Homework to Week 1

Statistics: Principle, Methods and R (II)

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Week 2, 6 March 2017

The homework is due on Monday, 13 March 2017. Please hand in the solutions to the teaching assistant He Siyuan at the beginning of the lecture.

1. Verify that the Laplace prior is conjugate with respect to the Laplace model. To unify the notation, suppose $X_1, \ldots, X_n$ are IID Laplace-distributed with location parameter μ and known scale parameter $b > 0$, i.e., the density function is

$$f(x;\mu) = \frac{1}{2b}\exp\big(-|x-\mu|/b\big).$$

   Suppose we put on μ the prior of Laplace distribution with location parameter α and scale β. Conduct your investigation from here.

2. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Let $\lambda \sim \text{Gamma}(\alpha, \beta)$ be the prior. Show that the posterior is also a Gamma. Find the posterior mean.

3. Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$. Find the maximum likelihood estimator and the Fisher information $I(\lambda)$.

4. In each of the following models, find the Bayes risk and the Bayes estimator, using L^2 loss.
   (a) $X \sim \text{Binomial}(n, p)$, $p \sim \text{Beta}(\alpha, \beta)$.
   (b) $X \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{Gamma}(\alpha, \beta)$.
   (c) $X \sim N(\theta, \sigma^2)$ where σ^2 is known and $\theta \sim N(a, b^2)$.

5. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{L^2} p.$$

6. Let $\lambda_n = 1/n$ for $n = 1, 2, \ldots$. Let $X_n \sim \text{Poisson}(\lambda_n)$.

(a) Show that $X_n \xrightarrow{P} 0$.

(b) Let $Y_n = nX_n$. Show that $Y_n \xrightarrow{P} 0$.

7. Suppose that $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$. Define

$$X_n = \begin{cases} X & \text{with probability } 1 - 1/n \\ e^n & \text{with probability } 1/n. \end{cases}$$

Does X_n converge to X in probability? Does X_n converge to X in distribution? Does $\mathbb{E}[X - X_n]^2$ converge to 0?