Robust Choice Model

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|N| = number of purchases made by ONE customer
|J| = number of product choices
dimension of β = number of attributes
dimension of α = |J|

Let $\hat{j}_n = \sum_{j \in J} j \times y_{nj}$, where $y_{nj} \in \{0, 1\}$ denotes the observed choice. We further denote $\boldsymbol{Z}^j = (z_{1j}, z_{2j}, \ldots, z_{|N|j})'$ be a vector in $\Re^{|N|}$ and matrix $\boldsymbol{Z} = (\boldsymbol{Z}^j)_{j \in J}$.

$$\max_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \quad \min_{\boldsymbol{Z} \in \mathscr{Z}(\Gamma)} \sum_{n \in N} \sum_{j \in J} z_{nj} \ln \frac{\exp(\alpha_j + x'_{nj}\boldsymbol{\beta})}{\sum_{\ell \in J} \exp(\alpha_\ell + x'_{n\ell}\boldsymbol{\beta})} \tag{1}$$

where

$$\mathscr{Z}(\Gamma) = \left\{ \boldsymbol{Z} \in \Re^{|N| \times |J|} \;\middle|\; \begin{array}{l} \sum_{j \in J} z_{nj} = 1, \quad n \in N, \\ \sum_{n \in N} z_{n\hat{j}_n} \geq |N| - \Gamma, \\ z_{nj} \geq 0, \quad n \in N, j \in J \end{array} \right\}.$$

The above problem is equivalent to

$$\max_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \quad G(\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{2}$$

and

$$\begin{aligned} G(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \quad & \min \quad \sum_{n \in N} \sum_{j \in J} z_{nj} \ln \frac{\exp(\alpha_j + x'_{nj}\boldsymbol{\beta})}{\sum_{\ell \in J} \exp(\alpha_\ell + x'_{n\ell}\boldsymbol{\beta})} \\ & \text{s.t.} \quad \boldsymbol{Z} \in \mathscr{Z}(\Gamma). \end{aligned} \tag{3}$$

Problem (3) can be written as

$$\begin{aligned} \min \quad & \sum_{n \in N} \sum_{j \in J} z_{nj} \left(-\ln \sum_{\ell \in J} \exp(\alpha_\ell - \alpha_j + (x_{n\ell} - x_{nj})'\boldsymbol{\beta}) \right) \\ \text{s.t.} \quad & \sum_{j \in J} z_{nj} = 1, \quad n \in N, \\ & \sum_{n \in N} z_{n\hat{j}_n} \geq |N| - \Gamma, \\ & z_{nj} \geq 0, \quad n \in N, j \in J. \end{aligned} \tag{4}$$

The dual of problem (4) is

$$
\begin{aligned}
\max \quad & \sum_{n \in N} a_n + b(|N| - \Gamma) \\
\text{s.t.} \quad & a_n + b \leq -\ln \textstyle\sum_{\ell \in J} \exp(\alpha_\ell - \alpha_{\hat{\jmath}_n} + (x_{n\ell} - x_{n\hat{\jmath}_n})'\boldsymbol{\beta}), \quad n \in N, \\
& a_n \leq -\ln \textstyle\sum_{\ell \in J} \exp(\alpha_\ell - \alpha_j + (x_{n\ell} - x_{nj})'\boldsymbol{\beta}), \quad j \in J \setminus \{\hat{\jmath}_n\}, \quad n \in N, \\
& b \geq 0.
\end{aligned}
\tag{5}
$$