

# Bernstein approximation of chance constrained problems: an example

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## Abstract

We study the example in [2] in more detail and repeat the computation using the new conic program solver.

## The model

We describe the chance constrained problem in detail. As in [2], consider the following chance constrained program

$$\begin{aligned} \max_{\tau \in \mathbb{R}} \quad & (\tau - 1) \quad \text{s.t.} \quad \mathbb{P} \left( \tau > \sum_{j=0}^n r_j x_j \right) \leq \alpha, \quad \sum_{j=0}^n x_j \leq 1 \\ & x_0, x_1, \dots, x_n \geq 0 \end{aligned} \quad (1)$$

where  $\alpha \in [0, 1]$  is a given constant. The assumptions are

1. The returns  $r_0, r_1, \dots, r_n$  satisfy  $r_0 = 1$  and  $\mathbb{E}(r_i) = 1 + \rho_i$  with  $0 \leq \rho_1 \leq \dots \leq \rho_n$ .
2. For  $1 \leq j \leq n$  and  $1 \leq l \leq q$ , one has  $r_j = \eta_j + \sum_{l=1}^q \gamma_{jl} \zeta_l$  where  $\eta_j \sim \mathcal{LN}(\mu_j, \sigma_j^2)$  (the individual noises) and  $\zeta_l \sim \mathcal{LN}(\nu_l, \theta_l^2)$ . All  $\eta_j$  and  $\zeta_l$  are independent of each other.
3. One has  $\nu_l = 0$ ,  $\theta_l = 0.1$  for all  $l$ ,  $\mu_j = \sigma_j$  for all  $j$ ,  $\sum_{l=1}^q \gamma_{jl} \exp \left( \nu_l + \frac{\theta_l^2}{2} \right) = \frac{\rho_j}{2}$  for all  $j$  and  $\sum_{j=1}^n \exp \left( \mu_j + \frac{\sigma_j^2}{2} \right) = 1 + \frac{\rho_j}{2}$ .

We see that the problem can be rewritten into (1.1) in [2] with  $m = 1$ . Denote

$$\tilde{x} = (\tau, x_0, x_1, \dots, x_n)^T.$$

The objective function is simply  $f(\tilde{x}) = -\tau$ , and the chance constraint is

$$\mathbb{P}(F(\tilde{x}, \xi) \leq 0) \geq 1 - \alpha$$

where

$$F(\tilde{x}, \xi) = g_0(\tilde{x}) + \sum_{j=1}^d \xi_j g_j(\tilde{x}), \quad d = n + q, \quad g_0(\tilde{x}) = \tau - x_0,$$

$$\begin{aligned}\xi_j &= \eta_j, \quad g_j(\tilde{x}) = -x_j, \quad 1 \leq j \leq n, \\ \xi_{n+l} &= \zeta_l, \quad g_{n+l}(\tilde{x}) = -\sum_{j=1}^n \gamma_{jl} x_j, \quad 1 \leq l \leq q.\end{aligned}$$

## The Bernstein approximation and standard form formulation

Here we construct the Bernstein approximation to (1), which is a convex optimization problem and reformulate it into a standard form conic program involving exponential cone constraints.

Note that the discretization scheme described in [2] has been adopted and all random variables  $\xi_j$ ,  $1 \leq j \leq d$  are now discrete with finite support. For each  $j$ , denote the support and the associated probability masses as  $\{(v_k^j, p_k^j)\}_{1 \leq k \leq N_j}$ . In other words, for each  $j$ ,  $k = 1, \dots, N_j$ , one has  $\mathbb{P}(\xi_j = v_k^j)$  and the moment generating function of  $\xi_j$  is  $M_j : z \rightarrow \sum_{k=1}^{N_j} p_k^j \exp(v_k^j z)$ .

The Bernstein approximation to (1) is therefore the following convex maximization problem

$$\begin{aligned}\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1, \dots, x_n \geq 0}} \quad & (\tau - 1) \quad \text{s.t.} \quad \sum_{j=0}^n x_j \leq 1, \quad \inf_{t>0} \left( g_0(\tilde{x}) + \sum_{j=1}^d t \Lambda_j(t^{-1} g_j(\tilde{x})) - t \log \alpha \right) \leq 0\end{aligned}\tag{2}$$

Note that problem (2) is equivalent to

$$\begin{aligned}\max_{\tau \in \mathbb{R}} \quad & (\tau - 1) \\ & x_0, x_1, \dots, x_n \geq 0 \\ & g_0, g_1, \dots, g_d \in \mathbb{R} \\ & s_1, \dots, s_d \in \mathbb{R} \\ \text{s.t.} \quad & \sum_{j=0}^n x_j \leq 1 \\ & g_0 + \sum_{j=1}^d s_j - t \log \alpha = 0 \\ & (*)_j : \quad s_j \geq t \Lambda_j\left(\frac{g_j}{t}\right), \quad j = 1, \dots, d \\ & g_0 = \tau - x_0 \\ & g_j = -x_j, \quad j = 1, \dots, n \\ & g_{n+l} = -\sum_{j=1}^n \gamma_{jl} x_j, \quad l = 1, \dots, q\end{aligned}\tag{3}$$

Since  $\Lambda_j(\cdot) = \log M_j(\cdot)$ , for  $j = 1, \dots, d$ , constraint  $(*)_j$  in (3) is equivalent to

$$\begin{aligned}
& \sum_{k=1}^{N_j} p_k^j \exp\left(v_k^j \cdot \frac{g_j}{t}\right) \leq \exp\left(\frac{s_j}{t}\right) \\
\Leftrightarrow & \sum_{k=1}^{N_j} p_k^j \exp\left(\frac{v_k^j g_j - s_j}{t}\right) \leq 1 \\
\Leftrightarrow & \sum_{k=1}^{N_j} p_k^j \cdot t \exp\left(\frac{v_k^j g_j - s_j}{t}\right) \leq t \\
\Leftrightarrow & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad t \exp\left(\frac{w_k^j}{t}\right) \leq u_k^j, \quad w_k^j = v_k^j g_j - s_j, \quad k = 1, \dots, N_j \\
\Leftrightarrow & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad [w_k^j; u_k^j; t] \in \mathcal{K}_{\text{exp}}, \quad w_k^j = v_k^j g_j - s_j, \quad k = 1, \dots, N_j.
\end{aligned}$$

Eventually, problem (2) can be reformulated into the standard form (PD') in [1], namely (note that  $d = n + q$  and the constant term in the objective has been dropped)

$$\begin{aligned}
& \min \quad -\tau \\
& \text{s.t.} \quad x_0 + x_1 + \dots + x_n + s_x = 1 \\
& \quad g_0 + \left(\sum_{j=1}^d s_j\right) - (\log \alpha) t_0 = 0 \\
& \quad g_0 - \tau + x_0 = 0 \\
& \quad g_j + x_j = 0, \quad j = 1, \dots, n \\
& \quad g_{n+l} + \sum_{j=1}^n \gamma_{jl} x_j = 0, \quad l = 1, \dots, q \\
& \quad w_k^j - v_k^j g_j + s_j = 0, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j \\
& \quad \sum_{k=1}^{N_j} p_k^j u_k^j - t_0 = 0, \quad j = 1, \dots, d \\
& \quad t_0 - t_k^j = 0, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j
\end{aligned} \tag{4}$$

with decision variables

$$\begin{aligned}
& \tau \in \mathbb{R} \\
& x_0, x_1, \dots, x_n, s_x \geq 0 \\
& g_0, g_1, \dots, g_d \in \mathbb{R} \\
& t_0 \geq 0 \\
& s_1, \dots, s_d \in \mathbb{R} \\
& [w_k^j; u_k^j; t_k^j] \in \mathcal{K}_{\text{exp}}, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j.
\end{aligned}$$

Note that we keep the slack variable  $s_x \geq 0$  in the first constraint, although it can be shown that there is always an optimal solution  $(x_0^*, x_1^*, \dots, x_n^*)$  with  $\sum_{j=0}^n x_j^* = 1$ .

## References

- [1] Y. Gao. Design and implementation of a homogeneous interior-point method for conic programming involving exponential cone constraints, 2006.
- [2] A. Nemirovsky and A. Shapiro. Convex approximation of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.