Bernstein approximation of chance constrained problems: an example

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Abstract

We study the example in [2] in more detail and repeat the computation using the new conic program solver.

The model

We describe the chance constrained problem in detail. As in [2], consider the following chance constrained program

$$\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1 \cdots, x_n \geq 0}} \quad (\tau - 1) \quad \text{s.t.} \;\; \mathbb{P}\left(\tau > \sum_{j=0}^{n} r_j x_j\right) \leq \alpha, \; \sum_{j=0}^{n} x_j \leq 1 \tag{1}$$

where $\alpha \in [0, 1]$ is a given constant. The assumptions are

1. The returns $r_0, r_1, \cdots, r_n$ satisfy $r_0 = 1$ and $\mathbb{E}(r_i) = 1 + \rho_i$ with $0 \leq \rho_1 \leq \cdots \leq \rho_n$.
2. For $1 \leq j \leq n$ and $1 \leq l \leq q$, one has $r_j = \eta_j + \sum_{l=1}^{q} \gamma_{jl}\zeta_l$ where $\eta_j \sim \mathcal{LN}(\mu_j, \sigma_j^2)$ (the individual noises) and $\zeta_l \sim \mathcal{LN}(\nu_l, \theta_l^2)$. All η_j and ζ_l are independent of each other.
3. One has $\nu_l = 0$, $\theta_l = 0.1$, $l = 1, \cdots, q$, $\mu_j = \sigma_j$, $j = 1 \cdots, n$ and

$$\mathbb{E}\left[\sum_{l=1}^{q} \gamma_{jl}\zeta_l\right] = \sum_{l=1}^{q} \gamma_{jl} \exp\left(\nu_l + \frac{\theta_l^2}{2}\right) = \frac{\rho_j}{2}, \; j = 1, \cdots, n$$
$$\mathbb{E}\left[\eta_j\right] = \exp\left(\mu_j + \frac{\sigma_j^2}{2}\right) = 1 + \frac{\rho_j}{2}, \; j = 1, \cdots, n$$

We see that the problem can be rewritten into (1.1) in [2] with $m = 1$. Denote

$$\tilde{x} = (\tau, x_0, x_1, \cdots, x_n)^T.$$

The objective function is simply $f(\tilde{x}) = -\tau$, and the chance constraint is

$$\mathbb{P}\left(F(\tilde{x}, \xi) \leq 0\right) \geq 1 - \alpha$$

where

$$F(\tilde{x}, \xi) = g_0(\tilde{x}) + \sum_{j=1}^{d} \xi_j g_j(\tilde{x}),\ d = n + q,\ g_0(\tilde{x}) = \tau - x_0,$$

$$\xi_j = \eta_j,\ g_j(\tilde{x}) = -x_j,\ 1 \le j \le n,$$

$$\xi_{n+l} = \zeta_l,\ g_{n+l}(\tilde{x}) = -\sum_{j=1}^{n} \gamma_{jl} x_j,\ 1 \le l \le q.$$

The Bernstein approximation and standard form formulation

Here we construct the Bernstein approximation to (1), which is a convex optimization problem and reformulate it into a standard form conic program involving exponential cone constraints.

Note that the discretization scheme described in [2] has been adopted and all random variables ξ_j, $1 \le j \le d$ are now discrete with finite support. For each j, denote the support and the associated probability masses as $\left\{(v_k^j, p_k^j) \mid k = 1, \cdots, N_j\right\}$. In other words, for each j, $k = 1, \cdots, N_j$, one has $\xi_j \in \left\{v_k^j \mid k = 1, \cdots, N_j\right\}$ and $\mathbb{P}\left(\xi_j = v_k^j\right) = p_k^j$ and the moment generating function of ξ_j is $M_j : z \to \sum_{k=1}^{N_j} p_k^j \exp\left(v_k^j z\right)$.

The Bernstein approximation to (1) is therefore the following convex maximization problem

$$\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1 \cdots, x_n \ge 0}} (\tau - 1) \quad \text{s.t.} \quad \sum_{j=0}^{n} x_j \le 1,\ \inf_{t>0} \left(g_0(\tilde{x}) + \sum_{j=1}^{d} t\Lambda_j\left(t^{-1} g_j(\tilde{x})\right) - t \log \alpha \right) \le 0 \tag{2}$$

Note that problem (2) is equivalent to

$$
\begin{aligned}
\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1, \cdots, x_n \geq 0 \\ g_0,\ g_1, \cdots, g_d \in \mathbb{R} \\ s_1, \cdots, s_d \in \mathbb{R}}} \quad & (\tau - 1) \\
\text{s.t.} \quad & \sum_{j=0}^{n} x_j \leq 1 \\
& g_0 + \sum_{j=1}^{d} s_j - t \log \alpha = 0 \\
(*)_j : \quad & s_j \geq t \Lambda_j \left(\frac{g_j}{t} \right),\ j = 1, \cdots, d \\
& g_0 = \tau - x_0 \\
& g_j = -x_j,\ j = 1, \cdots, n \\
& g_{n+l} = -\sum_{j=1}^{n} \gamma_{jl} x_j,\ l = 1, \cdots, q
\end{aligned}
\tag{3}
$$

Since $\Lambda_j(\cdot) = \log M_j(\cdot)$, for $j = 1, \cdots, d$, constraint $(*)_j$ in (3) is equivalent to

$$
\begin{aligned}
& \sum_{k=1}^{N_j} p_k^j \exp\left(v_k^j \cdot \frac{g_j}{t} \right) \leq \exp\left(\frac{s_j}{t} \right) \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j \exp\left(\frac{v_k^j g_j - s_j}{t} \right) \leq 1 \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j \cdot t \exp\left(\frac{v_k^j g_j - s_j}{t} \right) \leq t \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad t \exp\left(\frac{w_k^j}{t} \right) \leq u_k^j,\ w_k^j = v_k^j g_j - s_j,\ k = 1, \cdots, N_j \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad \left[w_k^j; u_k^j; t \right] \in \mathcal{K}_{\exp},\ w_k^j = v_k^j g_j - s_j,\ k = 1, \cdots, N_j.
\end{aligned}
$$

Eventually, problem (2) can be reformulated into the standard form (PD′) in [1], namely

(note that $d = n + q$ and the constant term in the objective has been dropped)

$$
\begin{aligned}
& \min \ -\tau \\
\text{s.t.} \quad & x_0 + x_1 + \cdots + x_n + s_x = 1 \\
& g_0 + \left(\sum_{j=1}^{d} s_j \right) - (\log \alpha)\, t_0 = 0 \\
& g_0 - \tau + x_0 = 0 \\
& g_j + x_j = 0, \ j = 1, \cdots, n \\
& g_{n+l} + \sum_{j=1}^{n} \gamma_{jl} x_j = 0, \ l = 1, \cdots, q \\
& w_k^j - v_k^j g_j + s_j = 0, \ j = 1, \cdots, d, \ k = 1, \cdots, N_j \\
& \sum_{k=1}^{N_j} p_k^j u_k^j - t_0 = 0, \ j = 1, \cdots, d \\
& t_0 - t_k^j = 0, \ j = 1, \cdots, d, \ k = 1, \cdots, N_j
\end{aligned}
\tag{4}
$$

with decision variables

$$
\begin{aligned}
& \tau \in \mathbb{R} \\
& x_0, x_1, \cdots, x_n, s_x \geq 0 \\
& g_0, g_1, \cdots, g_d \in \mathbb{R} \\
& t_0 \geq 0 \\
& s_1, \cdots, s_d \in \mathbb{R} \\
& \left[w_k^j ; u_k^j ; t_k^j \right] \in \mathcal{K}_{\exp}, \ j = 1, \cdots, d, \ k = 1, \cdots, N_j.
\end{aligned}
$$

Note that we keep the slack variable $s_x \geq 0$ in the first constraint, although it can be shown that there is always an optimal solution $(x_0^*, x_1^*, \cdots, x_n^*)$ with $\sum_{j=0}^{n} x_j^* = 1$.

The nominal problem

We define the *nominal problem* as the problem with all random variables replaced by constants equal to their respective mean values. Specifically, the nominal problem associated with (1) is

$$
\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1, \cdots, x_n \geq 0}} \quad (\tau - 1) \qquad \text{s.t.} \ \ \tau \leq \sum_{j=0}^{n} (1 + \rho_j)\, x_j, \ \sum_{j=0}^{n} x_j \leq 1. \tag{5}
$$

Since $\rho_1 \leq \cdots \leq \rho_n$, it can be easily seen that (5) has an optimal objective $(1 + \rho_n)$ with optimal solution $x_0 = x_1 = \cdots = x_{n-1} = 0, \ x_n = 1$.

References

[1] Y. Gao. Design and implementation of a homogeneous interior-point method for conic programming involving exponential cone constraints, 2006.

[2] A. Nemirovsky and A. Shapiro. Convex approximation of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.