StatsI — Exercise 3

1. Let the joint density function for X, Y be

$$f_{X,Y}(x,y) = \begin{cases} 8xy, & 0 < y < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Evaluate $E(X)$.

   (b) Find $E(Y|X)$, and also $E(Y)$ and $\mathrm{Cov}(X,Y)$.

   (c) Find $\mathrm{Var}(Y|X = 0.5)$.

2. Consider the random variables X and Y with joint density

$$f_{X,Y}(x,y) = \begin{cases} ke^{-\alpha x}e^{-\beta y}, & 0 < x < y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

   where α, β are known positive constants, and $k > 0$ is an unknown constant.

   (a) Evaluate k.

   (b) Write down $f_{Y|X}(y|x)$. Hence find $E(Y|X)$ and evaluate $\mathrm{Var}(E(Y|X))$.

   (c) Find the covariance of X and Y.

3. Let $Y|X \sim N(X, \sigma_2^2)$ and $X \sim N(\mu, \sigma_1^2)$. Show that $Y \sim N(\mu, \sigma_1^2 + \sigma_2^2)$.
   Hint: Find the MGF of Y.

4. Suppose $\mathbf{X} \sim N(\mu, \Sigma)$, verify that $\mathbf{X}$ has mean μ and covariance matrix Σ.

5. If X has the density function (sometimes called a Type II Beta distribution)

$$f_X(x) = \frac{x^{b-1}}{B(a,b)(1+x)^{a+b}} \qquad 0 < x < \infty,$$

   where $a, b > 0$ are constants. Find the density function of $Y = X/(1+X)$.

6. If X has the Weibull distribution

$$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau} \quad x > 0,$$

   where $c, \tau > 0$ are constants. What is the density function of $Y = cX^\tau$?

7. If the joint density function of X, Y is

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

   find the density function of $U = X/Y$.

8. (**A universal random generator**.) Let X be a r.v. taking values in (a,b), where $-\infty \le a < b \le \infty$. Let X have a strictly increasing CDF F (i.e. $F(x_1) < F(x_2)$ for any $a < x_1 < x_2 < b$).

   (a) Let $Y = F(X)$. Find the PDF of Y.

(b) Let $U \sim U(0, 1)$. Let $V = F^{-1}(U)$. Show that V and X have the same distribution.

(c) Write an R-function that takes $U(0, 1)$ random variables and generates random variables from the distribution with the PDF

$$f(x) = \begin{cases} 1/4 & x \in (-1, 0] \cup (1, 2) \\ 1/2 & x \in (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Plot a standardized histogram of a sample of size 5000 from this distribution and together with this PDF.

9. Generate random numbers $x_1, \cdots, x_{10,000}$ from $N(0, 1)$. Let $\bar{x}_n = n^{-1} \sum_{i=1}^{n} x_i$. Plot $\bar{x}_n$ against n for $n = 1, \cdots, 10000$. Repeat this exercise with Cauchy distribution now. Explain why there is such a difference.

10. Let X_1 and X_2 be independent $U[0, 1]$ random variablems. Find the probability density function for (a) $Y_1 = X_1 + X_2$, and (b) $Y_2 = X_1 - X_2$.

11. Let $X_1, \cdots, X_n$ be a sample from distribution with the probability density function $f(x)$. Find the density function for (a) $Y_1 = \max_{1 \le j \le n} X_j$, and (b) $Y_2 = \min_{1 \le j \le n} X_j$.