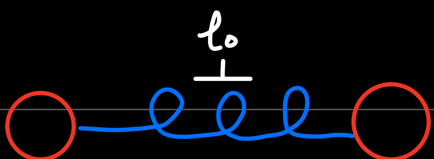
A Single Spring in 3D:

Support any Mesh
e.g. triangle, quad and Tet.



Particle 0 $\quad x_0(t) \in \mathbb{R}^3$

Particle 1 $\quad x_1(t) \in \mathbb{R}^3$

The generalized Coordinates for mass Spring System:

$$q = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\dot{q} = \begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

The Lagrangian: $L = T - V$

Spring & 2 masses form a System

Particle 0 $\quad \frac{1}{2} m \|v_0\|_2^2$ $\quad + \quad$ Particle 1 $\quad \frac{1}{2} m \|v_1\|_2^2$

Kinetic Energy for 3D Mass-Spring System:

Total Kinetic Energy:

$$\sum_{i=0}^{1} \frac{1}{2} m \|v_i\|_2^2 = \sum_{i=0}^{1} \frac{1}{2} m v_i^T v_i = \sum_{i=0}^{1} \frac{1}{2} v_i^T \underbrace{(mI)}_{M_i} v_i$$

per particle mass matrix

$$\sum_{i=0}^{1} \frac{1}{2} v_i^T \underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}}_{M_i} v_i$$

M_i = Per particle Mass Matrix

$$\frac{1}{2}\dot{q}^T \underbrace{M}_{\begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix}} \dot{q}$$

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}^T \begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = v_0^T M_0 v_0 + v_1^T M_1 v_1$$

Block Diagonal Mass Matrix

↳ Scalars → matrix → Block Matrix

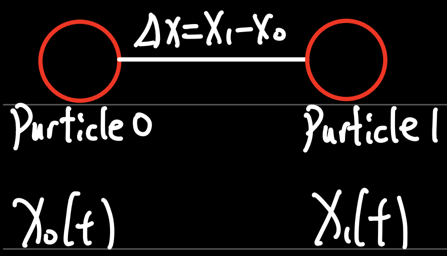
Potential Energy for a 3D Spring:

1. Spring should go back to original length when all external forces are removed. i.e. rest length should be a minimum. $l - l_0$ can be negative violating
2. Rigid Body (translation and/or rotation) should not change the energy.
3. Energy should depend on particle positions.

 (Simplify).

Strain: $\underbrace{l}_{\text{Deformed Length}} - l_0$ (CG's strain)

Potential Energy: $\frac{1}{2}k(\ell-\ell_0)^2$



$$\Delta x = \underbrace{\begin{pmatrix} -I & I \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}}_{q}$$

$$\ell = \sqrt{\Delta x^T \Delta x} = \sqrt{q^T B^T B q}$$

length in terms of generalized coordinates q.

The Lagrangian!

$$V = \frac{1}{2}k(\ell - \ell_0)^2$$

$$= \frac{1}{2}k\left(\sqrt{q^T B^T B q} - \ell_0\right)^2 \quad \text{(Use Symbolic Toolbox MATLAB)}$$

$$L = \underbrace{T}_{\frac{1}{2}\dot{q}^T M \dot{q}} - V$$

Euler-Lagrange Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} = \frac{\partial (T-V)}{\partial q} = \underbrace{-\frac{\partial V}{\partial q}}_{\text{Generalized Forces } f}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{q}^T M \dot{q}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{q}^T M \dot{q}\right) = \frac{d}{dt}(M\dot{q})$$

$$\frac{d}{dt}(M\dot{q}) = M\ddot{q}$$

Large Mass-Spring Systems in 3D: (Bunny System)

$$q = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \dot{q} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Total Kinetic Energy $\sum_{i=0}^{n-1} \frac{1}{2} v_i^T \underbrace{\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}}_{M_i} v_i$

$$\frac{1}{2}\dot{q}^T \underbrace{M}_{\begin{pmatrix} M_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{n-1} \end{pmatrix} \in \mathbb{R}^{3n \times 3n}} \dot{q}$$

Block Diagonal Matrix

Total Potential Energy : $V = \sum_{j=}^{m-1} \underbrace{V_j}_{\text{Potential Energy for each Spring.}} (x_A, x_B)$

Aside: Selection Matrices:

$$\begin{pmatrix} A_0 & \underbrace{A_1}_{\mathbb{R}^{3\times3}} & \cdots & A_n \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} A_i x_i$$

Select i^{th} particle position:

$A_i = I$

$A_{j \neq i} = 0$

$$\underbrace{\begin{pmatrix} 0 & I & \cdots & 0 \end{pmatrix}}_{S_1} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1$$

Total potential Energy $V = \sum_{j=0}^{m-1} V_j(x_A, x_B)$

$$q_j = \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \underbrace{\begin{pmatrix} S_1 \\ S_n \end{pmatrix}}_{E_j} q$$

Easier to Comput via B & q_j

$$\Rightarrow V = \sum_{j=0}^{m-1} V_j(q_j) = \sum_{j=0}^{m-1} \underbrace{V_j}_{\text{Potential Energy for each Spring}}(E_j q)$$

The Largrangian:

$$L = \underbrace{T}_{\frac{1}{2}\dot{q}^T M \dot{q}} - \overbrace{V}^{\sum_{j=0}^{m-1} V_j(E_j q)}$$

$$\Rightarrow M\ddot{q} = -\frac{\partial V}{\partial q}$$

Generalized forces:

$\frac{\partial}{\partial q}$: Gradients & derivatives are linear operators.

$$-\frac{\partial V}{\partial q} = -\frac{\partial}{\partial q}\sum_{j=0}^{m-1} V_j(E_j q)$$

$$-\frac{\partial}{\partial q}\sum_{j=0}^{m-1} V_j(E_j q) = -\sum_{j=0}^{m-1} \frac{\partial}{\partial q} V_j(E_j q)$$

$$-\sum_{j=0}^{m-1} \frac{\partial}{\partial q} V_j(E_j q) = -\sum_{j=0}^{m-1} E_j^T \frac{\partial V_j}{\partial q_j}(q_j)$$

let $q_j = E_j q$

6x1

3nx6

$$\Rightarrow \frac{\partial V_j}{\partial q_j} \cdot \boxed{\frac{\partial (E_j q)}{\partial q}}$$

$$\Downarrow$$

$$E_j^T \cdot \frac{\partial V_j}{\partial q_j}$$

$$\Rightarrow E_j^T \frac{\partial V_j}{\partial q_j}$$

per spring gradient

$$-\frac{\partial V}{\partial q} = -\sum_{j=0}^{m-1} E_j^T \underbrace{\frac{\partial V_j}{\partial q_j}(q_j)}_{\text{per spring gradient}} = \sum_{j=0}^{m-1} E_j^T \underbrace{f_j(q_j)}_{\text{per-spring generalized force.}}$$

$(3n \times 1)$ generalised force vector

Assembly of forces:

Accumulate forces on particles

force on particle A

$$f_j = \begin{pmatrix} f_A \\ f_B \end{pmatrix} \qquad E_j^T \qquad f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

force on particle B.

Accumulate forces back to global force vector.

Linear-Implicit Time Integration:

$$A\dot{r} = f(r).$$

Equations of motion: $M\ddot{q} = f$

$$M\dot{q}^{t+1} = M\dot{q}^{t} + \Delta t\, f(q^{t+1})$$

$$q^{t+1} = q^{t} + \Delta t\, \dot{q}^{t+1}$$

Backward Euler

$$M\dot{q}^{t+1} = M\dot{q}^{t} + \Delta t\, f(q^{t} + \Delta t\, \dot{q}^{t+1})$$

Substitute.

First order Approximation

$$M\dot{q}^{t+1} = M\dot{q}^{t} + \Delta t f(q^{t}) + \Delta t^2 \underbrace{\frac{\partial f}{\partial q}}_{\text{Stiffness Matrix } K.} \dot{q}^{t+1}$$

Rearrange

$$\left(M - \Delta t^2 \frac{\partial f}{\partial q}\right)\dot{q}^{t+1} = M\dot{q}^{t} + \Delta t f(q^{t})$$ – Solve

$$q^{t+1} = q^{t} + \Delta t\, \dot{q}^{t+1}$$ – update Position

The Stiffness Matrix:

By definition:

$$K = \frac{\partial f}{\partial q} \qquad f = -\frac{\partial}{\partial q}\sum_{j=0}^{m-1} V_j(E_j q)$$

$$K = -\frac{\partial^2}{\partial q^2}\sum_{j=0}^{m-1} V_j(E_j q)$$

$$\frac{\partial}{\partial q}\left(\overbrace{\frac{\partial V_j}{\partial q_j}}^{6\times1}\ \overbrace{\frac{\partial (E_j q)}{\partial q}}^{6\times3n}\right)$$

$$K = -\sum_{j=0}^{m-1}\frac{\partial^2}{\partial q^2} V_j(E_j q)$$

$$\frac{\partial^2 V_j}{\partial q \partial q_j} = \frac{\partial^2 V_j}{\partial q_j \partial q}.$$

$$K = \sum_{j=0}^{m-1}\left(-E_j^T \frac{\partial^2 V_j}{\partial q_j^2} E_j\right) \qquad = \frac{\partial}{\partial q}\left(E_j^T \frac{\partial V_j}{\partial q_j}\right)$$

$$= \sum_{j=0}^{m-1}\left(E_j^T K_j E_j\right) \qquad = E_j^T\left(\frac{\partial}{\partial q_j}\left(\frac{\partial V_j}{\partial q}\right)\right)$$

$$\boxed{K_j = -\frac{\partial^2 V_j}{\partial q_j^2}} \qquad = E_j^T\left(\frac{\partial}{\partial q_j}\left(\frac{\partial V_j}{\partial q_j}\cdot\frac{\partial E_j q}{\partial q}\right)\right)$$

$$= E_j^T \frac{\partial^2 V_j}{\partial^2 q_j^2} E_j$$

$3n\times 6 \quad 6\times 6 \quad 6\times 3n.$ ✓

Assembly of Stiffness Matrix:

Accumulate Matrices

$$K_j = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{AB}^T & K_{BB} \end{pmatrix} \qquad \begin{bmatrix} K_{00} & \cdots & K_{0n} \\ \vdots & \ddots & \vdots \\ K_{n0} & \cdots & K_{nn} \end{bmatrix}$$

$3\times 3 \quad 3\times 3 \quad 3\times 3 \quad 3\times 3.$

$3n\times 6 \quad 6\times 6 \quad 6\times 3n.$

Key: We know how to compute f_j & K_j.

Fixed Boundary Conditions:

$q_i = b_i$

$(\cdot)^T$: Small Compact to Large System.

degrees of freedom (DOF)

$\perp$

$\hat{q} = Pq$

Selection Matrix that selects Non-Fixed Points.

$$P^T\hat{q} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \\ \vdots \\ x_n \end{bmatrix}$$

Zero in fixed positions.

$$P^T\hat{q} = \underbrace{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \\ \vdots \\ x_n \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ b_i \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b}}_{q}$$

$$q = P^T\hat{q} + b$$

$$\dot{q} = P^T\dot{\hat{q}} + \cancel{\dot{b}}$$ Fixed boundary $\Rightarrow \dot{b} = 0$.

Substitute into discrete update equation.

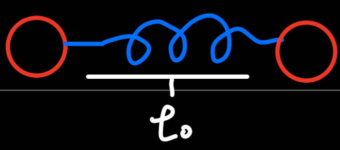
$$(M - \Delta t^2 K)\,\dot{q}^{t+1} = M\dot{q}^t + \Delta t f(q^t)$$

$$q^{t+1} = q^t + \Delta t\,\dot{q}^{t+1}.$$

$$\Rightarrow (M - \Delta t^2 K) P^T \dot{\hat{q}}^{t+1} = M P^T \dot{\hat{q}}^t + \Delta t f(q^t)$$

$$q^{t+1} = q^t + \Delta t P^T \dot{\hat{q}}^{t+1}$$

Summary:

1. 

$i = 1, \dots, n$ particles
$j = 1, \dots, m$ springs

Each Spring can only have 2 particles attached to it.

$q_0, q_1 \in \mathbb{R}^{3\times 1}$: Positions of the two particles

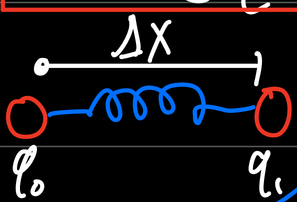
$\Rightarrow \Delta x = q_1 - q_0$: Deformed length of the Spring

a.k.a "ℓ". The key here is we want an expression for Potential energy : $V = \frac{1}{2}k(\ell - \ell_0)^2$ for each Spring $\Rightarrow$ want V_j.

Define $B = \begin{pmatrix} -I & I \end{pmatrix}$ and $q = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$

$\Delta x = Bq \Rightarrow \ell = \sqrt{\Delta x^T \Delta x} = \sqrt{q^T B^T B q}$

Tools.



$$V = \frac{1}{2}k(\ell - \ell_0)^2 = \frac{1}{2}k\left(\sqrt{q^T B^T B q} - \ell_0\right)$$

$\Leftarrow \forall j \in \{0, \dots, m-1\},\ V_j = \frac{1}{2}k\left(\sqrt{q_j^T B^T B q_j} - \ell_0\right)$

$\Rightarrow V = \sum_{j=0}^{m-1} V_j(q_0, q_1) = \sum_{j=0}^{m-1} V_j(q_j).$

Total Potential Energy : V

S_0, S_1 : Selection Matrix to extract q_0, q_1

e.g $S_0 \cdot q = q_0$ & $S_1 q = q_1$ where $S \in \mathbb{R}^{3\times 3n}$

E_j : Selection Matrix to extract q_j

e.g $q_j = E_j q$, where $E_j \in {}^{6\times 3n}$.

Conversely, $E_j^T q_j$ projects 6×1 vector back to $(3n \times 1)$ vector q to be "accumulated"!

P : $3(n-m) \times 3n$ Projection matrix s.t

$\hat{q} = Pq$: $3(n-m) \times 1$ Vector.

$$\boxed{\begin{aligned} &\Rightarrow q = P^T \hat{q} + b \text{ : Should not update } b. \\ &\Leftarrow \dot{q} = P^T \dot{\hat{q}} + \cancel{\dot{b}} \text{ : Fixed } b \Rightarrow \dot{b} = 0. \end{aligned}}$$

BC's

Generalized force $f = \frac{-\partial V}{\partial q}$: (Important)

↳ Global force

$$-\frac{\partial V}{\partial q} = -\frac{\partial}{\partial q}\sum_{j=0}^{m-1} V_j(E_j q)$$

$$= -\sum_{j=0}^{m-1} \frac{\partial}{\partial q} V_j(E_j q)$$

$$= -\sum_{j=0}^{m-1} \frac{\partial V_j}{\partial q_j} \cdot \frac{\partial (E_j q)}{\partial q}$$

$$= \sum_{j=0}^{m-1} E_j^T \left(-\frac{\partial V_j}{\partial q_j}\right)$$

$$= \sum_{j=0}^{m-1} E_j^T \boxed{f_j} \in \mathbb{R}^{6\times 1} \left(f_j = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \right)$$

Note: $V_j = \frac{1}{2} k \left(\sqrt{q_j^T B^T B q_j} - \ell_0\right) \Rightarrow$ MATLAB

for $\boxed{\frac{\partial V_j}{\partial q_j} \Leftarrow \text{gradient of } V_j.}$

Stiffness Matrix $K = -\frac{\partial^2 V}{\partial q^2}$: (Important)

Why? We need it for Linear-Implicit Euler Scheme!

(Something Hidden from: $M\ddot{q} = \frac{-\partial V}{\partial q}$)

$$K = \frac{\partial^2 V}{\partial q^2} = -\sum_{j=0}^{m-1} \frac{\partial^2}{\partial q^2}\left(V_j(E_j q)\right)$$

$$= -\sum_{j=0}^{m-1} \frac{\partial}{\partial q}\left(\frac{\partial V_j}{\partial q_j} \frac{\partial (E_j q)}{\partial q}\right)$$

$$= -\sum_{j=0}^{m-1} \frac{\partial}{\partial q}\left(E_j^T \frac{\partial V_j}{\partial q_j}\right) \quad 6\times 1 \quad 6\times 6.$$

$$= -\sum_{j=0}^{m-1} E_j^T \frac{\partial}{\partial q_j}\left(\frac{\partial V_j}{\partial q}\right) \quad \cancel{3n\times 6}. \quad 6\times 3n.$$

$$= -\sum_{j=0}^{m-1} E_j^T \frac{\partial}{\partial q_j}\left(\frac{\partial V_j}{\partial q_j} \frac{\partial (E_j q)}{\partial q}\right)$$

$$= \sum_{j=0}^{m-1} E_j^T \left(-\frac{\partial^2 V_j}{\partial q_j^2}\right) E_j$$

$$= \sum_{j=0}^{m-1} E_j^T \boxed{(K_j)} E_j$$

$$K_j = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} \quad \in \mathbb{R}^{3\times 3}$$

$$K_j \in \mathbb{R}^{6\times 6}$$

Note: $\frac{\partial^2 (\cdot)}{\partial x \partial r} = \frac{\partial^2 (\cdot)}{\partial r \partial x} \Rightarrow K_j$ is a Symmetric matrix

$K_j = -\frac{\partial^2 V_j}{\partial^2 q_j^2} \Rightarrow$ Hessian of V_j using MATLAB

Total Kinetic Energy: T

$$T = \sum_{i=0}^{n-1} \frac{1}{2} m \|\dot{q}_i\|^2 = \sum_{i=0}^{n-1} \frac{1}{2} m \dot{q}_i^T \dot{q}_i$$

$$= \sum_{i=0}^{n-1} \frac{1}{2} \dot{q}_i^T (mI) \dot{q}_i = \sum_{i=0}^{n-1} \frac{1}{2} \dot{q}_i^T M_i \dot{q}_i$$

$$\text{Define } \mathcal{M} = \begin{pmatrix} \mathcal{M}_0 & & \\ & \ddots & \\ & & \mathcal{M}_{n-1} \end{pmatrix}$$

$$\boxed{\Rightarrow T = \frac{1}{2}\dot{q}^T \mathcal{M} \dot{q} : \text{Total Kinetic Energy}}$$

<u>Euler-Lagrange:</u> $L = T - V$

$$L = \frac{1}{2}\dot{q}^T \mathcal{M}\dot{q} - V(q)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \mathcal{M}\ddot{q} \quad \& \quad \frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} = f$$

$$\Leftrightarrow \mathcal{M}\ddot{q} = f(q)$$

Let $v = \dot{q}$. $\Rightarrow \begin{cases} \mathcal{M}\dot{v} = f(q) \\ \dot{q} = v \end{cases} \Leftrightarrow A\dot{Y} = BY = f(Y)$

1. At $(t+1)$, $\dot{v} \approx \frac{1}{\Delta t}(\dot{q}^{t+1} - \dot{q}^t)$

$$\Rightarrow \mathcal{M}\frac{1}{\Delta t}(\dot{q}^{t+1} - \dot{q}^t) = f(q^{t+1})$$

$$\Rightarrow \mathcal{M}\dot{q}^{t+1} = \mathcal{M}\dot{q}^t + \Delta t\, f(q^{t+1}) \quad (1)$$

2. At $(t+1)$, $\dot{q} \approx \frac{1}{\Delta t}(q^{t+1} - q^t)$

$$\Rightarrow \frac{1}{\Delta t}(q^{t+1} - q^t) = \dot{q}^{t+1}$$

$$\Rightarrow q^{t+1} = q^t + \Delta t\, \dot{q}^{t+1} \quad (2)$$

Backward Euler

(1) + (2)

$\Rightarrow f(q^{t+1}) = f(q^t + \Delta t\, \dot{q}^{t+1})$

$\Rightarrow f(q^{t+1}) \approx f(q^t) + \Delta t\, \dot{q}^{t+1} \cdot \frac{\partial f}{\partial q}(q^t)$

$\Leftrightarrow$

$M\dot{q}^{t+1} = M\dot{q} + \Delta t\, f(q^t) + \Delta t^2 \dot{q}^{t+1} \frac{\partial f}{\partial q}(q^t)$

$\Leftrightarrow$

$\left(M - \Delta t^2 \frac{\partial f}{\partial q}(t)\right) \dot{q}^{t+1} = M\dot{q} + \Delta t\, f(q^t)$

$\Leftrightarrow (M - \Delta t^2 K)\dot{q}^{t+1} = M\dot{q} + f$

Since $\hat{q} = Pq$ and $\dot{q} = P^T\dot{\hat{q}} + \cancel{b}$

$\Rightarrow$

$$\boxed{\begin{aligned} &(M - \Delta t^2 \boxed{K}) P^T \dot{\hat{q}}^{t+1} = M P^T \dot{\hat{q}} + \boxed{f} \\ &q^{t+1} = q + \Delta t (P^T \dot{\hat{q}}^{t+1}) \end{aligned}}$$

Stiffness Matrix

force vector.