Homework to Week 2

Statistics: Principle, Methods and R (II)

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Week 2, 6 March 2017

The homework is due on Monday, 13 March 2017. Please hand in the solutions to the teaching assistant He Siyuan at the beginning of the lecture.

1. For the Bernoulli example in last week's exercise, use Beta$(2, 3)$ prior and calculate the 0.95 credible interval and 0.95 confidence interval with the CLT. You might need to find the credible interval numerically. Compare the two intervals. Run your own Bernoulli examples with 1000, 10000 and 100000 data points and calculate the 0.95 credible interval and confidence interval respectively. For the purpose of presentation, please make a 3×2 table with 3 credible intervals and 3 confidence intervals side by side. Do the credible and confidence intervals coincide as the number of data points $n \to \infty$?

2. Verify that the Laplace prior is *not* conjugate with respect to the Normal model. To unify the notation, suppose $X_1, \ldots, X_n$ are IID Normal-distributed with mean μ and **known** variance σ^2. Suppose we put the Laplace prior with location parameter a and scale parameter $b > 0$ on μ, i.e., the prior density function is

$$\pi(\mu) = \frac{1}{2b} \exp\Big(-\frac{|\mu - a|}{b})\Big).$$

   Conduct your investigation from here.

3. For $X \sim N(\mu, \sigma^2)$, calculate $\mathbb{E}|X|$ and $\mathbb{E}[|X|^2]$. Can you calculate $\mathbb{E}[|X|^k]$ for any positive integer k?

4. Verify that the Laplace prior is not conjugate with respect to the Normal model.

5. Let $X_1, \ldots, X_n \sim$ Poisson(λ). Let $\lambda \sim$ Gamma(α, β) be the prior. Show that the posterior is also a Gamma. Find the posterior mean and mode.

6. Let $X_1, \ldots, X_n \sim$ Poisson(λ). Find the maximum likelihood estimator and the Fisher information $I(\lambda)$.

7. Let $X_1, \dots, X_n \sim \text{Bernoulli}(p)$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{L^2} p.$$

8. Let $\lambda_n = 1/n$ for $n = 1, 2, \dots$. Let $X_n \sim \text{Poisson}(\lambda_n)$.

   (a) Show that $X_n \xrightarrow{P} 0$.

   (b) Let $Y_n = nX_n$. Show that $Y_n \xrightarrow{P} 0$.

9. Suppose that $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$. Define

$$X_n = \begin{cases} X & \text{with probability } 1 - 1/n \\ e^n & \text{with probability } 1/n. \end{cases}$$

   Does X_n converge to X in probability? Does X_n converge to X in distribution? Does $\mathbb{E}[X - X_n]^2$ converge to 0?