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Abstract

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For $\alpha > 0$, $k \geq 1$, define

$$I(\alpha, k) = \int_{\alpha}^{\infty} x^{k/2} e^{-x/2} dx.$$

Note that $I(\alpha, 0) = 2e^{-\alpha/2}$. Furthermore, it can be shown by integration by parts that, for $k \geq 0$,

$$I(\alpha, k) = 2\alpha^{k/2} e^{-\alpha/2} + kI(\alpha, k-2). \tag{1}$$

Let $Z \sim \chi^2(k)$, $k \geq 2$. The density of Z is

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}.$$

For $\alpha > 0$, $k \geq 2$, by (1),

$$\begin{aligned} E(Z \mid Z > \alpha) &= \frac{\int_{\alpha}^{\infty} x f(x) dx}{\int_{\alpha}^{\infty} f(x) dx} = \frac{I(\alpha, k)}{I(\alpha, k-2)} \\ &= \frac{2\alpha^{k/2} e^{-\alpha/2}}{I(\alpha, k-2)} + k \\ &= \frac{2\alpha^{k/2} e^{-\alpha/2}}{2\alpha^{k/2-1} e^{-\alpha/2} + kI(\alpha, k-4)} + k \\ &< \alpha + k. \end{aligned}$$

For $k = 1$, the above inequality needs to be modified slightly. It is clear that $E(Z|Z > \alpha) > \alpha$. When α is large, $k > 0$ is constant and $\alpha >> k$, by (1), (not sure for the part below)

$$I(\alpha, k) = \Theta(2\alpha^{k/2} e^{\alpha/2}). \tag{2}$$

Hence (not sure for the part below)

$$\begin{aligned} E(Z|Z > \alpha) &= \frac{2\alpha^{k/2} e^{-\alpha/2}}{2\alpha^{k/2-1} e^{-\alpha/2} + k\Theta(2\alpha^{k/2-2} e^{\alpha/2})} + k \\ &= \frac{\alpha}{1 + 2k\Theta(\alpha^{-2})} + k. \end{aligned}$$