with finitely many items, there is a large literature on equilibrium computation algorithms, some based on solving new equilibrium-capturing convex programs. For example, Devanur et al. (2008) established the first polynomial-time algorithm for exact equilibrium computation for a finite-dimensional linear Fisher market, based on a primal-dual algorithm for solving the Eisenberg-Gale convex program. Zhang (2011) proposed a distributed dynamics that converges to an equilibrium, which, as later analyzed in Birnbaum et al. (2011), is in fact a first-order optimization algorithm applied to a specific convex program due to Shmyrev (2009). Cheung et al. (2019) studied tâtonnement dynamics and show its equivalence to gradient descent on Eisenberg-Gale-type convex programs under the more general class of CES utilities. Gao and Kroer (2020) studied first-order methods based on (old and new) convex programs for Fisher market equilibria under commonly used utilities. Bei et al. (2019) studied earning and utility limits in markets with linear and spending-constraint utilities, and proposed a polynomial-time algorithm for computing an equilibrium.

Fair division. As stated previously, a Fisher market equilibrium on finitely many divisible items is known to be a strong fair division approach. There is also a literature on fair division of *indivisible* items via maximizing the Nash social welfare (NSW); the discrete analogue of the Eisenberg-Gale program. This was started by Caragiannis et al. (2016), who showed that the maximum NSW solution provides fairness guarantees in the indivisible divisible case as well, and proposed a practical algorithm based on mixed-integer linear programming. There are also several works on approximation algorithms for this settings, see e.g. Garg et al. (2018), Barman et al. (2018). Interestingly, in this paper, we show that our continuum setting allows us to construct allocations via convex programming, even for the indivisible setting.

EC.2. Proofs

Proof of Lemma 1

First, we state a lemma due to Dvoretzky et al. (1951, Theorems 1 and 4). Here, we assume that there is an underlying σ-algebra $\mathcal{M}$ on Θ. A measure is a countably additive set function mapping each measurable set in $\mathcal{M}$ to a nonnegative real number. A function f on Θ is measurable if, for any $c \in \mathbb{R}$, it holds that $\{f < c\} = \{\theta \in \Theta : f(\theta) < c\} \in \mathcal{M}$.

LEMMA EC.1. *Let μ_i be finite measures on Θ, $i \in m$. Let*

$$\mathcal{X} = \left\{ (x_1, \ldots, x_p) : \sum_j x_j = \mathbf{1}, \text{ a.e., } \textit{each } x_j \textit{ is a nonnegative measurable function on } \Theta \right\},$$
$$\mathcal{S} = \{(\Theta_1, \ldots, \Theta_p) \in \mathcal{M}^p : \ \Theta_i \cap \Theta_j = \emptyset \textit{ for all } i \neq j, \cup_i \Theta_i = \Theta\}.$$

Define

$$\mathcal{U} = \left\{ u \in \mathbb{R}^{m \times p} : u_{ij} = \int_\Theta x_j d\mu_i, \, (x_1, \ldots, x_p) \in \mathcal{X} \right\},$$
$$\mathcal{U}' = \left\{ u \in \mathbb{R}^{m \times p} : u_{ij} = \mu_i(\Theta_j), \, (\Theta_1, \ldots, \Theta_p) \in \mathcal{S} \right\}.$$

The set $\mathcal{U}$ is convex and compact. Furthermore, if each μ_i is atomless, then, $\mathcal{U}' = \mathcal{U}$.

Next, we prove Lemma 1. Note that v_i as measures are atomless, since they are absolutely continuous w.r.t. the Lebesgue measure on $\Theta \subseteq \mathbb{R}^d$. Define

$$\bar{U} = \left\{ u \in \mathbb{R}^{n \times (n+1)} : u_{ij} = \langle v_i, x_j \rangle, \, (i,j) \in [n] \times [n+1], \, x_i \in L^\infty(\Theta)_+ \sum_{i=1}^{n+1} x_i = \mathbf{1} \right\}$$

and

$$\bar{U}' = \left\{ u \in \mathbb{R}^{n \times (n+1)} : u_{ij} = v_i(\Theta_j), \, (i,j) \in [n] \times [n+1], \, \Theta_i \subseteq \Theta \text{ are measurable and a.e.-disjoint} \right\}.$$

By Lemma EC.1 (taking $m = n$ and $p = n+1$), we have

$$\bar{U} = \bar{U}'$$

and it is a convex and compact set in $\mathbb{R}^{n(n+1)}$. Therefore, their (Euclidean) projections on to the n dimensions corresponding to u_{ii}, $i \in [n]$ (corresponding to the utility values u_i) are also equal, that is,

$$U = U'.$$

The set $U = U'$ is convex and compact (in $\mathbb{R}^n$) since projection preserves convexity and compactness in a finite-dimensional Euclidean space.

Proof of Theorem 1

Proof Part (a). By Lemma 1, the set $U = U(v, \Theta) \subseteq \mathbb{R}^n_+$ is convex and compact. Let $\rho(u) = -\sum_i B_i \log u_i$. Taking $u_i^0 = \langle v_i, \mathbf{1}/n \rangle = \frac{1}{n} v_i(\Theta)$ for all i ensures $u^0 \in \mathcal{U}$ and $\rho(u^0)$ is finite. Since

$$u_i \leq \langle v_i, \mathbf{1} \rangle = v_i(\Theta) < \infty$$

for all $u \in U$, we have

$$\sum_i B_i \log u_i \leq M := \sum_i B_i \log v_i(\Theta) < \infty.$$

Hence, there exists $\epsilon > 0$ sufficiently small such that, for $u \in \mathcal{U}$, if some $u_i \le \epsilon$, then

$$\rho(u) \ge -B_i \log \epsilon - M > \rho(u^0).$$

In fact, it suffices to have $-B_i \log \epsilon > \rho(u^*) + M$ for all i, or simply

$$\epsilon < e^{-\frac{\rho(u^0)+M}{\|B\|_\infty}}.$$

Hence, removing all $u \in U$ such that $\min_i u_i < \epsilon$ does not affect the infimum of ρ over U, that is,

$$\inf_{u\in U} \rho(u) = \inf_{u\in U:\, u\ge\epsilon} \rho(u).$$

Since $\{u \in U : u \ge \epsilon\}$ is compact (as a closed subset of the compact set U) and ρ is continuous on it, by the extreme value theorem, there exists a minimizer $u^* \in U$ (such that $u \ge \epsilon$). By the definition of U', there exists x^* such that $x^*_i = \mathbf{1}_{\Theta_i}$ (where Θ_i are disjoint measurable subsets of Θ) and

$$u^*_i = \langle v_i, x^*_i\rangle = v_i(\Theta_i).$$

Finally, if $\sup_i \Theta_i \subsetneq \Theta$, then assign $\Theta \setminus (\cup_i \Theta_i)$ to buyer 1, that is, augment Θ_1 so that $\sup_i \Theta_i = \Theta$. This does not affect $\sum_i x_i \le \mathbf{1}$, nor does it affect optimality (since $v_i(S) \le v_i(T)$ if $S \subseteq T \subseteq \Theta$, all sets measurable). In fact, $\Theta_0 := \Theta \setminus (\cup_i \Theta_i)$ corresponds to the subset on which all buyers have valuation 0 a.e.

Proof of Part (b). See Lemma 2.

Proof of Part (c). See Theorem 2.

Proof of Lemma 2

Denote the objective function of (2) as $\psi(\beta)$. Notice that

$$\max_i \beta_i v_i \le \sum_i \beta_i v_i$$

and therefore

$$0 \le \left\langle \max_i \beta_i v_i, \mathbf{1}\right\rangle \le \sum_i \beta_i \langle v_i, \mathbf{1}\rangle,\ \forall\, \beta > 0.$$

For any $\lambda \in [0,1]$, $\beta, \beta' \in \mathbb{R}^n_+$, $\theta \in \Theta$,

$$\max_i\, (\lambda\beta_i + (1-\lambda)\beta'_i)\, v_i(\theta) \le \lambda \max_i \beta_i v_i(\theta) + (1-\lambda) \max_i \beta'_i v_i(\theta).$$

Therefore,

$$\left\langle \max_i\, (\lambda\beta_i + (1-\lambda)\beta'_i)\, v_i, \mathbf{1}\right\rangle \le \lambda \max_i \langle \beta_i v_i, \mathbf{1}\rangle + (1-\lambda)\langle \max_i \beta'_i v_i, \mathbf{1}\rangle.$$

In other words, the function

$$\beta \mapsto \left\langle \max_i \beta_i v_i, \mathbf{1}\right\rangle$$

is convex. Since $\beta \mapsto -\sum_i B_i \log \beta_i$ is strictly convex on $\mathbb{R}^n_{++}$, we know that ψ is real-valued, strictly convex and hence also continuous on $\mathbb{R}^d_{++}$. Furthermore, for any i, when $\beta_i \to 0$ or $\beta_i \to \infty$, we have $\psi(\beta) \to \infty$. Hence, for $\beta^0 = (1, \ldots, 1) > 0$, there exists $0 < \underline{\beta} < \bar{\beta} < \infty$ such that

$$\beta \notin [\underline{\beta}, \bar{\beta}] \Rightarrow \psi(\beta) > \psi(\beta^0).$$

Therefore, we can restrict β inside a closed interval without affecting the infimum:

$$\inf_{\beta \in \mathbb{R}^n_+} \psi(\beta) = \inf_{\beta \in [\underline{\beta}, \bar{\beta}]} \psi(\beta).$$

The right-hand side is the infimum of a continuous function on a compact set. Therefore, the infimum is attained at some $\beta^* \in [\underline{\beta}, \bar{\beta}]$. Clearly, $\beta^* > 0$. It is unique since ψ is strictly convex on $[\underline{\beta}, \bar{\beta}]$.

Finally, when solving $(\mathcal{D}_{\mathrm{EG}})$, for any fixed β, the objective is clearly minimized at $p = \max_i \beta_i v_i \in L^1(\Theta)_+$. Therefore, we can eliminate p in this way and obtain (2). In other words, for the optimal solution β^* of (2), setting $p^* := \max_i \beta^*_i v_i$ gives an optimal solution (p^*, β^*) of $(\mathcal{D}_{\mathrm{EG}})$, which is (a.e.) unique.

Tighter bounds for the optimal solution β^* are given in Lemma 4.

Proof of Lemma 3

Proof of Part (a) (weak duality). Introduce new variables $u = (u_i) \in \mathbb{R}^n_+$ and rewrite $(\mathcal{P}_{\mathrm{EG}})$ into

$$\begin{aligned} z^* = \sup_{x \in (L^\infty(\Theta)_+)^n,\, u \in \mathbb{R}^n_+} \ & \sum_i B_i \log u_i \\ \text{s.t.} \ & u_i \le \langle v_i, x_i \rangle, \\ & \sum_i x_i \le \mathbf{1}. \end{aligned} \tag{EC.1}$$

Let (x, u) be a feasible to (EC.1) and (p, β) be a feasible solution of $(\mathcal{D}_{\mathrm{EG}})$. Using the feasibility assumptions, we have $\beta_i(u_i - \langle v_i, x_i \rangle) \le 0$ and $\langle p, \sum_i x_i - \mathbf{1} \rangle \le 0$. Hence,

$$\begin{aligned} \sum_i B_i \log u_i &\le \sum_i B_i \log u_i - \sum_i \beta_i (u_i - \langle v_i, x_i \rangle) - \left\langle p, \sum_i x_i - \mathbf{1} \right\rangle \\ &= \sum_i (B_i \log u_i - \beta_i u_i) - \sum_i \langle p - \beta_i v_i, x_i \rangle + \langle p, \mathbf{1} \rangle \\ &\le \sum_i \left(B_i \log \frac{B_i}{\beta_i} - \beta_i \frac{B_i}{\beta_i} \right) - \sum_i \langle p - \beta_i v_i, x_i \rangle + \langle p, \mathbf{1} \rangle \\ &\le \sum_i (B_i \log B_i - B_i) + \langle p, \mathbf{1} \rangle - \sum_i B_i \log \beta_i \\ &= \langle p, \mathbf{1} \rangle - \sum_i B_i \log \beta_i - C, \end{aligned} \tag{EC.2}$$

where the second inequality is because $u_i = \frac{B_i}{\beta_i}$ maximizes the concave function

$$u_i \mapsto B_i \log u_i - \beta_i u_i$$

and the third inequality is because $p \geq \beta_i v_i$ a.e. for all i. Taking supremum on the left and infimum on the right yields

$$C + z^* \leq w^*.$$

Proof of Part (b). Suppose x^* is feasible to $(\mathcal{P}_{\rm EG})$ and attains z^*; (p^*, β^*) is feasible to $(\mathcal{D}_{\rm EG})$ and attains w^*. Then, (x^*, u^*), where $u_i^* := \langle v_i, x_i^* \rangle$, is feasible to (EC.1). Note that $C + z^* = w^*$ if and only if all inequalities in (EC.2) are tight (with $x = x^*$, $u = u^*$, $p = p^*$, $\beta = \beta^*$). The first inequality being tight implies (3), i.e.,

$$\left\langle p^*, \mathbf{1} - \sum_i x_i^* \right\rangle = 0.$$

The second inequality being tight implies (4), i.e.,

$$u^* = \frac{B_i}{\beta_i^*}.$$

The third inequality being tight implies (5), i.e.,

$$\langle p^* - \beta_i^* v_i, x_i^* \rangle = 0, \ \forall i.$$

Conversely, let x^* and (p^*, β^*) be feasible to $(\mathcal{P}_{\rm EG})$ and $(\mathcal{D}_{\rm EG})$, respectively. Then, (x^*, u^*), where $u_i^* := \langle v_i, x_i^* \rangle$ is feasible to (EC.1). If they satisfy (3)-(5), then all inequalities in (EC.2) are tight. Hence, both x^* and (p^*, β^*) must be optimal.

Proof of Theorem 2

Optimal solutions $\Rightarrow$ ME. We first show the forward direction. Let x^* and (p^*, β^*) be optimal solutions of $(\mathcal{P}_{\rm EG})$ and $(\mathcal{D}_{\rm EG})$, respectively. By Lemma 3, they satisfy (3)-(5). Here, (3) gives market clearance. It remains to verify buyer optimality and budget depletion, i.e., $x_i^* \in D_i(p^*)$ and $\langle p^*, x_i^* \rangle = B_i$. To this end, for each i, by (4) and (5),

$$\langle p^*, x_i^* \rangle = \beta_i^* \langle v_i, x_i^* \rangle = B_i.$$

In words, x_i^* depletes buyer i's budget B_i and the utility buyer i receives is

$$\langle v_i, x_i^* \rangle = \frac{B_i}{\beta_i^*}.$$

Consider any $x_i \in L^\infty(\Theta)_+$ such that $\langle p^*, x_i \rangle \leq B_i$. Feasibility of (p^*, β^*) implies $p^* \geq \beta_i^* v_i$. Then,

$$\langle v_i, x_i \rangle \leq \frac{1}{\beta_i^*} \langle p^*, x_i \rangle \leq \frac{B_i}{\beta_i^*} = \langle v_i, x_i^* \rangle.$$

Therefore,

$$x_i^* \in D_i(p^*).$$

Hence, (x^*, p^*) is a ME, where buyer i's equilibrium utility is clearly $u_i^* := \langle v_i, x_i^* \rangle = \frac{B_i}{\beta_i^*}$.

ME $\Rightarrow$ Optimal solutions. Conversely, let (x^*, p^*) be a ME and $\beta_i^* := \frac{B_i}{u_i^*}$, where $u_i^* := \langle v_i, x_i^* \rangle$. We first check that (p^*, β^*) is feasible to $(\mathcal{D}_{\rm EG})$. For any i, suppose there exists a measurable set $A \subseteq \Theta$ such that $p^*(A) < \beta_i^* v_i(A)$. Then, consider the allocation $x_i = \frac{B_i}{p^*(A)} \cdot \mathbf{1}_A$. We have

$$\langle p^*, x_i \rangle = B_i$$

and

$$\langle v_i, x_i \rangle = B_i \cdot \frac{v_i(A)}{p^*(A)} > \frac{B_i}{\beta_i^*} = u_i^* = \langle v_i, x_i^* \rangle,$$

which contradicts to buyer optimality $x_i^* \in D_i(p^*)$. Therefore, we must have

$$p^* \geq \beta_i^* v_i \text{ a.e., } \forall\, i.$$

Thus, (p^*, β^*) is feasible to $(\mathcal{D}_{\rm EG})$. We know that x^* is feasible to $(\mathcal{P}_{\rm EG})$, since ME already requires $\sum_i x_i^* \leq \mathbf{1}$. Furthermore, by the choices of x^* and (p^*, β^*), they satisfy (3)-(5). Therefore, by Lemma 3, they must be optimal to $(\mathcal{P}_{\rm EG})$ and $(\mathcal{D}_{\rm EG})$, respectively.

Proof of Corollary 1

Since (x^*, p^*) is a ME, by Theorem 2, x^* and (p^*, β^*) (where $\beta_i^* = \frac{B_i}{u_i^*}$, $u_i^* = \langle v_i, x_i^* \rangle$) are optimal solutions of $(\mathcal{P}_{\rm EG})$ and $(\mathcal{D}_{\rm EG})$, respectively. By Lemma 3, the KKT conditions (3)-(5) holds.

Proof of Corollary 2

Let $(\tilde{p}^*, \tilde{\beta}^*)$ be the (a.e.-unique) optimal solution of $(\mathcal{D}_{\rm EG})$. Since u_i^* are the equilibrium utilities, by Lemma 3, $\tilde{\beta}_i^* = \frac{B_i}{u_i^*}$ for all i. Hence, $\tilde{\beta}^* = \beta^*$ and therefore $\tilde{p} = \max_i \beta_i^* v_i$ a.e. Lemma 3 ensures that they satisfy (3)-(5). Then, since $\{\Theta_i\}$ is a pure allocation, we have $\Theta_i = \{p^* = \beta_i^* v_i\}$ a.e. (i.e., the symmetric difference of Θ_i and $\{p^* = \beta_i^* v_i\}$ has measure zero). Therefore, on each Θ_i, we must have $p^* = \beta_i^* v_i \geq \beta_j^* v_j$ a.e. for any $j \neq i$. Using this fact, we have

$$p^*(A) = \sum_i p^*(A \cap \Theta_i) = \sum_i \beta_i^* v_i(A \cap \Theta_i)$$

for any measurable set $A \subseteq \Theta$.

Proof of Corollary 3

If $\{\Theta_i\}$ is a (pure) equilibrium allocation, then, by Theorem 2, $p^* = \max_i \beta_i^* v_i$ is the equilibrium prices. By Corollary 1, $\langle p^* - \beta_i^* v_i, x_i^* \rangle = 0$, where $x_i^* = \mathbf{1}_{\Theta_i}$. In other words, $p^* = \beta_i^* v_i$ on Θ_i. Corollary 1 also implies (3), i.e., $\langle p^*, \mathbf{1} - \sum_i x_i^* \rangle = 0$. Since $\mathbf{1} - \sum_i x_i^* = \mathbf{1}_{\Theta_0}$, where $\Theta \setminus (\cup_i \Theta_i)$, we have $p^*(\Theta_0) = 0$.

Conversely, if $\{\Theta_i\}$ and β^*, p^* satisfy the said conditions, we can also verify similarly that (3)-(5) holds. Since $\{\Theta_i\}$ is feasible to $(\mathcal{P}_{\rm EG})$ and (p^*, β^*), by the construction, is feasible to $(\mathcal{D}_{\rm EG})$, by Part (b) Lemma 3, they are both optimal to $(\mathcal{P}_{\rm EG})$ and $(\mathcal{D}_{\rm EG})$, respectively. Hence, by Theorem 2, $\{\Theta_i\}$ is an equilibrium allocation.

Proof of Theorem 3

Pareto optimality. Since x^* is an equilibrium allocation, by Theorem 2, it is also an optimal solution of ($\mathcal{P}_{\rm EG}$). If there exists $\tilde{x} \in (L^\infty(\Theta)_+)^n$, $\sum_i \tilde{x}_i \leq 1$ such that $\langle v_i, \tilde{x}_i\rangle \geq \langle v_i, x^*_i\rangle$ for all i and at least one inequality is strict, then

$$\sum_i B_i \log\langle v_i, \tilde{x}_i\rangle > \sum_i B_i \log\langle v_i, x^*_i\rangle,$$

i.e., x^* is not an optimal solution of ($\mathcal{P}_{\rm EG}$), a contradiction. Therefore, x^* is Pareto optimal.

Envy-freeness. For any $j \neq i$, since $\langle p^*, x^*_i\rangle = B_i$ (Theorem 2) and $\langle p^*, x^*_j\rangle = B_j$, we have

$$\left\langle p^*, \frac{B_i}{B_j} x^*_j \right\rangle = B_i.$$

Since $x^*_i \in D_i(p^*)$ and $\frac{B_i}{B_j} x^*_j \geq 0$, we have

$$\langle v_i, x^*_i\rangle \geq \left\langle v_i, \frac{B_i}{B_j} x^*_j \right\rangle \Rightarrow \frac{\langle v_i, x^*_i\rangle}{B_i} \geq \frac{\langle v_i, x^*_j\rangle}{B_j}.$$

Therefore, x^* is envy-free.

Proportionality. By the market clearance condition of ME, we have

$$p^*(\Theta) = \langle p^*, \mathbf{1}\rangle = \sum_i \langle p^*, x^*_i\rangle = \|B\|_1.$$

Therefore, for each buyer i, it holds that

$$\left\langle p^*, \frac{B_i}{\|B\|_1}\mathbf{1} \right\rangle = \frac{B_i}{\|B\|_1} p^*(\Theta) = \frac{B_i}{\|B\|_1} = B_i.$$

In other words, buyer i can afford the bundle $\frac{B_i}{\|B\|_1}\mathbf{1}$. Hence, its equilibrium utility must be at least

$$\left\langle v_i, \frac{B_i}{\|B\|_1}\mathbf{1} \right\rangle = \frac{B_i}{\|B\|_1} v_i(\Theta).$$

Proof of Lemma 4

By the characterization of p^* in Corollary 2, clearly,

$$p^*(\Theta) = \sum_i p^*(\Theta_i) = \sum_i B_i = \|B\|_1,$$

where $\{\Theta_i\}$ is a pure equilibrium allocation. Clearly, we have

$$u^*_i = \langle v_i, x^*_i\rangle \leq v_i(\Theta) = 1.$$

On the other hand, since $\{\Theta_i\}$ is an equilibrium allocation, it is proportional (Theorem 3), that is, $\frac{B_i}{\|B\|_1}\mathbf{1}$ is a budget-feasible allocation for buyer i. Hence,

$$u^*_i \geq \left\langle v_i, \frac{B_i}{\|B\|_1}\mathbf{1} \right\rangle = \frac{B_i}{\|B\|_1} v_i(\Theta) = \frac{B_i}{\|B\|_1}.$$

The bounds on $\beta^*_i = \frac{B_i}{u^*_i}$ follow immediately.

Proof of Lemma 5

Since $\beta_i^* v_i$ are linear, the equilibrium price vector $p^* = \max_i \beta_i^* v_i$ is a piecewise linear function with at most n pieces. Each linear piece has a support interval corresponding to the "winning set" of a buyer $\{p^* = \beta_i^* v_i\}$. Since all v_i are linear, normalized and distinct, there is no tie, i.e., no $i \neq j$ such that $\beta_i^* v_i = \beta_j^* v_i$ on a set of positive measure (otherwise we must have $v_i = v_j$ on $[0,1]$). Since $B_i > 0$ for all i, each buyer must receive a positive equilibrium utility $u_i^* > 0$. Hence, p^* consists of exactly n linear pieces and each buyer must get a nonempty interval as its equilibrium allocation in order to receive a positive equilibrium utility (Lemma 4). Let the breakpoints of p^* be $a_0^* = 0 < a_1^* < \cdots < a_n^* = 1$, which are clearly unique since β^* is unique (Lemma 2). Hence, there exists a (unique) permutation σ of $[n]$ such that $\{p^* = \beta_i^* v_i\} = [a_{\sigma(i)-1}^*, a_{\sigma(i)}^*]$ for all i (every buyer gets exactly one of the n nonempty intervals).

We show that σ must be the identity map $\sigma(i) = i$. In other words, at equilibrium, the entire interval is divided into n intervals; these intervals are allocated to buyers $1, \ldots, n$ from left to right, respectively. To see this, first note that any $\beta_i^* v_i$ and $\beta_j^* v_j$, $i < j$, must intersect on $[0,1]$ (otherwise one of them is completely dominated by the other, which means $p^* = \max_i \beta_i^* v_i$ cannot have n pieces and one of the buyers can only receive a zero-measure set at equilibrium, a contradiction to the fact that each buyer gets a positive equilibrium utility $u_i^* > 0$). Since v_i, v_j are linear and a_k^* are breakpoints of $p^* = \max_\ell \beta_\ell^* v_\ell$, we want to show that $\beta_i^* v_i(0) > \beta_j^* v_j(0)$, which will imply that, at equilibrium the interval for i is on the left of the interval for j. Suppose $\beta_i^* v_i(0) \leq \beta_j^* v_j(0)$, which implies $\beta_i^* d_i \leq \beta_j^* d_j$. By Assumption 1, $d_i > d_j$, which implies $\beta_i^* < \beta_j^*$. Furthermore,

$$v_i(1) = c_i + d_i = 2 - d_i < 2 - d_j = v_j(1).$$

Hence,

$$\beta_i^* v_i(1) < \beta_j^* v_j(1).$$

In other words, $\beta_i^* v_i < \beta_j^* v_j$ on $(0,1]$ and buyer i gets zero utility, a contradiction. Therefore, each interval $[a_{i-1}^*, a_i^*]$ is precisely the "winning set" $\{p^* = \beta_i^* v_i\}$ of buyer i. by Assumption 1, we have $v_i > 0$ on $(0,1)$ for all i, which implies $p^* > 0$ on $(0,1)$ (since $\beta_i^* \geq B_i > 0$ for all i, by Lemma 4).Therefore, by the market clearance condition of ME, every buyer must be allocated all of its winning set $[a_{i-1}^*, a_i^*]$ (except possibly the endpoints, which have measure zero). Therefore, $\Theta_i = [a_{i-1}^*, a_i^*]$, $i \in [n]$ is the unique pure equilibrium allocation.

Proof of Lemma 6

Assume $u_i^\circ > 0$ for all i. If this does not hold, remove the buyers with $u_i^\circ = 0$, assign them zero budgets $B_i = 0$ and consider the market without these buyers. Since u° is on the Pareto frontier, we know that $(1+\delta)u^\circ \notin U$ for any $\delta > 0$. Hence, u° is on the *boundary* of the convex compact set U. By the supporting hyperplane theorem, there exists $\beta^\circ \in \mathbb{R}^n$ such that

$$\langle \beta^\circ, u^\circ \rangle = 1 \text{ and } \langle \beta^\circ, u \rangle \leq 1, \forall\, u \in U. \tag{EC.3}$$

We can verify that $\beta_i^\circ \geq 0$ for all i: otherwise, if $\beta_i^\circ < 0$ for some i, decreasing $u_i^\circ > 0$ makes $\langle \beta^\circ, u^\circ \rangle > 1$ while ensuring $u^\circ \in U$, which contradicts (EC.3).

By Lemma 1, there exists a pure allocation $\{\Theta_i\}$ such that $u_i^\circ = v_i(\Theta_i)$ for all i. W.l.o.g., assume that $\cup_i \Theta_i = \Theta$ a.e. Define p° as

$$p^\circ(\theta) = \sum_i \beta_i^\circ v_i(\theta) \mathbf{1}_{\Theta_i}(\theta).$$

In other words, for $\theta \in \Theta_i$, $p^\circ(\theta) = \beta_i^\circ v_i(\theta)$. Clearly, $p^\circ \in L^1(\Theta)_+$ (since each $v_i \in L^1(\Theta)_+$ and $\beta^\circ \geq 0$) and, for any measurable set $A \subseteq \Theta$,

$$p^\circ(A) = \sum_i \beta_i^\circ v_i(A \cap \Theta_i).$$

Next, we show that $p^\circ = \max_i \beta_i^\circ v_i$ a.e. It suffices to show that $\beta_i^\circ v_i \geq \beta_j^\circ v_j$ on each Θ_i for any $j \neq i$. Suppose not, i.e., there exists a measurable set $A \subseteq \Theta_j$ such that, for $\ell \neq j$,

$$\beta_j^\circ v_j(A) < \beta_\ell^\circ v_\ell(A).$$

Remove the set A from Θ_j and give it to buyer ℓ instead, i.e., $\Theta_j' = \Theta_j \setminus A$, $\Theta_\ell' = \Theta_\ell \cup A$, $\Theta_i' = \Theta_i$ for all $i \notin \{j, \ell\}$. Clearly, $\{\Theta_i'\}$ is still a feasible (pure) allocation. However, its utilities $u_i' = v_i(\Theta_i')$ satisfy

$$\sum_i \beta_i^\circ u_i' = \sum_i \beta_i^\circ u_i^\circ - \beta_j^\circ v_j(A) + \beta_\ell^\circ v_\ell(A) > \langle \beta^\circ, u^\circ \rangle,$$

which contradicts (EC.3). Hence,

$$p^\circ = \max_i \beta_i^\circ v_i \text{ a.e.}.$$

Now, we are ready to show that $(\{\Theta_i\}, p^\circ)$ is a ME for buyers with budgets $B_i = \beta_i^\circ u^\circ$. Market clearance is satisfied since we assume $\cup_i \Theta_i = \Theta$ almost everywhere. To verify buyer optimality, note that for each i and any $x_i \in L^\infty(\Theta)_+$ such that $\langle p^\circ, x_i \rangle \leq B_i = \beta^\circ u_i^\circ$, we have

$$\beta_i^\circ v_i \leq \max_i \beta_i^\circ v_i = p^\circ \text{ a.e.},$$

which implies

$$\beta_i^\circ \langle v_i, x_i \rangle \leq \langle p^\circ, x_i \rangle \leq B_i \;\Rightarrow\; \langle v_i, x_i \rangle \leq u_i^\circ = v_i(\Theta_i).$$

Hence, $(\{\Theta_i\}, p^\circ)$ is a ME under budgets B_i and $u_i^\circ = v_i(\Theta_i)$ are the corresponding equilibrium utilities.

Proof of Lemma 7

Denote $U = U(v,[0,1])$. Assume w.l.o.g. that all $u_i > 0$: otherwise, simply remove the buyers with $u_i = 0$ and set $a_{i-1} = a_i$ in the final partition, i.e., giving an empty interval to this buyer.

The case of distinct d_i. We prove this case and show that the general case with some d_i being identical follows easily. Let $u^\circ \in U$ be a Pareto optimal utility vector such that $u^\circ \geq u$ (by the definition of Pareto optimality, such u° exists). By Lemma 7, there exists $B_i^\circ > 0$, $i \in [n]$ such that u_i° are the equilibrium utilities of buyers with budgets B_i° and valuations v_i. By Lemma 5, there exists

$$a_0^\circ = 0 < a_1^\circ < \cdots < a_n^\circ = 1$$

such that $\Theta_i^\circ = [a_{i-1}^\circ, a_i^\circ]$, $i \in [n]$ is the unique equilibrium allocation under budgets B_i°. Let $a_0 = 0$. Let $a_1 \leq a_1^*$ be such that $v_i([a_0, a_1]) = u_i$. Such a_1 exists because (i) $a_1 \mapsto v_i([0,a_1]) = \frac{c_1}{2} \cdot a_1^2 + d_1 a_1$ is continuous and strictly increasing and (ii) $v_i([0, a_1^\circ]) = u_i^\circ$. Inductively, there exist $a_i \leq a_i^\circ$ such that $v_i([a_{i-1}, a_i]) = u_i$ for all $i \in [n]$. Here, for simplicity, always take $a_n = 1$ regardless of the value of u_n, which ensures $v_n([a_{n-1}, a_n]) \geq v_n([a_{n-1}^\circ, 1]) = u_n^\circ \geq u_n$ (since $a_{n-1} \leq a_{n-1}^\circ$).

Handling identical valuations $v_i = v_j$ ($d_i = d_j$), $i \neq j$. In fact, the above procedure easily extend to the case of some intercepts d_i being equal. We can merge the buyer with the same d_i, where the "aggregate buyer" $I \subseteq [n]$ has the same valuation $v_I = v_i$, $i \in I$, budget $B_I = \sum_i B_i$ and "target utility value" $U_I = \sum_i u_i$. After merging all identical buyers, (u_I) is still a set of feasible utilities given (distinct) valuations v_I, i.e., $(u_I) \in U((v_I),[0,1])$. By the above case, we can partition $[0,1]$ into intervals, each for one aggregate buyer I. Let buyer I receives interval $[l_I, h_I] \subseteq [0,1]$ such that $v_i([l_I, h_I]) = u_I$. Since v_I is linear, we can easily find breakpoints l_i, h_i on $[l_I, h_I]$ via "cut" operations such that

$$v_i([l_i, h_i]) = u_i, \ i \in I.$$

This is because all buyers $i \in I$ share the same valuation $v_i = v_I$.

Proof of Theorem 4

Define $a_0 = 0$ and $a_n = 1$. By Lemma 7, for any $u \in U = U(v,[0,1])$, there exists $0 \leq a_1 \leq \cdots \leq a_{n-1} \leq 1$ such that

$$u_i \leq \bar{u}_i := \frac{c_i}{2}(a_i^2 - a_{i-1}^2) + d_i(a_i - a_{i-1}), \ i \in [n]. \tag{EC.4}$$

For any $0 \leq a_1, \ldots, a_{n-1} \leq 1$, since $v_i \geq 0$ on $[0,1]$, we have

$$a_1 \leq \cdots \leq a_{n-1} \Leftrightarrow \bar{u}_i \geq 0, \ i \in [n].$$

Hence, $u \in U$ is equivalent to the following constraints involving auxiliary variables a_i:

$$\begin{aligned}
&u \geq 0, \\
&u_i \leq \frac{c_i}{2}(a_i^2 - a_{i-1}^2) + d_i(a_i - a_{i-1}),\, i \in [n], \\
&0 \leq a_i \leq 1,\, i \in [n-1].
\end{aligned}$$

Note that

$$\frac{c_i}{2}(a_i^2 - a_{i-1}^2) + d_i(a_i - a_{i-1}) = \left(\frac{c_i}{2}a_i^2 + d_i a_i\right) - \left(\frac{c_i}{2}a_{i-1}^2 + d_i a_{i-1}\right),\, i \in [n].$$

Consider auxiliary variables $z_i \leq \frac{c_i}{2}a_i^2 + d_i a_i$ and $w_i \leq -(\frac{c_{i+1}}{2}a_i^2 + d_{i+1}a_i)$, $i \in [n-1]$. The above inequalities are equivalent to

$$\begin{aligned}
&u \geq 0, \\
&u_1 \leq z_1, \\
&u_i \leq z_i + w_{i-1},\, i = 2, \ldots, n-1, \\
&u_n \leq 1 + w_{n-1}, \\
&z_i \leq \frac{c_i}{2}a_i^2 + d_i a_i,\ w_i \leq -\frac{c_{i+1}}{2}a_i^2 - d_{i+1}a_i,\, i \in [n-1], \\
&0 \leq a_i \leq 1,\, i \in [n-1].
\end{aligned} \tag{EC.5}$$

Since $a_i \in [0,1]$, $d_i \in [0,2]$, $\frac{c_i}{2} + d_i = 1$ for all i, we have

$$\begin{aligned}
&\frac{c_i}{2}a_i^2 + d_i a_i \in [0,1], \\
&-\frac{c_{i+1}}{2}a_i^2 - d_{i+1}a_i \in [-1,0], \\
&\left(\frac{c_i}{2}a_i^2 + d_i a_i\right) + \left(-\frac{c_{i+1}}{2}a_i^2 - d_{i+1}a_i\right) = (d_{i+1} - d_i)(a_i^2 - a_i) \geq 0,
\end{aligned} \tag{EC.6}$$

where the last inequality is because $d_i \geq d_{i+1}$, $\frac{c_i}{2} + d_i = 1$ (for all $i \in [n]$) and $a_i \in [0,1] \Rightarrow a_i^2 - a_i \leq 0$. By the first two inequalities in (EC.6), we can add additional constraints

$$0 \leq z_i \leq 1,\ -1 \leq w_i \leq 0,\ z_i + w_i \geq 0,\ i \in [n-1]$$

to the inequalities in (EC.5) without affecting the feasible region of u. Next, for each i, consider the set S_i of (z_i, w_i) satisfying the following inequalities together with some a_i:

$$\begin{aligned}
&z_i \leq \frac{c_i}{2}a_i^2 + d_i a_i,\ w_i \leq -\frac{c_{i+1}}{2}a_i^2 - d_{i+1}a_i,\ a_i \in [0,1], \\
&0 \leq z_i \leq 1,\, -1 \leq w_i \leq 0,\, z_i + w_i \geq 0.
\end{aligned} \tag{EC.7}$$

When $d_i = d_{i+1} \in [0,2]$, the parametric curve

$$\Gamma_i = \left\{ \begin{bmatrix} \frac{c_i}{2}\alpha^2 + d_i\alpha \\ -\frac{c_{i+1}}{2}\alpha^2 - d_{i+1}\alpha \end{bmatrix} : \alpha \in [0,1] \right\},$$

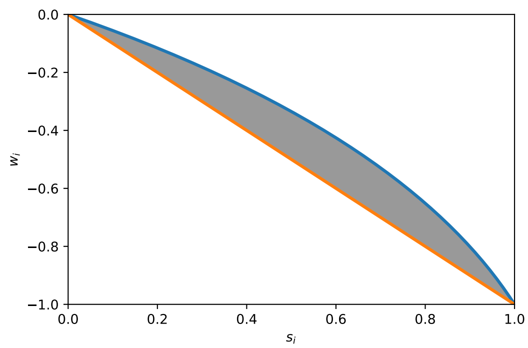

Figure EC.1 An illustration of the set S_i for (z_i, w_i), which is the region bounded by (i) the line segment between $(0,0)$ and $(1,-1)$ and (ii) the arc Γ_i (part of a quadratic curve) on the top-right of it. In this figure, we use $d_1 = 1.5$, $d_2 = 0.8$. When $d_{i+1} = d_i$, the region becomes the line segment itself.

is simply the line segment $(0,0)$ to $(1,-1)$. Together with the last inequality in (EC.7), we know that

$$S_i = \{(z_i, w_i) \in [0,1] \times [-1,0] : z_i + w_i = 0\}$$

is the same line segment as well.

When $d_i > d_{i+1}$, Γ_i is part of a quadratic curve connecting $(0,0)$ and $(1,-1)$. By the last inequality in (EC.6) (with $d_{i+1} - d_i < 0$ and $0 < a_i < 1$), Γ_i lies on the top-right of the line segment between the two points. In this case, the set S_i is the region between $z_i = 0$, $w_i = -1$ and Γ_i. See Figure EC.1 for an illustration. Let the *entire* curve be

$$\bar{\Gamma}_i := \left\{ \begin{bmatrix} \frac{c_i}{2}\alpha^2 + d_i\alpha \\ -\frac{c_{i+1}}{2}\alpha^2 - d_{i+1}\alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}.$$

It is a parabola, since it is the image of the standard parabola $\{(\alpha, \alpha^2) : \alpha \in \mathbb{R}\}$ under a linear transformation given by G_i:

$$G_i \begin{bmatrix} \alpha \\ \alpha^2 \end{bmatrix} = \begin{bmatrix} \frac{c_i}{2}\alpha^2 + d_i\alpha \\ -\frac{c_{i+1}}{2}\alpha^2 - d_{i+1}\alpha \end{bmatrix}.$$

Hence, the convex hull $\operatorname{conv}(\bar{\Gamma}_i)$—the set of convex combinations of any finite number of points on $\bar{\Gamma}_i$—is also the image of the epigraph of the standard parabola $\mathcal{C} = \operatorname{conv}(\{(\alpha, \alpha^2) : \alpha \in \mathbb{R}\})$ under the same linear transformation. By the convexity of $\operatorname{conv}(\bar{\Gamma}_i)$, we have

$$S_i = \operatorname{conv}(\bar{\Gamma}_i) \cap T_i,$$

where

$$T_i = \{(z_i, w_i) : z_i \in [0,1],\, w_i \in [-1,0],\, z_i + w_i \geq 0\}.$$

Therefore, the set S_i can be represented by the linear constraints in T_i and $(z_i, w_i) \in G_i\mathcal{C}$. The latter can be expanded with two additional variables s_i, t_i:

$$G_i \begin{bmatrix} s_i \\ t_i \end{bmatrix} = \begin{bmatrix} z_i \\ w_i \end{bmatrix}, \; \begin{bmatrix} s_i \\ t_i \end{bmatrix} \in \mathcal{C}.$$

Note that the above hold for both $d_{i+1}=d_i$ and $d_{i+1}>d_i$ (if $d_{i+1}=d_i$, then G_i maps the parabola $\{(t_1,t_1^2):t_1\in\mathbb{R}\}$ into the straight line $\{(z_i,w_i):z_i+w_i=0\}$ and the above characterization still works). Substituting this and the constraints in T_i into (EC.5), we obtain the desired set of constraints that characterize $u\in U(v,[0,1])$.

Note that we do not need to include $z_i+w_i\geq 0$ in the final set of constraints: this is the same as enlarging S_i to contain (z_i,w_i) such that $z_i+w_i<0$, $(z_i,w_i)\in[0,1]\times[-1,0]$, $(z_i,w_i)\in G_i\mathcal{C}$. Doing so not affect the feasible region of u.

Finally, we can also easily verify that S_i is the image under linear transformation G_i of the convex hull of the parabola segment $\mathcal{C}^0=\{(s_i,s_i^2):s_i\in[0,1]\}$:

$$\operatorname{conv}(\mathcal{C}^0)=\{0\leq s_i\leq 1,\ 0\leq t_i\leq 1,\ s_i^2\leq t_i\}.$$

Hence, the set of constraints (in particular, $z_i+w_i\geq 0$ and $(z_i,w_i)\in[0,1]\times[-1,0]$) imply

$$0\leq s_i,t_i\leq 1.$$

Proof of Theorem 5

Consider $\tilde{v}_i(\theta)=(h-l)^2c_i\theta+(h-l)(c_il+d_i)$, $\theta\in[0,1]$ and $\varphi(\theta)=\frac{\theta-l}{h-l}$. For any $[a,b]\subseteq[l,h]$, we have

$$\begin{aligned}\tilde{v}_i([\varphi(a),\varphi(b)])&=\frac{(h-l)^2c_i}{2}\left(\frac{(b-l)^2}{(h-l)^2}-\frac{(a-l)^2}{(h-l)^2}\right)+(h-l)(c_il+d_i)\left(\frac{b-l}{h-l}-\frac{a-l}{h-l}\right)\\&=(b-a)\left(\frac{c_i}{2}(a+b)+d_i\right)=v_i([a,b]).\end{aligned}$$

Therefore, for any $u\in\mathbb{R}^n_+$ such that $u_i=v_i([l_i,h_i])$ for a.e.-disjoint intervals $[l_i,u_i]\subseteq[l,u]$, we have

$$u_i=\tilde{v}_i([\varphi(l_i),\varphi(h_i)]),\ i\in[n]$$

and $[\varphi(l_i),\varphi(h_i)]\subseteq[0,1]$ are also a.e.-disjoint intervals. Let $\tilde{c}_i=(h-l)^2c_i$ and $\tilde{d}_i=(h-l)(c_il+d_i)$. By Lemma 7, we have

$$\begin{aligned}U(v,[l,h])&=\left\{u\in\mathbb{R}^n_+:u_i=\langle v_i,x_i\rangle,\ x_i\in L_\infty([l,h])_+,\ i\in[n],\ \sum_i x_i\leq\mathbf{1}\right\}\\&=\left\{u\in\mathbb{R}^n_+:u_i=v_i([l_i,h_i])\text{ for a.e. disjoint }[l_i,h_i]\subseteq[l,h],\ i\in[n]\right\}\\&=\left\{u\in\mathbb{R}^n_+:u_i=\tilde{v}_i([\tilde{l}_i,\tilde{h}_i])\text{ for a.e. disjoint }[\tilde{l}_i,\tilde{h}_i]\subseteq[0,1],\ i\in[n]\right\}\\&=\left\{u\in\mathbb{R}^n_+:u_i=\langle\tilde{v}_i,x_i\rangle,\ x_i\in L^1([0,1])_+,\ i\in[n],\ \sum_i x_i\leq\mathbf{1}\right\}\\&=U(\tilde{v},[0,1]).\end{aligned}$$

Furthermore, let $\hat{v}_i=\tilde{v}_i/\|\tilde{v}_i\|$, where

$$\|\tilde{v}_i\|=\tilde{v}_i([0,1])=\frac{(l-h)^2c_i}{2}+(l-h)(c_il+d_i)=\Lambda_i.$$

The coefficients of $\hat{v}_i$ are $\hat{c}_i = \tilde{c}_i/\|\tilde{v}_i\|$ and $\hat{d}_i = \tilde{d}_i/\|\tilde{v}_i\|$, which are the same as defined in the theorem statement. Then,

$$U(\tilde{v},[0,1]) = DU(\hat{v},[0,1]) = \{D\hat{u} : \hat{u} \in U(\hat{v},[0,1])\},$$

where $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $D_{ii} = \|\tilde{v}_i\| = \Lambda_i$. Let $P \in \{0,1\}^{n\times n}$ be the permutation matrix defined in the theorem statement. Then,

$$U(\hat{v},[0,1]) = PU(\hat{v}_\sigma,[0,1]),$$

since permutation does not affect the feasibility of x. By Theorem 4, $U(\hat{v}_\sigma,[0,1])$ can be represented by $O(n)$ linear and quadratic constraints using $O(n)$ auxiliary variables. Therefore,

$$U(v,[l,u]) = U(\tilde{v},[0,1]) = DPU(\hat{v}_\sigma,[0,1])$$

can also be represented by $O(n)$ linear and quadratic constraints using $O(n)$ auxiliary variables.

Proof of Theorem 6

First, by Lemma 1, each U_k is convex and compact. W.l.o.g., Assume not all v_i are 0 on $[a_{k-1},a_k]$ (otherwise, $U_k = \{\mathbf{0}\}$ is a singleton of the n-dimensional zero vector, and we can remove this k in all summations in the following analysis). For any $u \in U(v,[0,1])$, there exists $x \in L^\infty([0,1])^n_+$ such that $u_i = \langle v_i, x_i\rangle$. Let x_{ik} be the restriction of x_i on $[a_{k-1},a_k]$ and $u_{ik} = \langle v_i, x_{ik}\rangle$. Clearly, this makes the objective value of (7) at (u_{ik}) equal to that of $(\mathcal{P}_{\rm EG})$ at x. Conversely, for any (u_{ik}) feasible to (7), we can also find x feasible to $(\mathcal{P}_{\rm EG})$ that attains the same objective value. Therefore, (7) and $(\mathcal{P}_{\rm EG})$ have the same optimal objective value. In particular, the supremum of (7) is attained at some (u^*_{ik}). By the Pareto optimality of $(u^*_i) \in U(v,[0,1])$ (where u^*_i are the unique equilibrium utilities), an optimal solution (u^*_{ik}) of (7) must satisfy $u^*_i = \sum_k u^*_{ik}$ for all i.

By Theorems 4 and 5, each U_k can be represented by $O(n)$ variables and $O(n)$ (linear and quadratic) constraints (if some v_i is zero on $[a_{k-1},a_k]$, i.e., $v_i([a_{k-1},a_k]) = 0$, simply remove it from the set of buyers on this interval when representing the set U_k). The set $\mathcal{C}$ is the image of an transformation of the second-order cone $\mathcal{L}$:

$$(t_1,t_2) \in \mathcal{C} \;\Leftrightarrow\; t_1^2 \le t_2 \;\Leftrightarrow\; \sqrt{\left(\frac{1-t_2}{2}\right)^2 + t_1^2} \le \frac{1+t_2}{2} \;\Leftrightarrow\; \left(\frac{1+t_2}{2}, \frac{1-t_2}{2}, t_1\right) \in \mathcal{L}.$$

For any $u_i > 0$,

$$-\log u_i = \min_{q_i \ge -\log u_i} q_i = \min_{e^{q_i} \le u_i} (-q_i) = \min_{(u_i,1,q_i)\in\mathcal{E}} -q_i. \tag{EC.8}$$

In this way, introducing auxiliary variables s_i, t_i, u_i, $i \in [n]$ the objective $\max \sum_i B_i \log\left(\sum_k u_{ik}\right)$ can be written as $-\min -\sum_i B_i q_i$, with additional (linear and exponential cone) constraints

$$\left.\begin{aligned} u_i &= \textstyle\sum_k u_{ik}, \\ (u_i,1,q_i) &\in \mathcal{E}, \end{aligned}\right\} i \in [n].$$

Combining the above analysis, we arrive at the overall convex conic reformulation involving only linear and (convex) conic constraints. Recall that i and k are indices of buyers and linear segments of their valuations, respectively.

$$
\begin{aligned}
\min\ & -\sum_i B_i q_i \\
\text{s.t.}\ & \left.\begin{array}{l} (u_i, 1, q_i) \in \mathcal{E},\ q_i' = 1, \\ u_i = \sum_k u_{ik}, \\ u_{\sigma^k(i)k} = \Lambda_{\sigma^k(i)k} \hat{u}_{ik}, \end{array}\right\} \quad \forall\, i \in [n], \\
& \left.\begin{array}{l} \hat{u}_{1k} \le z_{1k}, \\ \hat{u}_{ik} \le z_{ik} + w_{i-1,k},\ \forall\, i = 2, \dots, n-1, \\ \hat{u}_{nk} \le 1 + w_{nk}, \\ G_{ik}(s_{ik}, t_{ik}) = (z_{ik}, w_{ik}),\ \left(\frac{1+t_{ik}}{2}, \frac{1-t_{ik}}{2}, s_{ik}\right) \in \mathcal{L}, \\ 0 \le z_{ik} \le 1,\ -1 \le w_{ik} \le 0,\ z_{ik} + w_{ik} \ge 0,\ \forall\, i \in [n], \end{array}\right\} \quad \forall\, k \in [K].
\end{aligned}
\tag{$\mathcal{CP}$}
$$

In the above, the first group of constraints involving objective transformation, decomposition of buyers' utilities and scaling of them over subintervals.The second group represents $(\hat{u}_{1k}, \dots, \hat{u}_{nk}) \in U(\hat{v}_k, [0,1])$, where $U(\hat{v}_k, [0,1])$ is the "standardized" version of U_k as described in Theorem 5. To complete the proof, we calculate the number of each type of variables in the final reformulation (8) below.

- The exponential cone variables are $(q_i, q_i', u_i) \in \mathcal{E},\ i \in [n]$ (with additional linear constraint $q_i' = 1$). Hence, $n_3 = O(n)$.
- For each (s_{ik}, t_{ik}), there is a second-order cone $\mathcal{L}$ (involving 3 conic variables, $t_{ik}' = \frac{1+t_{ik}}{2}$, $t_{ik}'' = \frac{1-t_{ik}}{2}$ and s_{ik}). Hence, $n_2 = O(nK)$.
- The linear cone (nonnegative) variables are u_{ik}, $\hat{u}_{ik}$, $i \in [n]$, $k \in [K]$, (z_{ik}, w_{ik}'), $i \in [n-1]$, $k \in [K]$ ($w_{ik}' = -w_{ik}' \ge 0$) and nonnegative auxiliary variables added to transform the $O(nK)$ linear inequality constraints (in the second group of constraints in ($\mathcal{CP}$)) into equality constraints. Hence, $n_1 = O(nK)$.
- The linear equality constraints (which form "$Ax = b$" in the standard form (8)) are those above in ($\mathcal{CP}$) plus additional ones involving auxiliary conic variables: $q_i'' = 1$, $t_{ik}' = \frac{1+t_{ik}}{2}$, $t_{ik}'' = \frac{1-t_{ik}}{2}$. Hence, $m = O(nK)$.
- It can be easily verified that, each of the linear constraints in ($\mathcal{CP}$) (except the linear constraints $u_i = \sum_k u_{ik}$, $k \in [K]$), involve only a constant number of variables. In particular, the linear equality constraints of $U_k = D^k P^k \hat{U}_k$ (i.e., last equality constraint in the first group of

constraints in ($\mathcal{CP}$)) only consist of (n) nonzeros, since D^k and P^k are diagonal and permutation matrices, respectively. Hence, the total number of nonzeros in all linear constraints is also $O(nK)$.

By (EC.8), we know that the minimum f^* of (8) is $-z^*$. Finally, the above reformulation does not affect feasible region of the variables (u_i) and (u_{ik}). Hence, in the optimal solution of the reformulation (8), these variables correspond to an optimal solution (u^*_{ik}) of (7).

Proof of Theorem 8

To make use of the above theorem, we first case our problem (7) into (9). Same as in the paragraph below (7), let the decision variables be (u_i), (u_{ik}), $(\hat{u}_{ik})$, (s_{ik}), (t_{ik}), (z_{ik}), (w_{ik}), with a total number of $O(nK)$. Here, the variables (u_i), (u_{ik}) correspond to those in (7). The variables $(\hat{u}_{1k}, \ldots, \hat{u}_{nk})$ are used to describe each "normalized" set of feasible utilities $\hat{U}_k$ such that $U_k = D^k P^k \hat{U}_k$. Denote the aggregate decision variable as x, which has $O(nK)$ dimensions. The (minimization) objective function is

$$f(x) = -\sum_i B_i \log u_i.$$

Next, we specify the feasible region. Recall that $u^*_i \in [B_i, 1]$ for all i at equilibrium (Lemma 4). Hence, we can add linear constraints

$$\min\{B_i, \epsilon/2\} \le u_i \le 1 \tag{EC.9}$$

without affecting the optimal solution. Then, "enlarge" the feasible sets given by the constraints by ϵ to ensure a nonempty interior of the feasible region.

- For each i, relax the equality constraint $u_i = \sum_k u_{ik}$ into $u_i \le \sum_k u_{ik} + \epsilon$ for all k (when there is no tolerance ϵ, the equality constraint can clearly be relaxed to an inequality without affecting the optimum, since at optimality the inequality must be tight).
- For each k, $(u_{1k}, \ldots, u_{nk}) \in U_k + [0, \epsilon]^n$.
- For each k, for every linear constraint describing $U_k = D^k P^k \hat{U}_k$, that is, $u_{\sigma^k(j)k} = \Lambda_{\sigma(j)k} \hat{u}_{jk}$, relax it into $u_{\sigma^k(j)k} = \Lambda_{\sigma(j)k} \hat{u}_{jk} + \epsilon$.
- For each k, in the linear and quadratic constraints for $\hat{U}_k$, add ϵ to *all* linear constraints involving $\hat{u}_{ik}$ and z_{ik} or w_{ik} (e.g., $\hat{u}_{ik} \le z_{ik} + w_{i-1,k} + \epsilon$).
- For each k, relax the constraints $G_{ik}(s_{ik}, t_{ik}) = (z_{ik}, w_{ik})$ (Theorem 4, with the "kth" copies of the variables and the G_i matrix) into $G_{ik}(s_{ik}, t_{ik}) = (z_{ik}, w_{ik}) + (\epsilon, \epsilon)$.
- For each i and k, keep the constraints $0 \le z_{ik} \le 1$, $-1 \le w_{ik} \le 1$, $(s_{ik}, t_{ik}) \in \mathcal{C}$ unchanged.

Constructing these constraints takes $O(nK \log n)$ time, where the $n \log n$ factor is due to sorting $\hat{d}_{1k}, \ldots, \hat{d}_{nk}$ for each k (see Theorem 5). It is negligible compared to the running time of the ellipsoid method. Now, it can be easily verified that the feasible region (described by all constraints above) contains a Euclidean ball of radius $r \geq \epsilon/2$. The objective function is also convex and continuous on the feasible region. Furthermore, the total number of constraints is $O(nK)$ and each constraint only involves a constant number of variables (except $u_i = \sum_k u_{ik}$ which involves K variables).

Next, we bound $R = \max_{x\in X} \|x\|$.

- Each of z_{ik}, w_{ik}, s_{ik}, t_{ik} have absolute values ≤ 1. There are $O(nK)$ such variables.
- The variables $\hat{u}_{ik}$ have absolute values $\leq 1+\epsilon$. There are $O(nK)$ such variables.
- The variables u_{ik} have absolute values $\leq 1+\epsilon$, since $\Lambda_{ik} = v_i([a_{k-1}, a_k]) \leq 1$. There are $O(nK)$ such variables.
- The variables u_i have absolute values ≤ 1, as we added the constraints (EC.9). There are n such variables.

Hence, $R = O(\sqrt{nK})$.

To bound V, first note that

$$f(x) \geq -\sum_i B_i \log 1 = 0.$$

Then, since $\|B\|_1 = 1$,

$$f(x) \leq -\sum_i B_i \log \min\{B_i, \epsilon/2\} \leq \sum_i B_i \log \max\{\kappa, 2/\epsilon\} \leq \log\kappa + \log\frac{2}{\epsilon}.$$

Therefore ($f^* = \min_{x\in X} f(x)$),

$$V = \max_{x\in X} f(x) - f^* \leq \log\kappa + \log\frac{2}{\epsilon}.$$

Hence, the overall ratio $\frac{VR}{\epsilon r}$ in the expression of the time complexity of the ellipsoid method in Theorem 7 is

$$\frac{VR}{\epsilon r} = O\left(\frac{\sqrt{nK}\left(\log\kappa + \log\frac{2}{\epsilon}\right)}{\epsilon^2}\right). \tag{EC.10}$$

The two oracles. The first-order oracle is trivial: the objective function is differentiable w.r.t. u_i and a subgradient simply consists of the derivatives $\frac{B_i}{u_i}$. This oracle takes $T_{\mathcal{G}} = O(n)$ time. Next, we describe the separation oracle. Given a solution x^0 (consisting of $O(nK)$ variables in total), it clearly takes $O(nK)$ time to verify whether all constraints are satisfied. Suppose not all constraints are satisfied. There are two cases.

- A linear constraint is violated, say, $g^\top x^0 > a$ while $g^\top x \leq a$ for all $x \in X$. Then, this constraint itself is a separating hyperplane.
- A quadratic constraint is violated, say, $(s_{ik}, t_{ik}) \notin \mathcal{C} = \{(t_1, t_2) : t_1^2 \leq t_2\}$. By elementary calculus, the line

  $$\{(t_1, t_2) : t_2 - s_{ik}^2 = (2s_{ik})(t_1 - s_{ik})\}$$

  is tangent to the curve $\{(t_1, t_2) : t_1^2 = t_2\}$ at the point (s_{ik}, s_{ik}^2) on the curve. Hence, it separates $\mathcal{C}$ and (s_{ik}, t_{ik}).

Since there are $O(nK)$ linear and quadratic constraints in total, the separation oracle described above takes $T_{\mathcal{S}} = O(nK)$ time.

By Theorem 7, the ellipsoid method finds a solution x_ϵ such that $f(x_\epsilon) - f^* \le \epsilon$ in $N(\epsilon)$ number of calls of the oracles and $O(1)n^2 N(\epsilon)$ additional arithmetic operations, where

$$N(\epsilon) = O(1)(nK)^2 \log\left(2 + \frac{VR}{\epsilon r}\right).$$

Combining the above, (EC.10) and the time complexity of the oracles, the overall time complexity for computing x_ϵ is

$$\begin{aligned}
&N(\epsilon)(T_{\mathcal{S}} + T_{\mathcal{G}}) + O(1)(nK)^2 N(\epsilon) \\
&= N(\epsilon)\left(O(n) + O(nK) + O((nK)^2)\right) \\
&= O\left((nK)^4 \log \frac{\sqrt{nK}\left(\log\kappa + \log\frac{2}{\epsilon}\right)}{\epsilon}\right) \\
&= O\left((nK)^4 \log \frac{nK\log\kappa}{\epsilon}\right). \qquad \text{(EC.11)}
\end{aligned}$$

Here, since the feasible region has been enlarged, the minimum $f^* \le -z^*$, where z^* is the true maximum of (7). Therefore, x_ϵ is also ϵ-close to the "true" minimum $-z^*$. Furthermore, we can transform the difference in objective value to the difference in utilities using strong convexity (where u_i are part of the solution x_ϵ):

$$\left(-\sum_i B_i \log u_i\right) - \left(-\sum_i B_i \log u_i^*\right) \ge \frac{\sigma}{2}\|u - u^*\|^2,$$

where a strong convexity modulus is $\sigma = \min_i B_i = \frac{1}{\kappa}$, since $u_i \le 1$. Therefore, for each i,

$$|u_i - u_i^*| \le \|u - u^*\| \le 2\epsilon\kappa. \qquad \text{(EC.12)}$$

To recover a fully feasible solution (u_{ik}) such that $(u_{1k}, \dots, u_{nk}) \in U_k$ and $\sum_k u_{ik} \ge u_i^* - \epsilon$, it suffices to "discount" x_ϵ as follows.

- Decrease each $\hat{u}_{ik}$ by ϵ (that is, $\hat{u}'_{ik} = \max\{\hat{u}_{ik} - \epsilon, 0\}$), which requires each u_{ik} $(i = \sigma^k(j))$ decrease by $\Lambda_{ik}\epsilon \le \epsilon$ to ensure $u_{ik} \le \Lambda_{ik}\hat{u}_{jk} + \epsilon$.
- Further decrease each u_{ik} by ϵ $(u_{ik} = \max\{u_{ik} - \epsilon, 0\})$, which makes u_i decrease by at most $(K+1)\epsilon$ to ensure $u_i \le \sum_k u_{ik}$.

We still use (u_{ik}) to denote the processed solution according to the above "discounting" procedure. This solution satisfies $(u_{1k}, \dots, u_{nk}) \in U_k$ for all k and $\sum_k u_{ik} \ge u_i - (K+1)\epsilon$ for all i. Combining this with (EC.12), we know that the processed solution (u_{ik}) approximately attains the equilibrium utilities:

$$\sum_i u_{ik} \ge u_i^* - 2\kappa\epsilon - (K+1)\epsilon.$$

Therefore, to ensure that $\sum_k u_{ik} \geq u_i^* - \epsilon$ for a given tolerance level $\epsilon > 0$, it suffices to replace ϵ by $\frac{\epsilon}{2\kappa+(K+1)}$, which slightly degrades the time complexity in (EC.11) (due to κ) and yields a final time complexity of

$$O\left((nK)^4\left(\log(nK) + \log\frac{\kappa}{\epsilon}\right)\right).$$

To construct a pure equilibrium allocation given (u_{ik}) is easy: for each k, since $(u_{1k}, \ldots, u_{nK}) \in U_k$, simply run Algorithm 1 on $[a_{k-1}, a_k]$, where no sorting is required as we reuse the sorting permutations in formulating the convex program (7). This produces a.e.-disjoint intervals $[l_{ik}, h_{ik}] \subseteq [a_{k-1}, a_k]$ such that $v_i([l_{ik}, h_{ik}]) \geq u_{ik}$. Therefore, the pure allocation $\Theta_i := \cup_k [l_{ik}, h_{ik}]$ satisfies

$$v_i(\Theta_i) \geq \sum_k u_{ik} \geq u_i^* - \epsilon.$$

Proof of Lemma 8

The first half is well-known, see, e.g., (Beck 2017, Theorem 3.50). In fact, by this theorem, the subdifferential (the convex set of all subgradients) is

$$\partial_\beta \phi(\beta,\theta) = \mathrm{conv}\left\{ v_i(\theta)\mathbf{e}^{(i)} : i \in \arg\max_i \beta_i v_i(\theta) \right\}.$$

For the second half, note that for any $\beta > 0$, since $g(\beta,\theta) \in \partial_\theta \phi(\beta,\theta)$ we have

$$\phi(\beta',\theta) - \phi(\beta,\theta) \geq \langle g(\beta,\theta), \beta' - \beta\rangle, \ \forall \beta' > 0.$$

Integrate w.r.t. θ over Θ on both sides yield

$$\phi(\beta') - \phi(\beta) \geq \int_\Theta \langle g(\beta,\theta), \beta' - \beta\rangle d\theta = \left\langle \int_\Theta g(\beta,\theta)d\theta, \beta' - \beta \right\rangle,$$

where the (component-wise) integral $\int_\Theta g(\beta,\theta)d\theta$ is well-defined and finite, since each component of $g(\beta,\theta)$ is uniformly bounded by the pointwise maximum $\max_i v_i$, which is integrable:

$$0 \leq \max_i v_i \leq \sum_i v_i \in L^1(\Theta).$$

Therefore, by the definition of subgradient,

$$\int_\Theta g(\beta,\theta)d\theta \in \partial\phi(\beta).$$

The integral can also be written as an expectation over $\theta \sim \mathrm{Unif}(\Theta)$, since the probability density of $\mathrm{Unif}(\Theta)$ is $\frac{1}{\mu(\Theta)}$ for all $\theta \in \Theta$.

Proof of Theorem 9

The bounds on $\mathbb{E}\|\beta^t - \beta^*\|^2$ and $= \mathbb{E}\|\tilde{\beta}^t - \beta^*\|^2$ are derived directly from the proof of (Xiao 2010, Corollary 4). Here, the regularizer $\Psi(\beta) = -\sum_i B_i \log \beta_i$ has domain $[B, \mathbf{1}] = \prod_i [B_i, 1]$. For $\beta_i \in [B_1, 1]$, we have

$$\frac{\partial^2}{\partial \beta_i^2}(-B_i \log \beta_i) = \frac{B_i}{\beta_i^2} \geq B_i.$$

Hence, Ψ (and the entire objective function) is strongly convex on $[B, \mathbf{1}]$ with modulus $\sigma = \min_i B_i$ (c.f. Lemma 4).

Based on Lemma 8, a subgradient can be computed as $g^t = g(\beta^t, \theta_t) = v_{i_t}(\theta_t) \cdot \mathbf{e}^{(i_t)}$, where $i_t \in \arg\max \beta_i^t v_i(\theta_t)$ (choosing the smallest index in a tie). Hence, we have

$$\|g^t\|^2 = v_{i_t}(\theta_t)^2 \leq \max_i v_i(\theta_t).$$

Therefore,

$$\mathbb{E}\|g_t\|^2 \leq \mathbb{E}_\theta[\max_i v_i(\theta)^2] = G^2.$$

Here, $\max_i v_i \in L^2(\Theta)$ since

$$0 \leq \max_i v_i \leq \sum_i v_i \in L^2(\Theta).$$

The second half is derived in a straightforward manner from the discussion in (Xiao 2010, pp. 2559). Note that

$$\Delta_t = \frac{G^3}{2\sigma}(6 + \log t)$$

is an upper bound on the regret in iteration t in an online optimization setting (Xiao 2010, §3.2, Eq. (20)). The constant V here is an upper bound on the difference between maximum and minimum attainable objective values of (10) (across all $\theta \in \Theta$). Since $B_i \leq \beta_i \leq 1$, we have ($\sigma = \min_i B_i$)

$$0 \geq \sum_i B_i \log \beta_i \geq \sum_i B_i \log B_i \geq \log \sigma$$

and

$$0 \leq \langle \max_i \beta_i v_i, \mathbf{1} \rangle \leq \left\langle \sum_i v_i, \mathbf{1} \right\rangle = n.$$

Hence,

$$V \leq n - \log \sigma.$$

Proof of Theorem 10

Proof of Part (i). Let $U = U(v, \Theta)$ be the set of feasible utilities defined in (1). By Lemma 1, we know that U is convex and compact. Hence, we can reformulate the convex program as one involving $u \in \mathbb{R}^n_+$ and $\delta \in \mathbb{R}^n_+$:

$$\begin{aligned} w^* = \sup & \sum_i (B_i \log u_i - \delta_i) \\ \text{s.t. } & u - \delta \in U, \\ & u, \delta \in \mathbb{R}^n_+. \end{aligned} \tag{EC.13}$$

Note that w^* is finite: taking $u = \delta = (1, \ldots, 1) \in \mathbb{R}^n$ gives an objective value of $-n$, while the objective is bounded above by $\sum_i B_i \log v_i(\Theta) < \infty$. Similar to the proof of Part (a) of Theorem 1, let $w_0 = -n \le w^*$ and consider the set

$$F = \left\{ (u, \delta) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : u - \delta \in U, \sum_i (B_i \log u_i - \delta_i) \ge w_0 \right\}.$$

The set F is convex and compact, on which the objective function is finite and continuous. Hence,

$$\sup \left\{ \sum_i (B_i \log u_i - \delta_i) : (u, \delta) \in F \right\}$$

is attained via some $(u^*, \delta^*) \in F$. The solution (u^*, δ^*) is also feasible to (EC.13) and attains its supremum, since F ensures feasibility and the level set constraint $\sum_i (B_i \log u_i - \delta_i) \ge w_0$ only excludes suboptimal solutions with objective value $< w_0$. Since $u^* \in U$, it can be attained by some feasible allocation $x^* \in (L^\infty(\Theta)_+)^n$, $\sum_i x_i^* \le \mathbf{1}$. Finally, Lemma 1 also ensures that $u^* \in U$ can be attained by a pure solution x^*, $x_i^* = \mathbf{1}_{\Theta_i}$ for a.e.-disjoint subsets $\Theta_i \subseteq \Theta$.

Proof of Part (ii) Note that, for any fixed $\beta \in \mathbb{R}^n_+$, setting $p = \max_i \beta_i v_i$ minimizes the objective of ($\mathcal{D}_{\text{QLEG}}$) subject to the constraints $p \ge \beta_i v_i$, $\forall i$. Hence, we can rewrite ($\mathcal{D}_{\text{QLEG}}$) in terms of $\beta \in \mathbb{R}^n_+$, which is a finite-dimensional convex program with a finite, strongly convex objective function (c.f. Part (b) of Theorem 1). The proof of Lemma 2 implies that there is a unique optimal solution β^* of ($\mathcal{D}_{\text{QLEG}}$), which means an optimal solution (p^*, β^*) must satisfy $p^* = \max_i \beta_i^* v_i$ a.e.

Before proving the next parts, we first establish weak duality.

Weak duality. Similar to the proof of Lemma 3, we first establish weak duality. For any (x, u, δ) feasible to ($\mathcal{P}_{\text{QLEG}}$) and any (p, β) feasible to ($\mathcal{D}_{\text{QLEG}}$),

$$\begin{aligned} \sum_i (B_i \log u_i - \delta_i) &\le \sum_i (B_i \log u_i - \delta_i) - \sum_i \beta_i (u_i - \langle v_i, x_i \rangle - \delta_i) - \left\langle p, \sum_i x_i - \mathbf{1} \right\rangle \\ &\le \sum_i \left(B_i \log \frac{B_i}{\beta_i} - \delta_i \right) - \sum_i \beta_i \left(\frac{B_i}{\beta_i} - \langle v_i, x_i \rangle - \delta_i \right) - \left\langle p, \sum_i x_i - \mathbf{1} \right\rangle \\ &= \sum_i (\beta_i - 1)\delta_i + \sum_i \langle \beta_i v_i - p, x_i \rangle + \langle p, \mathbf{1} \rangle - \sum_i B_i \log \beta_i + \sum_i B_i (\log B_i - 1) \\ &\le \langle p, \mathbf{1} \rangle - \sum_i B_i \log \beta_i + \sum_i B_i (\log B_i - 1), \end{aligned} \tag{EC.14}$$

where the second inequality is because $u_i = \frac{B_i}{\beta_i}$ maximizes $(u_i \mapsto B_i \log u_i - \beta_i u_i)$ and the other inequalities easily follow from the feasibility assumptions on (x, u, δ) and (p, β). Let the supremum of $(\mathcal{P}_{\rm QLEG})$ be z^* and the infimum of $(\mathcal{D}_{\rm QLEG})$ be w^*. Then, the above inequalities imply

$$z^* \leq w^* + \sum_i B_i (\log B_i - 1).$$

Strong duality and proof of Part (iii). We list the KKT conditions again for convenience.

$$\left\langle p^*, \mathbf{1} - \sum_i x_i^* \right\rangle = 0, \tag{EC.15}$$

$$u_i^* := \frac{B_i}{\beta_i^*},\ \forall i, \tag{EC.16}$$

$$\delta_i^* (1 - \beta_i^*) = 0,\ \forall i, \tag{EC.17}$$

$$\langle p^* - \beta_i^* v_i, x_i^* \rangle = 0,\ \forall i. \tag{EC.18}$$

Both primal and dual optima (z^* and w^*) are attained, by (x^*, δ^*) and (p^*, β^*), respectively, if and only if the inequalities in (EC.14) are all tight. When this happens, the first inequality being tight implies (EC.15), the second inequality being tight implies (EC.16), the last inequality being tight implies (EC.17) and (EC.18). Conversely, for feasible solutions (x^*, δ^*) and (p^*, β^*), the set of conditions imply that all inequalities in (EC.14) are tight, which ensure that both optima are attained.

The following two paragraphs complete the proof of Part (iv).

Optimality ⇒ QLME. Given optimal solutions (x^*, u^*, δ^*) and (p^*, β^*) of $(\mathcal{P}_{\rm QLEG})$ and $(\mathcal{D}_{\rm QLEG})$, respectively, by the above analysis, the KKT conditions (EC.15)-(EC.18) hold. Hence, market clearance (EC.15) holds. For any $x_i \in L^\infty(\Theta)_+$ such that $\langle p^*, x_i \rangle \leq B_i$, we need to show that $\langle v_i - p^*, x_i^* \rangle \geq \langle v_i - p^*, x_i \rangle$. There are two cases.

- Suppose $\beta_i^* = 1$. Then, (EC.18) implies

  $$\langle v_i - p^*, x_i^* \rangle = \langle \beta_i^* v_i - p^*, x_i^* \rangle + (1 - \beta_i^*) \langle v_i, x_i^* \rangle = (1 - \beta_i^*) \langle v_i, x_i^* \rangle = 0.$$

  Since $p^* \geq \beta_i^* v_i$, we have

  $$\langle v_i - p^*, x_i \rangle \leq \langle v_i - \beta_i^* v_i, x_i \rangle = (1 - \beta_i^*) \langle v_i, x_i \rangle = 0 = \langle v_i - p^*, x_i^* \rangle.$$

- If $\beta_i^* < 1$, then, (EC.17) implies $\delta_i^* = 0$. By the constraint $u_i \leq \langle v_i, x_i \rangle + \delta_i$ in $(\mathcal{P}_{\rm QLEG})$, we must have

  $$u_i^* = \langle v_i, x_i^* \rangle.$$

  Using $p^* \geq \beta_i^* v_i$ (feasibility w.r.t. $(\mathcal{D}_{\rm QLEG})$), we have

  $$\beta^* \langle v_i, x_i \rangle \leq \langle p^*, x_i \rangle \leq B_i = \beta_i^* u_i^* \ \Rightarrow\ \langle v_i, x_i \rangle \leq u_i^* = \langle v_i, x_i^* \rangle.$$

Hence,

$$\langle v_i - p^*, x_i\rangle \le \langle v_i - \beta_i^* v_i, x_i\rangle = (1-\beta_i^*)\langle v_i, x_i\rangle \le (1-\beta_i^*)\langle v_i, x_i^*\rangle = \langle v_i - p^*, x_i^*\rangle,$$

where the last equality is due to (EC.18).

Therefore, (x^*, p^*) is a QLME.

QLME $\Rightarrow$ optimality. Let (x^*, p^*) be a QLME. Then,

$$x^* \in \arg\max\left\{\langle v_i - p^*, x_i\rangle : \langle p^*, x_i\rangle \le B_i,\ x_i \in L^\infty(\Theta)_+\right\}.$$

For each i, construct β^*, u^* and δ^* as follows.

- If $\langle p^*, x_i^*\rangle < B_i$, then, by the above analysis, $\langle v_i, x_i^*\rangle = \langle p^*, x_i^*\rangle$. Set $\beta_i^* = 1$, $u_i^* = B_i$ and $\delta_i^* = u_i^* - \langle v_i, x_i^*\rangle > 0$.
- If $\langle p^*, x_i^*\rangle = B_i$, then

$$\langle v_i - p^*, x_i^*\rangle \ge 0 \Rightarrow \langle v_i, x_i^*\rangle \ge \langle p^*, x_i^*\rangle = B_i > 0.$$

Set $u_i^* = \langle v_i, x_i^*\rangle$, $\delta_i^* = 0$, and $\beta_i^* = \frac{B_i}{u_i^*} = \frac{B_i}{\langle v_i, x_i^*\rangle} \le 1$.

Finally, set $p^* = \max_i \beta_i^* v_i$. In this way, we have (x^*, u^*, δ^*) feasible to $(\mathcal{P}_{\rm QLEG})$ and (p^*, β^*) feasible to $(\mathcal{D}_{\rm QLEG})$ that satisfy (EC.15)-(EC.18) (where (EC.15) is due to the market clearance property of QLME and the other three are easily verified through our construction of β^*, u^* and δ^*). Hence, they make all inequalities in (EC.14) tight and are both optimal to $(\mathcal{P}_{\rm QLEG})$ and $(\mathcal{D}_{\rm QLEG})$, respectively.

EC.3. More details on the numerical examples

Linear v_i. The buyers' budgets are $B = (B_1, B_2, B_3, B_4) = (0.1, 0.3, 0.2, 0.4)$. Buyers have linear valuations v_i with intercepts $d = (1.2, 0.6, 0.3, 1.9)$, which give $c = (-0.4, 0.8, 1.4, -1.8)$ since v_i are normalized. The descending order of buyers by d_i is $\sigma = (4, 1, 2, 3)$, that is, buyer 4 should be allocated first (from left to right), and then buyer 1, and so on. Solving the convex program (7) (with only $K = 1$ subinterval being the entire interval $[0,1]$) yields equilibrium utilities $u^* = (0.1241, 0.3688, 0.2834, 0.5814)$. To partition $[0,1]$ into n intervals, first find $a_1^* \in [0,1]$ such that $v_3([0, a_1^*]) = \frac{c_3}{2}(a_1^*)^2 + d_3 a_1^* = u_1^*$ (since $\sigma(1) = 3$, i.e., d_3 is the largest and buyer 3 should get the leftmost interval); solving the quadratic equation gives $a_1^* = 0.3713$ (this is the "cut" operation introduced in §4.1). Next, find $a_2^* \in [a_1^*, 1]$ such that $v_1([a_1^*, a_2^*]) = u_2^* \;\Rightarrow\; a_2^* = 0.4921$ ($\sigma(2) = 1$). Similarly, $\sigma(3) = 2$, $v_2([a_2^*, a_3^*]) = u_2^* \;\Rightarrow\; a_3^* = 0.8199$. Figure 2 illustrates the equilibrium β^* (as in $\beta_i^* v_i$) and prices $p^* = \max_i \beta_i^* v_i$, where the breakpoints of p^* are precisely a_i^*. The allocation is as follows: buyer 1 gets the second interval $[l_1, h_1] = [0.3713, 0.4921]$ (since $\sigma(1) = 2$), buyer 2 gets the third interval $[l_2, h_2] = [0.4921, 0.8199]$, buyer 3 gets the fourth interval $[l_3, h_3] = [0.8199, 1.0]$ and buyer 4 gets the first interval $[l_4, h_4] = [0.0, 0.3713]$. Since all v_i are distinct, it is also the unique pure equilibrium allocation. As illustrated in Figure 2, the interval of buyer i is in fact its winning set, i.e., $[l_i, h_i] = \{p^* = \beta_i^* v_i\}$, where

$$\beta_i^* = B_i/u_i^* \;\Rightarrow\; \beta^* = (0.8058, 0.8135, 0.7057, 0.6880).$$

To verify that the primal solution (x^*, p^*), $x_i^* = \mathbf{1}_{[l_i, h_i]}$ is indeed a ME, it suffices to verify that $x_i^* = \mathbf{1}_{\Theta_i}$ (the pure allocation) and p^*, β^* satisfy the conditions in Corollary 3. Alternatively, by Theorem 2, we can also verify that the duality gap is zero, i.e., the primal and dual objective values are equal, after adding back the constant $\|B\|_1 - \sum_i B_i \log B_i$ to the dual objective ($\mathcal{D}_{\rm EG}$) (the constant is defined in Lemma 3). When computing the objective value of ($\mathcal{D}_{\rm EG}$), the term $\langle p^*, \mathbf{1}\rangle = p^*([0,1])$ can be decomposed, according to the pure equilibrium allocation given by $\Theta_i = [l_i, h_i]$, as

$$\langle p^*, \mathbf{1}\rangle = \sum_i \beta_i^* v_i([l_i, h_i]) = \sum_i \beta_i^* u_i^*.$$

Piecewise linear v_i. Here, we generate random budgets

$$B = (B_1, B_2, B_3, B_4) = (0.2270, 0.2584, 0.2642, 0.2505)$$

and random piecewise linear coefficients c_{ik}, d_{ik} such that $v_i(\theta) = c_{ik}\theta + d_{ik} \geq 0$ for $\theta \in [a_{k-1}, a_k]$, $i \in [n]$, $k \in [K]$. Here, a_k are the breakpoints of the predefined intervals corresponding to the linear pieces of v_i. The values of a_k, c_{ik}, d_{ik} are as follows:

$$\begin{aligned}
a &= (a_0, a_1, a_2, a_3) = (0, 0.3741, 0.8147, 1),\\
c = (c_{ik}) &= \begin{bmatrix} 1.2887 & 1.6253 & -0.4692 \\ -1.2494 & -0.2604 & -0.1476 \\ -0.4802 & -1.7084 & 1.1019 \\ -0.0501 & 2.5419 & 1.2096 \end{bmatrix},\\
d = (d_{ik}) &= \begin{bmatrix} 1.9391 & -0.2972 & 1.3209 \\ 0.4674 & 0.4864 & 0.1476 \\ 0.4137 & 1.3919 & -0.0462 \\ 0.4262 & 0.6464 & 0.8471 \end{bmatrix}.
\end{aligned}$$

Reformulating and solving the convex program (7) yield $u^* = (u_{ik}^*)$ as follows (e.g., the amount of utility buyer 3 receives from its allocation of interval $[a_1, a_2]$ is $u_{32}^* = 0.0191$):

$$u^* = \begin{bmatrix} 0.5845 & 0.0000 & 0.0000 \\ 0.0454 & 0.0579 & 0.0000 \\ 0.0000 & 0.0191 & 0.1767 \\ 0.0000 & 0.6089 & 0.0000 \end{bmatrix}.$$

Then, we partition each $[a_{k-1}, a_k]$ among the buyers as follows.

- For $k = 1$, since the buyers are sorted as $\sigma^1 = (2,3,4,1)$ in descending order of $\hat{d}_{1k}$, where $\hat{d}_{:,k} = (0.8894, 2.0, 1.2771.022)$, they also get intervals from the left $a_0 = 0$ to right $a_1 = 0.3741$ in this order. For buyer 2, $l_{21} = 0$ (the left endpoint of the subinterval of buyer 2 in interval $[a_0, a_1]$) and $v_2([0, h_{21}]) = u^*_{21} \Rightarrow h_{21} = 0.1148$. For buyer 3, since $u^*_{31} = 0$, $l_{31} = h_{31} = h_{21} = 0.1148$. The same is true for buyer 4. Buyer 1 gets $[l_{11}, h_{11}] = [0.1148, 1]$, which gives $v_1([l_{11}, h_{11}]) = 0.5845 = u^*_{11}$.
- Similarly, for $k = 2$, the order is $\sigma^k = (3,2,4,1)$; buyer 3 gets $[l_{32}, h_{32}] = [0.3741, 0.4001]$ with utility $u^*_{32} = 0.0579$, buyer 2 gets $[l_{22}, h_{22}] = [0.4001, 0.5604]$ with utility $u^*_{22} = 0.0191$, buyer 4 gets $[l_{42}, h_{42}] = [0.5604, 0.8147]$ with utility $u^*_{42} = 0.6089$, buyer 1 gets nothing.
- For $k = 3$, the order is $\sigma^3 = (2,1,4,3)$; only buyer 4 gets the entire interval, $[l_{43}, h_{43}] = [a_2, 1] = [0.8147, 1]$ with utility $u^*_{43} = 0.1767$. Similarly, we can also verify that it is a pure ME by either Corollary 3 or showing zero duality gap as in the linear example above.