

# Bernstein approximation of chance constrained problems: an example

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August 17, 2016

## Abstract

We study the example in [2] in more detail and repeat the computation using the new conic program solver.

## The model

We describe the chance constrained problem in detail. As in [2], consider the following chance constrained program

$$\begin{aligned} \max_{\tau \in \mathbb{R}} \quad & (\tau - 1) \quad \text{s.t.} \quad \mathbb{P} \left( \tau > \sum_{j=0}^n r_j x_j \right) \leq \alpha, \quad \sum_{j=0}^n x_j \leq 1 \\ & x_0, x_1, \dots, x_n \geq 0 \end{aligned} \quad (1)$$

where  $\alpha \in [0, 1]$  is a given constant. The assumptions are

1. The returns  $r_0, r_1, \dots, r_n$  satisfy  $r_0 = 1$  and  $\mathbb{E}(r_i) = 1 + \rho_i$  with  $0 \leq \rho_1 \leq \dots \leq \rho_n$ .
2. For  $1 \leq j \leq n$  and  $1 \leq l \leq q$ , one has  $r_j = \eta_j + \sum_{l=1}^q \gamma_{jl} \zeta_l$  where  $\eta_j \sim \mathcal{LN}(\mu_j, \sigma_j^2)$  (the individual noises) and  $\zeta_l \sim \mathcal{LN}(\nu_l, \theta_l^2)$ . All  $\eta_j$  and  $\zeta_l$  are independent of each other.
3. One has  $\nu_l = 0$ ,  $\theta_l = 0.1$ ,  $l = 1, \dots, q$ ,  $\mu_j = \sigma_j$ ,  $j = 1, \dots, n$  and

$$\begin{aligned} \mathbb{E} \left[ \sum_{l=1}^q \gamma_{jl} \zeta_l \right] &= \sum_{l=1}^q \gamma_{jl} \exp \left( \nu_l + \frac{\theta_l^2}{2} \right) = \frac{\rho_j}{2}, \quad j = 1, \dots, n \\ \mathbb{E} [\eta_j] &= \exp \left( \mu_j + \frac{\sigma_j^2}{2} \right) = 1 + \frac{\rho_j}{2}, \quad j = 1, \dots, n \end{aligned}$$

We see that the problem can be rewritten into (1.1) in [2] with  $m = 1$ . Denote

$$\tilde{x} = (\tau, x_0, x_1, \dots, x_n)^T.$$

The objective function is simply  $f(\tilde{x}) = -\tau$ , and the chance constraint is

$$\mathbb{P}(F(\tilde{x}, \xi) \leq 0) \geq 1 - \alpha$$

where

$$\begin{aligned}
F(\tilde{x}, \xi) &= g_0(\tilde{x}) + \sum_{j=1}^d \xi_j g_j(\tilde{x}), \quad d = n + q, \quad g_0(\tilde{x}) = \tau - x_0, \\
\xi_j &= \eta_j, \quad g_j(\tilde{x}) = -x_j, \quad 1 \leq j \leq n, \\
\xi_{n+l} &= \zeta_l, \quad g_{n+l}(\tilde{x}) = -\sum_{j=1}^n \gamma_{jl} x_j, \quad 1 \leq l \leq q.
\end{aligned}$$

## The Bernstein approximation and standard form formulation

Here we construct the Bernstein approximation to (1), which is a convex optimization problem and reformulate it into a standard form conic program involving exponential cone constraints.

Note that the discretization scheme described in [2] has been adopted and all random variables  $\xi_j$ ,  $1 \leq j \leq d$  are now discrete with finite support. For each  $j$ , denote the support and the associated probability masses as  $\{(v_k^j, p_k^j) \mid k = 1, \dots, N_j\}$ . In other words, for each  $j$ ,  $k = 1, \dots, N_j$ , one has  $\xi_j \in \{v_k^j \mid k = 1, \dots, N_j\}$  and  $\mathbb{P}(\xi_j = v_k^j) = p_k^j$  and the moment generating function of  $\xi_j$  is  $M_j : z \rightarrow \sum_{k=1}^{N_j} p_k^j \exp(v_k^j z)$ .

The Bernstein approximation to (1) is therefore the following convex maximization problem

$$\begin{aligned}
&\max_{\substack{\tau \in \mathbb{R} \\ x_0, x_1, \dots, x_n \geq 0}} (\tau - 1) \quad \text{s.t.} \quad \sum_{j=0}^n x_j \leq 1, \quad \inf_{t > 0} \left( g_0(\tilde{x}) + \sum_{j=1}^d t \Lambda_j(t^{-1} g_j(\tilde{x})) - t \log \alpha \right) \leq 0 \\
&\hspace{30em} (2)
\end{aligned}$$

Note that problem (2) is equivalent to

$$\begin{aligned}
& \max_{\tau \in \mathbb{R}} \quad (\tau - 1) \\
& x_0, x_1, \dots, x_n \geq 0 \\
& g_0, g_1, \dots, g_d \in \mathbb{R} \\
& s_1, \dots, s_d \in \mathbb{R} \\
& \text{s.t.} \quad \sum_{j=0}^n x_j \leq 1 \\
& g_0 + \sum_{j=1}^d s_j - t \log \alpha = 0 \\
& (*)_j : \quad s_j \geq t \Lambda_j \left( \frac{g_j}{t} \right), \quad j = 1, \dots, d \\
& g_0 = \tau - x_0 \\
& g_j = -x_j, \quad j = 1, \dots, n \\
& g_{n+l} = - \sum_{j=1}^n \gamma_{jl} x_j, \quad l = 1, \dots, q
\end{aligned} \tag{3}$$

Since  $\Lambda_j(\cdot) = \log M_j(\cdot)$ , for  $j = 1, \dots, d$ , constraint  $(*)_j$  in (3) is equivalent to

$$\begin{aligned}
& \sum_{k=1}^{N_j} p_k^j \exp \left( v_k^j \cdot \frac{g_j}{t} \right) \leq \exp \left( \frac{s_j}{t} \right) \\
& \Leftrightarrow \sum_{k=1}^{N_j} p_k^j \exp \left( \frac{v_k^j g_j - s_j}{t} \right) \leq 1 \\
& \Leftrightarrow \sum_{k=1}^{N_j} p_k^j \cdot t \exp \left( \frac{v_k^j g_j - s_j}{t} \right) \leq t \\
& \Leftrightarrow \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad t \exp \left( \frac{w_k^j}{t} \right) \leq u_k^j, \quad w_k^j = v_k^j g_j - s_j, \quad k = 1, \dots, N_j \\
& \Leftrightarrow \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad [w_k^j; u_k^j; t] \in \mathcal{K}_{\exp}, \quad w_k^j = v_k^j g_j - s_j, \quad k = 1, \dots, N_j.
\end{aligned}$$

Eventually, problem (2) can be reformulated into the standard form (PD') in [1], namely

(note that  $d = n + q$  and the constant term in the objective has been dropped)

$$\begin{aligned}
& \min \quad -\tau \\
& \text{s.t.} \quad x_0 + x_1 + \cdots + x_n + s_x = 1 \\
& \quad g_0 + \left( \sum_{j=1}^d s_j \right) - (\log \alpha) t_0 = 0 \\
& \quad g_0 - \tau + x_0 = 0 \\
& \quad g_j + x_j = 0, \quad j = 1, \dots, n \\
& \quad g_{n+l} + \sum_{j=1}^n \gamma_{jl} x_j = 0, \quad l = 1, \dots, q \\
& \quad w_k^j - v_k^j g_j + s_j = 0, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j \\
& \quad \sum_{k=1}^{N_j} p_k^j w_k^j - t_0 = 0, \quad j = 1, \dots, d \\
& \quad t_0 - t_k^j = 0, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j
\end{aligned} \tag{4}$$

with decision variables

$$\begin{aligned}
& \tau \in \mathbb{R} \\
& x_0, x_1, \dots, x_n, s_x \geq 0 \\
& g_0, g_1, \dots, g_d \in \mathbb{R} \\
& t_0 \geq 0 \\
& s_1, \dots, s_d \in \mathbb{R} \\
& [w_k^j; v_k^j; t_k^j] \in \mathcal{K}_{\text{exp}}, \quad j = 1, \dots, d, \quad k = 1, \dots, N_j.
\end{aligned}$$

Note that we keep the slack variable  $s_x \geq 0$  in the first constraint, although it can be shown that there is always an optimal solution  $(x_0^*, x_1^*, \dots, x_n^*)$  with  $\sum_{j=0}^n x_j^* = 1$ .

## The nominal problem

We define the *nominal problem* as the problem with all random variables replaced by constants equal to their respective mean values. Specifically, the nominal problem associated with (1) is

$$\begin{aligned}
& \max_{\tau \in \mathbb{R}} \quad (\tau - 1) \quad \text{s.t.} \quad \tau \leq \sum_{j=0}^n (1 + \rho_j) x_j, \quad \sum_{j=0}^n x_j \leq 1. \\
& x_0, x_1, \dots, x_n \geq 0
\end{aligned} \tag{5}$$

Since  $\rho_1 \leq \dots \leq \rho_n$ , it can be easily seen that (5) has an optimal objective  $(1 + \rho_n)$  with optimal solution  $x_0 = x_1 = \dots = x_{n-1} = 0, x_n = 1$ .

## References

- [1] Y. Gao. Design and implementation of a homogeneous interior-point method for conic programming involving exponential cone constraints, 2006.
- [2] A. Nemirovsky and A. Shapiro. Convex approximation of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.