The Geometry of Multiple Views

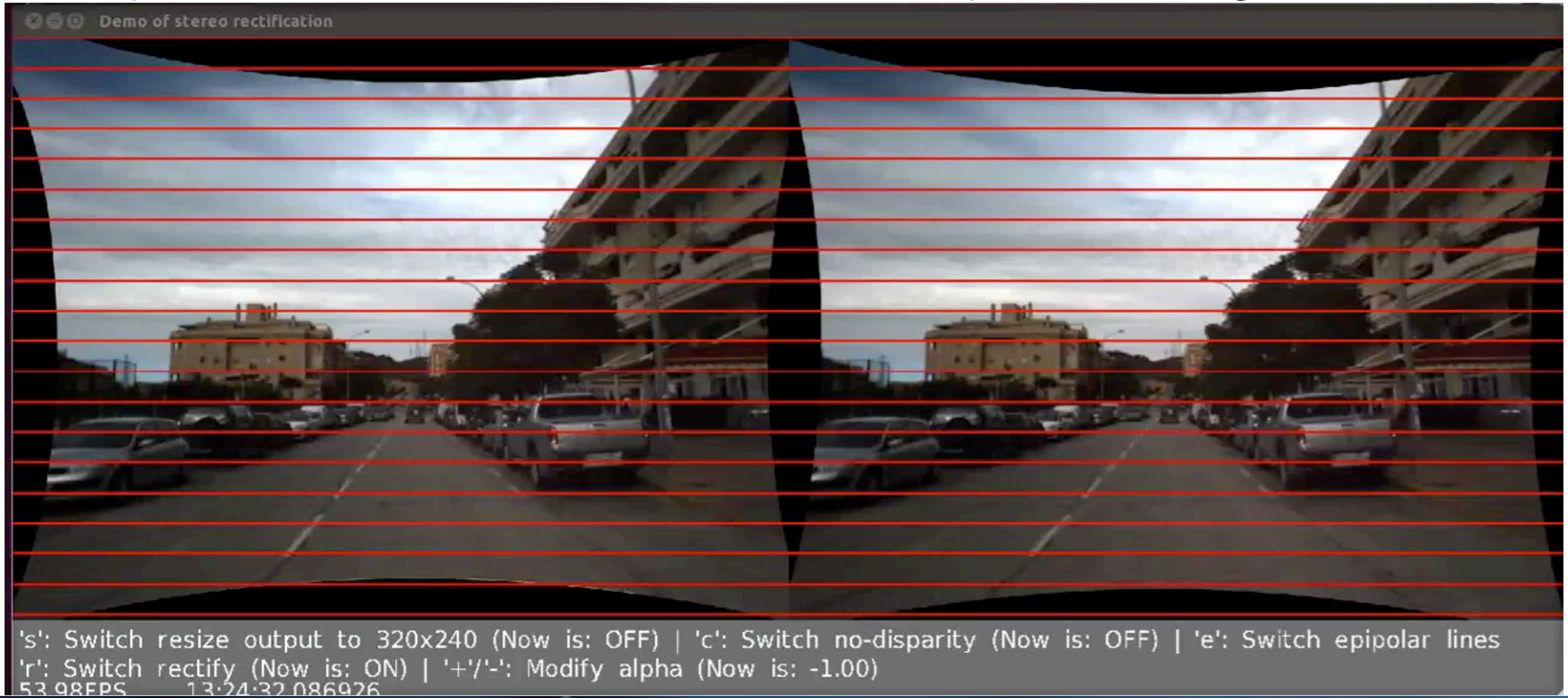
Gary Overett (Slides adapted from CMU 16-720 2014)

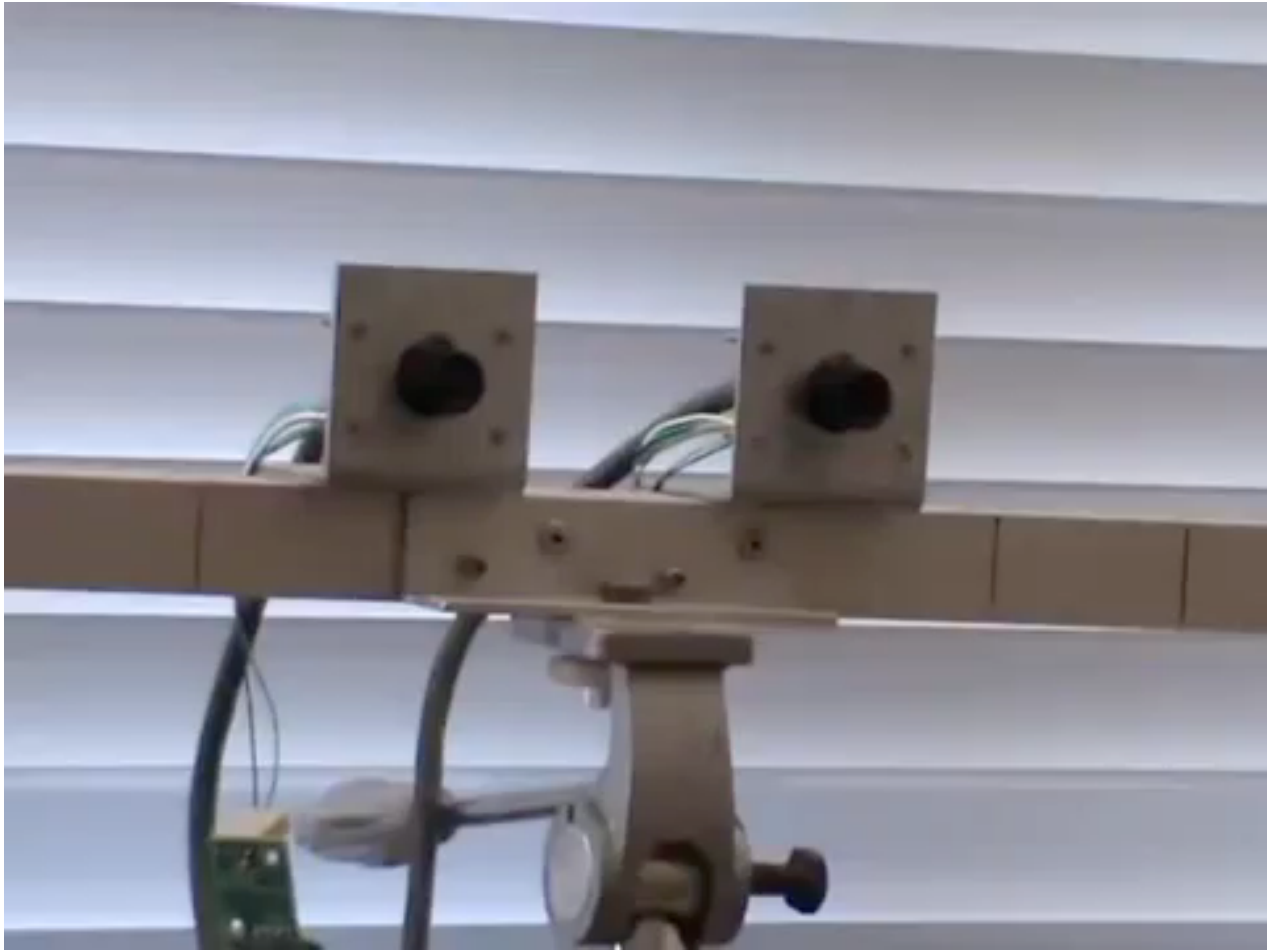
Szeliski Chapter 7

Forsyth & Ponce, Chapter 10

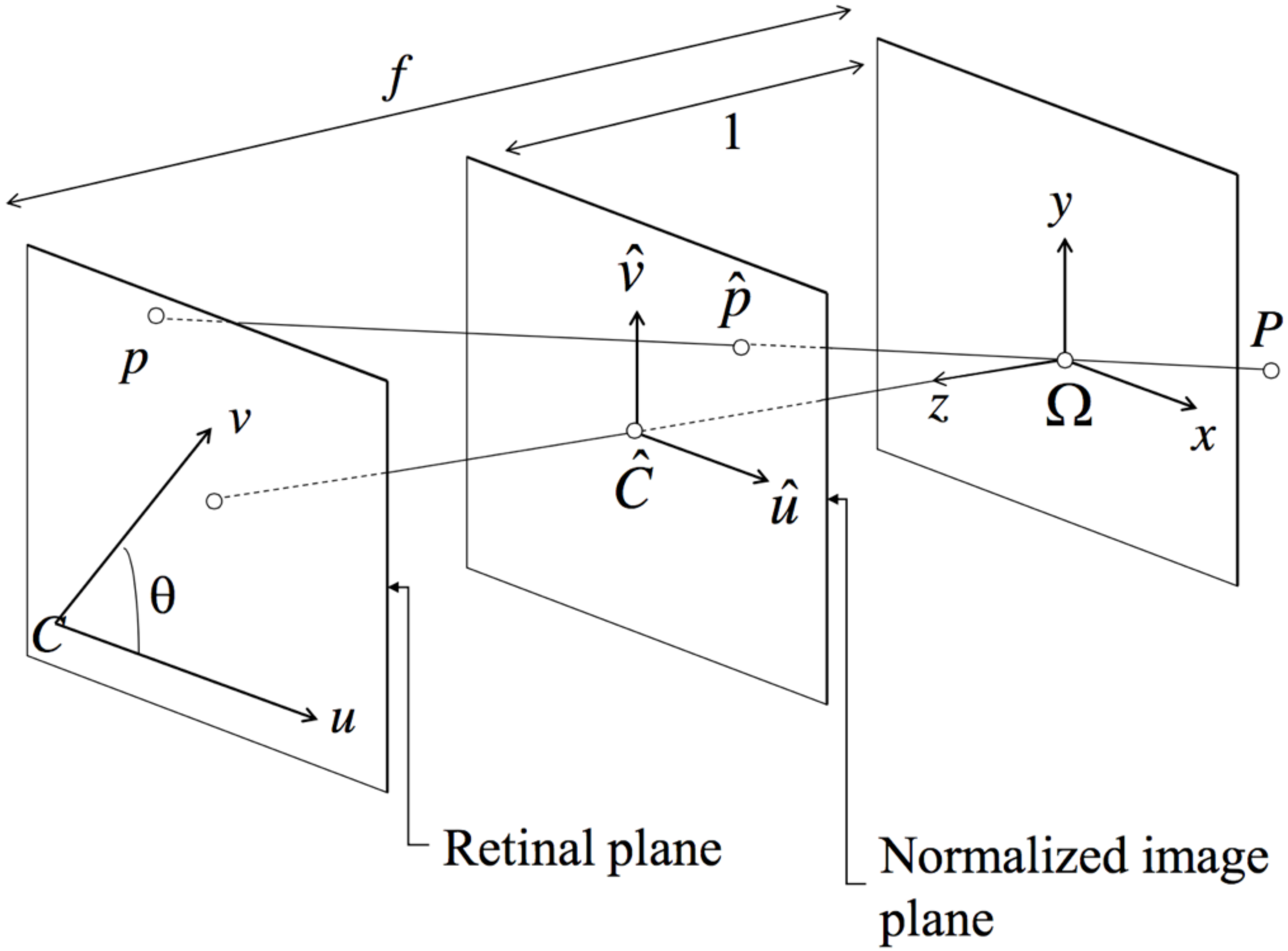
Faugeras & Luong. The Geometry of Multiple Images. MIT Press, 2001

Hartley & Zisserman. Multi View Geometry. Cambridge Press, 2000

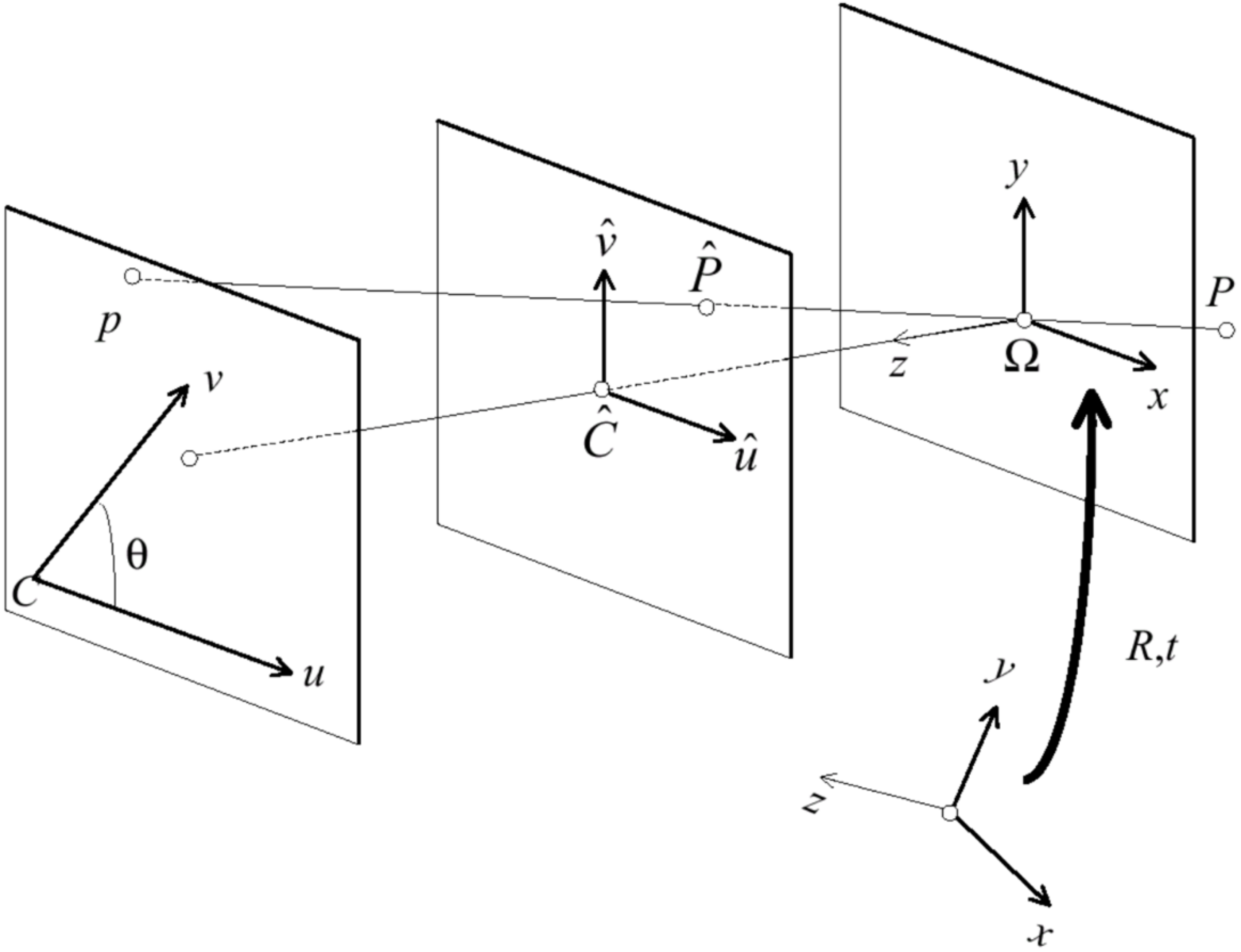




Normalized Image Projection



Normalized Image Projection



Alternative Notations

By rows:

$$M = \begin{bmatrix} m_1^T \\ m_2^T \\ m_3^T \end{bmatrix}$$

(3 × 4)

By components:

$$M = \underbrace{K}_{\substack{3\text{x}3 \\ \text{Intrinsic parameter matrix}}} \underbrace{\begin{bmatrix} R & t \end{bmatrix}}_{\substack{3\text{x}3 \;\; 3\text{x}1 \\ \text{Extrinsic parameter matrix}}}$$

By blocks:

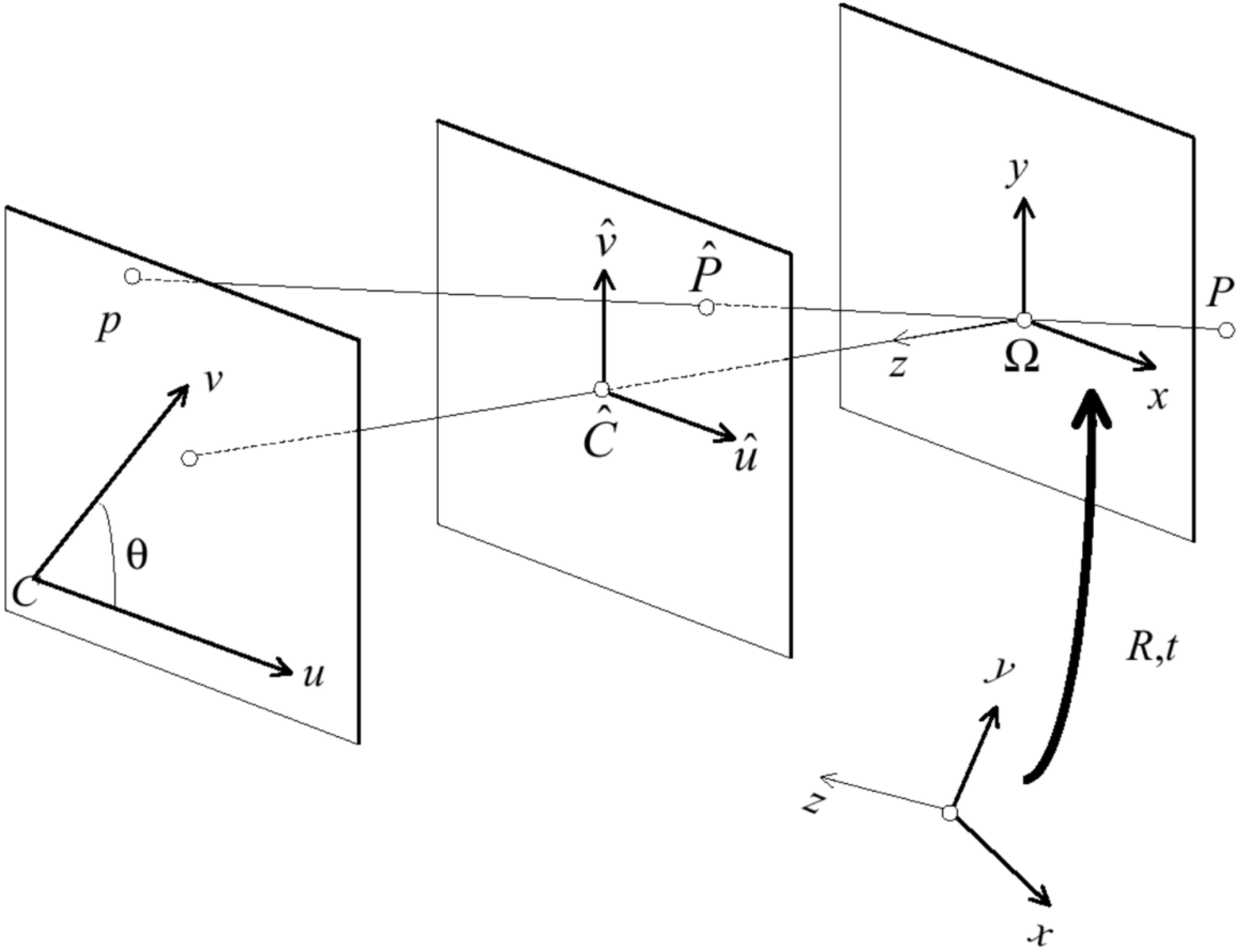
$$M = \begin{bmatrix} \underset{3\text{x}3}{A} & \underset{3\text{x}1}{b} \end{bmatrix}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

Q: Is a given 3x4 matrix M the projection matrix of some camera?
A: Yes, if and only if det(A) is not zero

Q: Is the decomposition unique?
A: There are multiple equivalent solutions

Normalized Image Projection

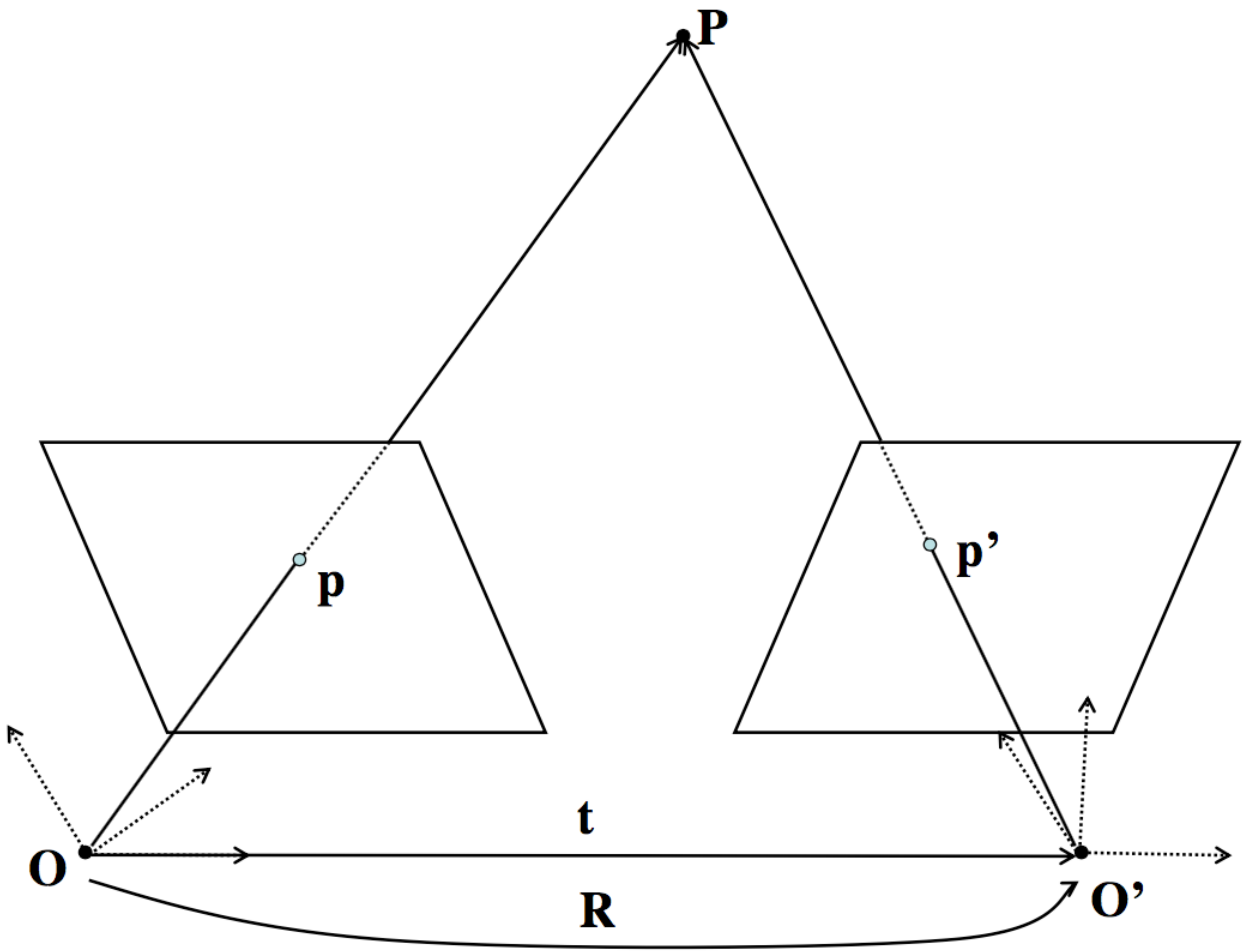


$$\hat{\mathbf{p}} \equiv [\mathbf{R}|\mathbf{t}]\mathbf{P}$$

$$\hat{\mathbf{p}} \equiv \mathbf{K}^{-1}\mathbf{p}$$

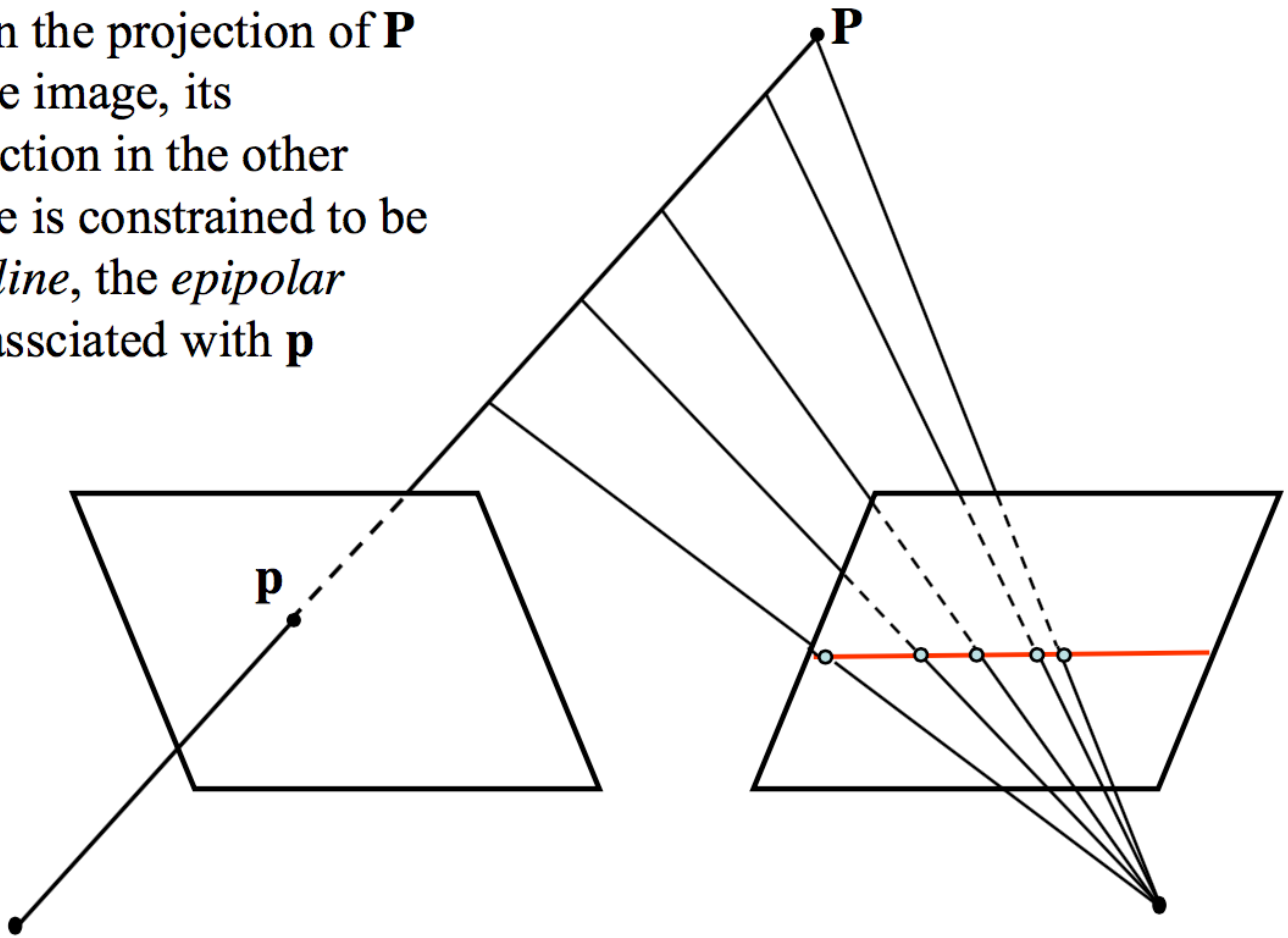
$$\mathbf{p} \equiv \mathbf{K}[\mathbf{R}|\mathbf{t}]\mathbf{P}$$

2 Camera Geometry

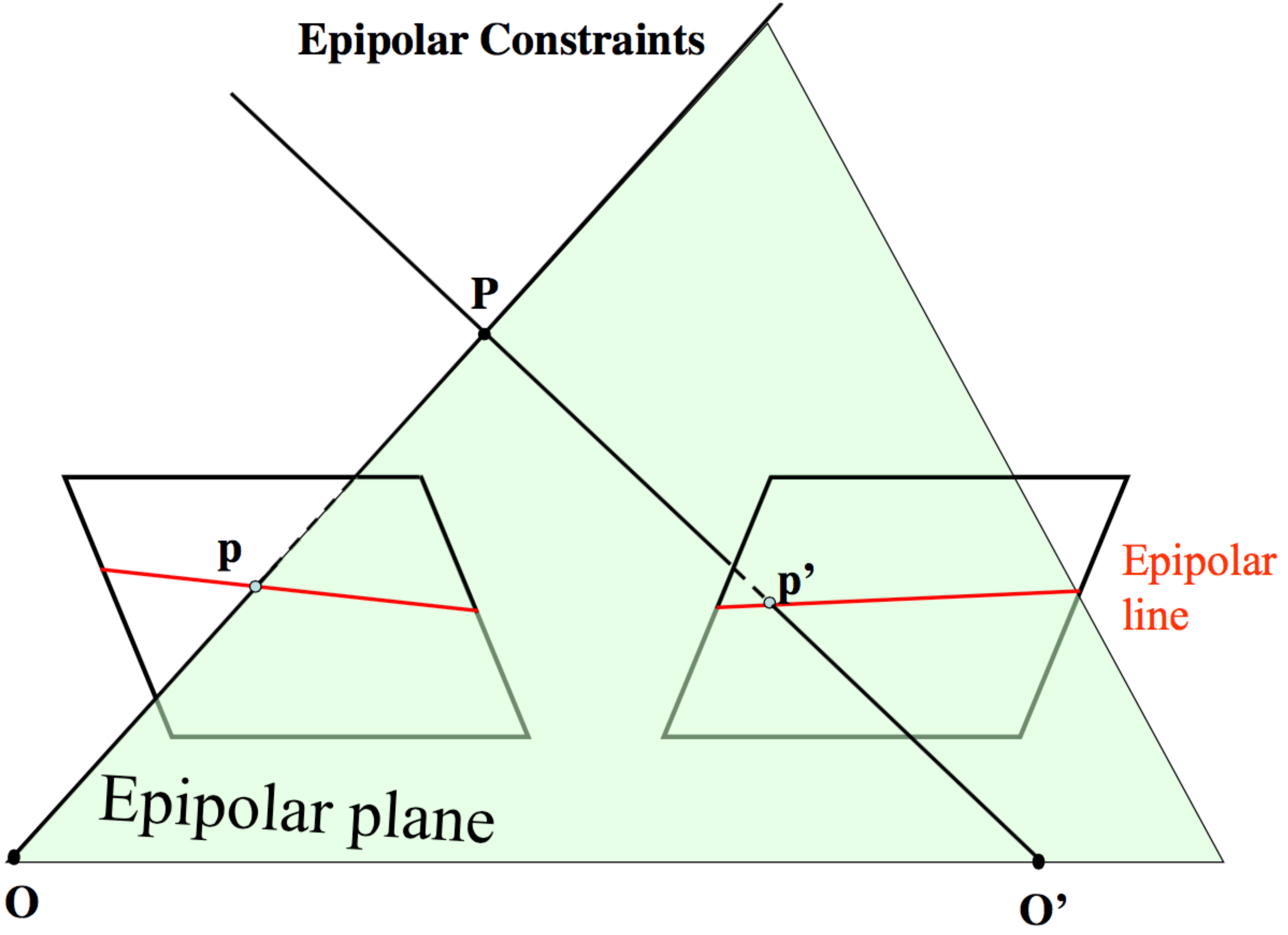


Epipolar Constraint

Given the projection of **P** in one image, its projection in the other image is constrained to be on a *line*, the *epipolar line* assciated with **p**

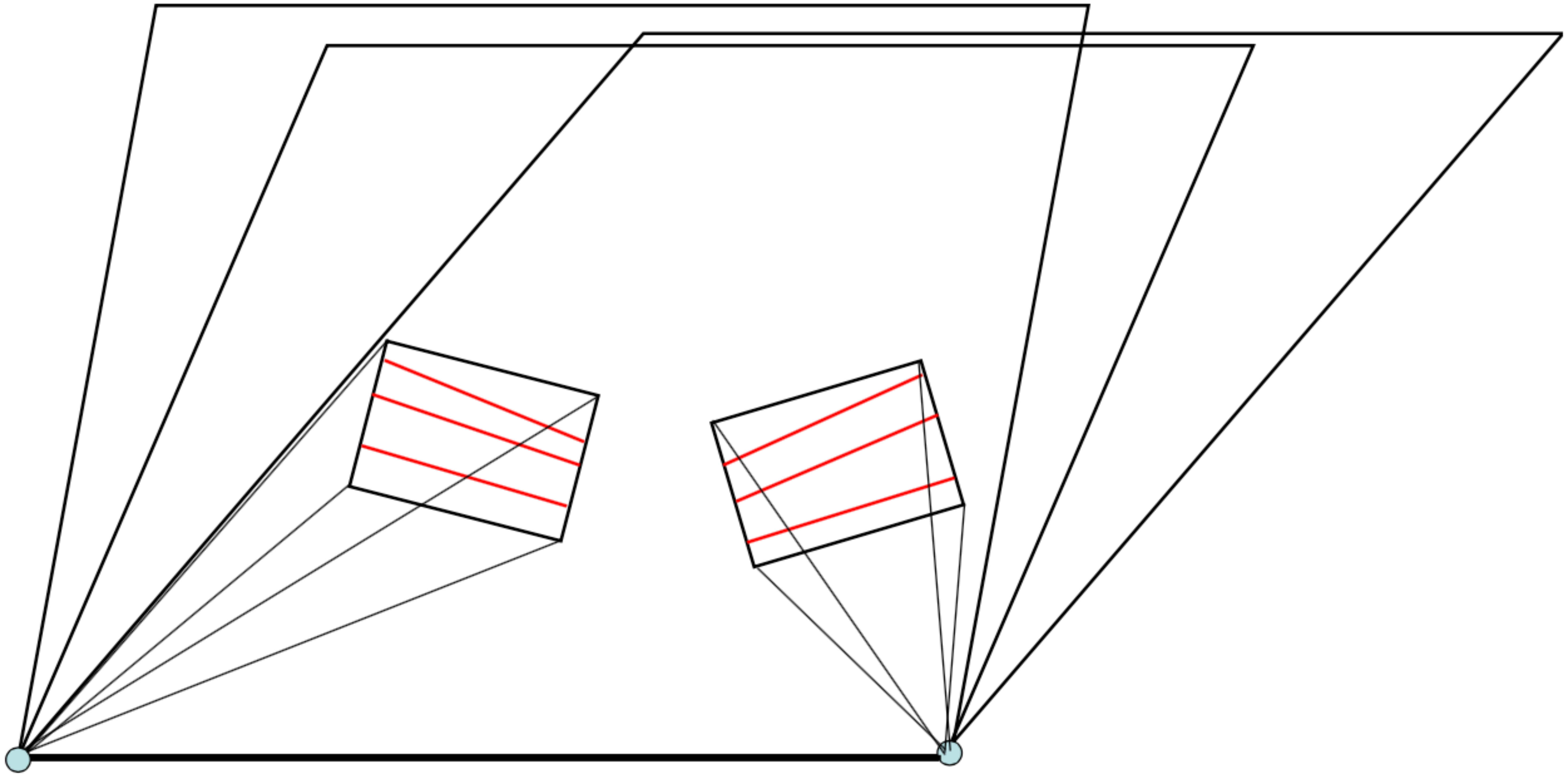


Epipolar Constraints

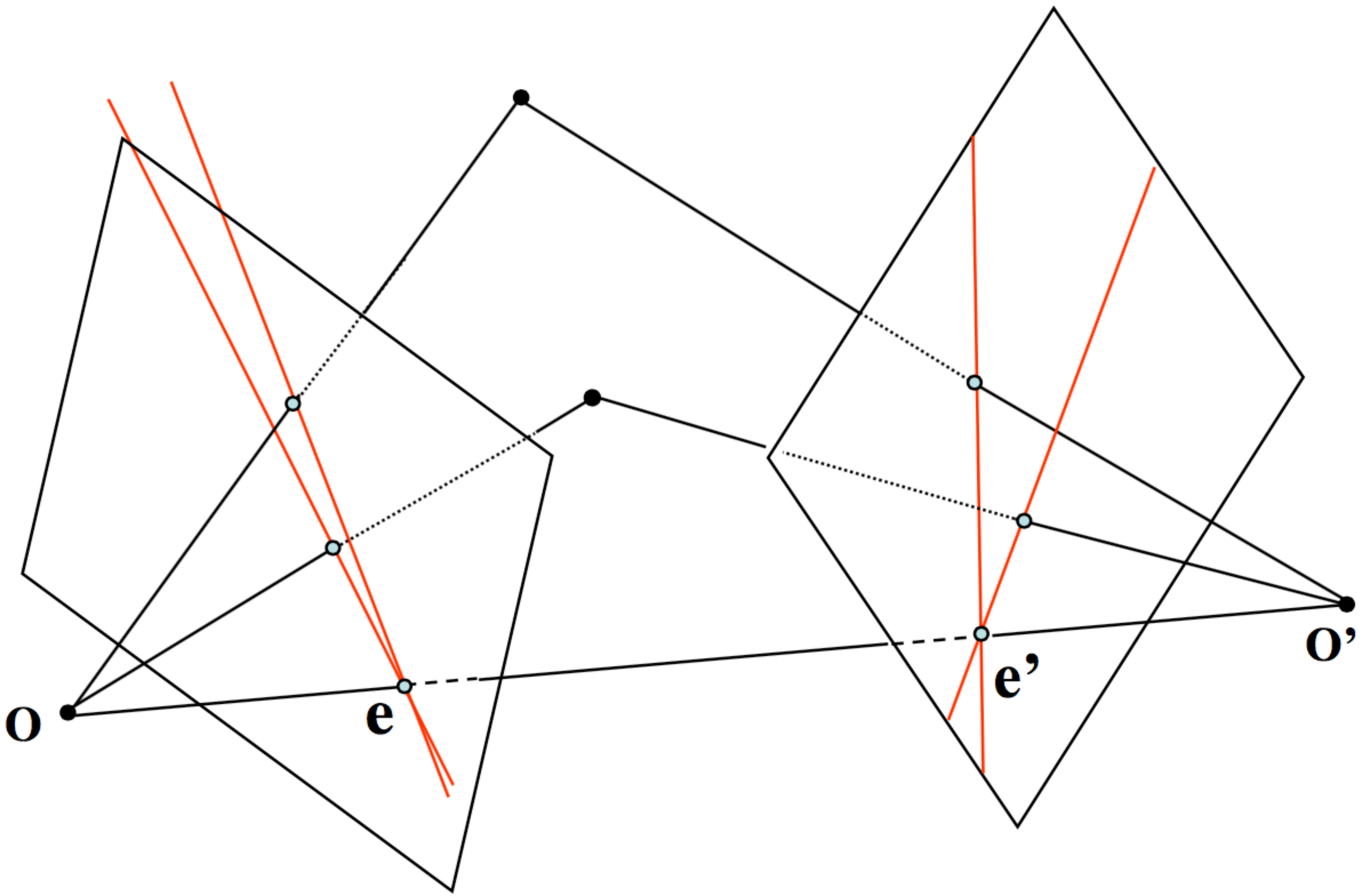


Epipolar Planes

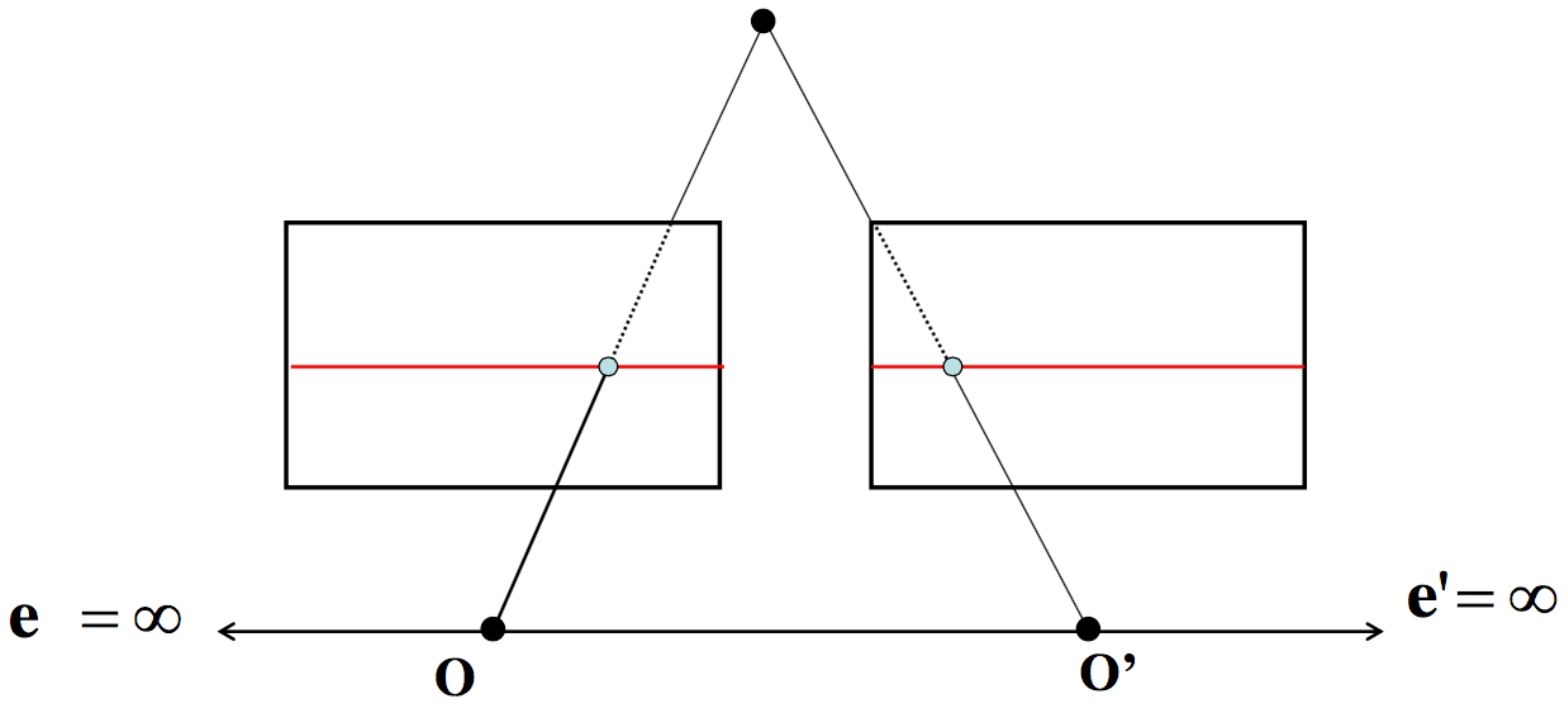
The epipolar lines are the intersection of the image planes with the epipolar lines



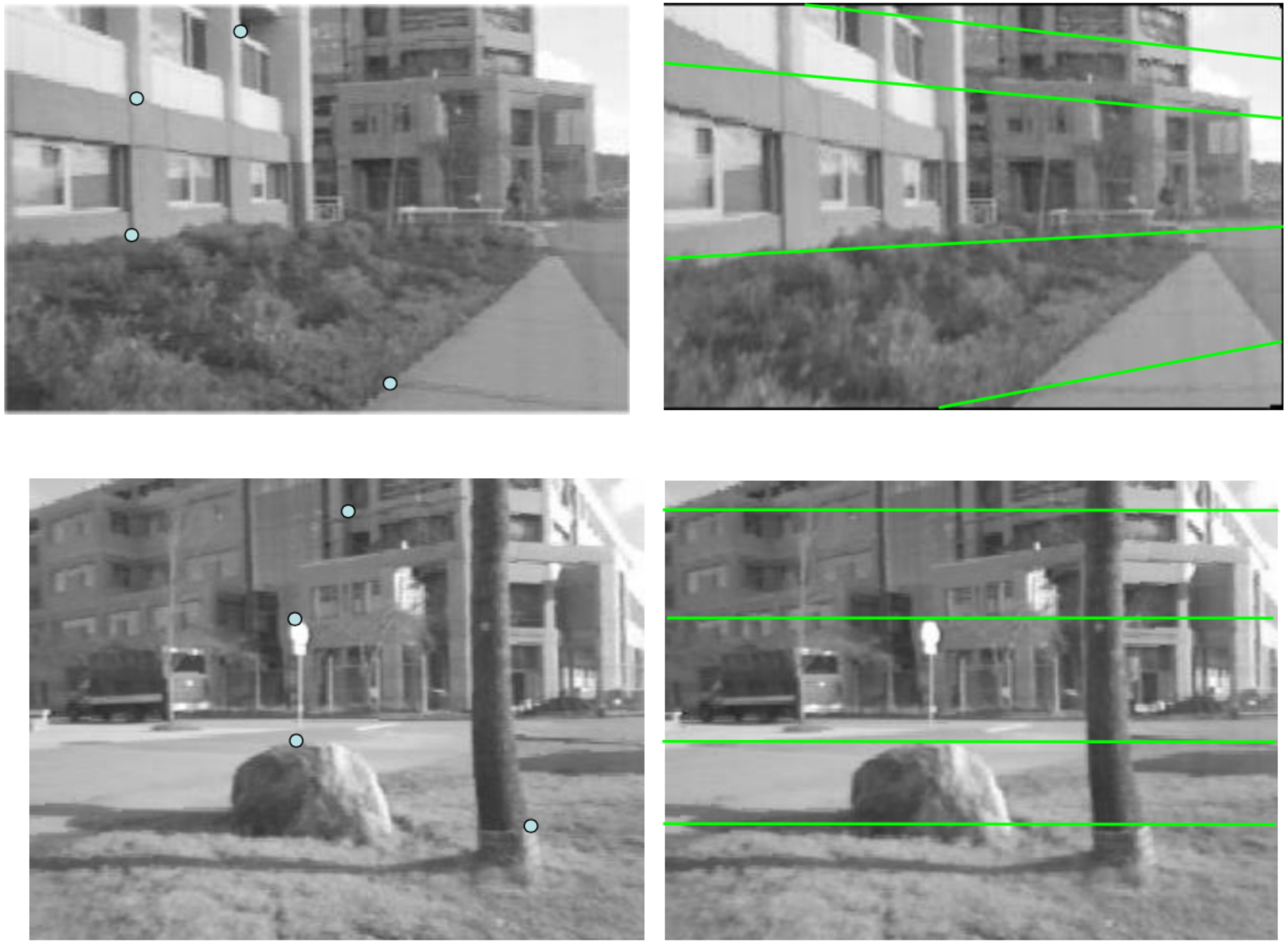
Epipoles are at the intersection of the epipolar lines



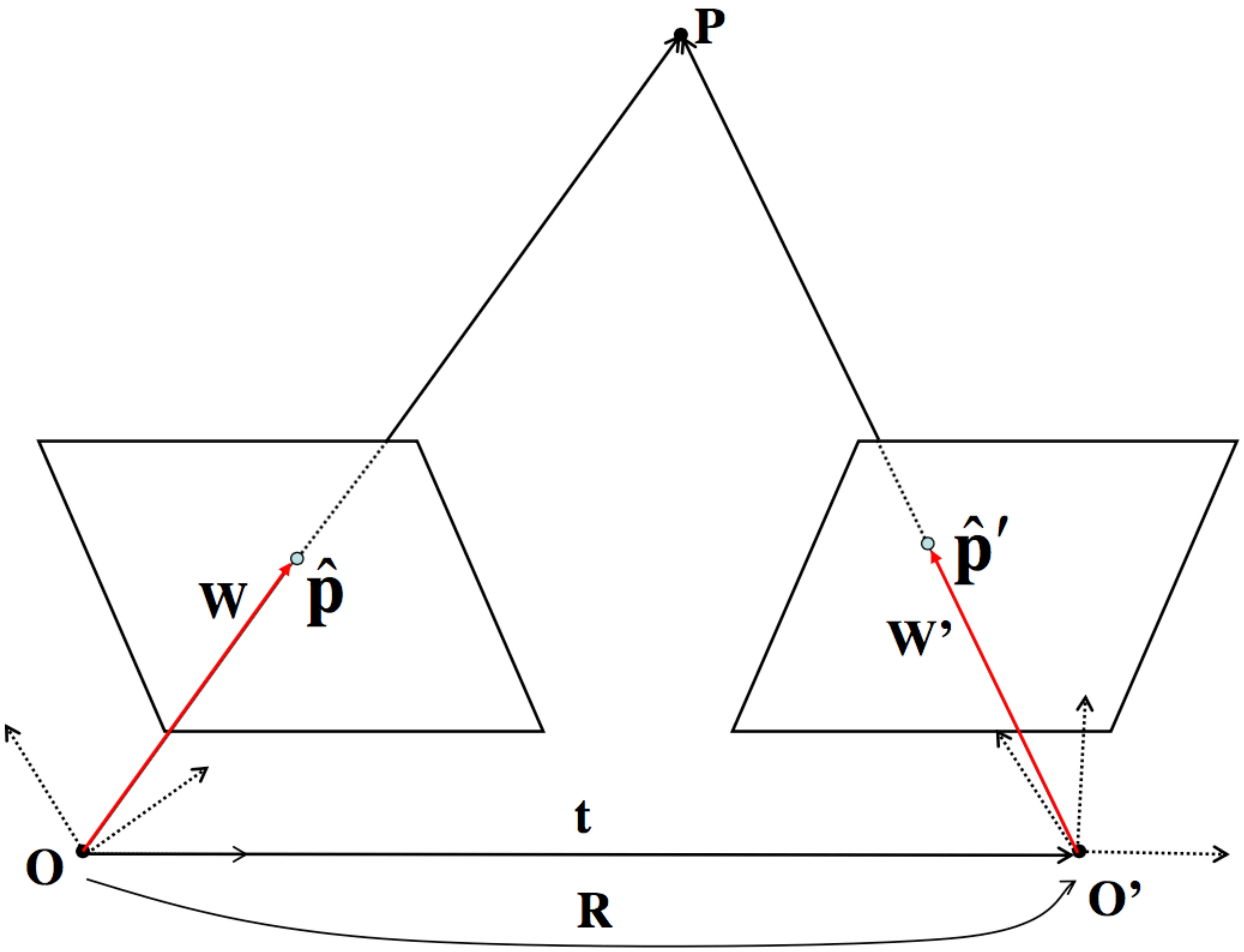
Special case: frontoparallel cameras. Epipoles are at infinity.



Epipolar Geometry Example



Epipolar Geometry



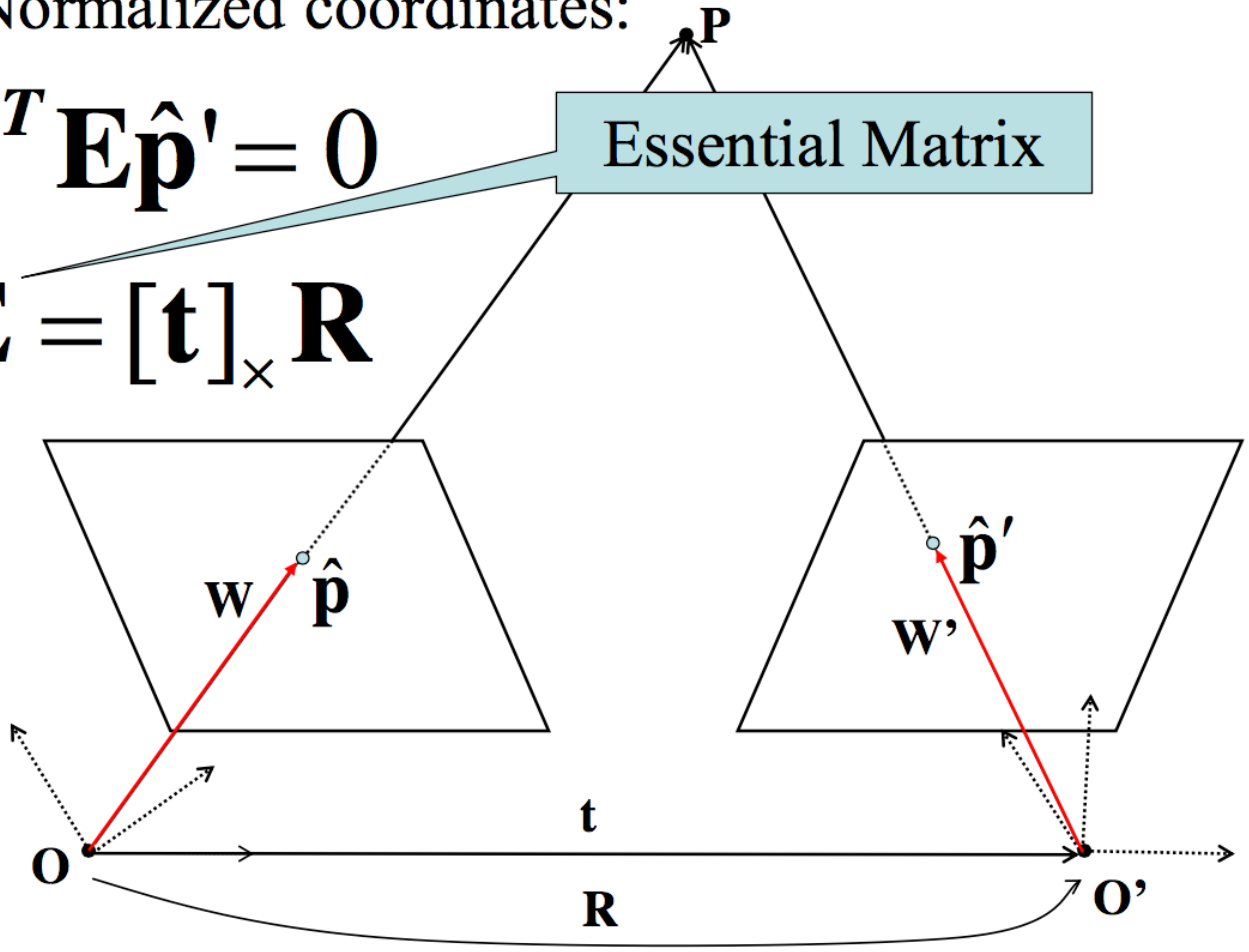
Epipolar Geometry

In Normalized coordinates:

$$\hat{\mathbf{p}}^T \mathbf{E} \hat{\mathbf{p}}' = 0$$

$$\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$$

Essential Matrix



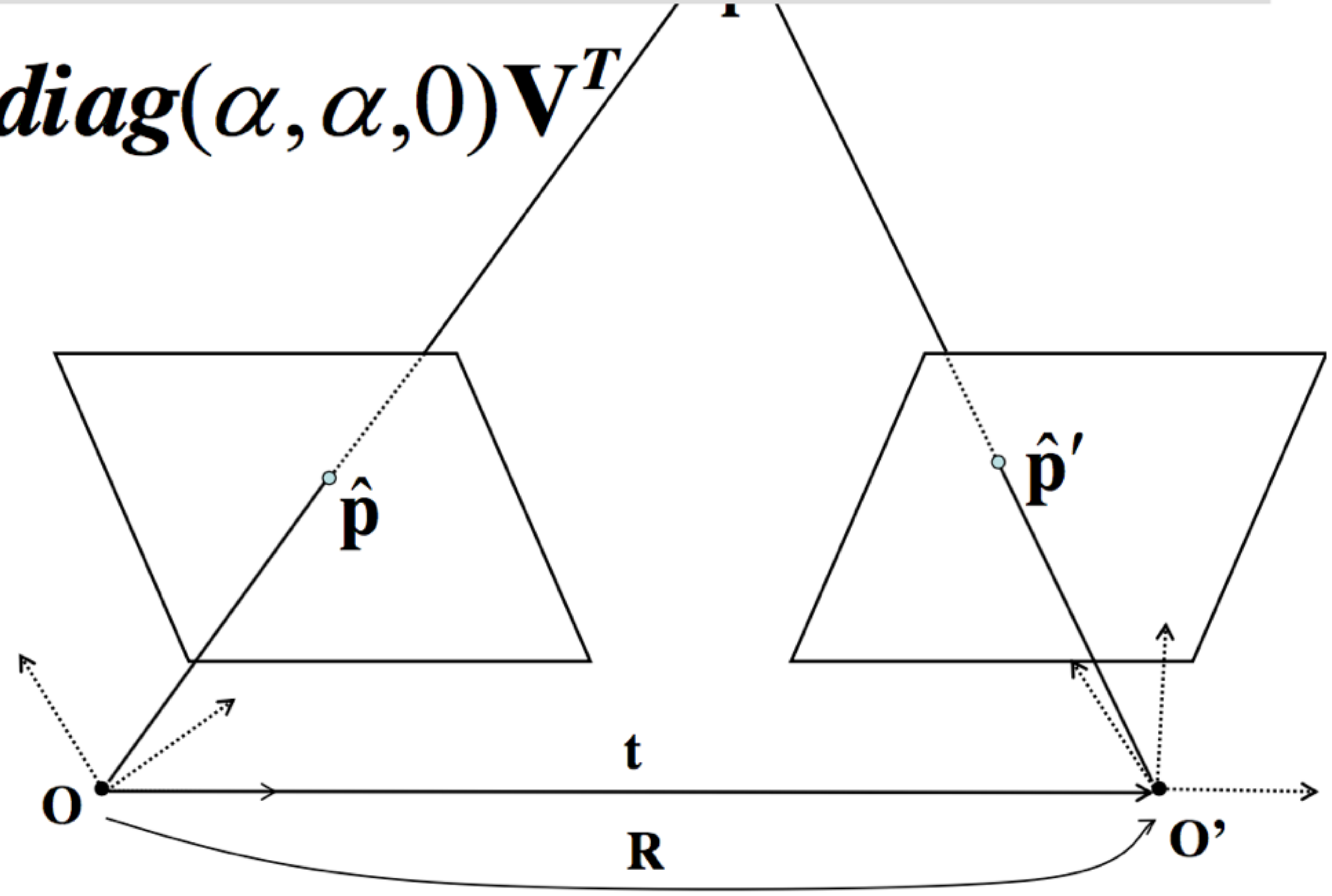
Properties of the Essential Matrix

Q: Can any 3x3 matrix **E** be an essential matrix?

A: No. Conditions:

- **E** must be singular → One singular value is 0
- The 2 non-zero singular values of **E** must be equal

$$\mathbf{E} = \mathbf{W}\boldsymbol{diag}(\alpha,\alpha,0)\mathbf{V}^T$$



Session 1

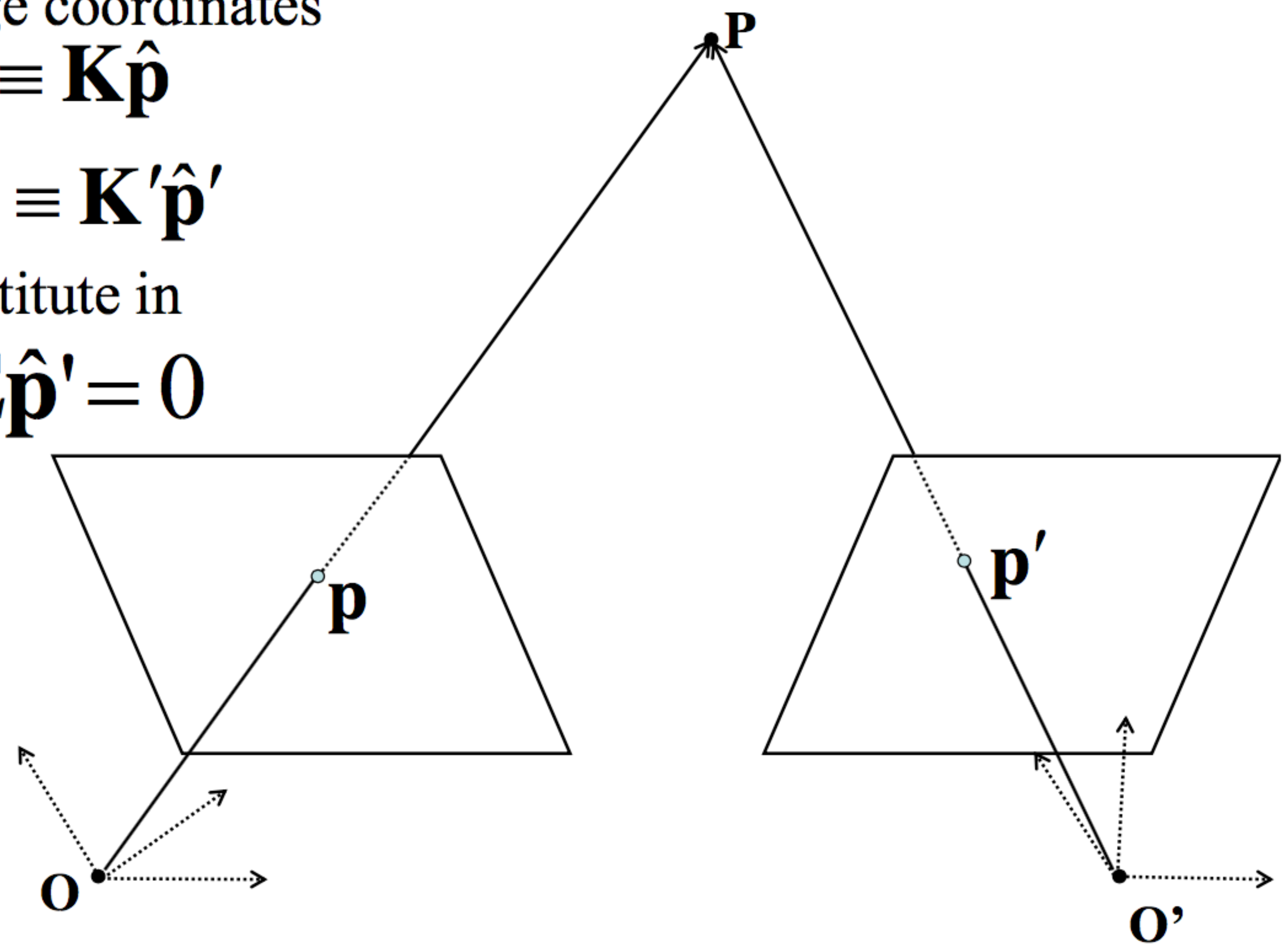
General Case: Write down conversion from normalized to real image coordinates

$$\mathbf{p} \equiv \mathbf{K}\hat{\mathbf{p}}$$

$$\mathbf{p}' \equiv \mathbf{K}'\hat{\mathbf{p}}'$$

and substitute in

$$\hat{\mathbf{p}}^T \mathbf{E}\hat{\mathbf{p}}' = 0$$



General Case:

$$\mathbf{p}^T \mathbf{F} \mathbf{p}' = 0$$

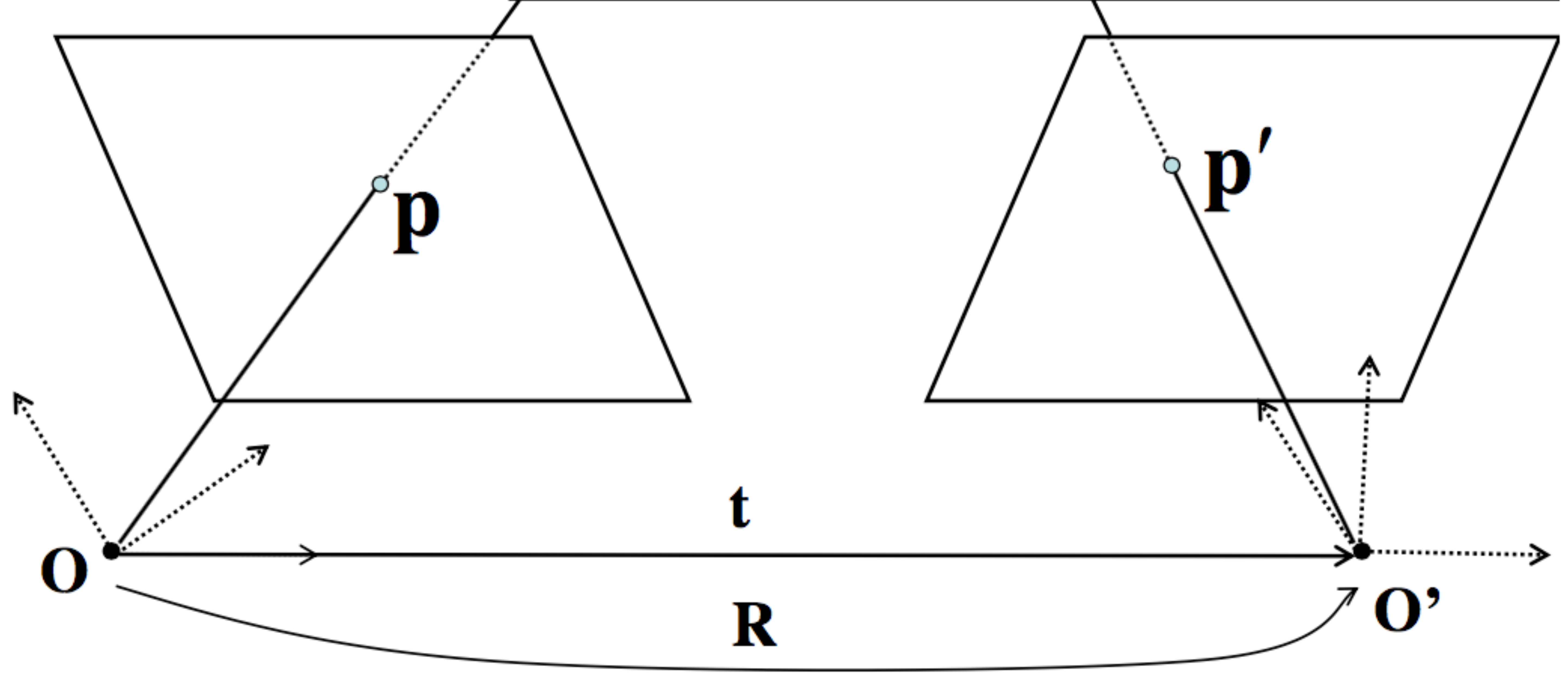
Fundamental Matrix

$$\mathbf{F} = \mathbf{K}^{-T} \mathbf{E} \mathbf{K}'^{-1}$$

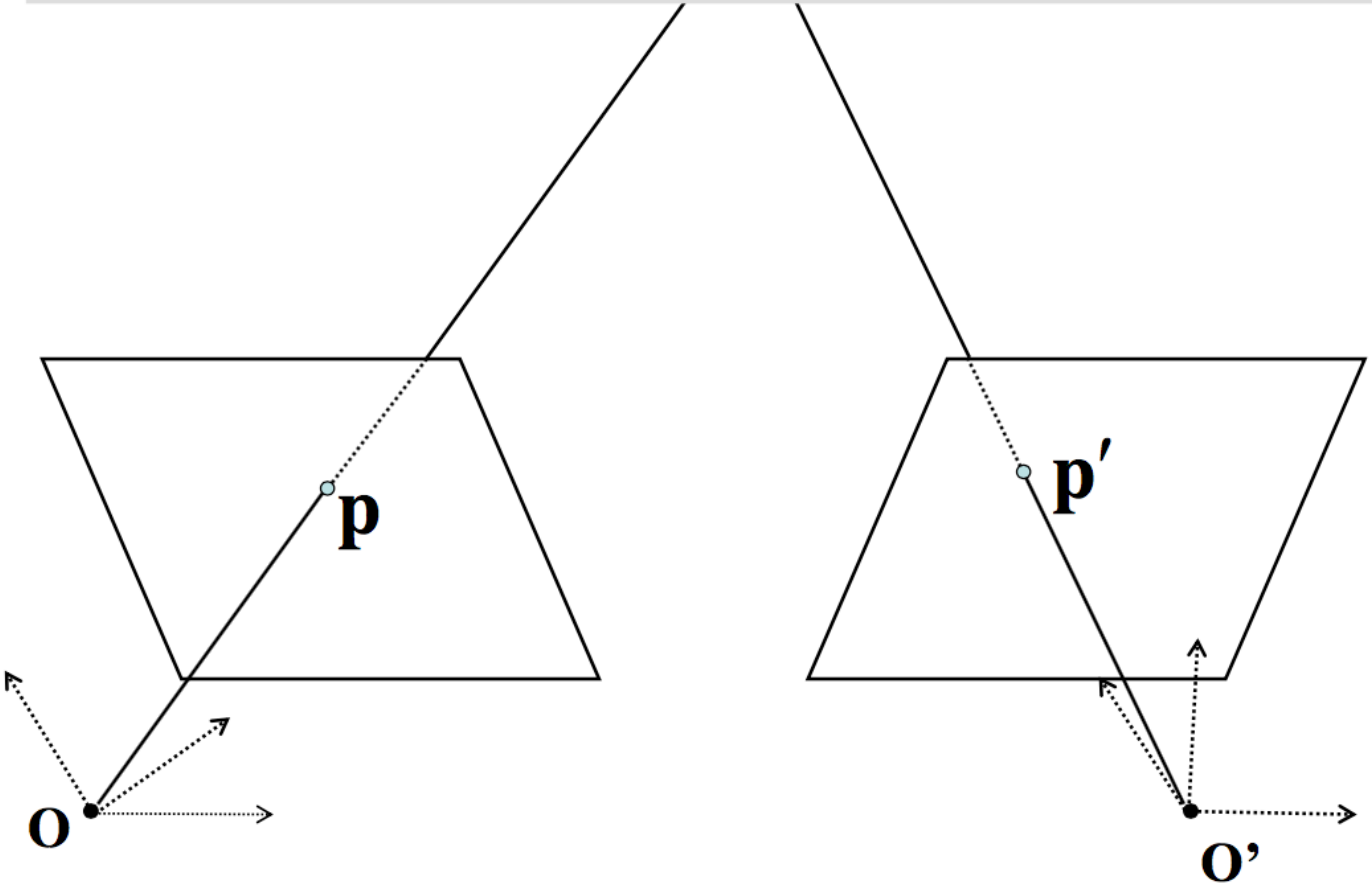
Relation between Fundamental and Essential Matrices

$$\mathbf{F}\mathbf{e}' = \mathbf{F}^T \mathbf{e} = 0$$

The epipole is the null space of **F**

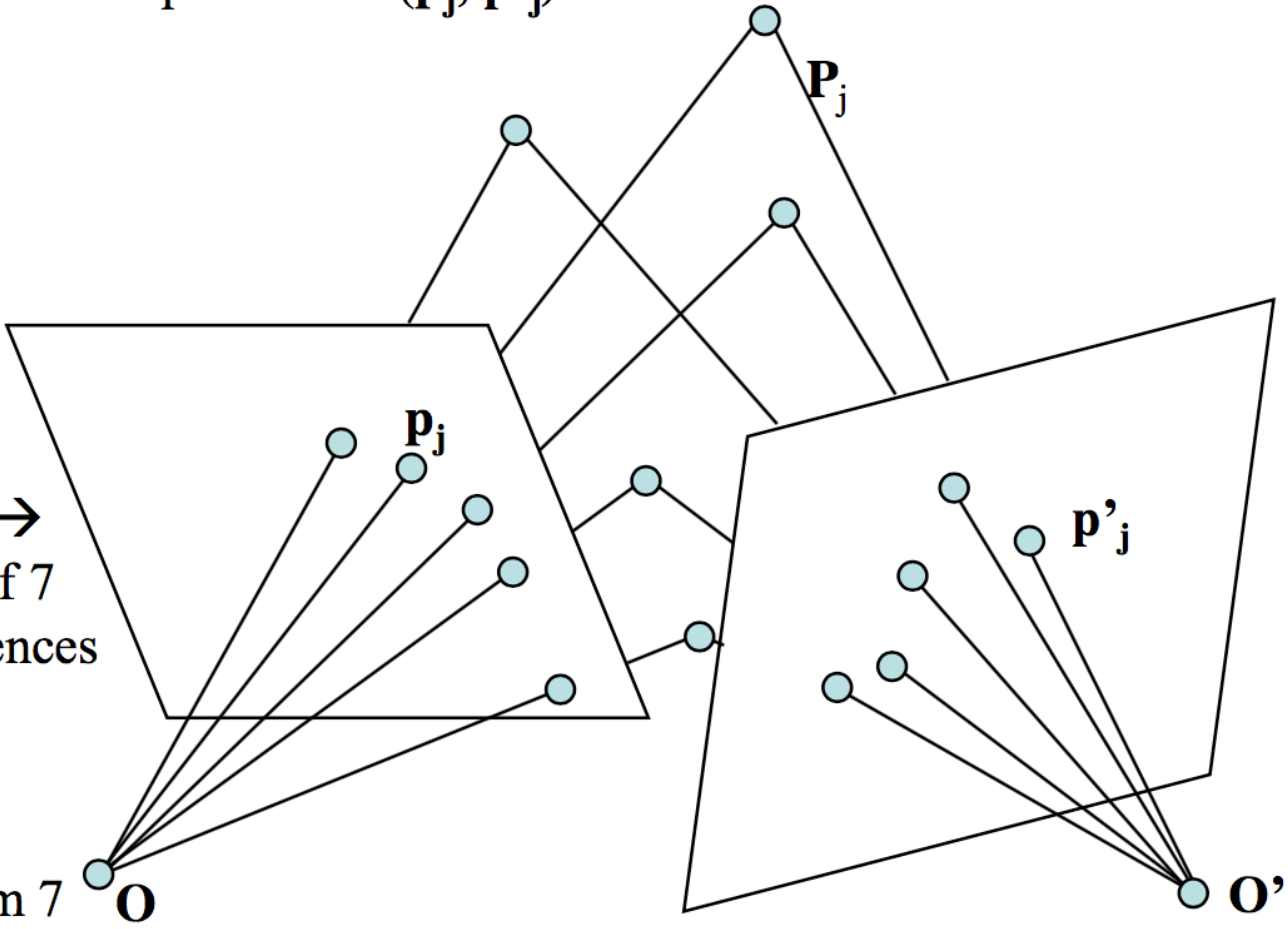


Q: Can any matrix **F** be a fundamental matrix
A: No, **F** must be singular
Q: Given 2 images, is **F** unique?
A: Absolutely not.



Estimating **F** from correspondences

Given: set of correspondences ($\mathbf{p}_j$, $\mathbf{p'}_j$)



F has 7 free parameters → Minimum of 7 correspondences

F can be computed directly from 7 correspondences

- Estimating from 7 correspondences:

$$\mathbf{p}_i^T \mathbf{F} \mathbf{p}'_i = 0 \quad i = 1,..,7$$

$$\mathbf{U}\mathbf{f} = 0$$

- $\mathbf{U}$ is a 7x9 matrix

- Two solutions: $\mathbf{f}_1, \mathbf{f}_2 \rightarrow \mathbf{F}_1, \mathbf{F}_2$

- General solution of the form: $\mathbf{F} = (1-\lambda)\mathbf{F}_1 + \lambda \mathbf{F}_2$

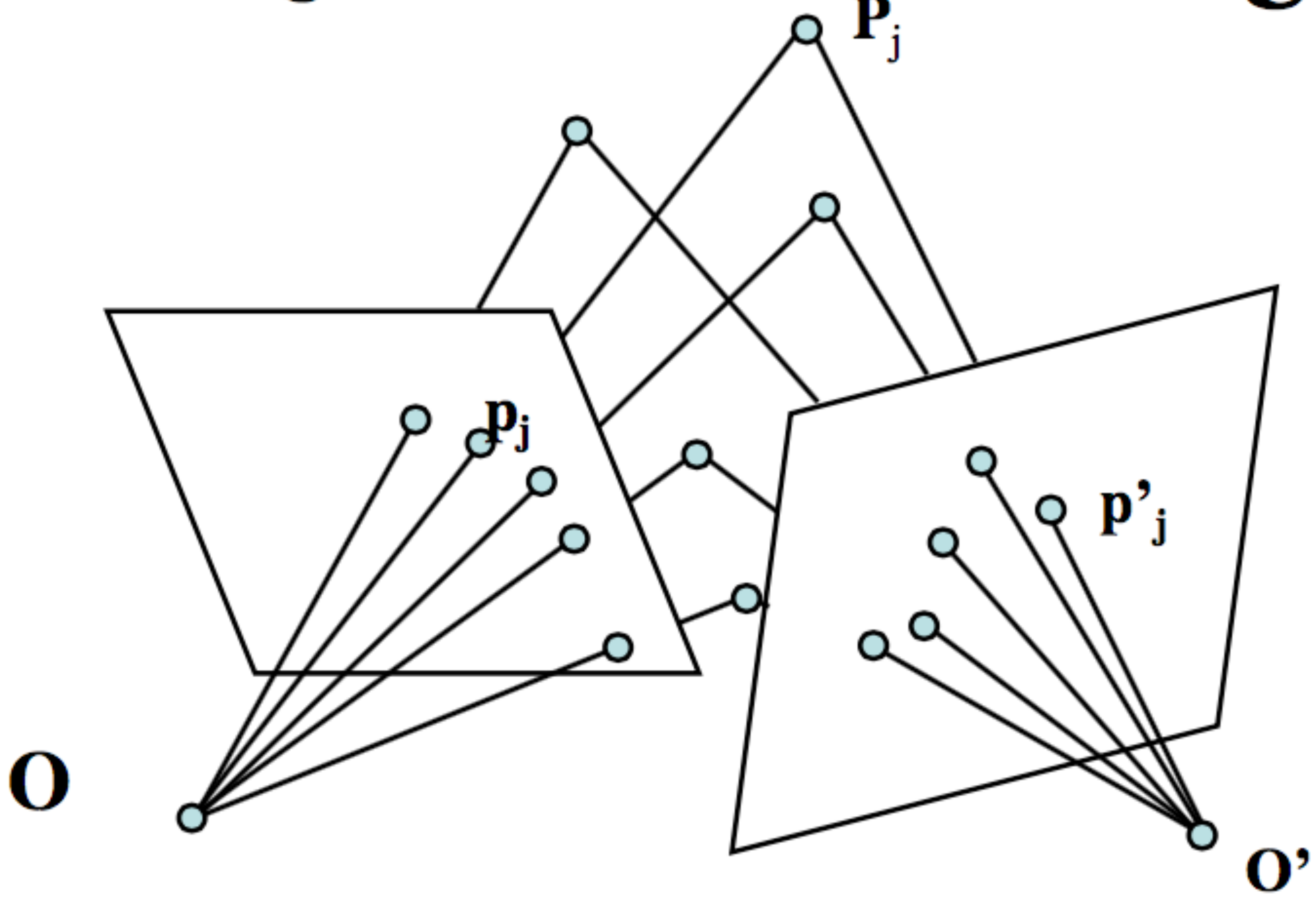
- Constraint: $\text{Det}(\mathbf{F}) = 0 \rightarrow 3^{rd}$ order polynomial in λ has at least one real solution

For every correspondence: $\mathbf{p}_i^T \mathbf{F} \mathbf{p}'_i$

Minimize over all correspondences: $\min \sum_i (\mathbf{p}_i^T \mathbf{F} \mathbf{p}'_i)^2$

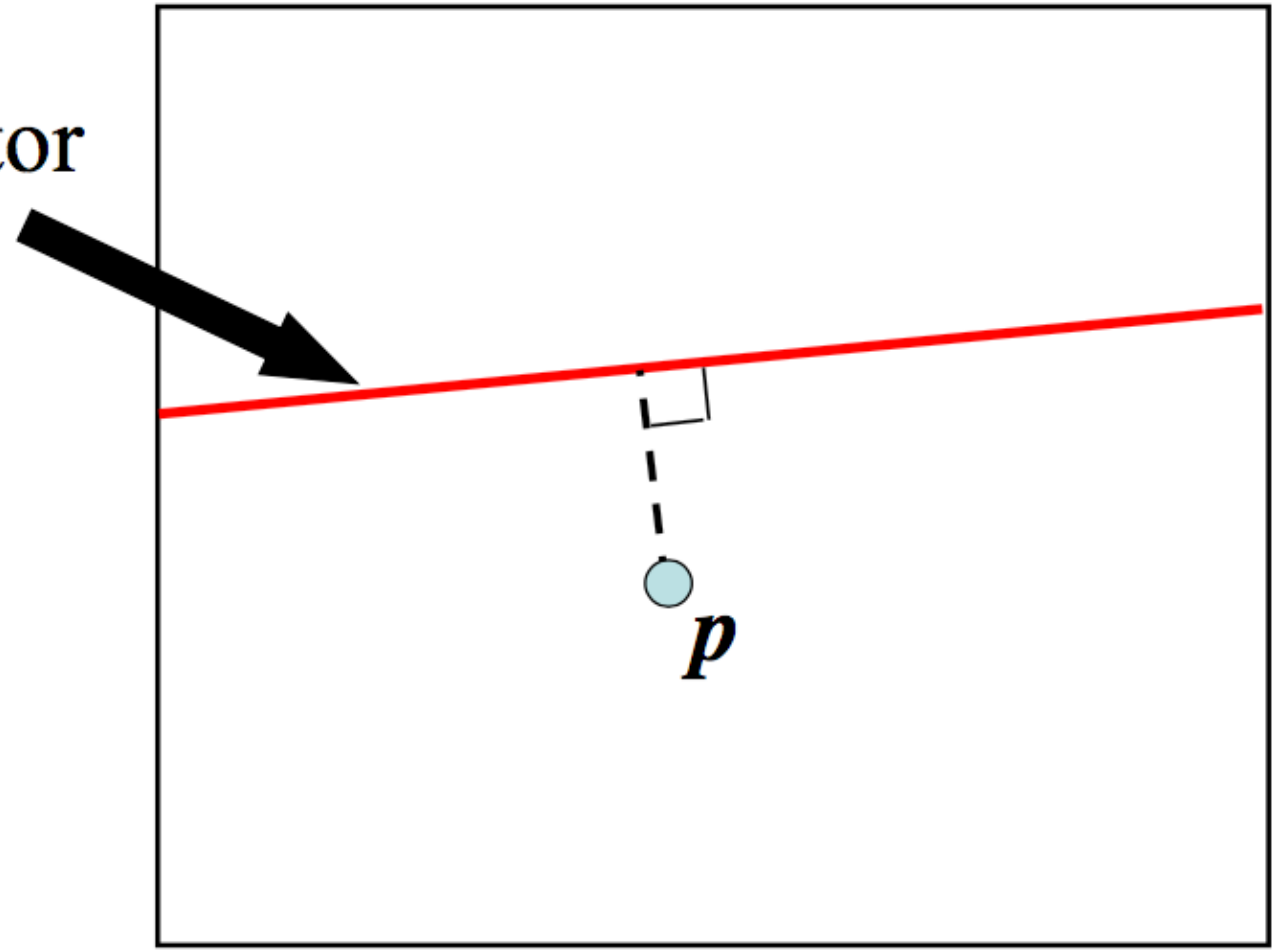
Equivalent to: $\min_{\|\mathbf{f}\|=1} \mathbf{f}^T \mathbf{U}^T \mathbf{U} \mathbf{f}$

Find min eigenvalue of 9x9 matrix: $\mathbf{U}^T \mathbf{U}$



Minimum
→8 correspondences
→*"8-point algorithm"*

Line described by vector of parameters: ***Fp'***



- Geometric distance between point and epipolar line:

$$d(p, Fp') = \frac{p^T F p'}{\sqrt{(Fp')_1^2 + (Fp')_2^2}}$$

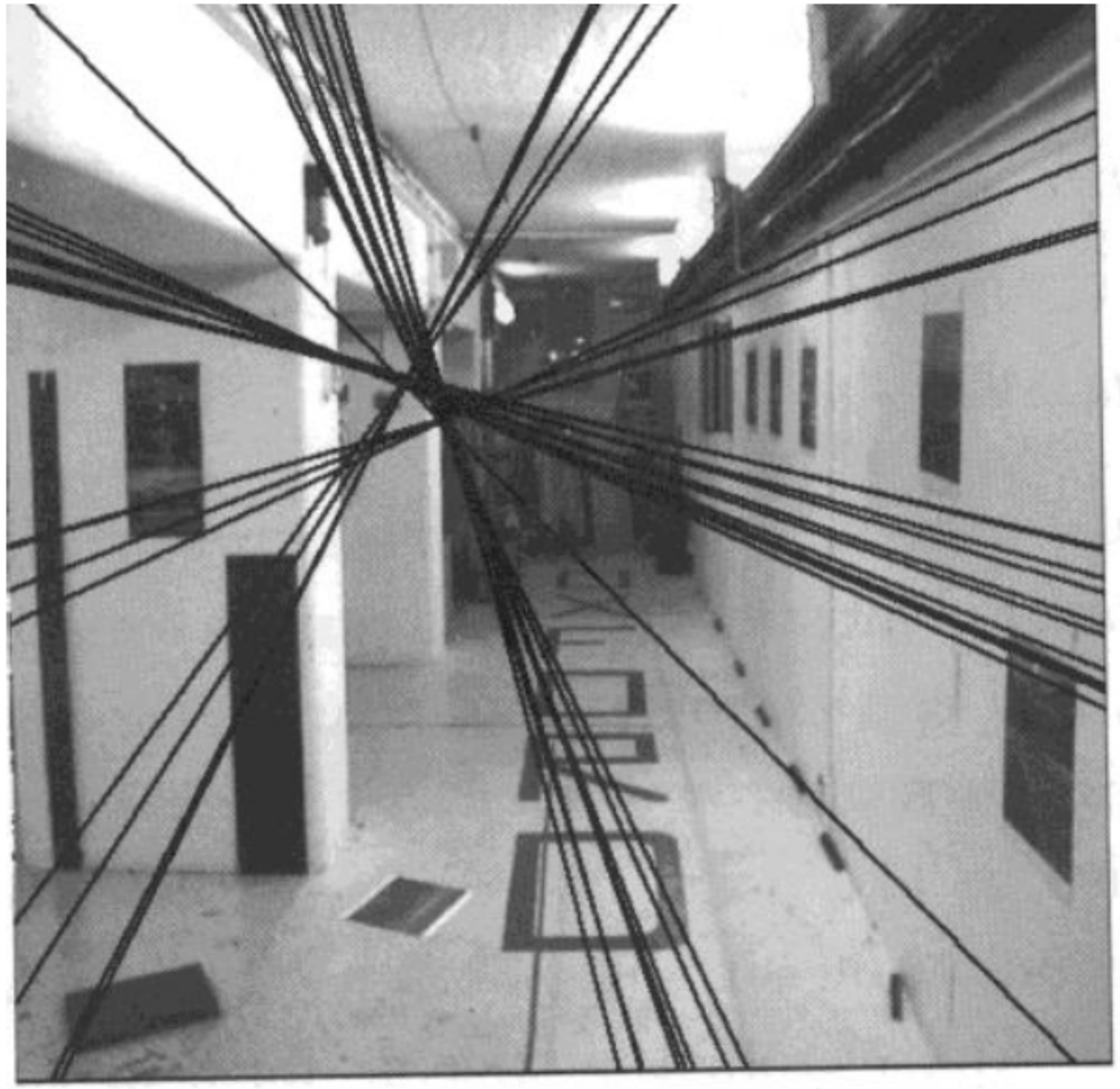
- Compute **U** from the correspondences
- Compute the eigenvector of $\mathbf{U}^{\mathrm{T}}\mathbf{U}$ corresponding to the smallest eigenvalue
- Make the resulting matrix **F** singular

$$\mathbf{F} = \mathbf{W}\,\mathrm{diag}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\mathbf{V}^{T} \rightarrow \mathbf{F} = \mathbf{W}\,\mathrm{diag}(\boldsymbol{a},\boldsymbol{b},0)\mathbf{V}^{T}$$

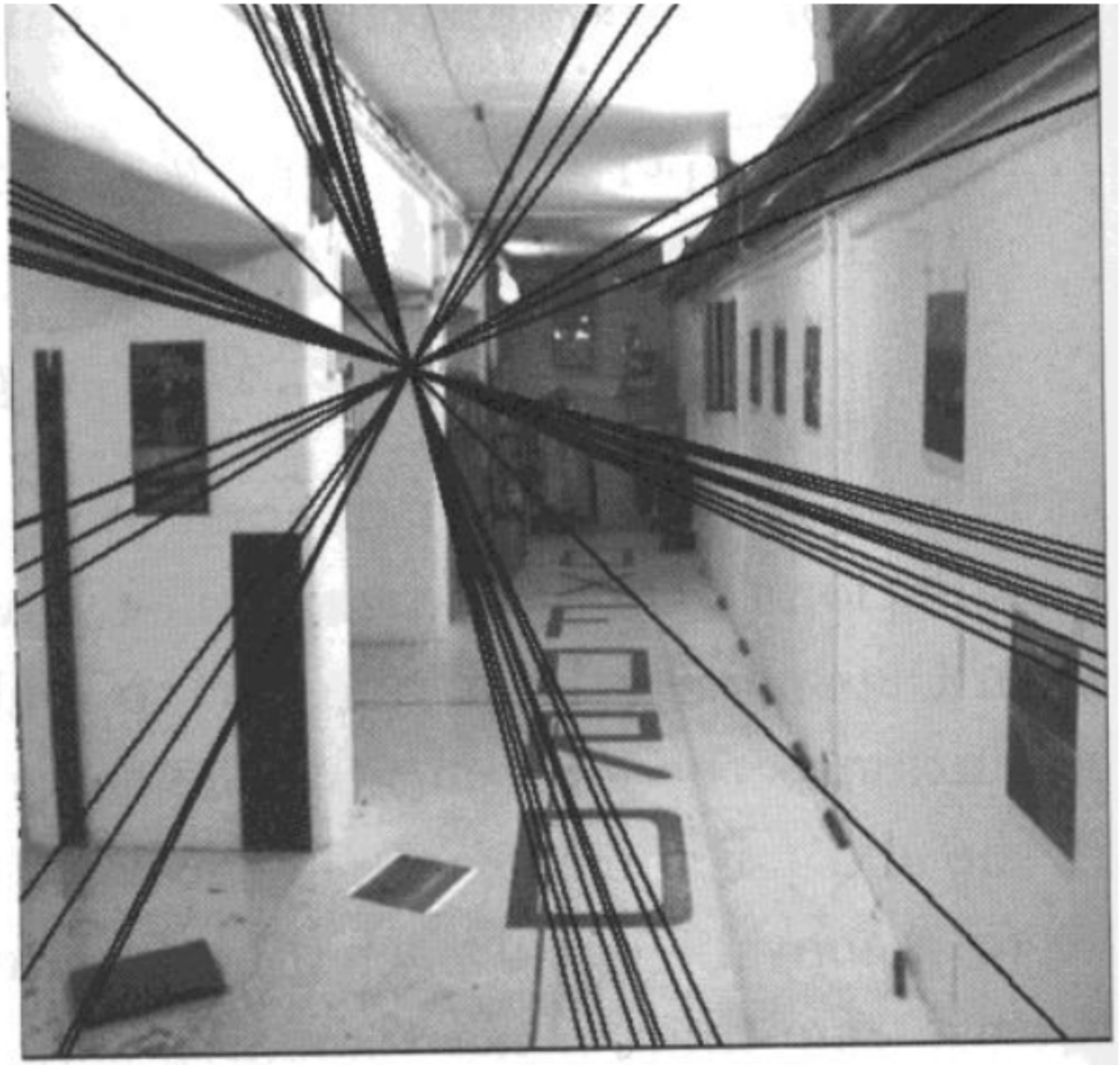
- Minimize the geometric error

$$\sum_{i} \boldsymbol{d}^{2}\left(\boldsymbol{p}_{i}, \boldsymbol{F}\boldsymbol{p}'_{i}\right) + \boldsymbol{d}^{2}\left(\boldsymbol{p}'_{i}, \boldsymbol{F}^{T}\boldsymbol{p}_{i}\right)$$

The importance of enforcing the constraint that **F** must be singular:



Incorrect **F** (non-singular matrix): The epipolar lines do not intersect



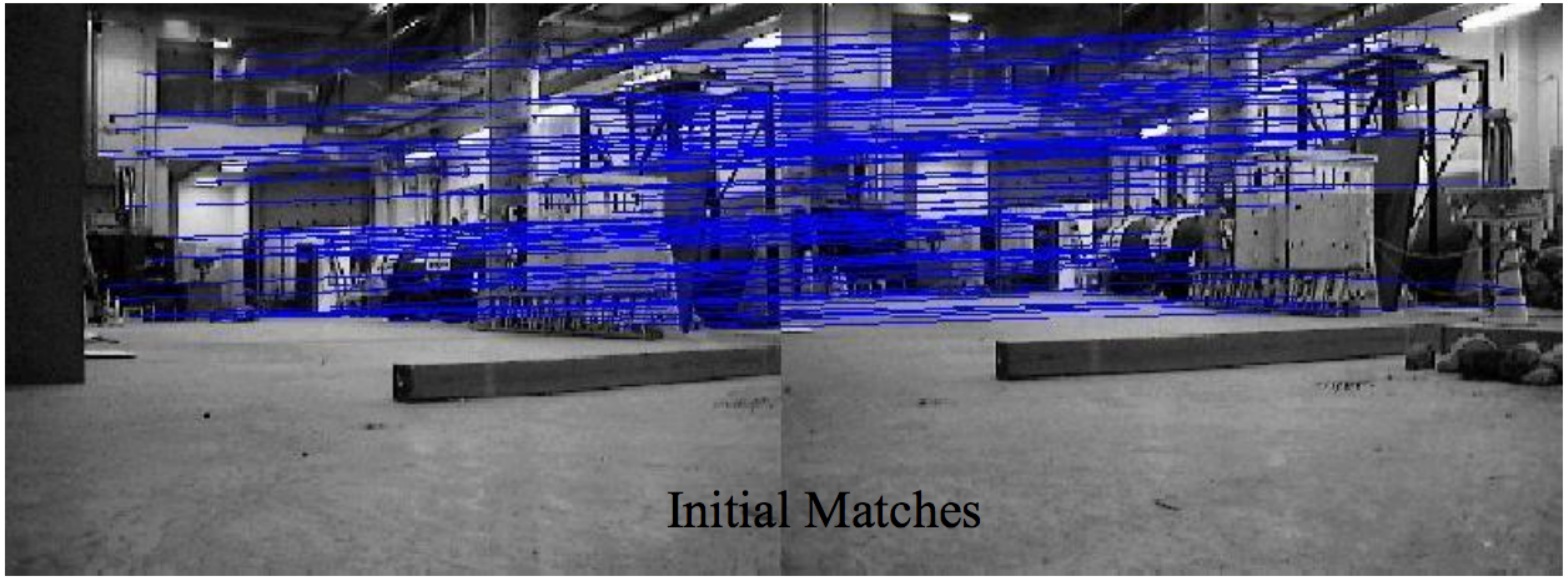
Correct **F**: All the epipolar lines Intersect at the epipole

Finding Correspondences

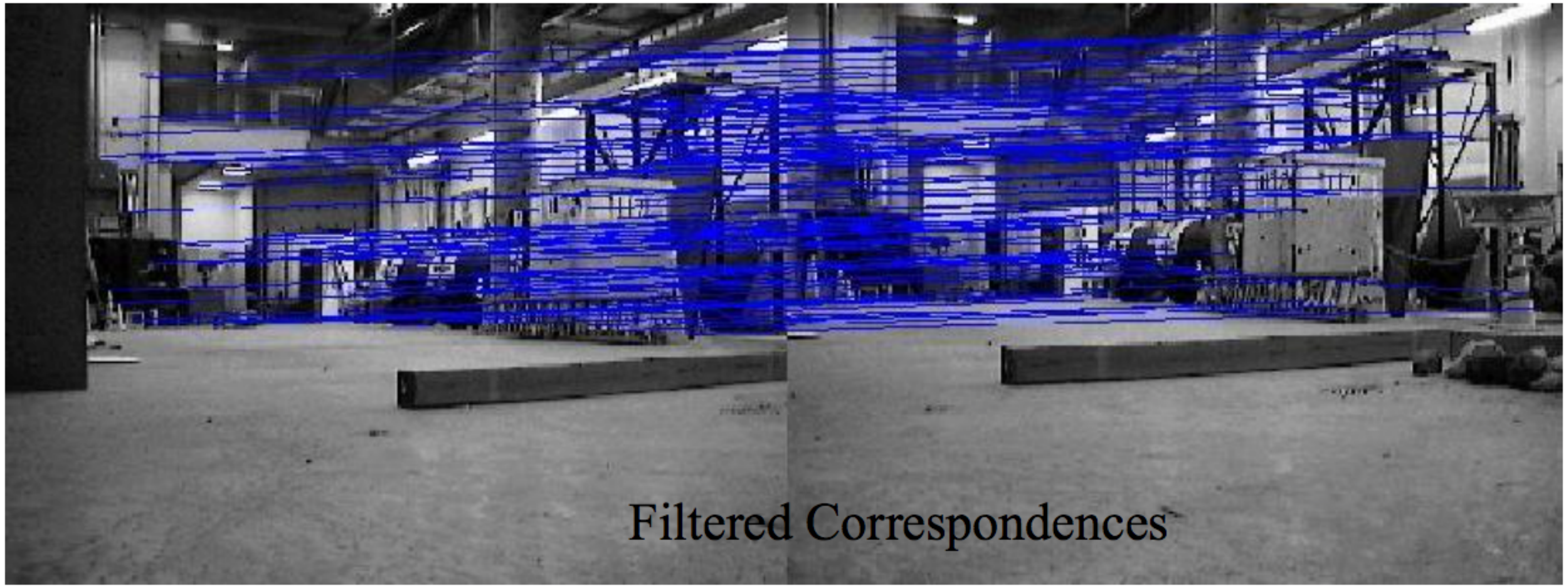
- The algorithms so far assume that we know the correspondences between the images
- In general, we do not know the correspondences and we must find them automatically
- Solution: RANSAC = RANdom SAmple Consensus

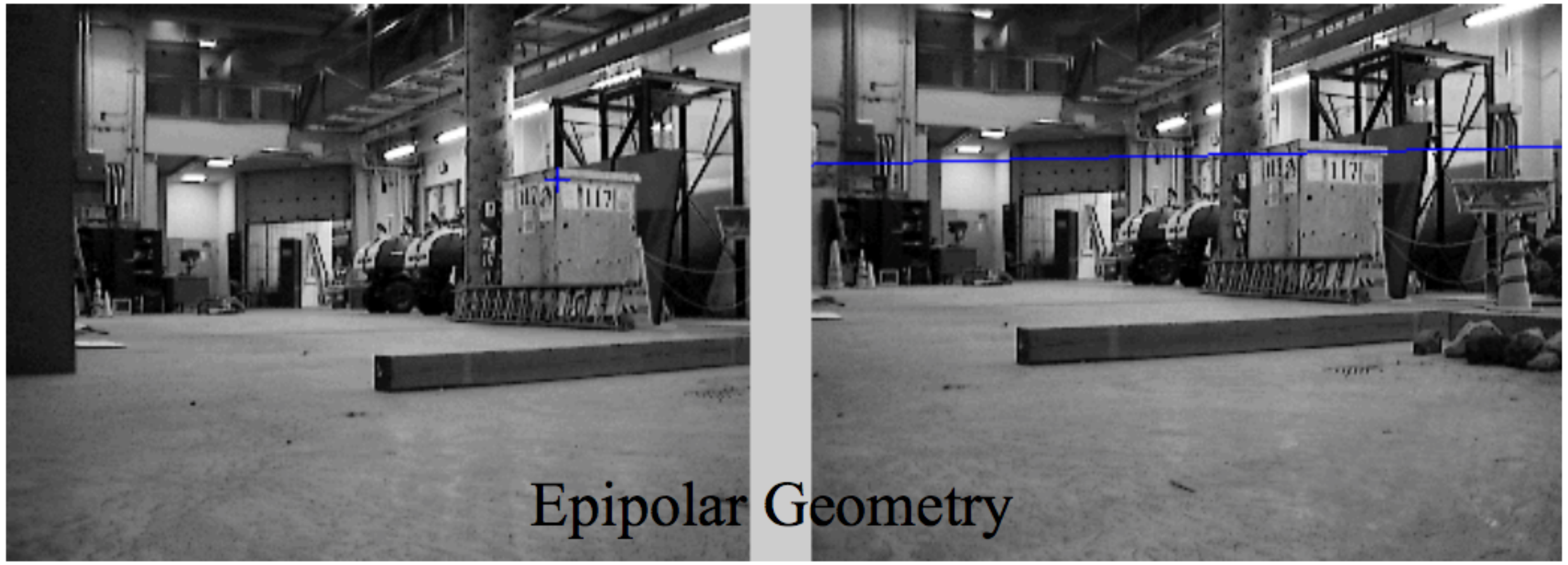
RANSAC

- Do *k* times:
  - Draw set with minimum number of correspondences
  - Fit *F* to the set
  - Count the number *d* of correspondences that are closer than *t* to the fitted epipolar lines
  - If $d > d_{min}$, recompute fit error using all the correspondences
- Return best fit found


Initial Matches


Features from one Image


Filtered Correspondences


Epipolar Geometry

Summary

Essential matrix

$$\hat{\mathbf{p}}^{T}\mathbf{E}\hat{\mathbf{p}}' = 0$$

$$\mathbf{E} = [\mathbf{t}]_{\times}\mathbf{R}$$

$$\mathbf{E} = \mathbf{W}\mathit{diag}(a, a, 0)\mathbf{V}^{T}$$

$$\mathbf{E}\hat{\mathbf{e}}' = \mathbf{E}^{T}\hat{\mathbf{e}} = \mathbf{0}$$

maps **p'** to epipolar line **Ep'**
in normalized camera coordinates
3x3 matrix of rank 2
defined up to scale
the 2 non-zero singular values are equal
5 parameters total
epipole is null space of **E**

Fundamental matrix

$$\mathbf{p}^{T}\mathbf{F}\mathbf{p}' = 0$$

$$\mathbf{F} = \mathbf{K}^{-T}\mathbf{E}\mathbf{K}'^{-1}$$

$$\mathbf{F}\mathbf{e}' = \mathbf{F}^{T}\mathbf{e} = \mathbf{0}$$

maps **p'** to epipolar line **Fp'**
in image coordinates
3x3 matrix of rank 2
7 parameters total
epipole is null space of **F**

Summary

Fundamental Matrix and Weak Calibration

$$\min \sum_i (\mathbf{p}_i^T \mathbf{F} \mathbf{p}'_i)^2$$

$$\min_{\|\mathbf{f}\|=1} \mathbf{f}^T \mathbf{U}^T \mathbf{U} \mathbf{f}$$

Linear: Basic 8-point algorithm: minimize sum of square *algebraic* errors.
Reduces to a min eigenvalue problem

$$\mathbf{F} = \mathbf{W}\,\mathrm{diag}(a, b, c)\mathbf{V}^T$$

$$\rightarrow \mathbf{F} = \mathbf{W}\,\mathrm{diag}(a, b, 0)\mathbf{V}^T$$

Enforce the constraint of rank 2 by setting smallest singular value to zero.

$$\sum_i d^2(\mathbf{p}_i, \mathbf{F}\mathbf{p}'_i) + d^2(\mathbf{p}'_i, \mathbf{F}^T\mathbf{p}_i)$$

Non-linear: Minimize sum of geometric errors.

Numerical issues: Rescale the coordinates between –1 and 1 (approx.)

The Classics

- Faugeras & Luong. The Geometry of Multiple Images. MIT Press, 2001
- Hartley & Zisserman. Multi View Geometry. Cambridge Press, 2000
- Xu & Zhang, Epipolar Geometry in Stereo, Motion, and Object Recognition, Kluwer Academic Publishers, 1996.
- www.dai.ed.ac.uk/Cvonline
- www-sop. inria.fr/robotvis/personnel/zzhang/CalibEnv/CalibEnv.html
- www.cs.huji.ac.il/labs/vision/research/project.html#geometry
- Longuet-Higgins, A computer algorithm for reconstructing a scene from 2 projections. Nature, 293, 1981.
- Hartley, In defence of the 8-point algorithm, PAMI 19(6), 1997.

- And, of course:
  16-822, Geometry-based Methods in Vision

Australia’s Contribution to CV!