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Infinite-Dimensional Fisher Markets and Tractable Fair Division

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Linear Fisher markets are a fundamental economic model with applications in Internet markets and fair division. In the finite-dimensional case of n buyers and m items, a market equilibrium can be computed using the Eisenberg-Gale convex program. Motivated by large-scale Internet advertising and fair division applications, this paper considers a generalization of a linear Fisher market where there is a finite set of buyers and a continuum of items. We introduce generalizations of the Eisenberg-Gale convex program and its dual to this infinite-dimensional setting, which leads to Banach-space optimization problems. We establish existence of optimal solutions, strong duality, as well as necessity and sufficiency of KKT-type conditions. All these properties are established via non-standard arguments, which circumvent the limitations of duality theory in optimization over infinite-dimensional Banach spaces. Furthermore, we show that there exists a pure equilibrium allocation, i.e., a division of the item space. When the item space is a closed interval and buyers have piecewise linear valuations, we show that the Eisenberg-Gale-type convex program over the infinite-dimensional allocations can be reformulated as a finite-dimensional convex conic program, which can be solved efficiently using off-the-shelf optimization software based on primal-dual interior-point methods. Based on our convex conic reformulation, we develop the first polynomial-time cake-cutting algorithm that achieves Pareto optimality, envy-freeness, and proportionality. For general buyer valuations or a very large number of buyers, we propose computing market equilibrium using stochastic dual averaging, which finds approximate equilibrium prices with high probability. Finally, we show that most of the above results and algorithms easily extend to the case of quasilinear utilities.

Key words: Market equilibrium, fair division, convex optimization

1. Introduction

Market equilibrium (ME) is a classical concept from economics, where the goal is to find an allocation of a set of items to a set of buyers, as well as corresponding prices, such that the market clears. One of the simplest equilibrium models is the (finite-dimensional) linear *Fisher market.* A Fisher market consists of a set of n buyers and m divisible items, where the utility for a buyer is linear in their allocation. Each buyer i has a budget B_i and valuation v_{ij} for each item j. A ME consists of an allocation (of items to buyers) and prices (of items) such that (i) each buyer receives a bundle of items that maximizes their utility subject to their budget constraint, and (ii) the market clears (all items such that $p_j > 0$ are exactly allocated). In spite of its simplicity, this model has several applications. Perhaps one of the most celebrated examples is the *competitive equilibrium from equal incomes* (CEEI), where m items are to be fairly divided among n agents. By giving each agent one unit of faux currency, the allocation from the resulting ME can be used as a fair division. This approach guarantees several fairness desiderata, such as envy-freeness and proportionality. Beyond fair division, linear Fisher markets also find applications in large-scale ad markets (Conitzer et al. 2018, 2019) and fair recommender systems (Kroer et al. 2019, Kroer and Peysakhovich 2019).

For the case of finite-dimensional linear Fisher markets, the Eisenberg-Gale convex program computes a market equilibrium (Eisenberg and Gale 1959, Eisenberg 1961). However, in settings like Internet ad markets and fair recommender systems, the number of items is often huge (Kroer et al. 2019, Kroer and Peysakhovich 2019), if not infinite or even uncountable (Balseiro et al. 2015). For example, each item can be characterized by a set of features. In that case, a natural model for an extremely large market, such as Internet ad auctions or recommender systems, is to assume that the items are drawn from some underlying distribution over a compact set of possible feature vectors.

Motivated by settings such as the above, where the item space is most easily modeled as a continuum, we study Fisher markets and its equilibria for a continuum of items. While equilibrium computation for finite-dimensional linear Fisher markets is well understood, nothing is known about computation of its infinite-dimensional analogue. We rectify this issue by developing infinite-dimensional convex programs over Banach spaces that generalize the Eisenberg-Gale convex program and its dual. We show that these convex programs lead to market equilibria, and give scalable first-order methods for solving the convex programs.

A problem closely related to our infinite-dimensional Fisher-market setting is the *cake-cutting* or *fair division* problem. There, the goal is to efficiently partition a "cake"—often modeled as a compact measurable space, or simply the unit interval $[0,1]$—among n agents so that certain fairness and efficiency properties are satisfied (Weller 1985, Brams and Taylor 1996, Cohler et al. 2011, Procaccia 2013, Cohler et al. 2011, Brams et al. 2012, Chen et al. 2013, Aziz and Ye 2014, Aziz and Mackenzie 2016, Deng et al. 2012). Focusing on the case of finding a division of a measurable space satisfying weak Pareto optimality and envy freeness, Weller (1985) shows the existence of a fair division. When all buyers have the same budget, our definition of a *pure* ME, i.e., where the allocation consist of indicator functions of a.e.-disjoint measurable sets, is equivalent to this notion of fair division. Thus, our convex programs yield solutions to the fair division setting of Weller (1985). Additionally, we also give an explicit characterization of the unique equilibrium prices based on a pure equilibrium allocation under arbitrary budgets. This generalizes the result of Weller (1985), which only holds for uniform budgets.

Under piecewise *constant* valuations over the $[0,1]$ interval, the equivalence of fair division and market equilibrium in certain setups has been discovered and utilized in the design of cake-cutting algorithms (Brams et al. 2012, Aziz and Ye 2014). For example, Aziz and Ye (2014) show that the special case with piecewise constant valuations can be (easily) reduced to a finite-dimensional Fisher market and hence captured by the classical Eisenberg-Gale framework. Our infinite-dimensional convex optimization characterization extends this connection from piecewise constant valuations to arbitrary valuations in the L^1 function space: we propose Eisenberg-Gale-type convex programs that characterize *all* ME. This includes pure ME which, under uniform budgets, correspond to fair divisions.

Beyond piecewise constant valuations, piecewise linear valuations have also been considered in fair division, although with different fairness and efficiency objectives. This setting is considerably more challenging, and e.g. Cohler et al. (2011, §4) focus on the case of two agents. Unlike the piecewise constant case, one cannot cut the unit interval *a priori* based on breakpoints of the pieces of buyers' valuations. Instead, as we will see, the correct such "cuts" inevitably depend on the equilibrium utility prices (price per unit utility) of each buyer. Nonetheless, we will show that in the piecewise linear case, it is possible to reformulate our general convex program as a finite-dimensional convex conic program involving second-order cones and exponential cones. We leverage this reformulation to give a polynomial-time procedure for computing a fair division for piecewise linear utilities in complete generality, for any number of agents. In addition to being polynomial-time computable in theory, our conic reformulation is also highly efficient numerically: it can be written as a sparse conic program that can be solved with standard convex optimization software. A key part of our finite-dimensional conic reformulation is to show that, given linear buyer valuations over a closed interval, the set of utilities attainable by feasible allocations of the item space can be described by a small number of linear and quadratic constraints. This result may be generalized to other classes of buyer valuations and may thus be of independent interest.

More related work in the areas of market equilibrium computation and fair division, proofs and additional details on numerical experiments can be found in the E-Companion available at `https://github.com/gao-yuan-hangzhou/fileshare` (stored in a GitHub repository).

1.1. Summary of contributions

Infinite-dimensional Fisher markets and equilibria. First, we propose the notion of a market equilibrium (ME) for an infinite-dimensional Fisher market with n buyers and a continuum of items $\Theta \subseteq \mathbb{R}^d$ (the case of Θ being a general finite measure space will be discussed at the end of §3). Here, buyers' valuations and allocations are nonnegative L^1 and L^∞ functions on Θ, respectively. A special case is *pure* allocations, where buyers get a.e.-disjoint measurable sets of items.

Market equilibrium and convex optimization duality. We then give two infinite-dimensional convex programs over Banach spaces of measurable functions on Θ (problems ($\mathcal{P}_{\rm EG}$) and ($\mathcal{D}_{\rm EG}$)), which generalize the EG convex program and its dual for finite-dimensional Fisher markets, and establish the existence of optimal solutions. Due to the lack of a compatible constraint qualification, general duality theory does not apply to these convex programs. Instead, we establish various duality properties directly through nonstandard arguments. Based on these duality properties and the existence of a minimizer in the "primal" convex program, we show that a pair of allocations and prices is a ME *if and only if* they are optimal solutions to the convex programs. Furthermore, we show that the primal convex program admits a *pure* optimal solution, meaning that buyers get disjoint subsets of items. and we conclude that there exists a pure equilibrium allocation, i.e., a *division* (modulo zero-value items) of the item space.

Properties of a market equilibrium. Based on the above results, we further show that a ME under the infinite-dimensional Fisher market satisfies (budget-weighted) proportionality, Pareto optimality and budget-weighted envy-freeness. Our results on the existence of ME and its fairness properties can be viewed as generalizations of those in Weller (1985), in which every buyer (agent) has the same budget.

All of the above results, except the existence of a pure equilibrium allocation, also hold when the item space Θ is discrete (finite or countably infinite).

Tractable reformulation under piecewise linear valuations. When the item space is a closed interval (e.g., $[0,1]$) and buyers have piecewise linear valuations, we show that equilibrium allocations can be computed efficiently via solving a convex conic reformulation of the infinite-dimensional Eisenberg-Gale-type convex program. This gives an efficient algorithm for computing a fair division under piecewise linear valuations, the first polynomial-time algorithm for this challenging problem to the best of our knowledge. The key in the reformulation is to show that, for linear valuations on a closed interval, the set of feasible utilities spanned by all feasible allocations can be described by a small number of linear and quadratic constraints with a few auxiliary variables.

Stochastic optimization for general valuations. For more general buyer valuations or a huge number of buyers, we propose solving a finite-dimensional convex reformulation of the dual of the infinite-dimensional EG for equilibrium utility prices using the stochastic dual averaging algorithm (SDA) and establish its convergence guarantees.

Extension to quasilinear utilities. Most of the above results easily extend to the setting where each buyer has a quasilinear utility. Specifically, we show that, in this case, a different pair of infinite-dimensional convex programs exhibit optimal solutions that correspond to quasilinear market equilibria. The convex conic reformulation can also be modified easily to capture pure quasilinear equilibrium allocations under piecewise linear valuations. Finally, SDA can also be easily modified to compute equilibrium utility prices in the quasilinear setting.

2. Infinite-Dimensional Fisher Markets

Measure-theoretic preliminaries. First, we introduce the measure-theoretic concepts that we will need. The following paragraph can be skimmed and referred back to later. The items will be represented by Θ, a compact subset of $\mathbb{R}^d$. Denote the Lebesgue measure on $\mathbb{R}^d$ as μ. Since Θ is compact, it is (Borel) measurable and $\mu(\Theta) < \infty$. In fact, any function f in this equivalence class give the same linear functional $g \mapsto \int_\Theta fgd\mu$. The suffix a.e. will be omitted unless the emphasis is necessary. For any set S of measurable functions on Θ, denote $S_+ = \{f \in S : f \geq 0\}$. For $f \in L^1(\Theta)$ and $g \in L^\infty(\Theta)$, denote $\langle f, g\rangle = \int_\Theta fgd\mu$. Since $L^\infty(\Theta)$ is the dual space of $L^1(\Theta)$, the integration $\int_\Theta fgd\mu$ is well-defined and is finite. Let $\mathbf{1}$ be the constant function taking value 1 on Θ. For any measurable set $A \subseteq \Theta$, denote $\mathbf{1}_A$ as the $\{0,1\}$-indicator function of A. For $q \in [1, \infty]$, let $L_q(\Theta)$ be the Banach space of L^q (integrable) functions on Θ with the usual L^q norm, that is, $\|f\| = \int_\Theta |f|^q d\mu$ if $1 \leq q < \infty$ and $\inf\{M > 0 : |f| \leq M \,\text{a.e. if}\; q = \infty$. Any $\tau \in L^1(\Theta)_+$ can also be viewed as a measure on Θ via $\mu_\tau(A) := \int_A \tau d\mu$ for any measurable set $A \subseteq \Theta$. In this work, we will denote the measure μ_τ simply as τ. Unless otherwise stated, any measure m used or constructed is absolutely continuous w.r.t. the Lebesgue measure μ and hence *atomless*. In other words, for any measurable set $A \subseteq \Theta$ such that $m(A) > 0$ and any $0 < c < m(A)$, there exists a measurable subset $B \subseteq A$ such that $m(B) = c$. Two measurable sets $A, B \subseteq \Theta$ are said to be *a.e.-disjoint* if $\mu(A \cap B) = 0$. We use equations and inequalities involving measurable functions to denote the corresponding (measurable) *preimages* in Θ. For example, $\{f \leq 0\} := \{\theta \in \Theta : f(\theta) \leq 0\}$ and $\{f \leq g\} := \{\theta \in \Theta : f(\theta) \leq g(\theta)\}$.

Fisher market. Here, we formally describe the infinite-dimensional (linear) Fisher market setup that we use throughout this work.

1. There are n buyers and an item space Θ, which is a compact subset of $\mathbb{R}^d$.
2. Each buyer has a valuation over the item space $v_i \in L^1(\Theta)_+$ (nonnegative L^1 functions on Θ).
3. The items' prices $p \in L^1(\Theta)_+$ live in the same space as valuations.
4. An allocation of items to a buyer i is denoted by $x_i \in L^\infty(\Theta)_+$. We use $x = (x_1, \ldots, x_n) \in (L^\infty(\Theta)_+)^n$ to denote the aggregate allocation. An allocation x is said to be a *pure* allocation (or a pure solution, when viewed as variables of a convex program) if for all i, $x_i = \mathbf{1}_{\Theta_i}$ for *a.e.-disjoint* measurable sets $\Theta_i \subseteq \Theta$ (where leftover is allowed, i.e., $\Theta \setminus (\cup_i \Theta_i) \neq \emptyset$). When x

is a pure allocation (solution), we also denote x as $\{\Theta_i\}$. An allocation is *mixed* if it is not pure, or equivalently, the set $\{0 < x_i < 1\} \subseteq \Theta$ has positive measure for some i.

5. Each buyer has a budget $B_i > 0$ and all items have unit supply, i.e., x is supply-feasible if $\sum_i x_i \le \mathbf{1}$. Without loss of generality, we also assume that $v_i(\Theta) = \|v_i\| > 0$ for all i (otherwise buyer i can be removed).

Note that the market is "linear" means the utility each buyer i receives from an allocation x_i is a linear functional $x_i \mapsto \langle v_i, x_i\rangle$. The valuation v_i itself, as an L^1 function on the item space Θ, may not be a linear function in $\theta \in \Theta$.

Example 1 (Valuations and allocations). Let there be $n = 2$ buyers and $\Theta = [0,1]$. Then, their allocations $x_1, x_2 \in L^\infty([0,1])_+$ are nonnegative measurable functions on $[0,1]$. An example of a pure allocation is $x_1 = \mathbf{1}_{[0,1/2]}$, $x_2 = \mathbf{1}_{[1/2,1]}$. We can also denote the pure allocation as $\{\Theta_i\}$, where $\Theta_1 = [0,1/2]$, $\Theta_2 = [1/2,1]$. Here, Θ_i are measurable and a.e.-disjoint. An example of a mixed allocation is $x_1(\theta) = 0.5 + 0.1\theta^2$ and $x_2(\theta) = 0.5 - 0.1\theta^2$, $\theta \in [0,1]$. In both cases, we have $x_1 + x_2 = \mathbf{1}$ a.e. (with both sides viewed as nonnegative measurable functions on $[0,1]$). Let buyer 1's valuation be $v_1(\theta) = \theta^2$. For the allocation $x_1 = \mathbf{1}_{[0,1/2]}$, the utility buyer 1 receives is $\langle v_1, x_1\rangle = v_1([0,1/2]) = \int_0^1 v_1(\theta)\mathbf{1}_{[0,1/2]}(\theta)d\theta = \int_0^{1/2}\theta^2 d\theta = \frac{1}{24}$; for the allocation $x_1(\theta) = 0.5 + 0.1\theta^2$, the utility is $\langle v_1, x_1\rangle = \int_0^1 v_1(\theta)x_1(\theta)d\theta = \int_0^1 \theta^2(0.5 + 0.1\theta)d\theta = \frac{23}{120}$. Note that v_1 can denote both the L^1 function $\theta \mapsto \theta^2$ and the induced measure $A \mapsto \int_A \theta^2 d\theta$; whether it is the function or the measure is always clear from the context.

Given prices $p \in L^1(\Theta)_+$, the *demand set* of buyer i is the set of utility-maximizing allocations subject to its budget constraint, that is, $D_i(p) = \arg\max\{\langle v_i, x_i\rangle : x \in L^\infty(\Theta)_+,\ \langle p, x_i\rangle \le B_i\}$. Generalizing its finite-dimensional counterpart (Nisan et al. 2007), a *market equilibrium* is defined as a pair $(x^*, p^*) \in (L^\infty(\Theta)_+)^n \times L^1(\Theta)_+$ satisfying the following.

- Buyer optimality: for every $i \in [n]$, $x_i^* \in D_i(p^*)$.
- Market clearance (up to zero-price items): $\sum_i x_i^* \le \mathbf{1}$ and $\langle p^*, \mathbf{1} - \sum_i x_i^*\rangle = 0$.

We say that $x^* \in (L^\infty(\Theta)_+)^n$ is an equilibrium allocation if (x^*, p^*) is a ME for some $p^* \in L^1(\Theta)_+$. A pair (x^*, p^*) is called a pure ME if it is a ME and x^* is a pure allocation. From the definition of market equilibrium, we can assume the following normalizations w.l.o.g. (i) $v_i(\Theta) = \|v_i\| = 1$ for all i, since $D_i(p)$ is invariant under scaling of v_i. (ii) $\|B\|_1 = 1$ since if (x^*, p^*) is a ME under $B = (B_i)$, then $(x^*, p^*/\|B\|_1)$ is a ME under normalized budgets $(B_i/\|B\|_1)$. (iii) The total supply of all items is $\mu(\Theta) = \|\mathbf{1}\| = 1$. Otherwise, we can scale the item space Θ via $\theta \mapsto \alpha\theta$ for some constant α or scale the measure μ.

3. Equilibrium and Duality

Due to intrinsic limitations of general infinite-dimensional convex optimization duality theory, we cannot start with a convex program and then derive its dual. Instead, we directly *propose* two infinite-dimensional convex programs, and then proceed to show from first principles that they exhibit optimal solutions and a strong-duality-like relationship. First, we give a direct generalization of the (finite-dimensional) Eisenberg-Gale convex program (Eisenberg 1961, Nisan et al. 2007):

$$z^* = \sup_{x \in (L^\infty(\Theta)_+)^n} \sum_i B_i \log\langle v_i, x_i\rangle \quad \text{s.t.} \quad \sum_i x_i \leq \mathbf{1}. \tag{$\mathcal{P}_{\rm EG}$}$$

Motivated by the dual of the finite-dimensional EG convex program (Cole et al. 2017, Lemma 3), we also consider the following convex program:

$$w^* = \inf_{p \in L^1(\Theta)_+,\, \beta \in \mathbb{R}^n_+} \left[\langle p, \mathbf{1}\rangle - \sum_i B_i \log \beta_i\right] \qquad \text{s.t. } p \geq \beta_i v_i \text{ a.e.}, \forall\, i. \tag{$\mathcal{D}_{\rm EG}$}$$

We state our central theoretical results in the following theorem. Parts of this theorem are stated in more detail in subsequent lemmas and theorems. All proofs can be found in EC.2 in the E-Companion.

THEOREM 1. *The following results hold regarding problems* ($\mathcal{P}_{\rm EG}$) *and* ($\mathcal{D}_{\rm EG}$).

(a) The supremum z^* *of* ($\mathcal{P}_{\rm EG}$) *is attained via a* pure *optimal solution* x^**, that is,* $x^* = (x_i^*)$ *and* $x_i^* = \mathbf{1}_{\Theta_i}$ *for a.e.-disjoint measurable subsets* $\Theta_i \subseteq \Theta$*.*

(b) The infimum w^* *of* ($\mathcal{D}_{\rm EG}$) *is attained via an optimal solution* (p^*, β^*)*, in which* $\beta^* \in \mathbb{R}^n_+$ *is unique and* $p^* = \max_i \beta_i^* v_i$ *a.e.*

(c) A pair of allocations and prices $(x^*, p^*) \in (L^\infty(\Theta)_+)^n \times L^1(\Theta)_+$ *is a ME if and only if* x^* *is an optimal solution of* ($\mathcal{P}_{\rm EG}$) *and* (p^*, β^*) *is the (a.e.-unique) optimal solution of* ($\mathcal{D}_{\rm EG}$)*.*

REMARK 1. If we view ($\mathcal{D}_{\rm EG}$) as the primal, then it can be shown that its Lagrange dual is ($\mathcal{P}_{\rm EG}$), and weak duality follows (see, e.g., (Ponstein 2004, §3)). However, we cannot conclude strong duality, or even primal or dual optimum attainment, since $L^1(\Theta)_+$ has an empty interior (Luenberger 1997, §8.8 Problem 1) and hence Slater's condition does not hold. If we choose the space for valuations and prices to be $L^\infty(\Theta)$ instead of $L^1(\Theta)$ for the space of allocations x_i (i.e., the underlying Banach space of ($\mathcal{P}_{\rm EG}$)), then ($\mathcal{D}_{\rm EG}$), with $p \in L^\infty(\Theta)_+$, does satisfy Slater's condition (Luenberger 1997, §8.8 Problem 2). However, its dual is ($\mathcal{P}_{\rm EG}$) but with the nonnegative cone $L^\infty(\Theta)_+$ (in which each x_i lies) replaced by the (much larger) cone $\{g \in L^\infty(\Theta)^* : \langle f, g\rangle \geq 0, \forall\, f \in L^\infty(\Theta)_+\} \subseteq L^\infty(\Theta)^*$. In this case, not every bounded linear functional $g \in L^\infty(\Theta)$ can be represented by a measurable function $\tilde{g}$ such that $\langle f, g\rangle = \int_\Theta \tilde{g} f d\mu$ (see, e.g., (Day 1973)). Therefore, we still cannot conclude that ($\mathcal{P}_{\rm EG}$) has an optimal solution in $(L^1(\Theta)_+)^n$ satisfying strong duality. Similar issues occur when ($\mathcal{P}_{\rm EG}$) is viewed as the primal instead.

We briefly explain the proof ideas of Theorem 1. Unlike the finite-dimensional case, the feasible region of ($\mathcal{P}_{\rm EG}$) here, although being closed and bounded in the Banach space $L^\infty(\Theta)$, is not compact. In fact, it is easy to construct an infinite sequence in the feasible region that does not have any convergent subsequence. This issue can be circumvented using the following lemma.

Lemma 1. *Define the set of feasible utilities as*

$$U = U(v,\Theta) = \left\{ (u_1,\dots,u_n) : u_i = \langle v_i, x_i\rangle,\, x\in (L^\infty(\Theta)_+)^n,\, \sum_i x_i \le \mathbf{1} \right\} \subseteq \mathbb{R}^n_+ \tag{1}$$

and the set of utilities attainable via pure allocations as

$$U' = U'(v,\Theta) = \{(v_1(\Theta_1),\dots,v_n(\Theta_n)) : \Theta_i \subseteq \Theta \textit{ measurable and a.e.-disjoint}\} \subseteq \mathbb{R}^n_+.$$

Then, $U = U'$ and this set is convex and compact.

Using Lemma 1, for part (a), we can show that there exists $u^*\in\mathbb{R}^n_{++}$ such that $z^* = \sum_i B_i \log u^*_i$, which then ensures that z^* is attained by some *pure* feasible solution $\{\Theta_i\}$ of $(\mathcal{P}_{\rm EG})$, that is, $\Theta_i\subseteq\Theta$ are a.e.disjoint and $v_i(\Theta_i) = u^*_i$.

Part (b) follows by reformulating $(\mathcal{D}_{\rm EG})$ into a finite-dimensional convex program in $\beta\in\mathbb{R}^n_+$. For a fixed $\beta>0$, setting $p = \max_i \beta_i v_i$ clearly minimizes the objective of $(\mathcal{D}_{\rm EG})$. Since $\beta\ge 0$ and $v_i\in L^1(\Theta)_+$, we have $0\le \max_i \beta_i v_i \le \|\beta\|_1 \sum_i v_i$, where the right-hand side is L^1-integrable since each v_i is. Hence, $\max_i \beta_i v_i \in L^1(\Theta)_+$ as well. Thus we can also reformulate $(\mathcal{D}_{\rm EG})$ as the following convex program:

$$\inf_{\beta\in\mathbb{R}^n_+} \left[\left\langle \max_i \beta_i v_i, \mathbf{1}\right\rangle - \sum_i B_i \log \beta_i \right]. \tag{2}$$

Lemma 2. *The infimum of (2) is attained via a unique minimizer $\beta^*>0$. The optimal solution (p^*,β^*) of $(\mathcal{D}_{\rm EG})$ has a unique β^* and satisfies $p^* = \max_i \beta^*_i v_i$ a.e.*

To show Part (c), we first establish weak duality and KKT conditions (necessary and sufficient for optimality) in the following lemma. As mentioned before, this is necessary due to the lack of general duality results in infinite-dimensional convex optimization. These conditions parallel those in KKT-type first-order optimality conditions in classical nonlinear optimization over Euclidean spaces (see, e.g., Nocedal and Wright (2006, §12.3) and Bertsekas (1999, §3.3.1)).

Lemma 3. *Let $C = \|B\|_1 - \sum_i B_i \log B_i$. We have*

(a) Weak duality: $C + z^* \le w^*$.

(b) KKT conditions: For x^* feasible to $(\mathcal{P}_{\rm EG})$ and (p^*,β^*) feasible to $(\mathcal{D}_{\rm EG})$, they are both optimal (i.e., attaining the optima z^* and w^* respectively) if and only if

$$\left\langle p^*, \mathbf{1} - \sum_i x^*_i \right\rangle = 0, \tag{3}$$

$$\langle v_i, x^*_i\rangle = u^*_i := \frac{B_i}{\beta^*_i},\ \forall\, i, \tag{4}$$

$$\langle p^* - \beta^*_i v_i, x^*_i\rangle = 0,\ \forall\, i. \tag{5}$$

Thus, we see that in spite of the general difficulties with duality theory in infinite dimensions, we have shown that ($\mathcal{P}_{\rm EG}$) and ($\mathcal{D}_{\rm EG}$) behave like duals of each other: strong duality holds, and KKT conditions hold if and only if a pair of feasible solutions are both optimal (see, e.g., Nisan et al. (2007, §5.2) for the finite-dimensional counterparts). Using Lemma 3, we can show the following theorem, which is an expanded version of Part (c) of Theorem 1 regarding the equivalence of market equilibrium and optimality w.r.t. the convex programs.

THEOREM 2. *Assume x^* and (p^*, β^*) are optimal solutions of ($\mathcal{P}_{\rm EG}$) and ($\mathcal{D}_{\rm EG}$), respectively. Then (x^*, p^*) is a ME, $\langle p^*, x_i^*\rangle = B_i$ for all i, and the equilibrium utility of buyer i is $u_i^* = \langle v_i, x_i^*\rangle = \frac{B_i}{\beta_i^*}$. Conversely, if (x^*, p^*) is a ME, then x^* is an optimal solution of ($\mathcal{P}_{\rm EG}$) and (p^*, β^*), where $\beta_i^* := \frac{B_i}{\langle v_i, x_i^*\rangle}$, is an optimal solution of ($\mathcal{D}_{\rm EG}$).*

We list some direct consequences of the results we have obtained so far. Below is a direct consequence of Theorem 2 and Part (a) of Lemma 3 on the structural properties of a market equilibrium.

COROLLARY 1. *Let (x^*, p^*) be a ME. Then, x^* and (p^*, β^*), where $\beta_i^* := \frac{B_i}{\langle v_i, x_i^*\rangle}$, satisfy* (3)-(5)*. In particular,* (5) *shows that a buyer's equilibrium allocation x_i^* must be zero a.e. outside its "winning" set of items $\{p^* = \beta_i^* v_i\}$.*

The equilibrium β^*, or equivalently, the second part of the unique optimal solution (p^*, β^*) of ($\mathcal{D}_{\rm EG}$)), is often known as the (equilibrium) *utility price*, that is, $\beta_i^* = \frac{B_i}{u_i^*}$ is the price each buyer i pays for a unit of utility. The above corollary shows that, at equilibrium, each buyer i only gets items where its $\beta_i^* v_i$ is the maximum among all buyers, that is, where $p^* = \beta_i^* v_i$. In other words, buyer i only pays for items with the lowest price per unit utility, or equivalently, the most utility per unit price. Since $p^* \geq \beta_i^* v_i$, under prices p^*, buyer i must pay at least β_i^* for each unit of utility. Given a pure optimal solution $\{\Theta_i\}$ of ($\mathcal{P}_{\rm EG}$), we can construct the (a.e.-unique) optimal solution (p^*, β^*) of ($\mathcal{D}_{\rm EG}$). In particular, such a construction ensures feasibility of (p^*, β^*).

COROLLARY 2. *Let $\{\Theta_i\}$ be a pure optimal solution of ($\mathcal{P}_{\rm EG}$), $u_i^* = v_i(\Theta_i)$ and $\beta_i^* = \frac{B_i}{u_i^*}$. (i) For each i, we have $\beta_i^* v_i \geq \beta_j^* v_j$ a.e. for all $j \neq i$ on Θ_i. (ii) Let $p^* := \max_i \beta_i^* v_i$. Then, $p^*(A) = \sum_i \beta_i^* v_i(A \cap \Theta_i)$ for any measurable set $A \subseteq \Theta$. (iii) The constructed (p^*, β^*) is an optimal solution of ($\mathcal{D}_{\rm EG}$) and satisfies* (3)-(5).

Given a pure allocation, we can also verify whether it is an equilibrium allocation using the following corollary.

COROLLARY 3. *A pure allocation $\{\Theta_i\}$ is an equilibrium allocation (with equilibrium prices p^*) if and only if the following conditions hold with $\beta_i^* := \frac{B_i}{v_i(\Theta_i)}$ and $p^* := \max_i \beta_i^* v_i$: (i) prices of items in Θ_i are given by $\beta_i^* v_i$: $p^* = \beta_i^* v_i$ on each Θ_i, $i \in [n]$; (ii) prices of leftover are zero: $p^*(\Theta \setminus (\cup_i \Theta_i)) = 0$.*

Fairness and efficiency properties of ME. Let $x \in (L^\infty(\Theta)_+)^n$, $\sum_i x_i \leq \mathbf{1}$ be an allocation. It is (strongly) *Pareto optimal* if there does *not* exist $\tilde{x} \in (L^\infty(\Theta)_+)^n$, $\sum_i \tilde{x}_i \leq \mathbf{1}$ such that $\langle v_i, \tilde{x}_i \rangle \geq \langle v_i, x_i \rangle$ for all i and the inequality is strict for at least one i (Cohler et al. 2011). It is *envy-free* (in a budget-weighted sense) if $\frac{1}{B_i}\langle v_i, x_i\rangle \geq \frac{1}{B_j}\langle v_i, x_j\rangle$ for any $j \neq i$ (Nisan et al. 2007, Kroer et al. 2019). When all $B_i = 1$, this is sometimes referred to as being "equitable" (Weller 1985). It is *proportional* if $\langle v_i, x_i\rangle \geq \frac{B_i}{\|B\|_1} v_i(\Theta)$ for all i, that is, each buyer gets at least the utility of its *proportional share* allocation, $x^{\mathrm{PS}} := \frac{B_i}{\|B\|_1}\mathbf{1}$. Similar to the finite-dimensional case, market equilibria in infinite-dimensional Fisher markets also exhibit these properties.

Theorem 3. *Let (x^*, p^*) be a ME. Then, x^* is Pareto optimal, envy-free and proportional.*

ME as generalized fair division. By Theorem 3, a pure ME $\{\Theta_i\}$ under uniform budgets ($B_i = 1/n$) is a fair division in the sense of Weller (1985), that is, a Pareto optimal and envy-free division (into a.e.-disjoint measurable subsets) of Θ. Furthermore, (Weller 1985, §3) shows that, there exist equilibrium prices p^* such that (i) $p^*(\Theta_i) = 1/n$ for all i, (ii) $v_i(\Theta_i) \geq v_i(A)$ for any measurable set $A \subseteq \Theta$ such that $p^*(A) \leq 1/n$ and (iii) for any measurable set $A \subseteq \Theta$, $p^*(A) = \frac{1}{n}\sum_i \frac{v_i(A\cap\Theta_i)}{v_i(\Theta_i)}$. Utilizing our results, when $B_i = 1/n$, and $\{\Theta_i\}$ is a pure ME, the first property above is a special case of $\langle p^*, x_i^*\rangle = B_i$ in Theorem 2 (with $x_i^* = \mathbf{1}_{\Theta_i}$); the second property above follows from the ME property $x_i^* \in D_i(p^*)$; the third property is a special case of Part (ii) in Corollary 2, since $\beta_i^* = \frac{B_i}{u_i^*} = \frac{1}{n}\cdot\frac{1}{v_i(\Theta)}$. Hence, ME under a continuum of items can be viewed as generalized fair division.

Bounds on equilibrium quantities. Using the KKT condition $u^* = \frac{B_i}{\beta_i^*}$ (Lemma 3) and an equilibrium allocation being proportional (Theorem 3), we can easily establish upper and lower bounds on equilibrium quantities. These bounds will be useful in subsequent convergence analysis of stochastic optimization in §5. Similar bounds hold in the finite-dimensional case (Gao and Kroer 2020). Recall that we assume $v_i(\Theta) = 1$ and $\|B\|_1 = 1$ w.l.o.g.

Lemma 4. *For any ME (x^*, p^*), we have $p^*(\Theta) = 1$. Furthermore, $B_i \leq u_i^* = \langle v_i, x_i^*\rangle \leq 1$ and hence $B_i \leq \beta_i^* := \frac{B_i}{u_i^*} \leq 1$ for all i.*

Discrete and other measurable item spaces. It can be easily verified that all the above results, except the existence of a pure ME, hold when Θ is a finite measure space (w.r.t. a measure μ), including the classical case of a finite item set Θ. To see this, note that Theorem 2 is based on Lemma 1, which is a consequence of Dvoretzky et al. (1951, Theorems 1 and 4; restated as Lemma EC.1 in the E-Companion). For a general finite measure space (where μ may not be *atomless* like the Lebesgue measure), only Dvoretzky et al. (1951, Theorem 1) holds and hence Lemma 1 partially holds (U is convex and compact, but $U' \neq U$ in general); nevertheless, this is sufficient to show the existence of ME and the convex programs capturing them.

4. Tractable Convex Optimization for Piecewise Linear Valuations

In this section we show that for the case where the item space is $\Theta = [0,1]$ and the buyers' valuations v_i are piecewise linear functions on $[0,1]$, it is possible to reformulate our infinite-dimensional convex program ($\mathcal{P}_{\rm EG}$) as a finite-dimensional convex conic program. This finite-dimensional program can be solved efficiently using off-the-shelf interior-point methods. Based on the optimal solution of this reformulation, an approximate pure equilibrium allocation can be constructed easily. This yields a highly practical approach for solving the case of piecewise linear valuations. Unfortunately, the current theory of interior-point methods does not allows us to immediately conclude that we have a polynomial-time algorithm. To complement our practical convex conic program, we use the ellipsoid method to show that there exists a theoretical algorithm that finds an ϵ-approximate pure equilibrium allocation in time polynomial in n, K and $\log\frac{\kappa}{\epsilon}$, where κ is the inverse of the smallest buyer budget $\min_i B_i$.

As discussed in §3, when all B_i are equal we are in the CEEI case, and a pure equilibrium allocation in which each buyer gets a union of intervals is a fair division in the sense of (Weller 1985). Hence, our method gives a polynomial-time algorithm for finding a fair division of the unit interval under piecewise linear valuations.

The key to our practical convex conic program is to show that when the buyers' valuations v_i are piecewise linear and $\Theta = [0,1]$, the set of feasible utilities $U(v,[0,1])$ defined in (1) can be represented by a small number of simple linear and quadratic constraints with a small number of auxiliary variables. The first subsection shows this result for a type of normalized linear valuations on the $[0,1]$ interval. The second subsection extends the characterization to general intervals $[a,b]$ and unnormalized linear valuations, which is necessary for handling piecewise linear valuations later. The next two subsections show our practical and theoretical algorithmic results, respectively, followed by numerical examples and experiments.

4.1. Characterization of the set $U(v,[0,1])$ under linear v_i

We first characterize the set of feasible utilities when each valuation v_i is *linear* over the unit interval $[0,1]$. We will show that it can be represented by $O(n)$ linear and quadratic constraints using $O(n)$ auxiliary variables. Before proceeding, we note that, for a nonnegative linear function $v_i: \theta \mapsto c_i\theta + d_i$ on $[0,1]$, the following operations take constant time:

- Eval: given $[a,b]$, compute its utility for buyer i: $v_i([a,b]) = \int_a^b v_i(\theta)d\theta = \frac{c_i}{2}(b^2-a^2) + d_i(b-a)$.
- Cut: given $a\in[0,1]$ and $0\le u_0 \le v_i([a,1])$, finding $b\in[a,1]$ such that $v_i([a,b]) = u_0$ amounts to solving a simple quadratic equation: $v_i([a,b]) = \frac{c_i}{2}(b^2-a^2) + d_i(b-a) = u_0$, which gives $b = \frac{-d_i + \sqrt{d_i^2 - c_i(c_i a^2 + 2d_i a + 2u_0)}}{c_i}$.

The names "eval" (evaluation of the utility of a given interval) and "cut" (finding a cut such that the utility of the resulting interval equals a given value) are customary in the cake cutting literature (Procaccia 2014, Robertson and Webb 1998).

Geometry of a pure equilibrium allocation. Recall that, by Corollary 1 and Eq. (5), if $\{\Theta_i\}$ is a pure equilibrium allocation, then it must hold that $\Theta_i \subseteq \{p^* = \beta_i^* v_i\}$ a.e., that is, each buyer only gets a non-zero allocation in regions where $\beta_i^* v_i$ is the maximum among all buyers. When the valuations v_i are linear functions on $[0,1]$, p^* must be a piecewise linear (p.w.l.) function with at most n pieces. We will use this to show that, in the case of linear valuations, a pure equilibrium allocation only needs to consist of $(n-1)$ cuts on $[0,1]$, resulting in n intervals, one for each buyer. If we are able to compute the equilibrium utilities u_i^* and an *ordering* of the buyers specifying who gets the first interval starting at 0, who gets the second interval, and so on, then an equilibrium allocation reduces to performing at most n "cut" operations to find the exact endpoints of these intervals. Such an ordering of the buyers at equilibrium can in fact be determined *a priori*. Intuitively, a buyer with a higher valuation on the left of the unit interval should always be assigned items on the left as well. This motivates us to consider an ordering based on the magnitudes of each valuation intercept $v_i(0) = d_i$. Since equilibrium allocations are invariant under arbitrary scaling of each v_i, such an ordering must be independent of the absolute magnitudes of buyers' valuations $\|v_i\|$. Hence, we consider normalized valuations such that for each i, $\|v_i\| = v_i([0,1]) = 1$, and we assume that the buyer indices are sorted by their intercepts d_i in descending order. We also assume that the valuations v_i are distinct. Due to normalization, this is equivalent to the intercepts d_i being distinct.

Assumption 1. *The item space is $\Theta = [0,1]$. The valuation of each buyer i is linear and non-negative: $v_i(\theta) = c_i\theta + d_i \geq 0$, $\theta \in [0,1]$. The valuations are normalized: $v_i(\Theta) = \|v_i\| = \int_0^1 v_i(\theta)d\theta = \frac{c_i}{2} + d_i = 1$. The intercepts of v_i satisfy $2 \geq d_1 > \cdots > d_n \geq 0$.*

The upper and lower bounds on d_i above are due to nonnegativity: $v_i(0) \geq 0$ and $v_i(1) \geq 0$ imply $d_i \geq 0$ and $0 \leq c_i + d_i = 2(1-d_i) + d_i \Rightarrow d_i \leq 2$. The following lemma shows that, under the above assumption, the equilibrium prices p^* are p.w.l. with exactly n linear pieces, corresponding to intervals that are the pure equilibrium allocations to the buyers.

Lemma 5. *Under Assumption 1 and budgets $B_i > 0$, the equilibrium prices $p^* = \max_i \beta_i^* v_i$ are piecewise linear with exactly n linear pieces. The (unique) breakpoints of the linear pieces $0 = a_0^* < a_1^* < \cdots < a_n^* = 1$ induce a pure allocation: buyer i receives $\Theta_i = [a_{i-1}^*, a_i^*]$, $i \in [n]$. This allocation is the unique equilibrium allocation.*

Recovering a pure allocation given feasible utilities. Based on Lemma 5, we can establish Lemma 7 below, which ensures that partitioning the interval into n subintervals is sufficient to attain any feasible utility $u \in U(v,[0,1])$. A key fact used in the proof of Lemma 7 is a variant of the well-known second welfare theorem for an exchange economy with finitely many divisible items, but for our infinite-dimensional setting. As a technical contribution, we state and prove a general second welfare theorem for the case of a continuum of items and general L^1 buyer valuations, which may be of independent interest. It states that any Pareto optimal allocation and its corresponding utilities are equilibrium allocations and equilibrium utilities of a Fisher market for some choice of buyer budgets. One technical challenge in establishing the lemma is the allocation space being a non-Euclidean Banach space. Hence, the set of feasible allocations cannot have a "tractable" dual space while having a nonempty interior at the same time (see Remark 1). This essentially rules out the use of a separation theorem in the allocation space, a key step in proving the classical finite-dimensional second welfare theorem. Instead, the proof relies on the convexity and compactness of the set of feasible utilities and the structure of an infinite-dimensional ME.

LEMMA 6. *Let each buyer i have valuation $v_i \in L^1(\Theta)_+$ where $\Theta \subseteq \mathbb{R}^d$ is a compact set. Assume $v_i(\Theta) > 0$ for all i. Let $u^\circ \in U$ be a Pareto optimal utility vector, that is, there is a Pareto optimal allocation $\{x_i^\circ\}$ such that $u_i^\circ = \langle v_i, x_i^\circ \rangle$. Then, there exists $B \in \mathbb{R}_+^n$ such that $\{x_i^\circ\}$ is an equilibrium allocation and u_i° are the corresponding equilibrium utilities of a Fisher market with n buyers, each having valuation v_i and budget B_i.*

Using the above two lemmas, when $\Theta = [0,1]$ and v_i are normalized linear valuations with descending intercepts $d_1 \geq \cdots \geq d_n$, we can show that any feasible utility vector $u \in U(v,[0,1])$ can be attained by a pure allocation of $[0,1]$ consisting of n intervals (some of which can have length zero). These intervals are allocated from left to right to buyers in the order of $1,\dots,n$.

LEMMA 7. *Let $v_i(\theta) = c_i\theta + d_i$ be normalized linear valuations on $[0,1]$ (i.e. $v_i([0,1]) = \frac{c_i}{2} + d_i = 1$) such that $2 \geq d_1 \geq \cdots \geq d_n \geq 0$. For any $u \in U(v,[0,1])$, there exists $a_0 = 0 \leq a_1 \leq \cdots \leq a_n = 1$ such that $v_i([a_{i-1}, a_i]) \geq u_i$ for all i.*

Suppose we are given a set of feasible utilities $u \in U(v,[0,1])$ and the valuations v_i are normalized and sorted in descending order of their intercept d_i. Then, we can find the breakpoints $a_0 = 0 \leq a_1 \leq \cdots \leq a_n = 1$ in Lemma 7 by performing $(n-1)$ "cut" operations sequentially from left to right, each time computing a new $a_i \in [a_{i-1}, 1]$ such that $v_i([a_{i-1}, a_i]) = u_i$ for $i = 1,\dots,n-1$ (and setting $a_n = 1$ ensures $v_n([a_{n-1}, a_n]) \geq u_n$). Later, we will present a general version of this procedure that handles unnormalized and unsorted linear valuations on an arbitrary interval $[l,h] \subseteq [0,1]$.

Convex conic representation of $U(v,[l,h])$. Through the next two theorems, we show that the set $U(v,[l,h])$, where $0 \leq l \leq h \leq 1$, can be represented by $O(n)$ number of linear and quadratic constraints using $O(n)$ auxiliary variables. We start with the case of sorted and normalized (but not necessarily distinct) v_i on $[0,1]$.

Theorem 4. *Let $v_i(\theta) = c_i\theta + d_i$, $\theta \in [0,1]$ and $v_i([0,1]) = \frac{c_i}{2} + d_i = 1$ for all i. Let $2 \geq d_1 \geq \cdots \geq d_n \geq 0$. Denote by $\mathcal{C} = \{(t_1, t_2) \in \mathbb{R}^2 : t_1^2 \leq t_2\}$, and let*

$$G_i = \begin{bmatrix} d_i & \frac{c_i}{2} \\ -d_{i+1} & -\frac{c_{i+1}}{2} \end{bmatrix} \in \mathbb{R}^{2\times 2},\ \forall\, i \in [n-1].$$

Then, a vector of utilities u is in $U(v,[0,1])$ if and only if it is part of a feasible solution to the following constraints together with auxiliary (real) variables z_i, w_i, s_i, t_i:

$$\begin{aligned} &u = (u_1, \ldots, u_n) \geq 0, \quad u_1 \leq z_1,\ u_i \leq z_i + w_{i-1}, \forall\, i = 2,\ldots,n-1,\ u_n \leq 1 + w_{n-1}, \\ &G_i \begin{bmatrix} s_i \\ t_i \end{bmatrix} = \begin{bmatrix} z_i \\ w_i \end{bmatrix},\ \begin{bmatrix} s_i \\ t_i \end{bmatrix} \in \mathcal{C},\ \forall\, i \in [n-1], \quad z_i \in [0,1], w_i \in [-1,0], z_i + w_i \geq 0,\ \forall\, i \in [n-1]. \end{aligned}$$

Furthermore, the above constraints imply $0 \leq s_i \leq 1$, $0 \leq t_i \leq 1$.

The key observation used in the proof is to express the *Pareto frontier* of utilities as the image of a linear transformation of the product of quadratic curves $\{(t_1,t_2) : t_1^2 \leq t_2\}$. In short, the above theorem shows that U is the first n dimensions of a convex compact set in $\mathbb{R}^{n+4(n-1)}$ represented by $O(n)$ linear and quadratic constraints. Next, we illustrate the derivation of the above characterization via the case of $n = 2$ buyers.

Example 2 (Representation of $U(v,[0,1])$ with $n = 2$ buyers). Under the assumptions of Theorem 4 (linear, normalized v_i sorted by d_i in descending order), let $2 \geq d_1 > d_2 \geq 0$. By Lemma 7, for any feasible utilities $u \in U(v,[0,1])$, we can find $a \in [0,1]$ such that $u_i \leq v_i([0,a])$, $u_2 \leq v_2([a,1])$. Therefore,

$$U = U(v,[0,1]) = \left\{ (u_1,u_2) \geq 0 : \exists\, a \in [0,1] \text{ s.t. } u_1 \leq \frac{c_1}{2}a^2 + d_1 a,\, u_2 \leq \frac{c_2}{2}(1-a^2) + d_2(1-a) \right\}.$$

Clearly, $u \in U$ if and only if $u \geq 0$ and there exist z_1, w_1, a such that

$$u_1 \leq z_1,\ u_2 \leq 1 + w_1,\ z_1 \leq \frac{c_1}{2}a^2 + d_1 a,\ w_1 \leq -\frac{c_2}{2}a^2 - d_2 a,\ a \in [0,1].$$

Since $a \in [0,1]$ and $\frac{c_i}{2} + d_i = 1$, $i = 1,2$, the above inequalities imply the bounds on (z_1, w_1):

$$\begin{aligned} &0 \leq u_1 \leq z_1 \leq \frac{c_1}{2} + d_1 \leq 1, \quad -1 \leq -\frac{c_2}{2} - d_2 \leq w_1 \leq u_2 - 1 \leq \frac{c_2}{2} + d_2 - 1 = 0, \\ &z_1 + w_1 = (d_1 - d_2)a + ((1-d_1) - (1-d_2))\, a^2 = (d_1 - d_2)(a - a^2) \geq 0, \end{aligned}$$

where the last inequality uses $d_1 \geq d_2$ and $1 \geq a \geq a^2 \geq 0$. Therefore, $u \in U$ if and only if there exists (z_1, w_1) and a such that

$$\begin{aligned} &u_1 \leq z_1,\ u_2 \leq 1 + w_1, \quad z_1 \leq \frac{c_1}{2}a^2 + d_1 a,\ w_1 \leq -\frac{c_2}{2}a^2 - d_2 a,\ a \in [0,1], \\ &0 \leq z_1 \leq 1,\ -1 \leq w_1 \leq 0,\ z_1 + w_1 \geq 0. \end{aligned} \tag{6}$$

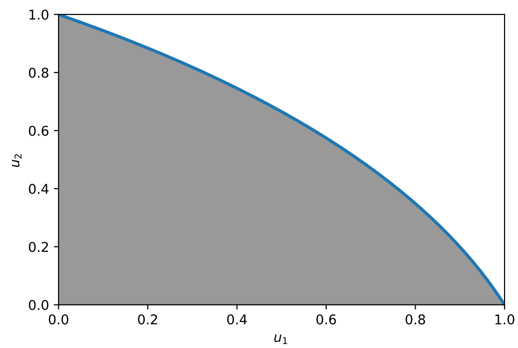

Figure 1 **An illustration of the Pareto frontier U° (the curve connecting $(0,1)$ and $(1,0)$) and $U(v,[0,1])$ (the set of feasible utilities, i.e., the shaded region bounded by the Pareto frontier and two axes) in Example 2. Here, $d_1 = 1.5$, $d_2 = 0.8$ and $c_i = 2(1-d_i)$, $i = 1, 2$.**

Let the Pareto frontier be $U^\circ = \left\{\left(\frac{c_1}{2}a^2 + \frac{d_1}{2}, \frac{c_2}{2}(1-a^2) + \frac{d_2}{2}(1-a)\right) : a \in [0,1]\right\}$. For specific values of d_1, d_2, the Pareto frontier U° and the set U are illustrated in Figure 1, where U is the region bounded by the axes and U°. The Pareto frontier U° is in fact the image under an affine transformation of a parabola segment $U^\circ = (0,1) + \Gamma$, where the parabola segment Γ is $\Gamma = \left\{\left(\frac{c_1}{2}a^2 + d_1 a, -\frac{c_2}{2}a^2 - d_2 a\right) : a \in [0,1]\right\} = G_1\{(s_1, s_1^2) : s_1 \in [0,1]\}$. The feasible region for (z_1, w_1) given by (6) can be described as $S_1 = \operatorname{conv}(\Gamma) = \operatorname{conv}(\bar{\Gamma}) \cap T_1$, where $\bar{\Gamma}$ is the *entire* curve of Γ ($a \in [0,1]$ in the parametric description replaced by $a \in \mathbb{R}$) and $T_1 = \{(z_1, w_1) \in [0,1] \times [-1, 0] : z_1 + w_1 \geq 0\}$ is the triangular region given by the implied linear inequalities in (6). Since $\bar{\Gamma} = G_1\{(s_1, s_1^2) : s_1 \in \mathbb{R}\}$, we have $\operatorname{conv}(\bar{\Gamma}) = G_1 \operatorname{conv}(\{(s_1, s_1^2) : s_1 \in \mathbb{R}\}) = G_1 \mathcal{C}$. Hence, the feasible region for (z_1, w_1) can be described by the linear inequalities in T_1 and $(z_1, w_1) = G_1(s_1, t_1)$, $(s_1, t_1) \in \mathcal{C}$.

We have arrived at all linear and quadratic constraints given in Theorem 4 for the case of $n = 2$. Furthermore, we can also verify that $S_1 = G_1 \operatorname{conv}(\{(s_1, s_1^2) : s_1 \in [0,1]\})$, where $\operatorname{conv}(\{(s_1, s_1^2) : s_1 \in [0,1]\}) = \{(s_1, t_1) : 0 \leq s_1 \leq 1, 0 \leq t_1 \leq 1, s_1^2 \leq t_1\}$. Therefore, we also know that $0 \leq s_1, t_1 \leq 1$.

4.2. The case of general linear v_i

When v_i are defined on a general interval $[l, h] \subseteq [0,1]$ and are neither normalized nor sorted in any particular order, we can still utilize Theorem 4 to characterize the set of feasible utilities. In this case, $U(v, [l,h])$ is the image of $U(v', [0,1])$ under a simple linear transformation, where $v' = (v_1', \ldots, v_n')$ is the corresponding set of linear valuations defined on $[0,1]$ that satisfy the assumptions of Theorem 4, i.e., normalized and sorted in descending order of their intercepts. The linear transformation is the composition of a permutation (represented by a permutation matrix) and a scaling (represented by a diagonal matrix). The following theorem describes the characterization of $U(v, [l,u])$ in detail.

THEOREM 5. *Let $v_i(\theta) = \frac{c_i}{2}\theta + d_i \geq 0$, $\theta \in [l,h] \subseteq [0,1]$. Assume*

$$\Lambda_i = v_i([l,h]) = \frac{c_i}{2}(h^2 - l^2) + d_i(h-l) > 0.$$

Denote $\hat{c}_i = (h-l)^2 c_i/\Lambda_i$, $\hat{d}_i = (h-l)(c_i l + d_i)/\Lambda_i$. Then, for each i, the normalized valuation $\hat{v}_i(\theta) = \hat{c}_i(\theta) + \hat{d}_i$ satisfies $\hat{v}_i([0,1]) = 1$. Let σ be a permutation of $[n]$ that sorts $\hat{d}_i$ in descending order, i.e., $\hat{d}_{\sigma(1)} \geq \cdots \geq \hat{d}_{\sigma(n)}$. Let $P \in \{0,1\}^{n\times n}$ be the (inverse) permutation matrix of σ with entries $P_{ij} = \mathbb{I}\{i = \sigma(j)\}$. Let $D \in \mathbb{R}^{n\times n}$ be a diagonal matrix with $D_{ii} = \Lambda_i := \frac{c_i}{2}(h^2 - l^2) + d_i(h-l)$. Then, $U(v,[l,h]) = DP\hat{U}^\sigma = \{DPu : u \in \hat{U}^\sigma\}$, where $\hat{U}^\sigma = U(\hat{v}_\sigma, [0,1])$ is the set of feasible utilities given (sorted and normalized) valuations $\hat{v}_{\sigma(j)}$, $j \in [n]$ on $[0,1]$ with coefficients $\hat{c}_{\sigma(j)}, \hat{d}_{\sigma(j)}$. In other words, for any feasible utility vector $u \in U(v,[l,h])$, there exists $u' \in \hat{U}^\sigma$ such that $u_{\sigma(j)} = \Lambda_{\sigma(j)} u'_j$, $\forall j \in [n]$, and vice versa.

In Theorem 5, the set $\hat{U}^\sigma$ satisfies the assumptions in Theorem 4 (i.e., normalized valuations sorted by intercepts) and hence can be represented by $O(n)$ linear and quadratic constraints with $O(n)$ auxiliary variables. Therefore, so is $U(v,[l,h])$, which requires n additional simple linear constraints, each only involving two utility variables $u_{\sigma(j)}$ (for the true utility) and u'_j (for the utility in the permuted and normalized space).

Recovering a pure allocation given feasible utilities (revisited). In §4.1, we showed how to find the breakpoints described in Lemma 7 given normalized and sorted v_i. Algorithm 1 solves the more general case of unnormalized, unsorted v_i defined on an arbitrary interval $[l,h] \subseteq [0,1]$. Its

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Algorithm 1 Partition $[l,h]$ according to $u \in U(v,[l,h])$

Input: coefficients c_i, d_i s.t. $v_i(\theta) = c_i\theta + d_i \geq 0$, $\theta \in [l,h]$, vector of feasible utilities $u \in U(v,[l,h])$.

Compute Λ_i, $\hat{c}_i$, $\hat{d}_i$, $i \in [n]$ as in Theorem 5.

Sort $\hat{d}_i$ in descending order and let σ be a permutation of $[n]$ such that $\hat{d}_{\sigma(1)} \geq \cdots \geq \hat{d}_{\sigma(n)}$.

Set $l_{\sigma(1)} = h_{\sigma(0)} = l$ (define $\sigma(0) = 0$) and $h_{\sigma(n)} = h$.

For $j = 1, \ldots, n-1$:

- Set $i = \sigma(j)$ (buyer i gets the jth interval, with left endpoint $l_i = l_{\sigma(j)} = h_{\sigma(j-1)}$).
- Find the right endpoint $h_i \in [l_i, h]$ such that $v_i([l_i, h_i]) = u_i$ by solving the following quadratic equation (the "cut" operation) $\frac{c_i}{2}(h_i^2 - l_i^2) + d_i(h_i - l_i) = u_i$.
- Set $l_{\sigma(j+1)} = h_i$.

Return: $[l_i, h_i]$, $i \in [n]$ s.t. $v_i([l_i, h_i]) \geq u_i$, where equality holds for $i = \sigma(1), \ldots, \sigma(n-1)$.

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running time is clearly $O(n \log n)$ due to the sorting step. The correctness of Algorithm 1 is a direct consequence of Theorem 5 and Lemma 7. In more detail, Lemma 5 shows that we can partition $[0,1]$ according to the transformed valuations $\hat{v}_{\sigma(j)}$ and utility values $u_{\sigma(j)}/\Lambda_{\sigma(j)}$, and then to get an allocation of $[l,h]$ we linearly transform these intervals back to $[l,h]$ and assign the buyers in the same order. This can be done by directly partitioning $[l,h]$ in the order of the ordering σ according to the scaled intercepts $\hat{d}_i$.

Next, we give an example that illustrates the representation of $U(v,[l,h])$ in Theorem 5 as well as the execution of Algorithm 1. It is taken from Example 5 in § 4.5 for recovering pure equilibrium allocations: the interval $[l,h]$ corresponds to the second predefined interval $[a_1,a_2]$ of the piecewise linear valuations v_i in that example.

Example 3 (Representation of $U(v,[l,h])$ and Algorithm 1). Consider $n=4$ buyers with valuations $v_i(\theta)=c_i\theta+d_i\geq 0$, $\theta\in[l,h]=[0.3741,0.8147]$. Here, the coefficients c_i and d_i of the valuations are

$$c=(1.6253,-0.2604,-1.7084,2.5419),\quad d=(-0.2972,0.4864,1.3919,0.6464).$$

The normalized coefficients $\hat{c}_i$, $\hat{d}_i$ and normalizing constants $\Lambda_i=v_i([l,h])$, as in Theorem 5, are

$$\hat{c}=(1.0708,-0.3460,-2.0,0.5192),$$
$$\hat{d}=(0.4646,1.1730,2.0,0.7404),$$
$$\Lambda=(0.2947,0.1461,0.1659,0.9506).$$

The order of descending $\hat{d}_i$ is $\sigma=(3,2,4,1)$, i.e., $\hat{d}_3\geq\hat{d}_2\geq\hat{d}_4\geq\hat{d}_1$. Therefore, the sorted arrays of $\hat{c}_i,\hat{d}_i$ are (e.g., $\hat{c}_{\sigma(1)}=\hat{c}_3=-2.0$):

$$\hat{c}_\sigma=(-2.0,-0.3460,0.5192,1.0708),\quad \hat{d}_\sigma=(2.0,1.1730,0.7404,0.4646).$$

Using $\hat{c},\hat{d}$, we get the transformed valuation $\hat{v}_{\sigma(j)}(\theta)=\hat{c}_{\sigma(j)}\theta+\hat{d}_{\sigma(j)}\geq 0$ for each $j=1,\ldots,4$, with a normalized value such that $\hat{v}_{\sigma(j)}([0,1])=1$. The elements of the diagonal matrix D are $D_{jj}=\Lambda_{\sigma(j)}=v_{\sigma(j)}([l,h])$, i.e.,

$$(D_{11},D_{22},D_{33},D_{44})=(\Lambda_3,\Lambda_2,\Lambda_4,\Lambda_1)=(0.1659,0.1461,0.9506,0.2947).$$

The statement $U(v,[l,h])=DP\hat{U}^\sigma$ in Theorem 5 is as follows:

$$u\in U(v,[l,h])\ \Leftrightarrow\ \exists\, u'\in\hat{U}^\sigma \text{ s.t. } u_3=\Lambda_3u_1',\ u_2=\Lambda_2u_2',\ u_4=\Lambda_4u_3',\ u_1=\Lambda_1u_4'.$$

Here, the set $\hat{U}^\sigma$ can be represented by $O(n)$ linear and quadratic constraints with $O(n)$ auxiliary variables as in Theorem 4, since $\hat{v}_{\sigma(j)}$ are normalized on $[0,1]$ and $\hat{d}_{\sigma(1)}\geq\ldots\hat{d}_{\sigma(4)}$. Next, suppose we are given $u\in U(v,[l,h])$ and need to find a partition of $[l,h]$ into 4 intervals that achieve u_i. Here, we use $u=(0.0000,0.0732,0.0036,0.5646)\in U(v,[l,h])$ (which is the equilibrium utility achieved on this interval in Example 5). We briefly describe the execution of Algorithm 1 on this instance. (a) The allocation order is $\sigma = (3,2,4,1)$, i.e. decreasing in $\hat{d}_i$. (b) Since $\sigma(1) = 3$, set $l_3 = l$ and find $h_3 \in [l,h]$ such that $v_3([l,h_3]) = u_3 = 0.0036$. This gives $[l_3,h_3] = [0.3741, 0.3789]$. Set $l_{\sigma(2)} = l_2 = h_3 = 0.3789$. (c) For $\sigma(2) = 2$ we find $h_2 \in [l_2,h]$ such that $v_2([l_2,h_2]) = u_2$, which gives $[l_2,h_2] = [0.3789, 0.5815]$. (d) Similarly, the next buyer is $\sigma(3) = 4$ and $[l_4,h_4] = [0.5815, 0.8147]$. (e) The last buyer is $\sigma(4) = 1$, which gets an empty interval ($l_1 = h_1 = h = 0.8147$), resulting in zero utility $u_1 = 0$.

4.3. Convex conic reformulation of $(\mathcal{P}_{\mathrm{EG}})$

We now show how to handle the general case of piecewise linear valuations on $[0,1]$. We first give a convex program whose variables are the utilities buyers receive from the subintervals defined by their linear pieces. Formally, the item space is $\Theta = [0,1]$ and each v_i is a piecewise linear valuation on $[0,1]$. Let the union of their breakpoints be $a_0 = 0 \le a_1 \le \cdots \le a_{K-1} \le a_K = 1$. For each buyer $i \in [n]$ and each subinterval $k \in [K]$, v_i is linear on $[a_{k-1}, a_k]$: $v_i(\theta) = c_{ik}\theta + d_{ik},\ \ \theta \in [a_{k-1}, a_k]$. For each k, let the set of feasible utilities with item space $[a_{k-1}, a_k]$ and valuations v_i be $U_k := U(v, [a_{k-1}, a_k])$, as defined in (1). Consider the following convex program, whose variables denote how much utility each buyer i receives from each linear segment k:

$$\sup_{(u_{ik}) \in \mathbb{R}_+^{n\times K}} \sum_{i=1}^n B_i \log\left(\sum_{k=1}^K u_{ik}\right) \text{ s.t. } (u_{1k}, \ldots, u_{nk}) \in U_k,\ \forall\, k \in [K]. \tag{7}$$

By Theorem 5, the set U_k is the image of a permutation and a scaling of another set of feasible utilities spanned by normalized valuations on $[0,1]$, that is, $U_k = D^k P^k \hat{U}_k$. Here, $\hat{U}_k$ is the set of feasible utilities spanned by $\hat{v}_{ik}$, where v_{ik} is the restriction of v_i on interval $[a_{k-1}, a_k]$ and $\hat{v}_{ik}$ (defined on $[0,1]$) are the normalized and sorted versions of v_{ik} as described in Theorem 5. We will adopt the notation of Theorem 5, but since we need a set $\hat{U}_k$ for each piece k, we use an additional subscript k to refer to the kth copy corresponding to the subinterval $[l,h] = [a_{k-1}, a_k]$. Thus, D^k is a diagonal matrix with diagonal entries $\Lambda_{ik} = v_i([a_{k-1}, a_k])$ (this corresponds to Λ_i in Theorem 5 with $[l,h] = [a_{k-1}, a_k]$), P^k is a permutation matrix corresponding to the permutation σ^k that sorts $\hat{d}_{ik}$ (this corresponds to the intercepts $\hat{d}_i$ in Theorem 5) in descending order. Both D^k and P^k depend on (c_{ik}, d_{ik}). The set $\hat{U}_k$ is the set of feasible utilities of normalized valuations as given in Theorem 4. We will use $(s_{ik}, t_{ik}, w_{ik}, z_{ik}, \hat{u}_{ik})$ to denote the variables $(s_i, t_i, w_i, z_i, u_i)$ in Theorem 4 corresponding to $\hat{U}_k$.

Next we will describe how the convex program 7 can be solved efficiently in practice using industry-grade interior-point methods. To that end, denote the 3-dimensional second-order (quadratic) cone as $\mathcal{L} = \{(t_1,t_2,t_3) : t_1 \ge \sqrt{t_2^2 + t_3^2}\}$ and the 3-dimensional exponential cone $\mathcal{E}$ as the *closure* of the set $\mathcal{E} = \left\{(t_1, t_2, t_3) : e^{t_3/t_2} \le t_1/t_2,\, t_2 > 0\right\}$ (see, e.g., (Chares 2009), (Dahl and Andersen 2019), (Serrano 2015)). Define the following standard-form convex conic program (Skajaa and Ye 2015, Dahl and Andersen 2019, Nemirovski 2004):

$$f^* = \min\, c^\top x \text{ s.t. } Ax = b,\ x \in \mathcal{K} := \mathbb{R}_+^{n_1} \times \mathcal{L}^{n_2} \times \mathcal{E}^{n_3} \tag{8}$$

where $A \in \mathbb{R}^{m\times \bar{n}}$, $c \in \mathbb{R}^{\bar{n}}$, $b \in \mathbb{R}^m$, $\bar{n} = n_1 + 3n_2 + 3n_3$ (we use $\bar{n}$ to denote the dimension to distinguish it from n, the number of buyers). Problem (7) can be reformulated into (8) via standard techniques. The following lemma summarizes the facts about the said reformulation. In additional to the small dimensions $\bar{n} = O(nK)$, $m = O(nK)$, the reformulation also ensures $\mathrm{nnz}(A) = O(\bar{n})$, where $\mathrm{nnz}(A)$ denotes the number of nonzeros in the matrix A. A complete convex conic reformulation can be found in the proof of the lemma.

Theorem 6. *The supremum of* (7) *is attained and is equal to* z^*, *the supremum of* $(\mathcal{P}_{\mathrm{EG}})$. *For any optimal solution* (u_{ik}^*) *of* (7), $u_i^* := \sum_{k=1}^K u_{ik}^*$ *is the equilibrium utility of buyer* i. *Each interval* $[a_{k-1}, a_k]$ *can be divided into* n *a.e.-disjoint subintervals* $[l_{ik}, h_{ik}]$ *such that* $u_{ik}^* = v_i([l_{ik}, h_{ik}])$. *Hence,* $\Theta_i := \cup_{k=1}^K [l_{ik}, h_{ik}]$, $i \in [n]$ *is an equilibrium allocation. Problem* (7) *can be reformulated into the standard form* (8) *with dimensions* $n_1 = O(nK)$, $n_2 = O(nK)$, $n_3 = O(n)$, $m = O(nK)$ *and* $\mathrm{nnz}(A) = O(nK)$. *The minimum of the reformulation is* $f^* = -z^*$. *An optimal solution of the reformulation contains an optimal solution of* (7), *that is,* $(u_{ik}) \in \mathbb{R}_+^{n\times K}$ *such that* $(u_{1k}, \dots, u_{nK}) \in U_k$ *for all* k *and* $\sum_k u_{ik} = u_i^*$ *for all* i.

Solving (8) *using an interior-point method.* The standard-form problem (8), in which $\mathcal{K}$ is the product of a nonnegative orthant, second-order cones, and exponential cones, can be solved via off-the-shelf optimization software based on interior-point methods even for very large instances using the *Mosek* solver (Mosek 2010, Dahl and Andersen 2019). In fact, modern optimization software usually do not require such a standard-form input and allows more general input formats. Although the theoretical time complexity of interior-point methods for a general convex optimization problem with computable self-concordant barrier functions has been well-studied (see e.g. (Nesterov and Nemirovski 1994), (Nesterov et al. 2018, §5), (Nemirovski 2004, §4)), to the best of our knowledge, clear-cut polynomial-time complexity results are only available for the case of *self-scaled* cones, that is, linear programming (LP), second-order cone programming (SOCP) and semidefinite programming (SDP).

For general convex optimization beyond self-scaled cones, there exist bounds on the number of Newton iterations that are roughly of the form "$O\left(\sqrt{\nu}\log\frac{M}{\epsilon}\right)$"—where ν is the barrier parameter, ϵ is the tolerance level (for duality gap and infeasibility) and M is an instance-dependent constant—for "theoretical versions" of various interior-point methods applied to strictly feasible problems (see, e.g., (Nesterov and Nemirovski 1994, §3 and §6), (Nesterov et al. 2018, 5.3.4), (Nemirovski 2004, §4 and §7)). In the various forms of this type of bound, the constant M depends critically on the geometry of the problem and the instance data, such as closeness of the initial solution to the boundary of the feasible region (Nemirovski 2004, Theorems 4.5.1 and 7.4.1). Whether we can extend the polynomial time complexity results for self-scaled cones to the case of exponential cones is beyond the scope of this work. For our purpose, it remains a challenge to bound this constant using the market data B_i, c_{ik}, d_{ik} as well. Furthermore, we point out that mature interior-point optimization software rarely implements all components of a theoretically-convergent method. Instead, highly sophisticated numerical linear algebra methods and stepsizing heuristics are used to speed up and stabilize the computation of search directions (see e.g. (Toh et al. 2012, Sturm 1999)). These techniques, necessary for practically efficient implementations, often invalidate the theoretical complexity guarantees (Dahl and Andersen 2019, Skajaa and Ye 2015). In conclusion, at this time one cannot derive polynomial-time solvability of the piecewise linear problem directly from our conic reformulation in (8). Nevertheless, we remark that solving the reformulation (8) of (7) using industry-grade interior-point optimization software (such as Mosek) is a highly efficient and stable approach for computing a pure equilibrium allocation: Mosek easily solves problems with hundreds of variables and constraints. After computing an optimal solution (u^*_{ik}) of (7) using Mosek, a pure allocation that attains these utilities can easily be found with Algorithm 1.

4.4. A polynomial-time algorithm for computing equilibrium allocations

Due to the problems mentioned in the previous section regarding polynomial-time solvability of conic programs involving exponential cones, we next investigate polynomial-time solvability of our problem via alternative methods. The results in this section are primarily of theoretical interest; in practice the conic program from the previous section is extremely efficient and preferable.

We are going to present an algorithm that finds an ϵ-approximate pure equilibrium allocation in Poly $\left(n, k, \log\frac{1}{\epsilon}\right)$ time. This method computes approximate equilibrium utility prices using the ellipsoid method for convex optimization and constructs a pure allocation via Algorithm 1.

Recall our assumptions that $\|B\|_1 = 1$ and $v_i([0,1]) = 1$ for all i; the unit interval is divided into K subintervals by breakpoints $a_0 = 0 < a_1 < \cdots < a_K = 1$; for each i, buyer i's valuation is $v_i(\theta) = c_{ik}\theta + d_{ik}$ on the kth subinterval $[a_{k-1}, a_k]$.

The ellipsoid method for convex optimizaton. First, we describe the ellipsoid method for generic convex optimization. We refer the readers to the survey (Bland et al. 1981) for the history of development of ellipsoid methods and further references. Here, we adopt the exposition in (Ben-Tal and Nemirovski 2019). Consider the following generic convex program (Ben-Tal and Nemirovski 2019, §4.1.4):

$$f^* := \min_x f(x) \text{ s.t. } x \in X \tag{9}$$

where f is convex and continuous (and hence subdifferentiable) on a compact region $X \subseteq \mathbb{R}^n$. Assume we have access to the following oracles: (i) The *separation* oracle $\mathcal{S}$: given any $x \in \mathbb{R}^n$, either report $x \in \operatorname{int} X$ or return a $g \neq 0$ (representing a separating hyperplane) such that $\langle g, x\rangle \geq \langle g, y\rangle$ for any $y \in X$ and (ii) the *first-order* or *subgradient* oracle $\mathcal{G}$: given $x \in \operatorname{int} X$ (the interior of X), return a subgradient $f'(x)$ of f at x, that is, $f(y) \geq f(x) + \langle f'(x), y - x\rangle$ for any y. The time complexity of the ellipsoid method is as follows.

THEOREM 7. *(Ben-Tal and Nemirovski 2019, Theorem 4.1.2) Let $V = \max_{x\in X} f(x) - f^*$, $R = \sup_{x\in X} \|x\|$, and $r > 0$ be the radius of a Euclidean ball contained in X. For any $\epsilon > 0$, a solution $x_\epsilon \in X$ such that $f(x_\epsilon) \leq f^* + \epsilon$ can be computed using no more than $N(\epsilon)$ calls to $\mathcal{S}$ and $\mathcal{G}$, followed by no more than $O(1)n^2 N(\epsilon)$ arithmetic operations to process the outputs of the oracles, where $N(\epsilon) = O(1)n^2 \log\left(2 + \frac{VR}{\epsilon r}\right)$.*

Using the above theorem, we can show that problem (7) can be solved in polynomial time.

THEOREM 8. *For any $0 < \epsilon < 1$, we can compute $(u_{ik}) \in \mathbb{R}_+^{n\times K}$ such that $(u_{1k}, \ldots, u_{nk}) \in U(v, [a_{k-1}, a_k])$, $\forall\, k \in [K]$ and $u_i := \sum_i u_{ik} \geq u_i^* - \epsilon$, $\forall\, i \in [n]$ in $O\left(n^4 K \left(\log(nK) + \log\frac{\kappa}{\epsilon}\right)\right)$ time, where $\kappa = \frac{1}{\min_i B_i}$. Furthermore, a pure equilibrium allocation $\{\Theta_i\}$, where buyer i receives an allocation $\Theta_i = \cup_k [l_{ik}, u_{ik}]$ of at most K intervals, with value $v_i(\Theta_i) \geq u_i^* - \epsilon$, can be constructed in $O(nK)$ additional time.*

At a high level, in order to use Theorem 7 to solve our problem (7) in polynomial time, we need to cast it into the form (9), construct efficient separation and first-order oracles, and bound the ratio $\frac{VR}{\epsilon r}$. To this end, all variables involved have absolute values bounded above by either absolute constants or κ. In order to ensure a nonzero radius $r > 0$ of the feasible region, we can simply "ϵ-perturb" the linear constraints and then "ϵ-discount" the solution obtained to ensure feasibility w.r.t. the original constraints.

We remark that the constant κ in Theorem 8 can be viewed as a "condition number" of problem (7): given a fixed accuracy level ϵ, the running time of the algorithm scales logarithmically (via the term $\log\kappa$) as κ grows. This aligns with our intuition about a "second-order" method such as an interior-point method or the ellipsoid method. In contrast, the running time of a first-order method—such as projected gradient descent—usually scales polynomially in the problem's condition number.

4.5. Numerical examples and experiments

To round out this section, we describe two specific examples of computing a pure equilibrium over $[0, 1]$ given linear and piecewise linear valuations, respectively. Then, we run our proposed method end-to-end on randomly generated large instances to demonstrate its scalability. More details can be found in EC.3.

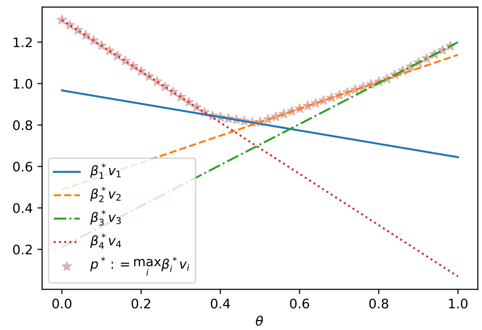

Figure 2 **The equilibrium prices p^* and $\beta_i^* v_i$ for 4 buyers with linear (normalized) v_i on $[0,1]$ (Example 4). The stars denote p^*, whose linear pieces correspond to the winning segments for each buyer.**

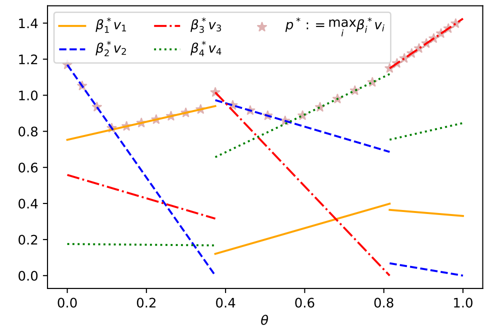

Figure 3 **The equilibrium prices p^* and $\beta_i^* v_i$ for $n=4$ buyers with piecewise linear v_i on $[0,1]$ (Example 5). The stars denote p^*, whose linear pieces correspond to the winning segments for each buyer. Note that each v_i, and hence p^*, are p.w.l. They are not necessarily continuous.**

Example 4 (Linear v_i). Let there be four buyers with budgets $B = (0.1, 0.3, 0.2, 0.4)$ and normalized linear v_i with intercepts $d = (1.2, 0.6, 0.3, 1.9)$, which implies $c = (-0.4, 0.8, 1.4, -1.8)$. Ordering the buyers by decreasing intercept gives $\sigma = (4, 1, 2, 3)$. Figure 2 illustrates the equilibrium prices p^* and the scaled valuation $\beta_i^* v_i$ for each buyer. Buyer 4 receives the leftmost interval $[0, 0.3713]$, buyer 1 receives $[0.3713, 0.4921]$, and so on. For each buyer, its allocated interval $[l_i, h_i]$ is precisely the segment where it "wins", i.e., $[l_i, h_i] = \{p^* = \beta_i^* v_i\}$. Since all buyers have distinct d_i and positive budgets, this is the unique pure allocation. The fact that this pure allocation is indeed an equilibrium allocation (with the corresponding β^*) can also be verified using Corollary 3.

Example 5 (Piecewise linear v_i). Let there be $n = 4$ buyers, each with piecewise linear valuations v_i with $K = 3$ pieces. Each buyers' linear pieces share endpoints $0 = a_0 < a_1 < a_2 < a_3 = 1$, with $a_1 = 0.3741$, $a_2 = 0.8147$. Solving the convex program (7) gives equilibrium utilities u_{ik}^* for all i, k, where u_{ik}^* is the amount of utility buyer i gets from segment $[a_{k-1}, a_k]$ (which can be 0 for some buyer-segment pairs). Similarly to the linear case above, we divide each segment $[a_{k-1}, a_k]$ among the buyers according to their u_{ik}^* and their ordering according $\hat{d}_i$ on interval k (c.f. Lemma 7). In the final pure allocation, each buyer i gets a union of at most K intervals, one from each $[a_{k-1}, a_k]$, which is a subset of its winning set $\{p^* = \beta_i^* v_i\}$. The solution is illustrated in Figure 3 using $\beta_i^* v_i$ and p^*. For example, on $[a_0, a_1]$, since $p^* > \beta_3^* v_3$ everywhere on the segment, buyer 3 does not get allocated anything from this interval, and thus $u_{31}^* = 0$. The same is true for buyer 4 on $[a_0, a_1]$.

n \ K	50	100	150	200
50	1.53 ± 0.05	3.29 ± 0.14	5.57 ± 0.91	6.49 ± 0.18
100	3.34 ± 0.14	6.50 ± 0.20	10.36 ± 0.40	13.62 ± 0.48
150	4.84 ± 0.12	9.86 ± 0.18	16.68 ± 1.46	20.50 ± 0.22
200	6.44 ± 0.07	13.18 ± 0.23	20.12 ± 0.33	28.65 ± 1.14

Table 1 Running times for each (n, K), mean and standard error across 8 seeds

Large-scale experiments. Next, we generate random instances with varying values of n and K. For each (n,K) combination and each random seed value in $\{1,\dots,8\}$, we perform the conic reformulation, solve the resulting convex program using Mosek and construct pure allocations based on the solution (u^*_{ik}). For each (n,K) combination, we record the mean running time and the standard error. The results are presented in Table 1. As can be seen, for fixed n, the running time scales approximately linearly in K, and similarly running time scales linearly with n for a fixed K. Thus our model should be scalable to very large instances. We remark that, in the experiment, around 2/3 of the running time arises from "model building", that is, constructing the sparse matrices and interacting with Mosek through its API, even after writing a highly vectorized implementation. Only around 1/3 of the time arises from "optimization", that is, running the interior-point method through `Model.solve()`. The time for constructing pure allocations from (u^*_{ik}) is negligible compared to model building and computation. As an additional note for practitioners, we also tried the CVXPY modeling language: it was vastly slower at reformulating the model before calling Mosek.

5. Stochastic optimization for general item spaces and valuations

Here, we consider the case of general valuations v_i on a convex compact set $\Theta \subseteq \mathbb{R}^d$ and show that we can use stochastic first-order methods to find approximate equilibrium utility prices $\tilde{\beta} \approx \beta^*$, which then gives approximate equilibrium prices $\tilde{p} = \max_i \tilde{\beta}_i v_i \approx p^*$.

Efficient subgradient computation. Our method will rely on oracle access to stochastic subgradients of $\phi(\beta) = \langle \max_i \beta_i v_i, \mathbf{1}\rangle$, the first term in the objective of (2). From the proof of Lemma 2, we know that the function ϕ is finite, convex and continuous on $\mathbb{R}^n_+$. Hence, it is subdifferentiable on $\mathbb{R}^n_{++}$ (Ben-Tal and Nemirovski 2019, Proposition C.6.5). The next lemma shows that ϕ can be viewed as the expectation of a stochastic function, and that an unbiased stochastic subgradient of ϕ can easily be computed.

LEMMA 8. *Let $\phi(\beta,\theta) = \max_i \beta_i v_i(\theta)$. For any $\theta \in \Theta \subseteq \mathbb{R}^d$, a subgradient of $\phi(\cdot,\theta)$ at β is $g(\beta,\theta) = v_{i^*}(\theta)\mathbf{e}^{(i^*)}$, where $i^* \in \arg\max_i \beta_i v_i(\theta)$ (taking the smallest index if there is a tie). Hence, a subgradient of ϕ at β is $\phi'(\beta) = \int_\Theta g(\beta,\theta)d\theta = \mu(\Theta)\cdot \mathbb{E}_\theta g(\beta,\theta) \in \partial\phi(\beta)$, where the expectation is taken over the uniform distribution $\theta \sim \mathrm{Unif}(\Theta)$.*

Using Lemma 8, we can solve (2) using a stochastic first-order method that only requires oracle access to stochastic subgradients. The structure of this problem is particularly suitable for the *stochastic dual averaging* (SDA) algorithm (Xiao 2010, Nesterov 2009). It solves problems of the form:

$$\min_{\beta} \ \mathbb{E}_\theta f(\beta,\theta) + \Psi(\beta), \tag{10}$$

where Ψ is a convex function—often known as a *regularizer* in the context of machine learning—with a closed, nonempty domain $\operatorname{dom}\Psi = \{\beta : \Psi(\beta) < \infty\}$. Now, assume that Ψ is strongly convex, $\theta \sim \mathcal{D}$ is a random variable with distribution $\mathcal{D}$, and $f(\cdot,\theta)$ is convex and subdifferentiable on $\operatorname{dom}\Psi$ for all $\theta \in \Theta$. The algorithm is shown in Algorithm 2.

Algorithm 2 *Stochastic dual averaging (SDA)*

Initialize: Choose $\beta^1 \in \operatorname{dom}\Psi$ and $\bar{g}^0 = 0$

For $t = 1, 2, \dots$*:*

Sample $\theta_t \sim \mathcal{D}$ *and compute* $g^t \in \partial_\beta f(\beta, \theta_t)$

Update the dual average $\bar{g}^t = \frac{t-1}{t}\bar{g}^{t-1} + \frac{1}{t} g^t$

Update the iterate $\beta^{t+1} = \arg\min_\beta \{\langle \bar{g}^t, \beta\rangle + \Psi(\beta)\} \quad (*)$

To solve (2), we set $f(\beta,\theta) = \max_i \beta_i v_i(\theta)$, $\mathcal{D} = \operatorname{Unif}(\Theta)$, where we assume $\mu(\Theta) = 1$ w.l.o.g. (otherwise, we can "shrink" the item space Θ by a scalar and consider $\Theta' = \{\alpha\theta : \theta \in \Theta\}$ or multiply the underlying measure μ by a scalar between 0 and 1; valuations v_i are then replaced by $v_i'(\theta') = v_i(\alpha\theta)$). Then, $\mathbb{E}_\theta[\max_i \beta_i v_i(\theta)] = \langle \max_i \beta_i v_i, \mathbf{1}\rangle$. By Lemma 8, we can choose $g^t = g(\beta,\theta_t) \in \partial_\beta f(\beta,\theta_t)$. Recall the bounds on β^* in Lemma 4, i.e., $\underline{\beta}_i = B_i \le \beta_i^* \le \bar{\beta}_i = 1$. Let the regularizer be

$$\Psi(\beta) = \begin{cases} -\sum_i B_i \log \beta_i & \text{if } \beta \in [\underline{\beta}, \bar{\beta}], \\ \infty & \text{o.w.} \end{cases}$$

Clearly, $\operatorname{dom}\Psi = [\underline{\beta}, \bar{\beta}]$ is closed and nonempty. Given these specifications, in Algorithm 2, the step $(*)$ yields a simple, explicit update: at iteration t, compute $\beta_i^{t+1} = \Pi_{[\underline{\beta}_i, \bar{\beta}_i]}\left(\frac{B_i}{\bar{g}_i^t}\right)$, $i \in [n]$, where $\Pi_{[a,b]}(c) = \min\{\max\{a,c\}, b\}$ is the projection onto a closed interval. This can be derived easily from its first-order optimality condition. Using the convergence results in (Xiao 2010) for strongly convex Ψ, we can show that the uniform average of all β^t generated by SDA converges to β^* both in mean square error (MSE) and with high probability, under mild additional boundedness assumptions on v_i.

THEOREM 9. *Assume $v_i \in L^2(\Theta)$, that is, $\langle v_i^2, \mathbf{1}\rangle = \mathbb{E}_\theta[v_i(\theta)^2] < \infty$ for all i. Let $G^2 = \mathbb{E}_\theta[\max_i v_i(\theta)^2] < \infty$, $\sigma = \min_i B_i > 0$. Let $\tilde{\beta}^t := \frac{1}{t}\sum_{\tau=1}^t \beta^\tau$. Then,*

$$\mathbb{E}\|\beta^t - \beta^*\|^2 \le \frac{6+\log t}{t} \times \frac{G^2}{\sigma^2},\ \mathbb{E}\|\tilde{\beta}^t - \beta^*\|^2 \le \frac{6(1+\log t) + \frac{1}{2}(\log t)^2}{t} \times \frac{G^2}{\sigma^2}.$$

Further assume that $v_i \le G$ a.e. for all i. Then, for any $\delta > 0$, with probability at least $1 - 4\delta \log t$, we have $\|\tilde{\beta}^t - \beta^*\|^2 \le \frac{2M_t}{\sigma}$, where

$$M_t = \frac{\Delta_t}{t} + \frac{4G}{t}\sqrt{\frac{\Delta_t \log(1/\delta)}{\sigma}} + \max\left\{\frac{16G^2}{\sigma}, 6V\right\}\frac{\log(1/\delta)}{t},\ \Delta_t = \frac{G^2}{2\sigma}(6+\log t),\ V = n + \log\frac{1}{\sigma}.$$

It can be seen that the above bounds grow as the strong convexity modulus of the objective function $\sigma = \min_i B_i$ decreases. For the CEEI case, $B_i = 1/n$ for all i and $\sigma = \min_i B_i = 1/n$. For the general case of heterogeneous buyer budgets, $\sigma \le 1/n$.

6. Extension to quasilinear utilities

We now discuss how the results in previous sections—convex optimization characterizations, a finite-dimensional reformulation under piecewise linear utilities and convergence guarantees of stochastic optimization—generalize to the case where each buyer has a quasilinear utility function. For the finite-dimensional case, there is a natural extension of EG to QL utilities, as shown by Chen et al. (2007), Cole et al. (2017). Furthermore, Conitzer et al. (2019) showed that budget management in an auction market with first-price auctions can be computed with the QL variant of EG.

A QL utility is one such that cost is deducted from the utility, that is, $u_i(x_i) = \langle v_i - p, x_i\rangle$, where $p \in L_1(\Theta)_+$ is the vector of prices of all items. A market equilibrium under QL utilities (QLME) is a pair of allocations $x^* = (x_i) \in (L_1(\Theta)_+)^n$ and prices $p^* \in L_1(\Theta)_+$ such that (i) for each buyer i, their allocation is optimal: $x_i^* \in \arg\max\{\langle v_i - p^*, x_i\rangle : \langle p^*, x_i\rangle \le B_i,\ x_i \in L_\infty(\Theta)_+\}$ and (ii) market clears: $\langle p^*, \mathbf{1} - \sum_i x_i^*\rangle = 0$. In the QL case, we cannot normalize both valuations and budgets, since buyers' budgets have value outside the current market. Without loss of generality, we can only assume that $\|B\|_1 = 1$ and $v_i(\Theta) > 0$ for all i (all buyers' v_i and B_i must be scaled by the same constant).

Here, we consider the infinite-dimensional "primal" Eisenberg-Gale convex program:

$$\begin{aligned} &\sup \sum_i (B_i \log u_i - \delta_i) \\ &\text{s.t.}\ \ u_i \le \langle v_i, x_i\rangle + \delta_i,\ \forall\, i \in [n],\ \sum_i x_i \le \mathbf{1},\ u_i \ge 0,\ \delta_i \ge 0,\ x_i \in L_1(\Theta)_+,\ \forall\, i \in [n]. \end{aligned} \tag{$\mathcal{P}_{\mathrm{QLEG}}$}$$

The "dual" is

$$\inf \langle p, \mathbf{1}\rangle - \sum_i B_i \log \beta_i \quad \text{s.t.} \quad p \geq \beta_i v_i,\ \beta_i \leq 1, \forall\, i \in [n],\ p \in L_1(\Theta)_+,\ \beta \in \mathbb{R}^d_+. \tag{$\mathcal{D}_{\text{QLEG}}$}$$

As in our earlier results, the "primal" and "dual" terminology should only be understood intuitively; the programs are not derived from each other via duality. However, the next theorem shows that they indeed behave like duals.

THEOREM 10. *The following results hold regarding* $(\mathcal{P}_{\text{QLEG}})$ *and* $(\mathcal{D}_{\text{QLEG}})$.

(i) The supremum of $(\mathcal{P}_{\text{QLEG}})$ *is attained via an optimal solution* (x^*, δ^*), *in which* $x^* = (x_i^*)$ *is a pure allocation, that is,* $x_i^* = \mathbf{1}_{\Theta_i}$ *for a.e.-disjoint measurable subsets* $\Theta_i \subseteq \Theta$.

(ii) The infimum of $(\mathcal{D}_{\text{QLEG}})$ *is attained via an optimal solution* (p^*, β^*), *in which* $\beta^* \in \mathbb{R}^n_+$ *is unique and* $p^* = \max_i \beta_i^* v_i$ *a.e.*

(iii) Given a feasible (x^*, δ^*) *to* $(\mathcal{P}_{\text{QLEG}})$ *and a feasible* (p^*, β^*) *to* $(\mathcal{D}_{\text{QLEG}})$, *they are both optimal solutions of their respective convex programs if and only if the following KKT conditions hold:* $\langle p^*, \mathbf{1} - \sum_i x_i^*\rangle = 0$, $u_i^* := \frac{B_i}{\beta_i^*}$, $\delta_i^*(1 - \beta_i^*) = 0$, $\langle p^* - \beta_i^* v_i, x_i^*\rangle = 0$, $\forall\, i$.

(iv) A pair of allocations and prices $(x^*, p^*) \in (L_\infty(\Theta)_+)^n \times L_1(\Theta)_+$ *is a QLME if and only if there exists* $\delta^* \in \mathbb{R}^n_+$ *and* $\beta^* \in \mathbb{R}^n_+$ *such that* (x^*, δ^*) *and* (p^*, β^*) *are optimal solutions of* $(\mathcal{P}_{\text{QLEG}})$ *and* $(\mathcal{D}_{\text{QLEG}})$, *respectively.*

Note that u_i^* above does *not* correspond to the equilibrium utility of buyer i, which is $\langle v_i - p^*, x_i^*\rangle$. Instead, by the definition of QLME and the above theorem, for each buyer i, there are two possibilities at equilibrium ($\Leftrightarrow$ primal and dual optimality). (i) If $\beta_i^* < 1$, then $\delta_i^* = 0$ and $u_i^* = \langle v_i, x_i^*\rangle$ in $(\mathcal{P}_{\text{QLEG}})$. Since $\langle p^* - \beta_i^* v_i, x_i^*\rangle = 0$, the equilibrium utility is $\langle v_i - p^*, x_i\rangle = (1 - \beta_i^*)\langle v_i, x_i^*\rangle = (1 - \beta_i^*) u_i^*$. (ii) If $\beta_i^* = 1$, then $\langle p^* - \beta_i^* v_i, x_i^*\rangle = 0$ implies the equilibrium utility is $\langle v_i - p^*, x_i^*\rangle = 0$.

Tractable convex optimization under piecewise linear valuations over $[0,1]$. Similar to §4, we can reformulate $(\mathcal{P}_{\text{QLEG}})$ into a tractable convex program using the same characterization of the set of feasible utilities u_i (which does not take prices into account) in Theorems 4 and (5). To reconstruct a pure allocation that achieves the equilibrium utilities, run Algorithm 1 on the subintervals corresponding to the linear pieces of the valuations.

Stochastic optimization. Similar to the case of linear utilities (Lemma 4), we can establish bounds on equilibrium quantities such as the equilibrium utility prices β^*. For a finite-dimensional Fisher market with buyers having QL utilities, (Gao and Kroer 2020, Lemma 2) gives bounds on equilibrium prices. Similar to their proof, we can show that $u_i^* \leq v_i(\Theta) + B_i \leq 1 + B_i$, i.e., $\beta_i^* = \frac{B_i}{u_i^*} \geq \frac{B_i}{v_i(\Theta) + B_i} > 0$. It follows that we can add bounds (together with the existing bound $\beta_i \leq 1$) $\frac{B_i}{v_i(\Theta)+B_i} \leq \beta_i \leq 1$, to $(\mathcal{D}_{\text{QLEG}})$ without affecting its (unique) optimal solution β^*. Hence, completely analogous to the linear case discussed in §5, we can use SDA to solve $(\mathcal{D}_{\text{QLEG}})$ with similar convergence guarantees.

7. Summary, discussion and future research

Motivated by applications in ad auctions and fair recommender systems, we considered a Fisher market with a continuum of items and the concept of a market equilibrium in this setting. By extending the finite-dimensional Eisenberg-Gale convex program and its dual, we proposed convex programs whose optimal solutions are ME, and vice versa. Optimality conditions for the convex programs parallel various structural properties of a market equilibrium. Due to the limitations of general duality theory for optimization over infinite-dimensional vector spaces, we established these properties via directly exploiting the problem structure. In particular, we showed that, under a continuum of items, a pure market equilibrium must exist, and an equilibrium allocation is guaranteed to be Pareto optimal, envy-free and proportional. Hence, when all buyers have the same budget, a pure equilibrium allocation is a fair division. Under piecewise linear buyer valuations over a closed interval, we showed that the infinite-dimensional Eisenberg-Gale convex program ($\mathcal{P}_{\rm EG}$) can be reformulated as a finite-dimensional convex program with linear and quadratic constraints, via simple characterizations of the set of feasible utilities and a sequence of reformulations. This yielded a highly scalable approach for computing a fair division under piecewise linear buyer valuations, and the first polynomial-time algorithm for this problem. We also showed that, for more general valuations, the finite-dimensional convex program (2) in utility prices β can be solved via subgradient-based stochastic optimization, for which we established mean-square convergence and high-probability convergence guarantees. Finally, we showed that most of the above results also extend to the case of quasilinear utilities: a class of utilities relevant to characterizing pacing equilibria in auction markets.

For future research, we would like to consider tractable convex optimization formulations for a multidimensional item space $\Theta = [a_i, b_i]^d$ representing linear features of items. This has application in low-rank market models in ad auctions with budget constraints (Conitzer et al. 2019). There, a low-rank model typically assumes a distribution over the space of possible item types (which captures the extremely large space of possible impressions). Modeling it as a finite-dimensional Fisher markets require discretizing the space of items, leading to a huge number of items. Our approach can potentially provide a much more compact convex optimization reformulation, and scale much better in both the number of items and the feature dimension. Another interesting direction is to find other structured classes of valuations for which tractable reformulations similar to the piecewise linear case exist.

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