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ON HOMOGENEOUS INTERIOR-POINT ALGORITHMS FOR SEMIDEFINITE PROGRAMMING*

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A simple homogeneous primal-dual feasibility model is proposed for semidefinite programming (SDP) problems. Two infeasible-interior-point algorithms are applied to the homogeneous formulation. The algorithms do not need a big M initialization. If the original SDP problem has a solution (X^*, y^*, S^*), then both algorithms find an ϵ-approximate solution (i.e., a solution with residual error less than or equal to ϵ) in at most $O(\sqrt{n}\ln(\rho^*\epsilon_0/\epsilon))$ steps, where $\rho^* = \mathrm{Tr}(X^* + S^*)$ and ϵ_0 is the residual error at the (normalized) starting point. A simple way of monitoring possible infeasibility of the original SDP problem is provided such that in at most $O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$ steps either an ϵ-approximate solution is obtained, or it is determined that there is no solution (X^*, y^*, S^*) with $\mathrm{Tr}(X^* + S^*)$ less than or equal to a given number $\rho > 0$. Numerical results on Mehrotra type primal-dual predictor-corrector algorithms show that the homogeneous algorithms outperform their non-homogeneous counterparts, with improvement of more than 20% in many cases, in terms of total CPU time.

Keywords: Semidefinite programming; homogeneous interior-point algorithm; polynomial complexity

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1 INTRODUCTION

This paper is concerned with the semidefinite programming (SDP) problem:

$$(P) \quad \min\{C \bullet X : A_i \bullet X = b_i, \ i = 1, \ldots, m, \ X \succeq 0\}, \tag{1.1}$$

and its associated dual problem:

$$(D) \ \max \left\{ b^T y : \sum_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0 \right\}, \tag{1.2}$$

where $C \in \mathcal{S}^n, A_i \in \mathcal{S}^n, i = 1, \ldots, m, \ b = (b_1, \ldots, b_m)^T \in R^m$ are given data, and $X \in \mathcal{S}^n_+$, $(y, S) \in R^m \times \mathcal{S}^n_+$ are the primal and dual variables, respectively. By $G \bullet H$ we denote the trace of $(G^T H)$. $\mathcal{S}^n$ is the set of all $n \times n$ symmetric matrices and $\mathcal{S}^n_+$ is the set of all symmetric positive semidefinite matrices. For simplicity we assume that $A_i, i = 1, \ldots, m$, are linearly independent.

We consider the primal-dual SDP problem:

$$A_i \bullet X = b_i, i = 1, \ldots, m, \tag{1.3a}$$

$$\sum_{i=1}^{m} y_i A_i + S = C, \tag{1.3b}$$

$$X \bullet S = 0, X \succeq 0, S \succeq 0. \tag{1.3c}$$

We mention that (1.3) provides sufficient conditions (but not always necessary) for an optimal solution of (1.1) and (1.2).

Recently, Kojima, Shindoh and Hara [10], Nesterov and Todd [15], and Monteiro [11] extended some interior-point methods from LP to SDP. In the latter paper Monteiro developed a new formulation of the primal-dual search direction originally introduced in [10]. All above mentioned methods, with the exception of the infeasible-interior-point potential-reduction method of Kojima, Shindoh and Hara [10], require a strictly feasible starting point and therefore are *feasible-interior-point methods*. Nesterov [14] proposed several infeasible-interior-point algorithms for nonlinear conic problems, which include SDP, by trying to find a recession direction of a shifted homogeneous primal-dual problem. More recently, Zhang [24], Kojima, Shida and Shindoh [8] and Potra and Sheng [17] independently proposed new infeasible-interior-point path-following algorithms for SDP.

In the present paper, we propose a simple homogeneous primal-dual feasibility model for the problem (1.3). This formulation is an extension of the homogeneous and self-dual feasibility model for linear programming developed by Goldman and Tucker [5, 21]. We mention that Ye, Todd and Mizuno [23] proposed a homogeneous algorithm for linear programming, and their results were simplified by Xu, Hung and Ye [22] who developed an efficient code by using Goldman and Tucker's homogeneous formulation for linear programming and an infeasible-interior-point method to solve it. Using a similar idea, we apply two interior-point algorithms to solve the homogeneous formulation of SDP from infeasible starting points. The algorithms do not need a big M initialization. If the original SDP problem has a solution (X^*, y^*, S^*), then both algorithms find an ϵ-approximate solution in at most $O(\sqrt{n}\ln(\rho^*\epsilon_0/\epsilon))$ steps, where $\rho^* = \mathrm{Tr}(X^* + S^*)$. We also provide a simple way to monitor possible infeasibility of the original SDP problem such that in at most $O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$ steps we can either get a ϵ-approximate solution or determine that there is no solution (X^*, y^*, S^*) with $\mathrm{Tr}(X^* + S^*)$ less than or equal to a given number $\rho > 0$. This is better than the complexity of the algorithm considered in our previous paper [17] where we assume that a solution exists and proved that the algorithm finds it in $O(n\ln(\epsilon_0/\epsilon))$ iterations if the starting point is large enough (big M initialization) or in $O(\sqrt{n}\ln(\epsilon_0/\epsilon))$ iterations if the starting point is almost feasible, where ϵ_0 denotes the residual error at the starting point.

We note that after the release of the first version of this paper [16] De Klerk, Roos and Terlaky [3] proposed a self-dual skew-symmetric embedding which has a known strictly feasible centered starting point. Their embedding problem can be solved by using any primal, dual, or primal-dual interior point method and thus yields either a primal-dual pair of optimal solutions with zero duality gap or shows that such solutions do not exist. In a subsequent paper De Klerk, Roos and Terlaky [4] further elaborated on the embedding idea; their results cover all possible duality situations, including problems having non-zero duality gap, weak infeasibility etc. De Klerk, Roos and Terlaky concentrate on the theoretical development of the embedding strategy while our presentation makes explicit use of the special structure of the embedding problem in developing interior point algorithms.

The model proposed in [16] was subsequently implemented in a *Mathematica* package in [2] and in a MATLAB package in [18].

In the present version we use some of the results obtained in those papers. In particular we describe how to compute the search directions of the homogeneous formulation of SDP. We also provide numerical results on the Mehrotra type predictor-corrector algorithms which show that the homogeneous algorithms outperform their non-homogeneous counterparts, with improvement of more than 20% in many cases, in terms of total CPU time.

The following notation and terminology are used throughout the paper:
R^p: the p-dimensional Euclidean space;
$R^{p\times q}$: the set of all $p \times q$ matrices with real entries;
$\mathcal{S}^p$: the set of all $p \times p$ symmetric matrices;
$\mathcal{S}^p_+$: the set of all $p \times p$ symmetric positive semidefinite matrices;
$\mathcal{S}^p_{++}$: the set of all $p \times p$ symmetric positive matrices;
m_{ij}: the (i, j)-th entry of a matrix M;
$\mathrm{Tr}(M)$: the trace of a $p \times p$ matrix M, $\mathrm{Tr}(M) = \sum_{i=1}^p m_{ii}$;
$M \succeq 0$: M is positive semidefinite;
$M \succ 0$: M is positive definite;
$\lambda_i(M)$, $i = 1, \ldots, n$: the eigenvalues of $M \in \mathcal{S}^n$;
$\lambda_{\max}(M)$, $\lambda_{\min}(M)$: the largest, smallest, eigenvalue of $M \in \mathcal{S}^n$;
$G \bullet H \equiv \mathrm{Tr}(G^T H)$;
$\|\cdot\|$: Euclidean norm of a vector and the corresponding norm of a matrix, i.e.,
$\|y\| \equiv \sqrt{\sum_{i=1}^p y_i^2}, \quad \|M\| \equiv \max\{\|My\| : \|y\| = 1\}$;
$\|M\|_F \equiv \sqrt{\sum_{i=1}^p \sum_{j=1}^q m_{ij}^2}$, $M \in R^{p\times q}$: Frobenius norm of a matrix.
$\mathbf{vec}(M) \equiv (m_{11}, m_{21}, \ldots, m_{p1}, m_{12}, \ldots, m_{pq})^T$, $M \in R^{p\times q}$.

2 A HOMOGENEOUS FEASIBILITY MODEL

Our homogeneous model is given by the following homogeneous system:

$$A_i \bullet X = \tau b_i, \quad i = 1, \ldots, m, \tag{2.1a}$$

$$\sum_{i=1}^m y_i A_i + S = \tau C, \tag{2.1b}$$

$$\kappa = b^T y - C \bullet X, \tag{2.1c}$$

$$X \succeq 0, S \succeq 0, \tau \geq 0, \kappa \geq 0. \tag{2.1d}$$

It is easily seen that (2.1a), (2.1b) and (2.1c) imply

$$X \bullet S + \tau\kappa = 0. \tag{2.2}$$

The above formulation is an extension of the homogeneous and self-dual feasibility model for linear programming developed by Goldman and Tucker [5, 21]. It is also closely related to the shifted homogeneous primal-dual model considered by Nesterov [13, 14]. The following theorem is easily checked.

THEOREM 2.1. *The SDP problem (1.3) has a solution if and only if the homogeneous system (2.1) has a solution*

$$(X^*, y^*, S^*, \tau^*, \kappa^*) \in \mathcal{S}^n_+ \times R^m \times \mathcal{S}^n_+ \times R_+ \times R_+,$$

such that $\tau^* > 0, \kappa^* = 0$.

We denote the solution set of the problem (1.3) by $\mathcal{F}^*$. Also, let us define the solution set of (2.1) by $\mathcal{H}^*$. Notice that $\mathcal{H}^*$ is not empty since (2.1) has a trivial zero solution.

The residues of (2.1a)–(2.1c) are denoted by:

$$r_i = b_i\tau - A_i \bullet X, \; i = 1, \ldots, m, \tag{2.3a}$$

$$R_d = \tau C - \sum_{i=1}^{m} y_i A_i - S, \tag{2.3b}$$

$$\gamma = b^T y - C \bullet X - \kappa. \tag{2.3c}$$

Throughout the paper we will use the notation:

$$\mu = \frac{X \bullet S + \tau\kappa}{n+1}. \tag{2.4}$$

Let us define

$$\mathcal{H}_{++} = \mathcal{S}^n_{++} \times R^m \times \mathcal{S}^n_{++} \times R_{++} \times R_{++}.$$

Then we can define the infeasible central path of the homogeneous problem (2.1) by

$$\begin{aligned} \mathcal{C} = \{&(X, y, S, \tau, \kappa) \in \mathcal{H}_{++} : \\ &XS = \mu I, \;\; \tau\kappa = \mu, \quad r_i = (\mu/\mu_0) r_i^0, \, i = 1, \ldots, m, \\ &R_d = (\mu/\mu_0) R_d^0, \;\; \gamma = (\mu/\mu_0)\gamma_0\}. \end{aligned}$$

In our algorithms, we will use the following neighborhood of this central path:

$$\begin{aligned}\mathcal{N}(\sigma) &= \{(X, y, S, \tau, \kappa) \in \mathcal{H}_{++} : \\ &\quad \times \left(\|X^{1/2}SX^{1/2} - \mu I\|_F^2 + (\tau\kappa - \mu)^2\right)^{1/2} \le \sigma\mu\} \\ &= \{(X, y, S, \tau, \kappa) \in \mathcal{H}_{++} : \\ &\quad \times \left(\sum_{i=1}^{m}(\lambda_i(XS) - \mu)^2 + (\tau\kappa - \mu)^2\right)^{1/2} \le \sigma\mu\},\end{aligned}$$

where σ is a constant such that $0 < \sigma < 1$.

3 SEARCH DIRECTIONS

The search direction $(U, w, V, \Delta\tau, \Delta\kappa)$ of our algorithms is defined by the following linear system:

$$X^{-1/2}(XV + US)X^{1/2} + X^{1/2}(VX + SU)X^{-1/2}$$
$$= 2(\xi\mu I - X^{1/2}SX^{1/2}), \tag{3.1a}$$
$$\kappa\Delta\tau + \tau\Delta\kappa = \xi\mu - \tau\kappa \tag{3.1b}$$
$$A_i \bullet U - b_i\Delta\tau = (1 - \xi)r_i, \quad i = 1, \dots, m, \tag{3.1c}$$
$$\sum_{i=1}^{m} w_i A_i + V - \Delta\tau C = (1 - \xi)R_d, \tag{3.1d}$$
$$\Delta\kappa - b^T w + C \bullet U = (1 - \xi)\gamma, \tag{3.1e}$$

where ξ is a parameter such that $\xi \in [0, 1]$.

Throughout the paper we will use the following notation:

$$\tilde{X} = \begin{pmatrix} X & 0 \\ 0 & \tau \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & \kappa \end{pmatrix}, \tag{3.2}$$

$$\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & \Delta\tau \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} V & 0 \\ 0 & \Delta\kappa \end{pmatrix}. \tag{3.3}$$

Then it is easily seen that

$$\mu = \frac{\tilde{X} \bullet \tilde{S}}{n + 1}.$$

LEMMA 3.1. *Suppose* $(X, y, S, \tau, \kappa) \in \mathcal{H}_{++}$. *Then the linear system (3.1) has a unique solution*

$$(U, w, V, \Delta\tau, \Delta\kappa) \in \mathcal{S}^n \times R^m \times \mathcal{S}^n \times R \times R.$$

Proof Let us consider the following homogeneous linear system

$$X^{-1/2}(XV + US)X^{1/2} + X^{1/2}(VX + SU)X^{-1/2} = 0 \qquad (3.4a)$$

$$\kappa\Delta\tau + \tau\Delta\kappa = 0, \qquad (3.4b)$$

$$A_i \bullet U - b_i\Delta\tau = 0, \; i = 1, \ldots, m, \qquad (3.4c)$$

$$\sum_{i=1}^{m} w_i A_i + V - \Delta\tau C = 0, \qquad (3.4d)$$

$$\Delta\kappa - b^T w + C \bullet U = 0. \qquad (3.4e)$$

In view of (3.4a), (3.4c), (3.4d) and the proof of Lemma 2.1 of Monteiro [11], it follows that U and V are symmetric. From (3.4c)–(3.4e), we have $U \bullet V + \tau\kappa = 0$, i.e., $\tilde{U} \bullet \tilde{V} = 0$ with $\tilde{U}, \tilde{V}$ given by (3.2) and (3.3). According to (3.4a) and (3.4b), we have

$$\tilde{X}^{-1/2}(\tilde{X}\tilde{V} + \tilde{U}\tilde{S})\tilde{X}^{1/2} + \tilde{X}^{1/2}(\tilde{V}\tilde{X} + \tilde{S}\tilde{U})\tilde{X}^{-1/2} = 0,$$

or equivalently,

$$2\tilde{X}^{1/2}\tilde{V}\tilde{X}^{1/2} + [\tilde{X}^{-1/2}\tilde{U}\tilde{S}\tilde{X}^{1/2} + \tilde{X}^{1/2}\tilde{S}\tilde{U}\tilde{X}^{-1/2}] = 0. \qquad (3.5)$$

Therefore we obtain

$$[\tilde{X}^{-1/2}\tilde{U}\tilde{X}^{-1/2}] \bullet [\tilde{X}^{-1/2}\tilde{U}\tilde{S}\tilde{X}^{1/2} + \tilde{X}^{1/2}\tilde{S}\tilde{U}\tilde{X}^{-1/2}] = 0,$$

which implies

$$[\tilde{X}^{-1/2}\tilde{U}\tilde{X}^{-1/2}] \bullet [\tilde{X}^{-1/2}\tilde{U}\tilde{S}\tilde{X}^{1/2}] = 0.$$

Hence

$$\mathrm{Tr}(\tilde{X}^{-1/2}\tilde{U}\tilde{S}\tilde{U}\tilde{X}^{-1/2}) = 0,$$

i.e.,

$$\mathrm{Tr}([\tilde{X}^{-1/2}\tilde{U}\tilde{S}^{1/2}][\tilde{X}^{-1/2}\tilde{U}\tilde{S}^{1/2}]^T) = 0.$$

It follows from the above relation that $\tilde{X}^{-1/2}\tilde{U}\tilde{S}^{1/2} = 0$ which gives $\tilde{U} = 0$. Moreover, from (3.5) we obtain $\tilde{X}^{1/2}\tilde{V}\tilde{X}^{1/2} = 0$ which implies

$\tilde{V} = 0$. So, we deduce that $U = V = 0$, $\Delta\tau = \Delta\kappa = 0$. Consequently, the coefficient matrix of the linear system (3.1), which has $2n^2 + m + 2$ linear equations and $2n^2+m+2$ unknowns, is nonsingular, which implies the existence of a unique solution of (3.1). The symmetry of U and V determined by (3.1) can be deduced as in Lemma 2.1 of Monteiro [11]. □

LEMMA 3.2. *If $(U, w, V, \Delta\tau, \Delta\kappa)$ is a solution of the linear system (3.1), then,*

$$U \bullet V + \Delta\tau\Delta\kappa = 0.$$

Proof Let us write

$$(U', w', V', \Delta\tau', \Delta\kappa') = ((U + (1-\xi)X,\ w + (1-\xi)y,\ V + (1-\xi)S,$$
$$\tau + (1-\xi)\Delta\tau, \kappa + (1-\xi)\Delta\kappa).$$

Then from (3.1c)-(3.1e), we have

$$A_i \bullet U' - b_i \Delta\tau' = 0,\ \ i = 1, \ldots, m,$$
$$\sum_{i=1}^{m} w_i' A_i + V' - \Delta\tau' C = 0,$$
$$\Delta\kappa' - b^T w' + C \bullet U' = 0,$$

and therefore

$$U' \bullet V' + \Delta\tau'\Delta\kappa' = 0.$$

Hence, using the notation of (3.2) and (3.3), we get

$$\begin{aligned}(\tilde{U} &+ (1-\xi)\tilde{X}) \bullet (\tilde{V} + (1-\xi)\tilde{S}) \\ &= (U + (1-\xi)X) \bullet (V + (1-\xi)S) + (\tau + (1-\xi)\Delta\tau)(\kappa + (1-\xi)\Delta\kappa) \\ &= 0. \end{aligned} \tag{3.6}$$

In view of (3.1a) and (3.1b), we obtain

$$\frac{1}{2}[\tilde{X}^{-1/2}(\tilde{X}\tilde{V} + \tilde{U}\tilde{S})\tilde{X}^{1/2} + \tilde{X}^{1/2}(\tilde{V}\tilde{X} + \tilde{S}\tilde{U})\tilde{X}^{-1/2}] = (\xi\mu I - \tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}). \tag{3.7}$$

By taking the trace of both sides of (3.7), we have

$$\tilde{X} \bullet \tilde{V} + \tilde{U} \bullet \tilde{S} = (\xi - 1)\tilde{X} \bullet \tilde{S}. \tag{3.8}$$

Then by expanding (3.6) and using (3.8) we get $\tilde{U} \bullet \tilde{V} = 0$, i.e.,

$$U \bullet V + \Delta\tau\Delta\kappa = 0.$$ □

LEMMA 3.3. *Let $(X, y, S, \tau, \kappa) \in \mathcal{N}(\sigma)$ for some $\sigma \in [0, 1)$. Suppose that $(U, w, V, \Delta\tau, \Delta\kappa)$ is a solution of the linear system (3.1). Then,*

$$\|\tilde{X}^{-1/2}\tilde{U}\tilde{V}\tilde{X}^{1/2}\| \le \frac{\|\xi\mu I - \tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}\|_F^2}{2(1-\sigma)^2\mu}.$$

Proof Using the notation introduced in (3.2), (3.3) and Lemma 3.2, we have

$$2\tilde{X}^{1/2}\tilde{V}\tilde{X}^{1/2} + \tilde{X}^{-1/2}\tilde{U}\tilde{S}\tilde{X}^{1/2} + \tilde{X}^{1/2}\tilde{S}\tilde{U}\tilde{X}^{-1/2} = 2(\xi\mu I - \tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}), \tag{3.9}$$

and

$$\tilde{U} \bullet \tilde{V} = 0. \tag{3.10}$$

Since $(X, y, S, \tau, \kappa) \in \mathcal{N}(\sigma)$, we obtain

$$\|\tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2} - \mu I\|_F \le \sigma\mu. \tag{3.11}$$

Then the desired inequality follows from (3.9), (3.10), (3.11) and a manipulation similar to that used in the proof of Lemma 4.4 of Monteiro [11]. □

4 A GENERIC HOMOGENEOUS ALGORITHM

Generic Homogeneous Algorithm

Let $(X^0, y^0, S^0, \tau_0, \kappa_0) = (I, 0, I, 1, 1)$.

Repeat until a stopping criterion is satisfied:

- Choose $\xi \in [0, 1]$ and compute the solution $(U, w, V, \Delta\tau, \Delta\kappa)$ of the linear system (3.1).
- Compute a steplength $\bar{\theta}$ such that

$$X + \bar{\theta}U \succ 0,\ \ S + \bar{\theta}V \succ 0,\ \ \tau + \bar{\theta}\Delta\tau > 0,\ \ \kappa + \bar{\theta}\Delta\kappa > 0.$$

- Update the iterate

$$(X^+, y^+, S^+, \tau_+, \kappa_+) = (X, y, S, \tau, \kappa) + \bar{\theta}(U, w, V, \Delta\tau, \Delta\kappa).$$

We will define a stopping criterion later in this section.

THEOREM 4.1. *The iterates generated by the Generic Homogeneous Algorithm satisfy*:

$$\mu_+ = \frac{X^+ \bullet S^+ + \tau_+\kappa_+}{n+1} = (1-(1-\xi)\bar{\theta})\mu, \tag{4.1a}$$

$$r_i^+ = (1-(1-\xi)\bar{\theta})r_i, \ i = 1, \dots, m, \tag{4.1b}$$

$$R_d = (1-(1-\xi)\bar{\theta})R_d, \ \gamma_+ = (1-(1-\xi)\bar{\theta})\gamma. \tag{4.1c}$$

Proof By the definition of the Generic Homogeneous Algorithm, Lemma 3.2 and (3.8), we have

$$\begin{aligned} X^+ \bullet S^+ + \tau_+\kappa_+ &= \tilde{X}^+ \bullet \tilde{S}^+ \\ &= (\tilde{X} + \bar{\theta}\tilde{U}) \bullet (\tilde{S} + \bar{\theta}\tilde{V}) \\ &= \tilde{X} \bullet \tilde{S} + \bar{\theta}(\tilde{X} \bullet \tilde{V} + \tilde{U} \bullet \tilde{S}) + \tilde{U} \bullet \tilde{V} \\ &= (1-(1-\xi)\bar{\theta})(n+1)\mu, \end{aligned}$$

which implies (4.1a). Finally, (4.1b) and (4.1c) can be easily deduced from (3.1c)–(3.1e). □

By using the above theorem and the fact that $\mu_0 = 1$, we deduce the following corollary.

COROLLARY 4.2. *Let* $(X^k, y^k, S^k, \tau_k, \kappa_k)$ *be generated by Generic Homogeneous Algorithm. Then,*

$$r_i^k = \mu_k r_i^0, \ i = 1, \dots, m,$$

$$R_d^k = \mu_k R_d^0, \quad \gamma_k = \mu_k \gamma_0.$$

Let us define a linear manifold

$$\begin{aligned} \mathcal{H}_0 = \{ &(X', y', S', \tau', \kappa') \in \mathcal{S}^n \times R^m \times \mathcal{S}^n \times R \times R : \\ &A_i \bullet X' - \tau' b_i = 0, \ i = 1, \dots, m, \\ &\sum_{i=1}^m y_i' A_i + S' - \tau' C = 0, \\ &\kappa' - b^T y' + C \bullet X' = 0 \} \end{aligned}$$

Then it is easily seen that if $(X', y', S', \tau', \kappa') \in \mathcal{H}_0$, then

$$X' \bullet S' + \tau'\kappa' = 0.$$

The following lemma shows that the iterates $(X^k, y^k, S^k, \tau_k, \kappa_k)$ are bounded.

LEMMA 4.3. *If* $(X^k, y^k, S^k, \tau_k, \kappa_k)$ *are generated by the Generic Homogeneous Algorithm, then*

$$\mathrm{Tr}(X^k + S^k) + \tau_k + \kappa_k = (n+1)(1+\mu). \tag{4.2}$$

Proof For simplicity, we omit the index k. From Corollary 4.2, we have

$$(X - \mu X^0,\ y - \mu y^0,\ S - \mu S^0,\ \tau - \mu\tau_0,\ \kappa - \mu\kappa_0) \in \mathcal{H}_0.$$

Therefore,

$$(S - \mu S^0) \bullet (X - \mu X^0) + (\tau - \mu\tau_0)(\kappa - \mu\kappa_0) = 0. \tag{4.3}$$

The desired result follows by expanding (4.3) and using the fact that $X^0 = S^0 = I$ and $\tau_0 = \kappa_0 = 1$. □

THEOREM 4.4. *Let* $(X^k, y^k, S^k, \tau_k, \kappa_k)$ *be generated by the Generic Homogeneous Algorithm and let* $\rho > 0$ *be a given number. Suppose that the following two conditions are satisfied:*

(a) there exists a solution (X^*, y^*, S^*) *of the original problem (1.3) such that* $\mathrm{Tr}(X^* + S^*) \le \rho$;

(b) there exists a constant $\omega \in (0, 1)$ *such that* $\tau_k\kappa_k \ge \omega\mu_k$, $\forall k$.

Then,

$$\tau_k \ge \frac{\omega}{\rho + 1}, \quad \forall k.$$

Proof Let $\tau_* = 1$, $\kappa_* = 0$. Obviously, $(X^*, y^*, S^*, \tau_*, \kappa_*)$ is a solution of (2.1). From the proof of Lemma 4.3, for any real number η, we have

$$(X - \mu X^0 + \eta X^*,\ y - \mu y^0 + \eta y^*,\ S - \mu S^0 + \eta S^*,$$
$$\tau - \mu\tau_0 + \eta\tau_*,\ \kappa - \mu\kappa_0 + \eta\kappa_*) \in \mathcal{H}_0.$$

Therefore, we get

$$(S - \mu S^0 + \eta S^*) \bullet (X - \mu X^0 + \eta X^*) + (\tau - \mu\tau_0 + \eta\tau_*)(\kappa - \mu\kappa_0 + \eta\kappa_*) = 0.$$

By expanding the above equality and noting that η is an arbitrary real number and that $X^* \bullet S^* + \tau_*\kappa_* = 0$, we obtain

$$\begin{aligned} S \bullet X^* + S^* \bullet X + \tau\kappa_* + \tau_*\kappa &= \mu(S^0 \bullet X^* + S^* \bullet X^0 + \tau_0\kappa_* + \tau_*\kappa_0) \\ &= \mu[\mathrm{Tr}(X^* + S^*) + 1]. \end{aligned}$$

Then, from condition (a) and $\tau_* = 1$, we deduce that

$$\kappa \le \mu[\mathrm{Tr}(X^* + S^*) + 1] \le \mu(\rho + 1). \tag{4.4}$$

Hence, by (4.4) and condition (b), we have

$$\tau \geq \frac{\omega\mu}{\kappa} \geq \frac{\omega}{\rho+1}. \qquad \square$$

COROLLARY 4.5. *Under condition (b) of Theorem 4.4 suppose that* $\mu_k \to 0$ *as* $k \to \infty$. *Then*

(i) the primal-dual problem (1.3) has a solution if and only if $\kappa_k \to 0$ *as* $k \to \infty$ *and there exists a constant* $\tau_* > 0$ *such that* $\tau_k \geq \tau_*$, $\forall k$;

(ii) (1.3) has no solution if and only if $\tau_k \to 0$ *as* $k \to \infty$.

Proof Part (i) is an immediate consequence of Theorem 4.4. From Lemma 4.3, the sequence $\{\tau_k\}$ is bounded. Let us observe that if (1.3) has no solution then $\{\tau_k\}$ does not have a positive accumulation point (otherwise, there exists a subsequence $\{(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k)\}$ converging to a solution of (1.3)), and therefore $\tau_k \to 0$. Conversely, if $\tau_k \to 0$, then by Theorem 4.4 (1.3) has no solution. $\square$

Let us define the stopping criterion of the Generic Homogeneous Algorithm:

Stopping Criterion

$$E_k \equiv \max\{\|R_d^k\|/\tau_k,\ |r_i^k|/\tau_k, i = 1, \ldots, m,\ (X^k \bullet S^k + \tau_k\kappa_k)/\tau_k^2\} \leq \epsilon, \tag{4.5}$$

where $\epsilon > 0$ is the tolerance. Let us define the set of ϵ-approximate solutions of (1.3) by

$$\mathcal{F}_\epsilon = \{(X, y, S) \in \mathcal{S}_+^n \times R^m \times \mathcal{S}_+^n : \\ X \bullet S \leq \epsilon,\quad |b_i - A_i \bullet X| \leq \epsilon,\ i = 1, \ldots, m, \\ \left\| C - \sum_{i=1}^m y_i A_i - S \right\| \leq \epsilon\}.$$

Obviously, if (4.5) is satisfied, then $(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k)$ is an ϵ-approximate solution of the original problem (1.3). Let us observe that if our algorithm starts at a point that is a multiple of $(I, 0, I, 1, 1)$, say $(\nu I, 0, \nu I, \nu, \nu)$, then we obtain an iteration sequence

$$(\check{X}^k, \check{y}^k, \check{S}^k, \check{\tau}_k, \check{\kappa}_k) = (\nu X^k, \nu y^k, \nu S^k, \nu\tau_k, \nu\kappa_k).$$

However, we note that

$$(\check{X}^k/\check{\tau}_k, \check{y}^k/\check{\tau}_k, \check{S}^k/\check{\tau}_k) = (X^k/\tau_k, y^k/\tau_k, S^k/\tau_k).$$

Therefore, we see that the sequence $(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k)$ does not depend on the magnitude of the starting point. This explains why we do not need a big M initialization. If we define the residual error at the point (X, y, S, τ, κ) by

$$\mathrm{res}((X, y, S, \tau, \kappa)) = \max\{X \bullet S + \tau\kappa, \ \|R_d\|_F, \ |r_i|, i = 1, \ldots, m\}, \quad (4.6)$$

then for our starting point we have

$$\begin{aligned}\epsilon_0 &\equiv \mathrm{res}((I, 0, I, 1, 1)) \\ &= \max\{n + 1, \|C - I\|_F, |b_i - \mathrm{Tr}(A_i)|, i = 1, \ldots, m\},\end{aligned}$$

while for a $(\rho + 1)$ multiple of our starting point we have

$$\begin{aligned}\chi(\rho) &\equiv \mathrm{res}((\rho + 1)(I, 0, I, 1, 1)) \\ &= \max\{(\rho + 1)^2(n + 1), (\rho + 1)\|C - I\|_F, \\ &\quad (\rho + 1)|b_i - \mathrm{Tr}(A_i)|, i = 1, \ldots, m\} \\ &\leq \epsilon_0(\rho + 1)^2. \qquad (4.7)\end{aligned}$$

Using the above introduced quantity we obtain the following result.

LEMMA 4.6. *If $\tau_k \geq \omega/(\rho + 1)$, then the following estimate holds:*

$$E_k \leq \mu_k \chi(\rho)/\omega^2,$$

where $\chi(\rho)$ is the residual error defined by (4.7).

Proof From (4.5) and Corollary 4.2, we get

$$\begin{aligned}E_k &= \max\{\mu_k\|R_d^0\|_F/\tau_k, \ \mu_k|r_i^0|/\tau_k, i = 1, \ldots, m, \ \mu_k(n + 1)/\tau_k^2\} \\ &= \mu_k \max\{\|R_d^0\|_F/\tau_k, \ |r_i^0|/\tau_k, i = 1, \ldots, m, \ (n + 1)/\tau_k^2\} \\ &\leq \mu_k \max\left\{\frac{\|R_d^0\|_F}{\omega/(\rho + 1)}, \ \frac{|r_i^0|}{\omega/(\rho + 1)}, i = 1, \ldots, m, \ \frac{n + 1}{(\omega/(\rho + 1))^2}\right\} \\ &\leq (\mu_k/\omega^2)\max\{(\rho + 1)\|R_d^0\|_F, \ (\rho + 1)|r_i^0|, \\ &\quad i = 1, \ldots, m, \ (\rho + 1)^2(n + 1)\} \\ &= \mu_k\chi(\rho)/\omega^2.\end{aligned}$$

□

From Theorem 4.4 and Lemma 4.6, we see that the quantity τ_k is a useful certificate for monitoring possible infeasibility of the problem (1.3). Theorem 4.4 further allows the following corollary:

COROLLARY 4.7. *Under condition (b) of Theorem 4.4 if*

$$\tau_k < \omega/(\rho + 1), \text{ for some } k, \tag{4.8}$$

then the primal-dual problem (1.3) has no solution (X^*, y^*, S^*) *such that*

$$\mathrm{Tr}(X^* + S^*) \leq \rho.$$

In the next two sections, we will describe two examples of the Generic Homogeneous Algorithm, which achieves an $O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$-iteration complexity.

5 A PREDICTOR-CORRECTOR ALGORITHM

In this section, we apply the recent infeasible-interior-point predictor-corrector algorithm of Potra and Sheng [17] to the homogeneous system (2.1) and show that it has $O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$-iteration complexity. Let α, β be two positive constants satisfying the inequalities

$$\frac{\beta^2}{2(1-\beta)^2} \leq \alpha < \beta < \frac{\beta}{1-\beta} < 1. \tag{5.1}$$

For example, $\alpha = 0.25$, $\beta = 0.41$ verify (5.1).

Algorithm 5.1.
$(X, y, S, \tau, \kappa) \leftarrow (I, 0, I, 1, 1)$;
Repeat until stopping criterion (4.5) is satisfied or τ is sufficiently small:
(Predictor step)
Solve the linear system (3.1.) with $\xi = 0$;
Compute

$$\bar{\theta} = \max\{\tilde{\theta} \in [0, 1] :$$
$$\times \left(\sum_{i=1}^{n} (\lambda_i(X(\theta)S(\theta)) - (1-\theta)\mu I)^2 + (\tau(\theta)\kappa(\theta) - (1-\theta)\mu)^2 \right)^{1/2}$$
$$\leq \beta(1-\theta)\mu, \quad \forall \theta \in [0, \tilde{\theta}]\},$$

where

$$(X(\theta), S(\theta), \tau(\theta), \kappa(\theta)) = (X, S, \tau, \kappa) + \theta(U, V, \Delta\tau, \Delta\kappa),$$

$(X, y, S, \tau, \kappa) \leftarrow (X, y, S, \tau, \kappa) + \overline{\theta}(U, w, V, \Delta\tau, \Delta\kappa)$;
(Corrector step)
Solve the linear system (3.1) with $\xi = 1$;
$(X, y, S, \tau, \kappa) \leftarrow (X, y, S, \tau, \kappa) + (U, w, V, \Delta\tau, \Delta\kappa)$.

Using a proof similar to that of Theorem 2.6 of Potra and Sheng [17], we can show that after the corrector step,

$$(X^k, y^k, S^k, \tau_k, \kappa_k) \in \mathcal{N}(\alpha), \forall k.$$

Then, a proof similar to that of Lemma 2.5 of Potra and Sheng [17] gives the following lower bound for the steplength $\overline{\theta}$:

$$\overline{\theta} \geq \frac{2}{\sqrt{1 + 4\delta/(\beta - \alpha)} + 1}, \tag{5.2}$$

where

$$\delta \equiv \frac{1}{\mu}\|\tilde{X}^{-1/2}\tilde{U}\tilde{V}\tilde{X}^{1/2}\|_F. \tag{5.3}$$

Hence, in view of Lemma 3.3, we obtain

$$\begin{aligned}\delta &\leq \frac{\|\tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}\|_F^2}{2(1-\alpha)^2\mu^2} \\ &= \frac{\|\mu I\|_F^2 + \|\mu I - \tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}\|_F^2}{2(1-\alpha)^2\mu^2} \\ &\leq \frac{(n+1)\mu^2 + \alpha^2\mu^2}{2(1-\alpha)^2\mu^2} = \frac{n+1+\alpha^2}{2(1-\alpha)^2} = O(n),\end{aligned}$$

which implies

$$1 - \overline{\theta} \leq 1 - \frac{1}{O(\sqrt{n})}.$$

Here we have used the relation

$$(\mu I) \bullet [\mu I - \tilde{X}^{1/2}\tilde{S}\tilde{X}^{1/2}] = 0.$$

Then, from Theorem 4.1, we have

$$\mu_{k+1} = (1 - \overline{\theta}_k)\mu_k \leq \left(1 - \frac{1}{O(\sqrt{n})}\right)\mu_k. \tag{5.4}$$

Hence, condition (b) of Theorem 4.4 is satisfied with $\omega = 1 - \alpha$. Therefore, in view of (5.4), Theorem 4.4 and Lemma 4.6, we see that in at most $O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$ iterations, either the inequality $\tau_k \geq (1-\alpha)/\rho$ holds all the time or it is violated for some $k \leq O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$. In the former case, we get an ϵ-approximate solution of (1.3) while in the later case (1.3) has no solution (X^*, y^*, S^*) such that $\mathrm{Tr}(X^* + S^*) \leq \rho$. To summarize, we obtain the following polynomial complexity result.

THEOREM 5.2.
(i) If $\mathcal{F}^*$ is not empty, then Algorithm 5.1 terminates with an ϵ-approximate solution

$$(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k) \in \mathcal{F}_\epsilon$$

in a finite number of steps $k = K_\epsilon < \infty$.
(ii) If $\rho^* = \mathrm{Tr}(X^* + S^*)$ for some $(X^*, y^*, S^*) \in \mathcal{F}^*$, then $K_\epsilon = O(\sqrt{n}\ln(\rho^*\epsilon_0/\epsilon))$.
(iii) For any choice of $\rho > 0$ there is an index $k = \widehat{K}_\epsilon = O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$ such that either
(iiia) $(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k) \in \mathcal{F}_\epsilon$,
or,
(iiib) $\tau_k < (1-\alpha)/(\rho+1)$,
and in the latter case there is no solution $(X^*, y^*, S^*) \in \mathcal{F}^*$ with $\rho \geq \mathrm{Tr}(X^* + S^*)$.

6 A SHORT-STEP ALGORITHM

In this section, we extend a recent short-step feasible path-following algorithm by Kojima, Shindoh and Hara [10] and simplified by Monteiro [11] for solving the homogeneous system from infeasible starting points.

Algorithm 6.1.
$(X, y, S, \tau, \kappa) \leftarrow (I, 0, I, 1, 1)$;
Repeat until the stopping criterion (4.5) is satisfied or τ is sufficiently small:
Solve the linear system (3.1) with $\xi = 1 - 0.3/\sqrt{n+1}$;
$(X, y, S, \tau, \kappa) \leftarrow (X, y, S, \tau, \kappa) + (U, w, V, \Delta\tau, \Delta\kappa)$.

By an analogous analysis to that used in Theorem 4.1 of Monteiro [11], we can prove that

$$(X^k, y^k, S^k, \tau_k, \kappa_k) \in \mathcal{N}(0.3), \quad \forall k.$$

From Theorem 4.1, we get

$$\mu_{k+1} = (1 - 0.3/\sqrt{n+1})\mu_k.$$

Obviously, condition (b) of Theorem 4.4 is satisfied with $\omega = 0.7$. Hence, similar to Theorem 5.2, we obtain the following complexity result for Algorithm 6.1.

THEOREM 6.2.

(i) If $\mathcal{F}^*$ is not empty, then Algorithm 6.1 terminates with an ϵ-approximate solution

$$(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k) \in \mathcal{F}_\epsilon$$

in a finite number of steps $k = K_\epsilon < \infty$.

(ii) If $\rho^* = \mathrm{Tr}(X^* + S^*)$ for some $(X^*, y^*, S^*) \in \mathcal{F}^*$, then $K_\epsilon = O(\sqrt{n}\ln(\rho^*\epsilon_0/\epsilon))$.

(iii) For any choice of $\rho > 0$ there is an index $k = \widehat{K}_\epsilon = O(\sqrt{n}\ln(\rho\epsilon_0/\epsilon))$ such that either

(iiia) $(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k) \in \mathcal{F}_\epsilon$,

or,

(iiib) $\tau_k < 0.7/(\rho + 1)$,

and in the latter case there is no solution $(X^*, y^*, S^*) \in \mathcal{F}^*$ with $\rho \geq \mathrm{Tr}(X^* + S^*)$.

7 COMPUTATION OF HOMOGENEOUS SEARCH DIRECTIONS

The search direction described in Section 3 and analyzed in the above sections has now been known as one of the two HKM directions [10,6,11]. Another HKM direction is obtained from (3.1) with (3.1a) replaced by

$$S^{1/2}(XV + US)S^{-1/2} + S^{-1/2}(VX + SU)S^{1/2} = 2(\xi\mu I - S^{1/2}XS^{1/2}).$$

Other two commonly used directions are the AHO direction [1] and the NT direction [15]. These directions all belong to the MZ family of directions [24,12]. The MZ family of directions of the homogeneous model is defined by the linear system:

$$H_P(XV + US) = \xi\mu I - H_P(XS), \tag{7.1a}$$

$$\kappa\Delta\tau + \tau\Delta\kappa = \xi\mu - \tau\kappa, \tag{7.1b}$$

$$A_i \bullet U - b_i \Delta\tau = (1-\xi) r_i, \quad i = 1, \ldots, m, \tag{7.1c}$$

$$\sum_{i=1}^{m} w_i A_i + V - \Delta\tau C = (1-\xi) R_d, \tag{7.1d}$$

$$\Delta\kappa - b^T w + C \bullet U = (1-\xi)\gamma, \tag{7.1e}$$

where $H_P(\cdot)$ is the symmetrization operator of Zhang [24]:

$$H_P(M) = \frac{1}{2}[PMP^{-1} + (PMP^{-1})^T], \quad \forall M \in R^{n\times n}.$$

Let us describe the simple procedure developed in [2] to compute the direction $(U, w, V, \Delta\tau, \Delta\kappa)$ defined by (7.1).

Let

$$\begin{aligned} \tilde{A}^T &= [\mathrm{vec}(A_1), \mathrm{vec}(A_2), \ldots, \mathrm{vec}(A_m), -\mathrm{vec}(C)], \\ b^T &= [b_1, b_2, \ldots, b_m], \\ r_p^T &= [r_1, r_2, \ldots, r_m], \\ \gamma_c &= \xi\mu - \tau\kappa. \end{aligned}$$

Further, let $E, F \in R^{n^2\times n^2}$, $R_c \in R^{n\times n}$ be such that (7.1a) has the equivalent vector form:

$$E\mathrm{vec}(U) + F\mathrm{vec}(V) = \mathrm{vec}(R_c). \tag{7.2}$$

From (7.1b) and (7), we have

$$\Delta\kappa = \gamma_c/\tau - (\kappa/\tau)\Delta\tau.$$

Putting the above expression for $\Delta\kappa$ into (7.1e), we obtain

$$-C \bullet U = -(\kappa/\tau)\Delta\tau - b^T w + \gamma_c/\tau - (1-\xi)\gamma. \tag{7.3}$$

By (7.1d), we get

$$\tilde{A}^T \begin{pmatrix} w \\ \Delta\tau \end{pmatrix} + \mathrm{vec}(V) = (1-\xi)\mathrm{vec}(R_d). \tag{7.4}$$

Therefore, from (7.2), (7.3) and (7.4), we deduce

$$\begin{aligned} \tilde{A}E^{-1}F\tilde{A}^T \begin{pmatrix} w \\ \Delta\tau \end{pmatrix} &= -\tilde{A}E^{-1}F\mathrm{vec}(V) + (1-\xi)\tilde{A}E^{-1}F\mathrm{vec}(R_d) \\ &= \tilde{A}(\mathrm{vec}(U) - E^{-1}\mathrm{vec}(R_c)) + (1-\xi)\tilde{A}E^{-1}F\mathrm{vec}(R_d) \end{aligned}$$

$$
\begin{aligned}
&= \tilde{A}\mathrm{vec}(U) + \tilde{A}E^{-1}[(1-\xi)F\mathrm{vec}(R_d) - \mathrm{vec}(R_c)] \\
&= \begin{pmatrix} \Delta\tau b + (1-\xi)r_p \\ -C \bullet U \end{pmatrix} \\
&\quad + \tilde{A}E^{-1}[(1-\xi)F\mathrm{vec}(R_d) - \mathrm{vec}(R_c)] \\
&= \begin{pmatrix} \Delta\tau b + (1-\xi)r_p \\ \gamma_c/\tau - (1-\xi)\gamma - (\kappa/\tau)\Delta\tau - b^T w \end{pmatrix} \\
&\quad + \tilde{A}E^{-1}[(1-\xi)F\mathrm{vec}(R_d) - \mathrm{vec}(R_c)] \\
&= \begin{pmatrix} 0 & b \\ -b^T & -\kappa/\tau \end{pmatrix}\begin{pmatrix} w \\ \Delta\tau \end{pmatrix} + \begin{pmatrix} (1-\xi)r_p \\ \gamma_c/\tau - (1-\xi)\gamma \end{pmatrix} \\
&\quad + \tilde{A}E^{-1}[(1-\xi)F\mathrm{vec}(R_d) - \mathrm{vec}(R_c)].
\end{aligned}
$$

Then, we have the following procedure.

Procedure 7.1.

- *Compute w and $\Delta\tau$ by solving the linear system*

$$
\left[\tilde{A}E^{-1}F\tilde{A}^T + \begin{pmatrix} 0 & -b \\ b^T & \kappa/\tau \end{pmatrix}\right]\begin{pmatrix} w \\ \Delta\tau \end{pmatrix} = r_h, \tag{7.5}
$$

where

$$
r_h = \tilde{A}E^{-1}[(1-\xi)Fvec(R_d) - vec(R_c)] + \begin{pmatrix} (1-\xi)r_p \\ \xi\mu/\tau - \kappa - (1-\xi)\gamma \end{pmatrix}.
$$

- *Compute V, U and $\Delta\kappa$ as follows*:

$$
\begin{aligned}
V &= (1-\xi)R_d - \sum_{i=1}^{m} w_i A_i + \Delta\tau C, \\
vec(U) &= E^{-1}(R_c - Fvec(V)), \\
\Delta\kappa &= (\xi\mu - \kappa\Delta\tau)/\tau - \kappa.
\end{aligned}
$$

8 NUMERICAL RESULTS

In this section, we present numerical results on the implementation of Mehrotra predictor-corrector algorithms. The reader is referred to [19] for details about the implementation of Mehrotra predictor-corrector algorithms. We use two MATLAB packages: SDPT3 and SDPHA. SDPT3 was developed by Toh, Todd, and Tütüncü [20] and implements Mehrotra infeasible-interior-point algorithms. SDPHA was authored by Potra, Sheng and Brixius [18] and implements Mehrotra type homogeneous algorithms.

Our computations were performed on an HP-UX9000 workstation. We used the stopping criterion

$$\max\left\{\frac{\|r_p\|}{\|b\|+1}, \frac{\|R_d\|_F}{\|C\|_F+1}, \frac{X \bullet S}{\max\{1, (C \bullet X + b^T y)/2\}}\right\} \leq 10^{-9}. \quad (8.1)$$

The left side of the above inequality represents the "relative error". The numerical results are given in Tables 5.4–5.8 for the performance of SDPT3 and SDPHA while using the AHO, HKM and NT directions, respectively. We tested the following problems:

1. the matrix norm minimization problem,
2. the problem of computing the Chebyshev polynomial of a matrix,
3. the Max-Cut problem,
4. the Chebyshev approximation in the complex plane,
5. the logarithmic Chebyshev approximation problem,
6. the problem of "Controller Design in Active Noise Control".

All these problems are taken from [19] and [20] except problem 6 which comes from: Julia A. Olkin, SIAG/OPT Views-and-News, No. 8, Fall 1996.

The results in Tables 8.1–8.3 use the starting point

$$(X^0, y^0, S^0) = (1000I, \mathbf{0}, 1000I),$$

for infeasible-interior algorithms in SDPT3.

Table 8.4 compares the performance of SDPT3 and SDPHA by solving a small problem of $m = n = 2$ which has a solution that is

TABLE 8.1 Comparing SDPHA with SDPT3 using the AHO direction

Problem	*n*	*m*	*SDPT3*			*SDPHA*		
			its.	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*
1	20	50	13	77.1	3.18e-11	10	58.7	1.66e-10
2	50	11	12	125.3	4.59e-11	8	83.4	8.59e-11
3	20	20	11	17.7	1.55e-9	9	14.9	5.93e-10
4	90	11	13	109.4	4.78e-11	9	78.0	1.45e-10
5	60	10	15	40.3	5.48e-11	11	30.1	1.16e-10
6	20	20	9	13.4	3.48e-10	6	9.8	6.06e-11

TABLE 8.2 Comparing SDPHA with SDPT3 using the HKM direction

Problem	n	m	*SDPT3*			*SDPHA*		
			its.	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*
1	20	50	13	47.7	2.23e-9	11	37.1	4.56e-10
2	50	11	13	74.5	3.83e-10	9	53.4	3.46e-10
3	20	20	12	12.2	3.49e-8	11	9.9	3.50e-10
4	90	11	14	108.6	1.49e-10	11	86.8	1.80e-11
5	60	10	16	22.9	1.09e-10	12	16.0	3.81e-10
6	20	20	9	8.7	3.81e-10	6	6.0	5.28e-11

TABLE 8.3 Comparing SDPHA with SDPT3 using the NT direction

Problem	n	m	*SDPT3*			*SDPHA*		
			its.	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*
1	20	50	13	48.4	2.66e-9	11	39.3	1.64e-10
2	50	11	13	84.2	7.09e-10	10	66.5	3.57e-11
3	20	20	12	13.0	3.25e-8	10	10.2	6.85e-9
4	40	11	14	143.5	6.29e-10	10	106.5	9.39e-11
5	60	6	15	25.0	1.57e-9	12	19.5	1.98e-10
6	20	20	9	9.3	3.81e-10	6	6.7	5.28e-11

TABLE 8.4 Comparing SDPHA with SDPT3 by solving a problem of $n = m = 2$ and using various starting points

SDPT3 ρ	*AHO*			*HKM*			*NT*		
	its.	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*
10	*	*	*	*	*	*	*	*	*
10^3	21+	2.4+	2.72e-6	*	*	*	*	*	*
10^5	16	1.9	1.36e-11	16	1.4	1.19e-10	18	1.9	1.03e-10
10^7	16	1.8	8.10e-10	15	1.3	6.72e-10	16	1.7	2.53e-11
10^9	19	2.2	3.27e-10	20+	1.7+	3.72e-9	20	2.1	1.25e-10
10^{11}	21	2.5	2.04e-10	21	1.8	3.08e-10	21	2.2	1.88e-10
tune-up	*	*	*	12	1.0	9.93e-10	14	1.5	2.52e-10
SDPHA	12	1.9	4.02e-10	12	1.3	6.04e-10	12	1.6	6.21e-10

large in magnitude. The problem is actually a scaled version of the Kojima-Shida-Shindoh problem [9]:

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix},$$

$$b = \begin{pmatrix} -2 \cdot 10^6 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 10^6 \end{pmatrix}.$$

whose solution is:

$$X^* = \begin{pmatrix} 10^6 & 0 \\ 0 & 0 \end{pmatrix},$$

$$y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S^* = \begin{pmatrix} 0 & 0 \\ 0 & 10^6 \end{pmatrix}.$$

We implemented the infeasible-interior-point algorithms in SDPT3 using the following starting point:

$$(X^0, y^0, S^0) = (\rho I, \ \mathbf{0}, \ \rho I).$$

Results are given for $\rho = 10, 10^3, \ldots, 10^{11}$, and also for the tune-up starting point suggested in SDPT3. We note that the symbol "*" means the algorithm fails while "+" indicates the algorithm stops without satisfying the stopping rule (8.1). Finally, Table 8.5 gives the same comparison by solving an educational testing problem (ETP) [19,20] with $n = 30$, $m = 15$ which is scaled such that it has a solution with $\|X^*\|_F \geq 10^3$, $\|S^*\|_F \geq 10^3$.

Our numerical results indicate that the homogeneous algorithms generally take fewer iterations to capture a desired solution and that the homogeneous algorithms outperform their non-homogeneous versions, with an improvement of more than 20% in many cases, in terms of total CPU time.

TABLE 8.5 Comparing SDPHA with SDPT3 by solving a ETP problem of $n = 30$, $m = 15$ and using various starting points

SDPT3 ρ	*AHO*			*HKM*			*NT*		
	its.	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*	*its.*	*cpu sec*	*gap*
10^2	18	29.6	3.19e-5	22	21.0	5.46e-5	20	20.9	4.67e-5
10^4	13	22.0	1.91e-5	15	14.8	3.05e-5	15+	15.9+	1.40e-4
10^6	16	26.2	3.07e-6	17	16.8	4.87e-5	17	18.3	5.70e-5
10^8	18	29.4	2.24e-6	19	18.0	6.47e-5	19	19.5	8.31e-5
10^{10}	20	32.5	2.83e-6	21	19.8	7.42e-5	21	21.6	7.11e-5
10^{12}	22	35.6	3.38e-6	23	21.6	8.69e-5	23	23.6	5.94e-5
tune-up	13	21.2	2.62e-6	15	14.2	2.88e-5	15+	15.5+	1.23e-4
SDPHA	12	20.8	2.83e-6	14	12.2	3.03e-5	13	13.3	3.11e-5

9 FURTHER REMARKS

For simplicity of analysis, we have chosen the starting point $(I, 0, I, 1, 1)$. Indeed, we can use any starting point $(X^0, y^0, S^0, \tau_0, \kappa_0) \in \mathcal{H}_{++}$ near the infeasible central path because we are actually interested in the sequence $(X^k/\tau_k, y^k/\tau_k, S^k/\tau_k)$ which does not depend on the magnitude of the starting point.

As a final comment we note that we can also extend the long-step algorithm for linear programming by Kojima Mizuno and Yoshise [7] to the homogeneous system (2.1). By using the proof techniques and the results of Monteiro [11] we can prove that the iteration complexity of this extended algorithm is $O(n^{1.5}\ln(\rho\epsilon_0/\epsilon))$.

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