

## StatsI — Exercise 3

1. Let the joint density function for  $X, Y$  be

$$f_{X,Y}(x, y) = \begin{cases} 8xy, & 0 < y < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Evaluate  $E(X)$ .
- (b) Find  $E(Y|X)$ , and also  $E(Y)$  and  $\text{Cov}(X, Y)$ .
- (c) Find  $\text{Var}(Y|X = 0.5)$ .

2. Consider the random variables  $X$  and  $Y$  with joint density

$$f_{X,Y}(x, y) = \begin{cases} ke^{-\alpha x}e^{-\beta y}, & 0 < x < y < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\alpha, \beta$  are known positive constants, and  $k > 0$  is an unknown constant.

- (a) Evaluate  $k$ .
- (b) Write down  $f_{Y|X}(y|x)$ . Hence find  $E(Y|X)$  and evaluate  $\text{Var}(E(Y|X))$ .
- (c) Find the covariance of  $X$  and  $Y$ .

3. Let  $Y|X \sim N(X, \sigma_2^2)$  and  $X \sim N(\mu, \sigma_1^2)$ . Show that  $Y \sim N(\mu, \sigma_1^2 + \sigma_2^2)$ .

**Hint:** Find the MGF of  $Y$ .

4. Suppose  $\mathbf{X} \sim N(\mu, \Sigma)$ , verify that  $\mathbf{X}$  has mean  $\mu$  and covariance matrix  $\Sigma$ .
5. If  $X$  has the density function (sometimes called a Type II Beta distribution)

$$f_X(x) = \frac{x^{b-1}}{B(a, b)(1+x)^{a+b}} \quad 0 < x < \infty,$$

where  $a, b > 0$  are constants. Find the density function of  $Y = X/(1+X)$ .

6. If  $X$  has the Weibull distribution

$$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau} \quad x > 0,$$

where  $c, \tau > 0$  are constants. What is the density function of  $Y = cX^\tau$ ?

7. If the joint density function of  $X, Y$  is

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 < x < y < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

find the density function of  $U = X/Y$ .

8. (**A universal random generator.**) Let  $X$  be a r.v. taking values in  $(a, b)$ , where  $-\infty \leq a < b \leq \infty$ . Let  $X$  have a strictly increasing CDF  $F$  (i.e.  $F(x_1) < F(x_2)$  for any  $a < x_1 < x_2 < b$ ).

- (a) Let  $Y = F(X)$ . Find the PDF of  $Y$ .

- (b) Let  $U \sim U(0, 1)$ . Let  $V = F^{-1}(U)$ . Show that  $V$  and  $X$  have the same distribution.
- (c) Write an R-function that takes  $U(0, 1)$  random variables and generates random variables from the distribution with the PDF

$$f(x) = \begin{cases} 1/4 & x \in (-1, 0] \cup (1, 2) \\ 1/2 & x \in (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Plot a standardized histogram of a sample of size 5000 from this distribution and together with this PDF.

9. Generate random numbers  $x_1, \dots, x_{10,000}$  from  $N(0, 1)$ . Let  $\bar{x}_n = n^{-1} \sum_{i=1}^n x_i$ . Plot  $\bar{x}_n$  against  $n$  for  $n = 1, \dots, 10000$ . Repeat this exercise with Cauchy distribution now. Explain why there is such a difference.
10. Let  $X_1$  and  $X_2$  be independent  $U[0, 1]$  random variables. Find the probability density function for (a)  $Y_1 = X_1 + X_2$ , and (b)  $Y_2 = X_1 - X_2$ .
11. Let  $X_1, \dots, X_n$  be a sample from distribution with the probability density function  $f(x)$ . Find the density function for (a)  $Y_1 = \max_{1 \leq j \leq n} X_j$ , and (b)  $Y_2 = \min_{1 \leq j \leq n} X_j$ .