Physical System: SI Unit System.

$\vec{x}(t)$: position in Space. [m]

$\vec{v}(t) = \frac{d\vec{x}}{dt}(t)$: Velocity in Space $[\frac{m}{s}]$

$\vec{a}(t) = \frac{d^2\vec{x}}{dt^2}(t)$: acceleration in Space. $[\frac{m}{s^2}]$

m: Mass [kg]

Newton's 3 laws:

1. Every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by an action of an external force.

2. The force acting on an object is equal to time rate change of the momentum.

3. For every action, there is an equal and opposite reaction.

Momentum: $m\vec{v}$

Force: $f = \frac{d}{dt}(m\vec{v})$. Since mass can neither be created nor destroyed, it must be a constant. $\Rightarrow f = m\frac{d\vec{v}}{dt} = m\vec{a} \Leftarrow$ 2nd Law.

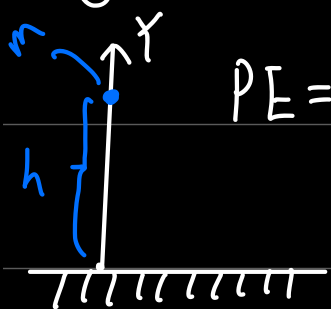
Variational Mechanics: $(KE + PE \Rightarrow q(t))$.

1. Kinetic Energy: Energy due to motion.

2. Potential Energy: Energy "held" within an object due to its position, internal stresses, electrical charges etc...

"Has the potential to become kinetic energy"

e.g.:



$PE = mgh$

height above ground.

Scalar acceleration

$$\overbrace{e\left(f(t), \dot{f}(t), \dots\right)}^{\text{Function of time and derivatives}} \longmapsto \mathbb{R}$$

functional

Real Numbers

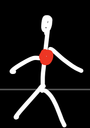
Path of physical objects in Space.

Generalized Coordinates:

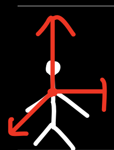
$$x(t) = f(\underbrace{q(t)}_{\text{generalized Coordinates}})$$

$$\frac{dx}{dt} = \overbrace{\frac{df}{dq}}^{\text{Jacobian}} \underbrace{\dot{q}(t)}_{\text{generalized Velocity}}$$

e.g.

 $X(t) = q(t)$ $\Rightarrow$ Jacobian is an identity map.

$q(t) \in \mathbb{R}^3$

 $$x(t) = \overbrace{RX + P}^{q}$$

Strength of Variational Approach.

The Lagrangian: (Joseph-Louis Lagrange)

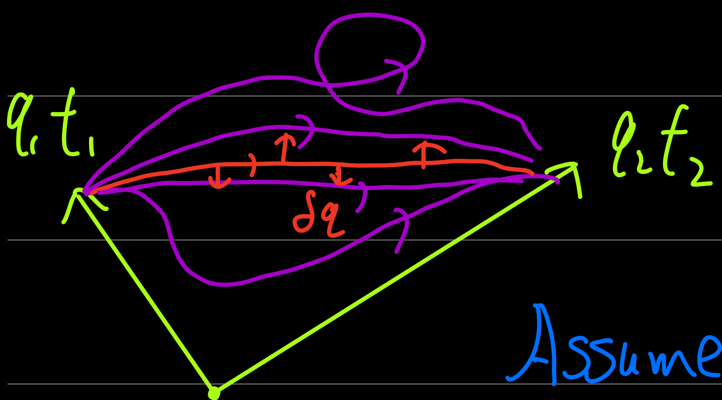
$$\underbrace{T}_{\text{Kinetic Energy}} - \overbrace{V}^{\text{Potential Energy}}$$

$$L = T - V$$

Potential Energy

Kinetic Energy

Principle of Least Action: (Stationary Point)



Assume you know the end points, find the path between them by finding a stationary point of Action.

$$\underbrace{S}_{\text{Action}}(q(t), \dot{q}(t)) = \int_{t_1}^{t_2} \overbrace{T(q(t), \dot{q}(t)) - V(q(t), \dot{q}(t))}^{\text{Lagrangian}}\, dt$$

Minimize by finding a flat spot

Hunt for a flat spot by perturbing the trajectory and seeing if S changes.

Perturbation / Small push.

$$S(q+\delta q, \dot{q}+\underset{\perp}{\delta \dot{q}}) = S(q(t), \dot{q}(t))$$

$\Rightarrow$ Correct Sln is $q(t)$, which is Stationary

$$\Leftrightarrow \boxed{S(q+\delta q, \dot{q}+\delta\dot{q}) = S(q, \dot{q})}.$$

Key Take Away: The Solution to the least action problem $\Leftrightarrow$ Finding a path $q(t)$ to make $S(q,\dot{q})$ Stationary $\Leftrightarrow$ $\delta S = 0$ $\square$.

Note: q minimizing S s.t $\delta S \neq 0$ $\Rightarrow$ instability in physical System $\Rightarrow$ Can not happen in reality.

Calculus of Variation:

$$\boxed{S(q(t), \dot{q}(t)) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t))\, dt}$$

$$S(q+\delta q, \dot{q}+\delta\dot{q}) = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q})\, dt.$$

Apply Taylor Expansion:

$$\approx \underbrace{\int_{t_1}^{t_2} L(q,\dot{q})\, dt}_{S(q(t),\dot{q}(t))} + \boxed{\underbrace{\int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\, dt}_{\delta S(q,\dot{q})}}$$

$\Rightarrow \delta S(q(t), \dot{q}(t)) = 0 \Leftrightarrow$ find $q, \dot{q}$

Back to Calculus of Variation! $\int f g' = -\int f' g + fg$.

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q + \underbrace{\frac{\partial L}{\partial \dot{q}}\delta \dot{q}}_{\text{Apply integration by parts}} dt = 0$$

We know BCs

$\Rightarrow \delta q = 0$ @ t_0 & t_1.

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q \, dt + \underbrace{\cancel{\frac{\partial L}{\partial \dot{q}}\delta q \Big|_{t_0}^{t_1}}}_{\text{Uh, Oh, Boundary Conditions.}} = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q \, dt = 0$$

$$\Leftrightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \quad \text{Euler-Largrange Equation}}$$

$\hookrightarrow q$ is a physically valid trajectory

or equation of motion.

Summary!

① $S(q,\dot{q})$, $\delta S(q,\dot{q})$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$, BCs $\Rightarrow \delta q = 0$.

② $q, \dot{q}$ s.t $\delta S \neq 0$ is not a physically valid sln even it results in a lower S.

③ Potential Energy has the "potential" to become kinetic energy.

④ Variation Mechanics provides a unifying principle for particle systems, Deformable Objects, Fluids, Rigid Bodies & More!