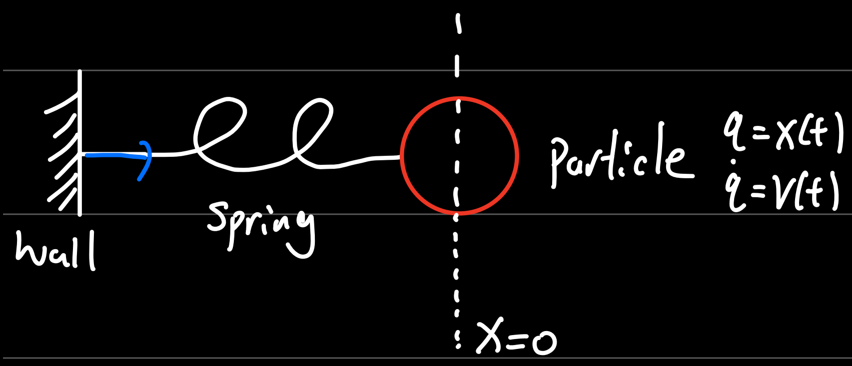
Mass-Spring System in 1D:



Hooke's Law – Force is linearly proportional to stretch in Spring

$$f = -kx$$

Potential Energy is the negative of the mechanical work

$$W = \int \underbrace{-kx(t)}_{\text{Force}} \overbrace{v(t)\,dt}^{\text{Displacement}} = \int -kx(t)\,dx = -\frac{1}{2}kx^2$$

$$\Rightarrow V = -W = \frac{1}{2}kx^2$$

Applying Euler-Lagrange: $L = T - V$

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}(m\dot{q})$$

$$\frac{\partial L}{\partial q} = -kq$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

$$\Longrightarrow \frac{d}{dt}(m\dot{q}) = -kq$$

$$m\ddot{q} = -kq$$ (Equation of motion)

Time Integration:

Initial Conditions.

$$q_0 = q(t_0) \quad \dot{q}_0 = \dot{q}(t_0)$$

Input: $\underbrace{\ddot{q} = f(q, \dot{q})}_{\text{ordinary Differential Equation (ODE)}}$

Output: $\underbrace{q^{t+1} = f(q^t, q^{t+1}, \ldots, \dot{q}^t, \dot{q}^{t+1})}_{\boxed{\text{Discrete Update Equation.}}}$

The Coupled First Order System: (Rewrite in Matrix Form)

$m\ddot{q} = -kq$ Second-order ODE

$\dot{q} = v$ Introduce Velocity

$m\dot{v} = -kq$ First Order ODE $\Leftrightarrow \begin{cases} m\dot{v} = -kq \\ \dot{q} = v \end{cases}$

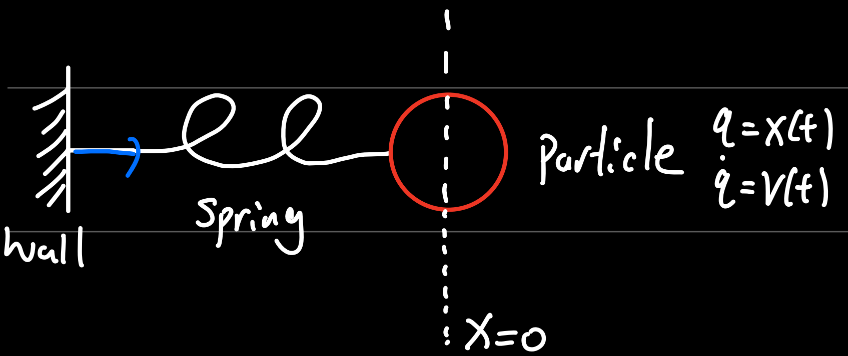
Rewrite in Matrix Form:

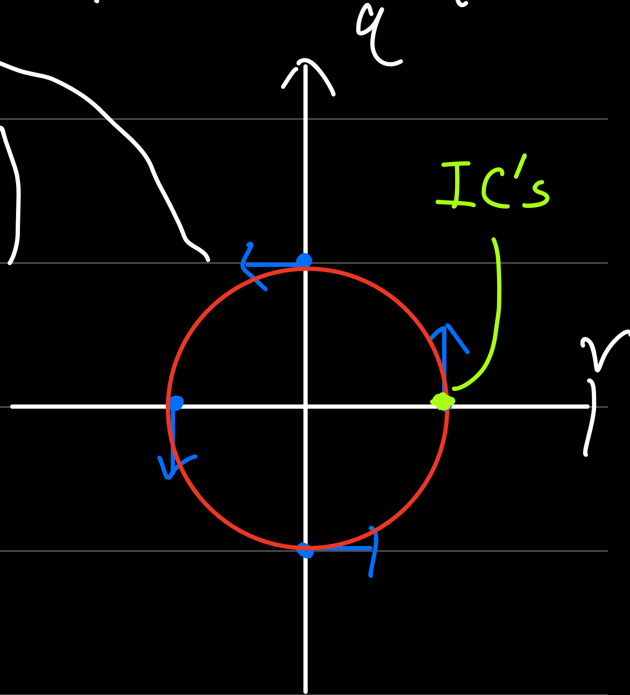
$$\Rightarrow \underbrace{\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}}_{A} \underbrace{\frac{d}{dt}\begin{pmatrix} v \\ q \end{pmatrix}}_{\dot{Y}} = \overbrace{\begin{pmatrix} 0 & -k \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} v \\ q \end{pmatrix}}_{Y}}^{f(Y)}$$

$$\boxed{\overbrace{A\dot{Y} = f(Y)}^{\text{1st order ODE}}}$$

The Phase Space: (Phase Space Trajectory tells Stability).

$$\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} 0 & -k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix}$$





$\underset{(v=0)}{q>0, q<0}$: v decreases and increases respectively by observing $m\dot{v} = -kq$. $\overset{(q=0)}{v>0, v<0}$: q increases and decreases respectively by physical intuition.

Inward ⟺ losing energy (Inaccurate)
Outward ⟺ gaining energy (Instability)
⟹ Aim for both accuracy + stability.
⟺ Doesn't gain and lose energy.

Types of Integration Algorithms:

Explicit: Next time step can be computed entirely using values from the current time step or before.

Implicit: Next time step is computed using values from future!

Concerns: 1. Performance 2. Stability 3. Accuracy

2\. Stability → Phase Space
3\. Accuracy → Visual

Forward Euler Integration: *

$A\dot{Y} = f(Y)$

Replace derivative with finite difference: $\dot{Y} \approx \frac{1}{\Delta t}(Y^{t+1} - Y^t)$

$A\frac{1}{\Delta t}(Y^{t+1} - Y^t) = \underline{f(Y^t)}$

Evaluated at current time step.

$Y^{t+1} = Y^t + \Delta t A^{-1} f(Y^t)$

$\Leftrightarrow v^{t+1} = v^t - \Delta t \frac{k}{m} q^t$

$q^{t+1} = q^t + \Delta t v^t$

Runge-Kutta Time Integration: *

Integrate using average "slope"

Example using two slopes:

$Y^{t+1} = Y^t + \Delta t A^{-1}(\alpha f(Y^{t+a}) + \beta f(Y^{t+b}))$

a, b: time coefficients

α, β: averaging coefficients

$f(Y^{t+a}), f(Y^{t+b})$: 2 slopes.

Use Forward Euler to estimate Y^{t+a}:

$$\hat{Y}^{t+a} = Y^t + a\Delta t A^{-1} f(Y^t) \quad \text{(Scaled by } \alpha\text{)}$$

<u>Heun's Method: $a=0$, $b=1$, $\alpha=\beta=\frac{1}{2}$.</u>

$$Y^{t+1} = Y^t + \frac{\Delta t}{2} A^{-1}\left(f(Y^t) + f(\hat{Y}^{t+1})\right)$$

Using $\hat{Y}^{t+1} = Y^t + 1 \cdot \Delta t A^{-1} f(Y^t)$

↳ rewrite into standard form

$$k_1 = A^{-1} f(Y^t)$$

$$k_2 = A^{-1} f(Y^t + \Delta t \cdot k_1)$$

$$Y^{t+1} = Y^t + \frac{\Delta t}{2}(k_1 + k_2).$$

<u>Fourth order Runge-Kutta:</u>

$$k_1 = A^{-1} f(Y^t)$$

$$k_2 = A^{-1} f\left(Y^t + \frac{\Delta t}{2} \cdot k_1\right)$$

$$k_3 = A^{-1} f\left(Y^t + \frac{\Delta t}{2} \cdot k_2\right)$$

$$k_4 = A^{-1} f(Y^t + \Delta t \cdot k_3)$$

$$Y^{t+1} = Y^t + \frac{\Delta t}{6}(k_1 + 2k_2 + 2 \cdot k_3 + k_4).$$

Backward Euler Time Integration:

$$A\dot{Y} = f(Y)$$

$$\underbrace{\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}}_{A} \underbrace{\frac{d}{dt}\begin{pmatrix} v \\ q \end{pmatrix}}_{\dot{Y}} = \overbrace{\underbrace{\begin{pmatrix} 0 & -k \\ 1 & 0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} v \\ q \end{pmatrix}}_{Y}}^{f(Y)} \Rightarrow A\dot{Y} = BY$$

Replace first time derivative with finite difference:

$$A\frac{1}{\Delta t}(Y^{t+1} - Y^{t}) = f(Y^{t+1}) = BY^{t+1}$$

Note: Evaluating at $(t+1)$ $\Leftrightarrow$ $\boxed{\dot{Y} \approx Y^{t+1} - Y^{t}}$ is using backward Euler

$$Y^{t+1} = Y^{t} + \Delta t A^{-1} B Y^{t+1}$$

$$\Leftrightarrow (I - \Delta t A^{-1} B) Y^{t+1} = Y^{t}$$

$$v^{t+1} + \Delta t \frac{k}{m} q^{t+1} = v^{t}$$

$$q^{t+1} - \Delta t v^{t+1} = q^{t} \Leftrightarrow q^{t+1} = q^{t} + \Delta t v^{t+1}$$

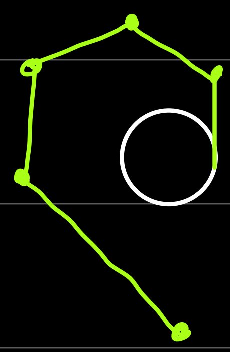
$$\Rightarrow v^{t+1} + \Delta t \frac{k}{m}(q^{t} + \Delta t v^{t+1}) = v^{t}$$

$$v^{t+1}\left(1 + \Delta t^{2} \frac{k}{m}\right) = v^{t} - \Delta t \frac{k}{m} q^{t}$$

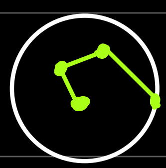
$$q^{t+1} = q^{t} + \Delta t \boxed{v^{t+1}}$$

Phase Space Trajectory:

1. Explicit ⇒ exploding



2. Implicit ⇒ damping



Sympletic Euler Time Integration!

$v^{t+1} = v^t - \Delta t \frac{k}{m} q^t$ (v^{t+1} using Forward Euler)

$q^{t+1} = q^t + \Delta t \, v^{t+1}$ (q^{t+1} using Backward Euler).

↳ Cancel out exploding and damping

⇒ area preserving ⇔ energy conservation!

Summary:

1. $T=\frac{1}{2}m\dot{q}^2$, $V=\frac{1}{2}kq^2 \Rightarrow L=T-V=\frac{1}{2}m\dot{q}^2-\frac{1}{2}kq^2$

$\Rightarrow \frac{\partial L}{\partial q}=-kq$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}=\frac{d}{dt}(m\dot{q})=m\ddot{q}$

$\Rightarrow m\ddot{q}=-kq$ (Second order ODE from Euler-Largrange)

2. $v=\dot{q} \Rightarrow m\dot{v}=-kq \Rightarrow \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} v \\ q \end{pmatrix}=\begin{pmatrix} 0 & -k \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix}$

as $A\dot{Y}=BY=f(Y)$ (1st order ODE)

3. Forward Euler at $\underline{t}$: $\dot{Y}\approx\frac{1}{\Delta t}(Y^{t+1}-Y^t)$

Backward Euler at $\underline{t+1}$: $\dot{Y}\approx\frac{1}{\Delta t}(Y^{t+1}-Y^t)$

Symplectic Euler: Velocity using Forward Euler

& Position using Backward Euler.

4. Runge-Kutta: Ignore proof, just go Wikipedia for general scheme & Analysis such as. order of accuracy, Consistency & stability. RK4: 4th order accurate scheme but still explicit.

Test Derivation: Not part of the note!

Given: $m\ddot{q} = -kq$ 2nd order ODE

Let $r = \dot{q} \Rightarrow m\dot{r} = -kq \Rightarrow \begin{cases} m\dot{r} = -kq \\ \dot{q} = r \end{cases}$ 1st order ODEs

$$\Rightarrow \underbrace{\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\frac{d}{dt}\begin{bmatrix} r \\ q \end{bmatrix}}_{\dot{Y}} = \underbrace{\begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} r \\ q \end{bmatrix}}_{Y}$$

Rearranging:

$$\dot{Y} = A^{-1} B Y$$

$$\Leftrightarrow \frac{dY}{dt} = \begin{bmatrix} \frac{dr}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{-k}{m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix}$$

$$\Rightarrow \frac{dY}{dt} = CY \Rightarrow \frac{dY}{dt} = f(Y)$$

Forward Euler:

$$\frac{1}{\Delta t}(Y^{t+1} - Y^t) = A^{-1} B Y$$

$$\Leftrightarrow Y^{t+1} = Y^t + \Delta t A^{-1} B Y$$

$$\Leftrightarrow \begin{bmatrix} r^{t+1} \\ q^{t+1} \end{bmatrix} = \begin{bmatrix} r^t \\ q^t \end{bmatrix} + \Delta t \begin{bmatrix} 0 & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r^t \\ q^t \end{bmatrix}$$

$$= \begin{bmatrix} r^t \\ q^t \end{bmatrix} + \Delta t \begin{bmatrix} -\frac{k}{m} q^t \\ r^t \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} v^{t+1} = v^t - \Delta t \frac{k}{m} q^t \\ q^{t+1} = q^t + \Delta t\, v^t \end{cases} \quad \text{(update Equ)}$$

Backward Euler:

$$\frac{1}{\Delta t}(Y^{t+1} - Y^t) = A^{-1} B Y^{t+1}$$

$$Y^{t+1} = Y^t + \Delta t A^{-1} B Y^{t+1}$$

$$\Rightarrow (I - \Delta t A^{-1} B) Y^{t+1} = Y^t$$

$$\Leftrightarrow \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \Delta t \begin{bmatrix} 0 & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \right) \begin{pmatrix} v^{t+1} \\ q^{t+1} \end{pmatrix} = \begin{pmatrix} v^t \\ q^t \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & \Delta t \frac{k}{m} \\ -\Delta t & 1 \end{pmatrix} \begin{pmatrix} v^{t+1} \\ q^{t+1} \end{pmatrix} = \begin{pmatrix} v^t \\ q^t \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} v^{t+1} + \Delta t \frac{k}{m} q^{t+1} \\ -\Delta t\, v^{t+1} + q^{t+1} \end{pmatrix} = \begin{pmatrix} v^t \\ q^t \end{pmatrix}$$

$$v^{t+1} = v^t - \Delta t \frac{k}{m} q^{t+1}$$

$$q^{t+1} = q^t + \Delta t\, v^{t+1}$$

$$v^{t+1} = v^t - \Delta t \frac{k}{m} \left(q^t + \Delta t\, v^{t+1} \right)$$

$$\hookrightarrow v^{t+1} = v^t - \Delta t \frac{k}{m} q^t - \Delta t^2 \frac{k}{m} v^{t+1}$$

$$v^{t+1}\left(1 + \Delta t^2 \frac{k}{m}\right) = v^t - \Delta t \frac{k}{m} q^t$$

$$q^{t+1} = q^t + \Delta t\, v^{t+1}$$

(update Equ)

RK4:

$$A\dot{Y} = BY \Leftrightarrow \dot{Y} = A^{-1}BY$$ (t never appears on right hand side $\Rightarrow f(Y)$)

$$\Rightarrow \frac{dY}{dt} = f(t, Y(t))$$

is reduced to:

$$\frac{dY}{dt} = f(Y)$$

$$\Rightarrow \frac{dY}{dt} = \begin{bmatrix} dv/dt \\ dq/dt \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v \\ q \end{bmatrix}}_{f(Y)} = \begin{bmatrix} -\frac{k}{m}q \\ r \end{bmatrix}$$

$$k_1 = f(Y) = \begin{bmatrix} -\frac{k}{m}q \\ r \end{bmatrix}$$

$$k_2 = f(Y + \frac{\Delta t}{2}\cdot k_1) = \begin{bmatrix} -\frac{k}{m}(q + \frac{\Delta t}{2}k_{1r}) \\ r + \frac{\Delta t}{2}k_{1q} \end{bmatrix}$$

$$k_3 = f(Y + \frac{\Delta t}{2}\cdot k_2) = \begin{bmatrix} -\frac{k}{m}(q + \frac{\Delta t}{2}\cdot k_{2r}) \\ r + \frac{\Delta t}{2}k_{2q} \end{bmatrix}$$

$$k_4 = f(Y + \Delta t\cdot k_3) = \begin{bmatrix} -\frac{k}{m}(q + \Delta t\cdot k_{3r}) \\ r + \Delta t\cdot k_{3q} \end{bmatrix}$$

$$\Rightarrow Y^{t+1} = Y^t + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Note: key is to probe $\frac{dY}{dt} = f(t, Y(t))$. If right hand side doesn't involve t, then $\frac{dY}{dt} = f(Y(t))$. Otherwise, we have $k_2 = f(t+\frac{\Delta t}{2}, Y+\frac{\Delta t}{2}k_1)$ ... so on. Similarly, if right hand side doesn't not have Y, then $\frac{dY}{dt} = f(t)$. $\Rightarrow$

$Y = \int f(t)dt = F(t) + C$. Since $Y(t_0) = Y_0$ we have.

$\Rightarrow Y = F(t) + (Y_0 - F(t_0))$.

<u>Symplectic Euler:</u>

Steal v^{t+1} from Forward Euler

Steal q^{t+1} from Backward Euler.