16720J: Homework 1 - Planar Homographies

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3 Planar Homographies: Theory (40pts)

3.1 (10pts)

Prove that there exists an $\boldsymbol{H}$ that the satisfies homography equation.

The easiest way to show this is to assume $\boldsymbol{Z}$=0. Then the points in the plane are of the form $[\boldsymbol{X}\,\boldsymbol{Y}\,\mathbf{0}\,\mathbf{1}]^{\boldsymbol{T}}$

Therefore, the original equations for $\boldsymbol{p1}$ and $\boldsymbol{p2}$:

$$p_1 \equiv \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}_1 \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

$$p_2 \equiv \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}_2 \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

can be written as:

$$p_1 \equiv \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_1 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \tag{3}$$

$$p_2 \equiv \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_2 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \tag{4}$$

therefore there exists an $\boldsymbol{H}$ where

$$\begin{aligned} p_2 &\equiv \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_2 \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_1^{-1} \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_1 \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} \\ &\equiv \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_2 \begin{bmatrix} m_{11} & m_{12} & m_{14} \\ m_{21} & m_{22} & m_{24} \\ m_{31} & m_{32} & m_{34} \end{bmatrix}_1^{-1} p1 \equiv Hp1 \end{aligned} \tag{5}$$

When does this fail?
The projection matrix of Cameras are not invertible.

3.2 (10pts)

Prove that there exists an $\boldsymbol{H}$ that satisfies homography equation given two cameras separated by a pure rotation.

$$p_1 = K_1[I\,0]P$$
$$p_2 = K_2[R\,0]P \tag{6}$$

Because two cameras are only separated by pure rotation, if we set the angle to be θ, then we can simplfied the equation as follows:

$$p_1 = K_1 P$$
$$p_2 = K_2 R_\theta P \tag{7}$$

The intrinsic parameters K can be defined as the following:

$$K \equiv \begin{bmatrix} \alpha & -\alpha cos\theta & \mu_0 \\ 0 & \frac{\beta}{sin\theta} & \nu_0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

K is a Row-Echelon Form matrix. The Rank of K is 3, which means K is Full Rank. Therefore, K is invertible. And the extrinsic parameters can be defined as:

$$R \equiv \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

Each row and column is unit vector and pairwise orthogonal, so R is orthogonal matrix, which suggests it can be inversed.

Therefore, p_2 can be expressed as:

$$p_2 \equiv K_2 R_\theta {K_1}^{-1} K_1 P \equiv K_2 R_\theta {K_1}^{-1} p_1 \tag{10}$$

so

$$H \equiv K_2 R {K_1}^{-1} \tag{11}$$

Obviously, det(H) is not zero because all of the factors are not zero(inversible). So there exists an H as planar homography.

3.3 (5pts)

From Section 3.2

$$H \equiv K R_\theta K^{-1} \tag{12}$$

So,

$$H^2 \equiv K R_\theta K^{-1} K R_\theta K^{-1} \equiv K R_\theta^2 K^{-1} \tag{13}$$

$$R_\theta^2 \equiv \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} cos2\theta & -sin2\theta & 0 \\ sin2\theta & cos2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

therefore

$$H^2 \equiv KR_{2\theta}K^{-1} \tag{15}$$

3.4 (5pts)

Why is the planar homography not completely sufficient to map any arbitrary scene image to another viewpoint?

Because when we talk about planar homography, we always regard a camera imaging of a 3D scene as if the scene were painted on a plane infinitely far away from the camera, that is, Z is set to 0. The depth of the image is ignored. So we can not always map any arbitrary scene image to another viewpoint only with planar homography.

3.5 (5pts)

We have a set of points $\boldsymbol{p_1^i}$ in an image taken by camera $\boldsymbol{C_1}$ and points $\boldsymbol{p_2^i}$ in an image taken by $\boldsymbol{C_2}$. Suppose we know there exists an unknown homography $\boldsymbol{H}$ such that

$$\boldsymbol{p_1^i \equiv H p_2^i} \tag{16}$$

Assume the points are homogeneous coordinates in the form $\boldsymbol{p_j^i = (x_j^i, y_j^i, 1)^T}$. For a single point pair, write a matrix equation of the form

$$\boldsymbol{Ah = 0} \tag{17}$$

Where $\boldsymbol{h}$ is a vector of the elements of $\boldsymbol{H}$ and $\boldsymbol{A}$ is a matrix composed of the point coordinates.

1. How many elements are there in h?

Nine. Because h is the eigenvector of H.

2. How many point pairs are required to solve this system?

Four. Because H is a 3*3 matrix with 9 unknown variable, and they are subject to $|H|$ = 1.So the freedom is 8. And each point pair provides two equations linear in the H matrix elements. So we need 4 pairs to solve the problem. If the number is over 4, the matrix will be over determined

3. Show how to estimate the elements in h using the Rayleigh Quotient Theorem mentioned in class to find a solution to minimize this homogeneous linear least squares system.

The problem is to find a vector h to minimize $|Ah|^2$

Because $|Ah|^2 = \sum |A_i h|^2 = \sum h^T {A_i}^T A_i h = h^T A^T A h$.

So the problem is to solve Min $h^T A^T A h$, subject to $|h| = 1$.

And according to Rayleigh Quotient Theorem, the minimum of $h^T A^T A h$ is reached at h is the eigenvector of $A^T A$ corresponding to the smallest eigenvalue.

So if the point pairs is 4, h can be obtained directly from the SVD of A.

4 Planar Homographies: Implementation (30pts)

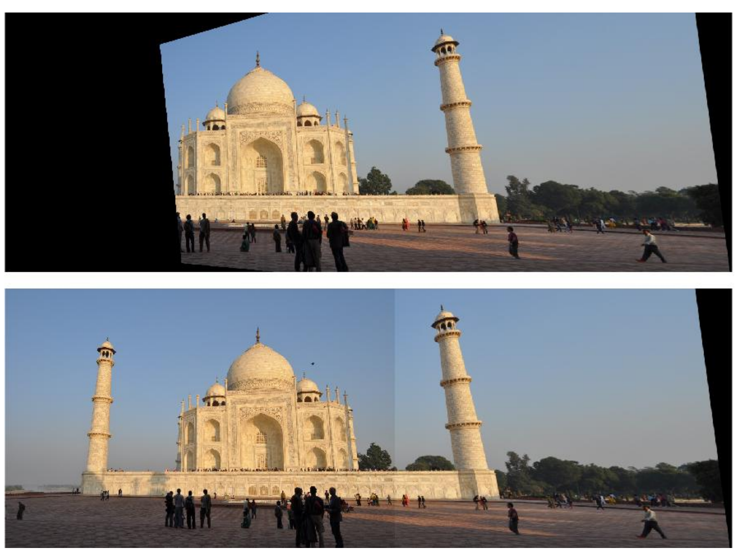
Figure 1: q42checker output.

5 Panoramas (30pts)

5.1 (15pts)

The Root Mean Squared Error between corresponding points has been calculated as 1.2344, which can be seen in the console when running the q5checker.m.

Figure 2: q51checker output.



5.2 (15pts)

Figure 3: q52checker output.