Bernstein approximation of chance constrained problems: an example

Yuan Gao

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Abstract

We study the example in [2] in more detail and repeat the computation using the new conic program solver.

The model

We describe the chance constrained problem in detail. As in [2], consider the following chance constrained program

$$\max_{x_0, x_1 \cdots, x_n, \tau} (\tau - 1) \quad \text{s.t.} \;\; \mathbb{P}\left(\tau > \sum_{j=0}^{n} r_j x_j\right) \leq \alpha, \;\; \sum_{j=0}^{n} x_j \leq 1, \;\; x_j \geq 0, \forall j \tag{1}$$

where $\alpha \in [0,1]$ is a given constant. The assumptions are

1. The returns $r_0, r_1, \cdots, r_n$ satisfy $r_0 = 1$ and $\mathbb{E}(r_i) = 1 + \rho_i$ with $0 \leq \rho_1 \leq \cdots \leq \rho_n$.
2. For $1 \leq j \leq n$ and $1 \leq l \leq q$, one has $r_j = \eta_j + \sum_{l=1}^{q} \gamma_{jl} \zeta_l$ where $\eta_j \sim \mathcal{LN}(\mu_j, \sigma_j^2)$ (the individual noises) and $\zeta_l \sim \mathcal{LN}(\nu_l, \theta_l^2)$. All η_j and ζ_l are independent of each other.
3. One has $\nu_l = 0$, $\theta_l = 0.1$ for all l, $\mu_j = \sigma_j$ for all j, $\sum_{l=1}^{q} \gamma_{jl} \exp\left(\nu_l + \dfrac{\theta_l^2}{2}\right) = \dfrac{\rho_j}{2}$ for all j and $\sum_{j=1}^{n} \exp\left(\mu_j + \dfrac{\sigma_j^2}{2}\right) = 1 + \dfrac{\rho_j}{2}$.

We see that the problem can be rewritten into (1.1) in [2] with $m = 1$. Denote

$$\tilde{x} = (\tau, x_0, x_1, \cdots, x_n)^T.$$

The objective function is simply $f(\tilde{x}) = -\tau$, and the chance constraint is

$$\mathbb{P}\left(F(\tilde{x}, \xi) \leq 0\right) \geq 1 - \alpha$$

where

$$F(\tilde{x}, \xi) = g_0(\tilde{x}) + \sum_{j=1}^{d} \xi_j g_j(\tilde{x}), \;\; d = n + q, \;\; g_0(\tilde{x}) = \tau - x_0,$$

$$\xi_j = \eta_j,\ g_j(\tilde{x}) = -x_j,\ 1 \leq j \leq n,$$

$$\xi_{n+l} = \zeta_l,\ g_{n+l}(\tilde{x}) = -\sum_{j=1}^{n} \gamma_{jl} x_j,\ 1 \leq l \leq q.$$

The Bernstein approximation and standard form formulation

Here we construct the Bernstein approximation to (1), which is a convex optimization problem and reformulate it into a standard form conic program involving exponential cone constraints.

Note that the discretization scheme described in [2] has been adopted and all random variables $\xi_j,\ 1 \leq j \leq d$ are now discrete with finite support. For each j, denote the support and the associated probability masses as $\left\{(v_k^j, p_k^j)\right\}_{1 \leq k \leq N_j}$. In other words, for each $j,\ k = 1, \cdots, N_j$, one has $\mathbb{P}\left(\xi_j = v_k^j\right)$ and the moment generating function of ξ_j is $M_j : z \to \sum_{k=1}^{N_j} p_k^j \exp\left(v_k^j z\right)$.

The Bernstein approximation to (1) is therefore the following convex maximization problem

$$\begin{aligned}
&\max_{\tau, x_0, x_1, \cdots, x_n} (\tau - 1) \\
\text{s.t.}\quad &\sum_{j=0}^{n} x_j \leq 1,\ x_j \geq 0, \forall j \\
&\inf_{t>0} \left(g_0(\tilde{x}) + \sum_{j=1}^{d} t\Lambda_j\left(t^{-1} g_j(\tilde{x})\right) - t \log \alpha \right) \leq 0
\end{aligned} \tag{2}$$

Note that problem (2) is equivalent to

$$\begin{aligned}
&\max_{\tau, x_0, x_1, \cdots, x_n, s_1, \cdots, s_d} (\tau - 1) \\
\text{s.t.}\quad &\sum_{j=0}^{n} x_j \leq 1,\ x_j \geq 0, \forall j \\
&g_0 + \sum_{j=1}^{d} s_j - t \log \alpha = 0 \\
(*)_j :\quad &s_j \geq t\Lambda_j\left(\frac{g_j}{t}\right),\ j = 1, \cdots, d \\
&g_0 = \tau - x_0 \\
&g_j = -x_j,\ j = 1, \cdots, n \\
&g_{n+l} = -\sum_{j=1}^{n} \gamma_{jl} x_j,\ l = 1, \cdots, q
\end{aligned} \tag{3}$$

Since $\Lambda_j(\cdot) = \log M_j(\cdot)$, for $j = 1, \cdots, d$, constraint $(*)_j$ is equivalent to

$$
\begin{aligned}
& \sum_{k=1}^{N_j} p_k^j \exp\left(v_k^j \cdot \frac{g_j}{t}\right) \le \exp\left(\frac{s_j}{t}\right) \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j \exp\left(\frac{v_k^j g_j - s_j}{t}\right) \le 1 \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j \cdot t \exp\left(\frac{v_k^j g_j - s_j}{t}\right) \le t \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad t \exp\left(\frac{w_k^j}{t}\right) \le u_k^j,\ w_k^j = v_k^j g_j - s_j,\ k = 1, \cdots, N_j \\
\Leftrightarrow\ & \sum_{k=1}^{N_j} p_k^j u_k^j = t, \quad \left[w_k^j; u_k^j; t\right] \in \mathcal{K}_{\exp},\ w_k^j = v_k^j g_j - s_j,\ k = 1, \cdots, N_j
\end{aligned}
$$

Eventually, problem (2) can be reformulated into the standard form (PD$'$) in [1], namely (note that $d = n + q$ and the constant term in the objective has been dropped)

$$
\begin{aligned}
& \min\ -\tau \\
\text{s.t.}\ & x_0 + x_1 + \cdots + x_n + s_x = 1 \\
& g_0 + \left(\sum_{j=1}^{d} s_j\right) - (\log\alpha)\, t_0 = 0 \\
& g_0 - \tau + x_0 = 0 \\
& g_j + x_j = 0,\ j = 1, \cdots, n \\
& g_{n+l} + \sum_{j=1}^{n} \gamma_{jl} x_j = 0,\ l = 1, \cdots, q \\
& w_k^j - v_k^j g_j + s_j = 0,\ j = 1, \cdots, d,\ k = 1, \cdots, N_j \\
& \sum_{k=1}^{N_j} p_k^j u_k^j - t_0 = 0,\ j = 1, \cdots, d \\
& t_0 - t_k^j = 0,\ j = 1, \cdots, d,\ k = 1, \cdots, N_j
\end{aligned}
\tag{4}
$$

where the decision variables are

$$
\begin{aligned}
& \tau \in \mathbb{R} \\
& x_0, x_1, \cdots, x_n, s_x \ge 0 \\
& g_0, g_1, \cdots, g_d \in \mathbb{R} \\
& t_0 \ge 0 \\
& s_1, \cdots, s_d \in \mathbb{R} \\
& [w_k^j; u_k^j; t_k^j] \in \mathcal{K}_{\exp},\ j = 1, \cdots, d,\ k = 1, \cdots, N_j.
\end{aligned}
$$

Note that we keep the slack variable $s_x \geq 0$ in the first constraint, although it can be shown that there is always an optimal solution $(x_0^*, x_1^*, \cdots, x_n^*)$ with $\sum_{j=0}^n x_j^* = 1$.

References

[1] Y. Gao. Design and implementation of a homogeneous interior-point method for conic programming involving exponential cone constraints, 2006.

[2] A. Nemirovsky and A. Shapiro. Convex approximation of chance constrained programs. *SIAM Journal on Optimization*, 17(4):969–996, 2006.