StatsI — Exercise 2

1. (R Experiment.) Consider tossing a fair six-sided die. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Then $P(A) = 1/2$, $P(B) = 2/3$ and $P(AB) = P(A)P(B)$. Hence A and B are two independent events.

   (a) Using R to toss such a die n times and to verify $P(AB) = P(A)P(B)$. You may estimate, e.g. $P(A)$ by the proportion of times of A occurred in the n tosses.

   (b) Now find two events A and B that are not independent. Repeat (a) above.

   (**Hint**. You may draw a random number x from $U(0, 1)$, the event $\{i\}$ occurs if $x \in (\frac{i-1}{6}, \frac{i}{6})$ for $i = 1, \cdots, 6$.)

2. Find the distribution functions corresponding to the following density functions:

(a)	$f_X(x) = 1/[\pi(1 + x^2)]$	$-\infty < x < \infty$	(Cauchy)
(b)	$f_X(x) = e^{-x}/(1 + e^{-x})^2$	$-\infty < x < \infty$	(Logistic)
(c)	$f_X(x) = (a - 1)/(1 + x)^a$	$0 < x < \infty,\ a > 1$	(Pareto)
(d)	$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau}$	$0 < x < \infty, \tau > 0, c > 0$	(Weibull).

3. Find (without using moment generating functions) the mean and the variance for the following distributions

   (a) $f_X(x) = \begin{cases} e^{-kx}x^{r-1}k^r/(r-1)! & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad$ r positive integer, $k > 0$
   (Gamma Distribution)

   (b) $P(X = x) = e^{-\lambda}\lambda^x/x!$ $\quad$ $x = 0, 1, 2, \ldots, \lambda > 0$
   (Poisson Distribution)

   (c) $f_X(x) = \frac{a-1}{(1+x)^a}$ $\quad$ $x > 0, a > 3$
   (Pareto)

   (d) $P(X = x) = \frac{(a+x-1)!}{x!(a-1)!}\left[\frac{b}{1+b}\right]^a\left[\frac{1}{1+b}\right]^x$ $\quad$ $x = 0, 1, 2, \ldots, a > 0, b > 0$
   (Negative Binomial)

4. A random variable has 'no memory' if for all x and for $y > 0$

$$P[X > x + y \mid X > x] = P[X > y].$$

   Show that if X has either the exponential distribution, or a geometric distribution with $P(X = x) = q^{x-1}p$, then X has no memory. Interpret this property.

5. Let $X \sim N(\mu, \sigma^2)$ and $Y = e^X$.

   (a) Find the density of Y. Also find the mean and variance of Y using the moment generating function of X.

   (b) Draw a sample $x_1, \cdots, x_{10,000}$ from $N(0, 1)$ using R. Apply the transformation $y_i = e^{x_i}$. Compute the sample mean and the sample variance

$$\bar{y} = \frac{1}{10000}\sum_{i=1}^{10000} y_i, \qquad s^2 = \frac{1}{9999}\sum_{i=1}^{10000}(y_i - \bar{y})^2.$$

   Draw a histogram of $\{y_i\}$. Compare your findings with the results obtained in (a) above.

   (Y has the Lognormal distribution, popular as a skew distribution for positive variables.)