Chapter 3. Random Variables and Distributions

Basic idea of introducing random variables: represent outcomes and/or random events by numbers.

3.1 Random variables and Distributions.

Definition. A random variable is a function defined on the sample space Ω, which assigns a real number $X(\omega)$ to each outcome $\omega \in \Omega$.

Example 1. Flip a coin 10 times. We may define random variables (r.v.s) as follows:

X_1 = no. of heads,
X_2 = no. of flips required to have the first head,

X_3 = no. of 'HT'-pairs,
X_4 = no. of tails.

For $\omega = HHTHHHTHHTT$, $X_1(\omega) = 6,\ X_2(\omega) = 1,\ X_3(\omega) = 3,\ X_4(\omega) = 4.$ Note $X_1 \equiv 10 - X_4$.

Remark. The values of a r.v. varies and cannot be pre-determined before an outcome occurs.

Definition. For any r.v. X, its (cumulative) distribution function (CDF) is defined as $F_X(x) = P(X \leq x)$.

Example 2. Toss a fair coin twice and let X be the number of heads. Then

$$P(X = 0) = P(X = 2) = 1/4, \quad P(X = 1) = 1/2.$$

Hence its CDF is $F_X(x) = \begin{cases} 0 & x < 0, \\ 1/4 & 0 \leq x < 1, \\ 3/4 & 1 \leq x < 2, \\ 1 & x \geq 2. \end{cases}$

Note. (i) $F_X(x)$ is right continuous, non-decreasing, and defined for all $x \in (-\infty, \infty)$. For example, $F_X(1.1) = 0.75$.

(ii) The CDF is a non-random function.

(iii) If $F(\cdot)$ is the CDF of r.v. X, we simply write $X \sim F$.

Properties of CDF. A function $F(\cdot)$ is a CDF iff

(i) F is non-decreasing: $x_1 < x_2$ implies $F(x_1) \leq F(x_2)$,
(ii) F is normalized: $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$,
(iii) F is right continuous: $\lim_{y\downarrow x} F(y) = F(x)$.

Probabilities from CDF

(a) $P(X > x) = 1 - F(x)$
(b) $P(x < X \leq y) = F(y) - F(x)$
(c) $P(X < x) = \lim_{h\downarrow 0} F(x - h) \equiv F(x-)$
(d) $P(X = x) = F(x) - F(x-)$.

Note. It is helpful for understanding (c) & (b) to revisit Example 2.

3.2 Discrete random variables

If r.v. X only takes some isolated values, X is called a discrete r.v. Its CDF is called a discrete distribution.

Definition. For a discrete r.v. X taking values $\{x_1, x_2, \cdots\}$, we define the probability function (or probability mass function) as

$$f_X(x_i) = P(X = x_i), \quad i = 1, 2, \cdots .$$

Obviously, $f_X(x_i) \geq 0$ and $\sum_i f_X(x_i) = 1$.

It is often more convenient to list a probability function in a table:

X	x_1	x_2	$\cdots\cdots$
Probability	$f_X(x_1)$	$f_X(x_2)$	$\cdots\cdots$

Example 2 (continue). The probability function is be tabulated:

X	0	1	2
Probability	1/4	1/2	1/4

Expectation or Mean EX or $E(X)$: a measure for the 'center', 'average value' of a r.v. X, and is often denoted by μ.

For a discrete r.v. X with probability function $f_X(x)$,

$$\mu = EX = \sum_i x_i f_X(x_i).$$

Variance Var(X): a measure for variation, uncertainty or 'risk' of a r.v. X, is often denoted by σ^2, while σ is called **standard deviation** of X.

For a discrete r.v. X with probability function $f_X(x)$,

$$\sigma^2 = \mathrm{Var}(X) = \sum_i (x_i - \mu)^2 f_X(x_i) = \sum_i x_i^2 f_X(x_i) - \mu^2.$$

The k-th moment of X: $\mu_k \equiv E(X^k) = \sum_i x_i^k f_X(x_i)$, $k = 1, 2, \cdots$.

Obviously, $\mu = \mu_1$, and $\sigma^2 = \mu_2 - \mu_1^2$.

Some important discrete distributions

Convention. We often use upper case letters $X, Y, Z, \cdots$ to denote r.v.s, and lower case letters $x, y, z, \cdots$ to denote the values of r.v.s. In contrast letters a, b or A, B are often used to denote (non-random) constants.

Degenerate distribution: $X \equiv a$, i.e. $F_X(x) = 1$ for any $x \geq a$, and 0 otherwise.

It is easy to see that $\mu = a$ and $\sigma^2 = 0$.

Bernoulli distribution: X is binary, $P(X = 1) = p$ and $P(X = 0) = 1 - p$, where $p \in [0, 1]$ is a constant. It represents the outcome of flipping a coin.

$$\mu = 1 \cdot p + 0 \cdot (1 - p) = p, \quad \sigma^2 = p(1 - p).$$

Note. Bernoulli trial refers to an experiment of flipping a coin repeatedly.

Binomial distribution Bin(n, p): X takes values 0, 1, $\cdots$, n only with the probability function

$$f_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots , n.$$

Theorem. If we toss a coin n times, let X be the number of heads. Then $X \sim \text{Bin}(n, p)$, where p is the probability that head occurs in tossing the coin once.

Proof. Let $\omega = HTHHT \cdots H$ denote an outcome of n tosses. Then $X = k$ iff there are k 'H' and $(n-k)$ 'T' in ω. Therefore the probability of such a ω is $p^k(1-p)^{n-k}$. Since those k H's may occur in any n positions of the sequence, there are $\left(\begin{array}{c} n \\ k \end{array} \right)$ such ω's. Hence

$$P(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots , n.$$

Let us check if the probability function above is well defined. Obviously $P(X = k) \geq 0$, furthermore

$$\sum_{k=0}^{n} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \{p + (1-p)\}^n = 1^n = 1.$$

Let us work out the mean and the variance for $X \sim \mathrm{Bin}(n, 1-p)$.

$$\begin{aligned}
\mu &= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{j=0}^{n-1} np \frac{(n-1)!}{j!(n-1-j)!} p^j (1-p)^{n-1-j} \\
&= np \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} p^j (1-p)^{m-j} = np.
\end{aligned}$$

Note that $\sigma^2 = E(X^2) - \mu^2 = E\{X(X-1)\} + \mu - \mu^2$. We need to work out

$$E\{X(X-1)\} = \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

$$= \sum_{k=2}^{n} n(n-1)p^2 \frac{(n-2)!}{(k-2)!\{(n-2)-(k-2)\}}p^{k-2}(1-p)^{\{(n-2)-(k-2)\}}$$

$$= n(n-1)p^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!\{(n-2)-j)\}}p^j(1-p)^{\{(n-2)-j\}} = n(n-1)p^2.$$

This gives $\sigma^2 = n(n-1)p^2 + np - (np)^2 = np(1-p)$.

By the above theorem, we can see immediately

(i) If $X \sim \mathrm{Bin}(n,p)$, $n - X \sim \mathrm{Bin}(n, 1-p)$.
(ii) If $X \sim \mathrm{Bin}(n,p)$, $Y \sim \mathrm{Bin}(m,p)$, and X and Y are independent, then $X + Y \sim \mathrm{Bin}(n+m, p)$.

Furthermore, let $Y_i = 1$ if the i-th toss yields H, and 0 otherwise. Then $Y_1, \cdots, Y_n$ are *independent* Bernoulli r.v.s with mean p and variance $p(1-p)$. Since $X = Y_1 + \cdots + Y_n$, we notice

$$EX = \sum_{i=1}^{n} EY_i = np, \quad \mathrm{Var}(X) = \sum_{i=1}^{n} \mathrm{Var}(Y_i) = np(1-p).$$

This is a much easier way to derived the means and variances for binomial distributions, which is based on the following general properties.

(i) For any r.v.s $\xi_1, \cdots, \xi_n$, and any constants $a_1, \cdots, a_n$,

$$E\Big(\sum_{i=1}^{n} a_i \xi_i\Big) = \sum_{i=1}^{n} a_i E(\xi_i).$$

(ii) If, in addition, $\xi_1, \cdots, \xi_n$ are *independent*,

$$\mathrm{Var}\Big(\sum_{i=1}^{n} a_i \xi_i\Big) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(\xi_i).$$

Independence of random variables. The r.v.s $\xi_1, \cdots, \xi_n$ are independent if

$$P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \times \cdots \times P(\xi_n \le x_n)$$

for any $x_1, \cdots, x_n$.

Moment generate function (MGF) of r.v. X:

$$\psi_X(t) = E(e^{tX}), \quad t \in (-\infty, \infty).$$

(i) It is easy to see that $\psi'_X(0) = E(X) = \mu$. In general $\psi_X^{(k)}(0) = E(X^k) = \mu_k$.

(ii) If $Y = a + bX$, $\psi_Y(t) = E(e^{(a+bX)t}) = e^{at}\psi_X(bt)$.

(iii) If $X_1, \cdots, X_n$ are independent, $\psi_{\sum_i X_i}(t) = \prod_{i=1}^n \psi_{X_i}(t)$, and vice versa

If X is discrete, $\psi_X(t) = \sum_i e^{x_i t} f_X(x_i)$.

To generate a r.v. from Bin(n, p), we can flip a coin (with p-probability for H) n times, and count the number of heads. However R can do the flipping for us much more efficiently:

```
> rbinom(10, 100, 0.1)  # generate 10 random numbers from \Bin(100, 0.1)
 [1]  8 11  9  9 18  7  5  5  3  7
> rbinom(10, 100, 0.1)  # do it again, obtain different numbers
 [1] 11 13  6  7 11  9  9  9 12 10
> x <- rbinom(10, 100, 0.7); x; mean(x)
 [1] 66 77 67 66 64 68 70 68 72 72
[1] 69                # mean close to np=70
> x <- rbinom(10, 100, 0.7); x; mean(x)
 [1] 70 73 72 70 68 69 70 66 79 71
[1] 70.8
```

Note that `rbinom(10000, 1, 0.5)` is equivalent to toss a fair coin 10000 times:

```
> y <- rbinom(10000, 1, 0.5); length(y); table(y)
[1] 10000
y
   0    1
4990 5010   # about a half times with head
```

You may try with smaller sample size, such as

```
> y <- rbinom(10, 1, 0.5); length(y); table(y)
[1] 10
y
0 1
3 7  # 7 heads and 3 tails
```

Also try out `pbinom` (CDF), `dbinom` (probability function), `qbinom` (quantile) for Binomial distributions.

Geometric Distribution Geom(p): X takes all positive integer values with probability function

$$P(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, \cdots .$$

Obviously, X is the number of tosses required in a Bernoulli trial to obtain the first head.

$$\mu = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = -p\,\frac{d}{dp}\sum_{k=1}^{\infty}(1-p)^k = -p\,\frac{d}{dp}(1/p) = 1/p,$$

and it can be shown that $\sigma^2 = (1-p)/p^2$.

Using the MGF provides an alternative way to find mean and variance: for

$t < -\log(1-p)$ (i.e. $e^t(1-p) < 1$),

$$\begin{aligned} \psi_X(t) &= E(e^{tX}) = \sum_{i=1}^{\infty} e^{ti}(1-p)^{i-1}p = \frac{p}{1-p}\sum_{i=1}^{\infty}\{e^t(1-p)\}^i \\ &= \frac{p}{1-p}\frac{e^t(1-p)}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)} = \frac{p}{e^{-t}-1+p}. \end{aligned}$$

Now $\mu = \psi_X'(0) = \left[\frac{pe^{-t}}{(e^{-t}-1+p)^2}\right]_{t=0} = 1/p$, and

$$\mu_2 = \psi_X''(0) = \left[\frac{2pe^{-2t}}{(e^{-t}-1+p)^3} - \frac{pe^{-t}}{(e^{-t}-1+p)^2}\right]_{t=0} = 2/p^2 - 1/p.$$

Hence $\sigma^2 = \mu_2 - \mu^2 = (1-p)/p^2$.

The R functions for Geom(p): `rgeom`, `dgeom`, `pgeom` and `qgeom`.

Poisson Distribution Poisson(λ): X takes all non-negative integers with probability function

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \qquad k = 0, 1, 2, \cdots,$$

where $\lambda > 0$ is a constant, called parameter.

The MGF $X \sim$ Poisson(λ):

$$\psi_X(t) = \sum_{k=0}^{\infty} e^{kt} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!} = e^{-\lambda} e^{e^t \lambda} = \exp\{\lambda(e^t - 1)\}.$$

Hence

$$\mu = \psi_X'(0) = [\exp\{\lambda(e^t - 1)\} \lambda e^t]_{t=0} = \lambda,$$

$$\mu_2 = \psi_X''(0) = [\exp\{\lambda(e^t - 1)\} \lambda e^t + \exp\{\lambda(e^t - 1)\} (\lambda e^t)^2]_{t=0} = \lambda + \lambda^2.$$

Therefore $\sigma^2 = \mu_2 - \mu^2 = \lambda$.

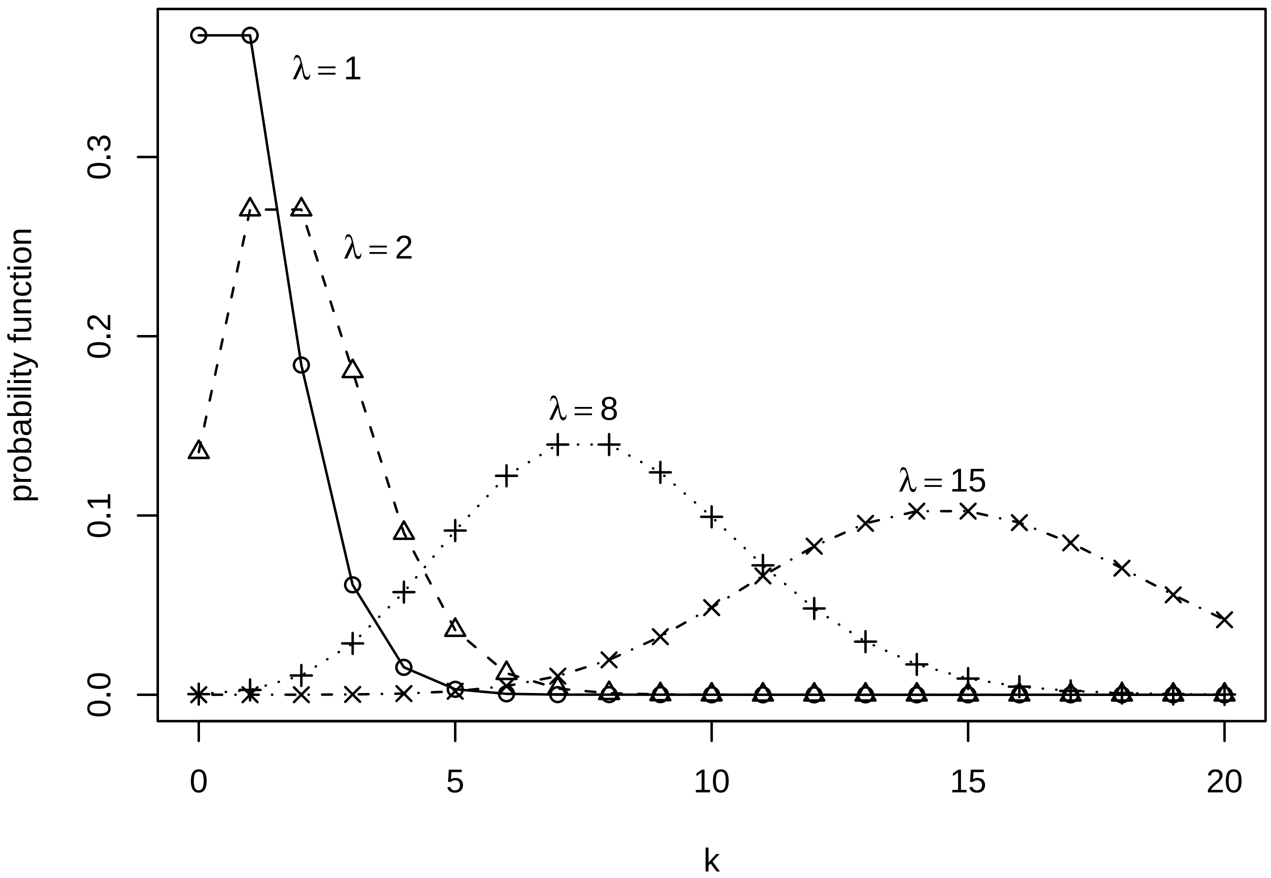
Remark. For Poisson distributions, $\mu = \sigma^2$.

The R functions for Poisson(λ): `rpois, dpois, ppois` and `qpois`.

To understand the role of the parameter λ, we plot the probability function of Poisson(λ) for different values of λ.

```
> x <- c(0:20)
> plot(x,dpois(x,1),type='o',xlab='k', ylab='probability function')
> text(2.5,0.35, expression(lambda==1))
> lines(x,dpois(x,2),typ='o',lty=2, pch=2)
> text(3.5,0.25, expression(lambda==2))
> lines(x,dpois(x,8),typ='o',lty=3, pch=3)
> text(7.5,0.16, expression(lambda==8))
> lines(x,dpois(x,15),typ='o',lty=4, pch=4)
> text(14.5, 0.12, expression(lambda==15))
```

Plots of $\lambda^k e^{-\lambda}/k!$ against k



Three ways of computing probability and distribution functions:

- calculators — for simple calculation
- statistical tables — for, e.g. the final exam
- R — for serious tasks such as real application

3.2 Continuous random variables

A r.v. X is *continuous* if there exists a function $f_X(\cdot) \geq 0$ such that

$$P(a < X < b) = \int_a^b f_X(x)dx, \quad \forall a < b.$$

We $f_X(\cdot)$ the *probability density function* (PDF) or, simply, density function. Obviously

$$F_X(x) = \int_{-\infty}^x f_X(u)du.$$

Properties of continuous random variables

(i) $F_X(x) = P(X \leq x) = P(X < x)$, i.e. $P(X = x) = 0 \neq f_X(x)$.

(ii) The PDF $f_X(\cdot) \geq 0$, and $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

(iii) $\mu = E(X) = \int_{-\infty}^{\infty} x f_X(x)dx$,

$$\sigma^2 = \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx = \int x^2 f_X(x)dx - \mu^2.$$

Furthermore the MGF of X is equal to

$$\psi_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx.$$

Some important continuous distributions

Uniform distribution $U(a,b)$: X takes any values between a and b equally likely. Its PDF is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Then the CDF is

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 0 & x < a, \\ \frac{1}{b-a}\int_a^x du = \frac{x-a}{b-a} & a \le x \le b, \\ 1 & x > b. \end{cases},$$

and

$$\mu = \int_a^b \frac{x dx}{b-a} = \frac{a+b}{2}, \quad \mu_2 = \int_a^b \frac{x^2 dx}{b-a} = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

Hence $\sigma^2 = \mu_2 - \mu^2 = (b-a)^2/12$.

R-functions related to uniform distributions: `runif, dunif, punif, qunif`.

Quantile. For a given CDF $F(\cdot)$, its quantile function is defined as

$$F^{-1}(p) = \inf\{x : F(x) \ge p\}, \quad p \in [0, 1]$$

```r
> x <- c(1, 2.5, 4)
> punif(x, 2, 3)     # CDF of U(2, 3) at 1, 2.5 and 4
[1] 0 0.5 1
> dunif(x, 2, 3)     # PDF of U(2, 3) at 1, 2.5 and 4
[1] 0 1 0
> qunif(0.5, 2, 3)    # quantile of U(2, 3) at p=0.5
[1] 2.5
```

Normal Distribution $N(\mu, \sigma^2)$: the PDF is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{-\frac{1}{2\sigma^2}(x-\mu)^2\Big\}, \quad -\infty < x < \infty,$$

where $\mu \in (-\infty, \infty)$ is the 'centre' (or mean) of the distribution, and $\sigma > 0$ is the 'spread' (standard deviation).

Remarks. (i) The most important distribution in statistics: Many phenomena in nature have approximately normal distributions. Furthermore, it provides asymptotic approximations for the distributions of sample means (Central Limit Theorem).

(ii) If $X \sim N(\mu, \sigma^2)$, $EX = \mu$, $\mathrm{Var}(X) = \sigma^2$, and $\psi_X(t) = e^{\mu t + \sigma^2 t^2/2}$.

We compute $\psi_X(t)$ below, the idea is applicable in general.

$$
\begin{aligned}
\psi_X(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int e^{tx}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx = \frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{1}{2\sigma^2}(x^2-2\mu x-2tx\sigma^2+\mu^2)}dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{1}{2\sigma^2}[\{x-(\mu+t\sigma^2)\}^2-(\mu+t\sigma^2)^2+\mu^2]}dx \\
&= e^{\frac{1}{2\sigma^2}\{(\mu+t\sigma^2)^2-\mu^2\}}\frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{1}{2\sigma^2}\{x-(\mu+t\sigma^2)\}^2}dx \\
&= e^{\frac{1}{2\sigma^2}\{(\mu+t\sigma^2)^2-\mu^2\}} = e^{\mu t+t^2\sigma^2/2}
\end{aligned}
$$

(iii) Standard normal distribution: $N(0,1)$.

If $X \sim N(\mu,\sigma^2)$, $Z \equiv (X-\mu)/\sigma \sim N(0,1)$. Hence

$$
P(a < X < b) = P\big(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\big) = \Phi\big(\frac{b-\mu}{\sigma}\big) - \Phi\big(\frac{a-\mu}{\sigma}\big),
$$

where

$$\Phi(x) = P(Z < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

is the CDF of $N(0, 1)$. Its values are tabulated in all statistical tables.

Example 3. Let $X \sim N(3, 5)$.

$$P(X > 1) = 1 - P(X < 1) = 1 - P\Big(Z < \frac{1-3}{\sqrt{5}}\Big) = 1 - \Phi(-0.8944) = 0.81.$$

Now find $x = \Phi^{-1}(0.2)$, i.e. x satisfies the equation

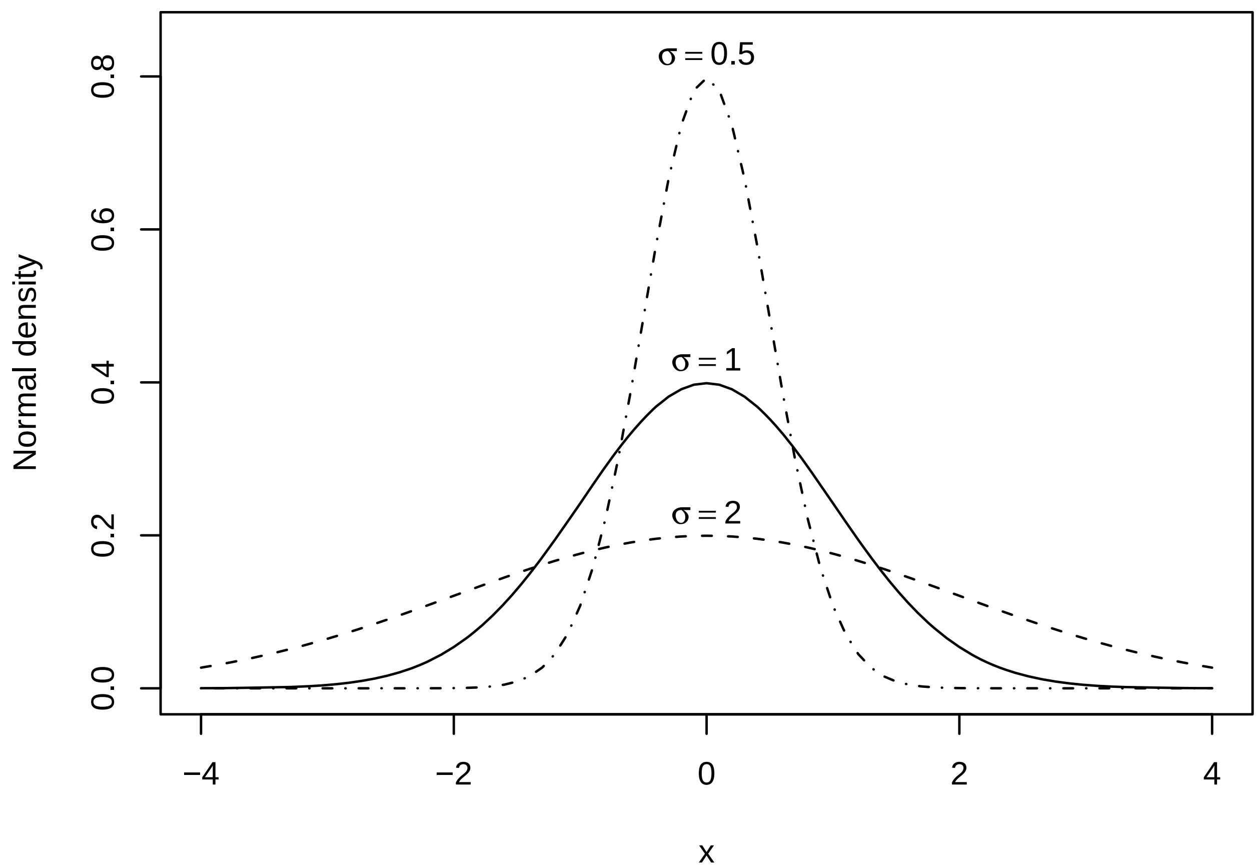
$$0.2 = P(X < x) = P\Big(Z < \frac{x-3}{\sqrt{5}}\Big).$$

From the normal table, $\Phi(-0.8416) = 0.2$. Therefore $(x-3)/\sqrt{5} = -0.8416$, leading to the solution $x = 1.1181$.

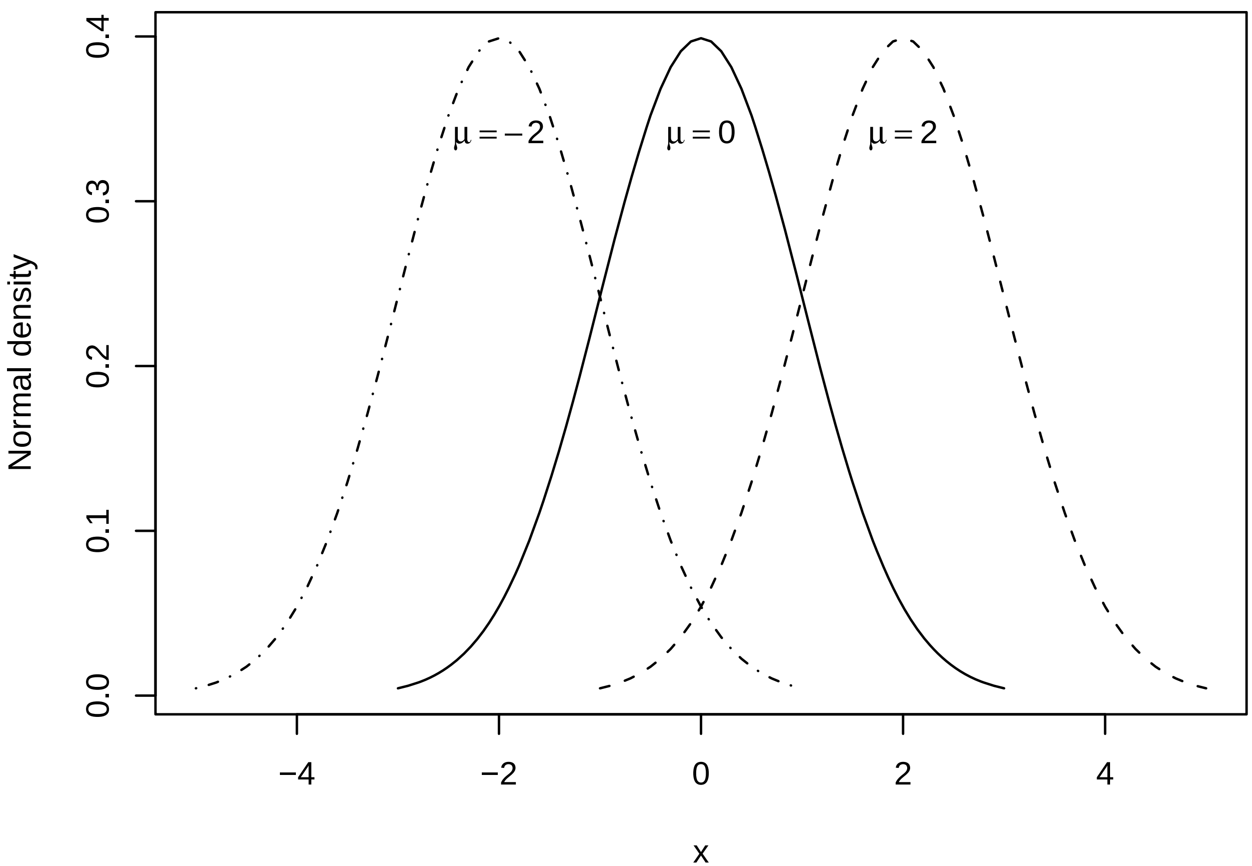
Note. You may check the answers using R:

```
1 - pnorm(1, 3, sqrt(5)),
qnorm(0.2, 3, sqrt(5))
```

Density functions of $N(0, \sigma^2)$



Density functions of $N(\mu, 1)$



Below are the R codes which produce the two normal density plots.

```
x <- seq(-4, 4, 0.1)      # x = (-4, -3.9, -3.8, ..., 3.9, 4)
plot(x, dnorm(x, 0, 1), type='l', xlab='x', ylab='Normal density',
                                   ylim=c(0, 0.85))
text(0,0.43, expression(sigma==1))
lines(x, dnorm(x, 0, 2), lty=2)
text(0,0.23, expression(sigma==sqrt(2)))
lines(x, dnorm(x, 0, 0.5), lty=4)
text(0,0.83, expression(sigma==sqrt(0.5)))

x <- seq(-3, 3, 0.1)
plot(x, dnorm(x, 0, 1), type='l', xlab='x', ylab='Normal density',
                           xlim=c(-5, 5))
text(0,0.34, expression(mu==0))
lines(x+2, dnorm(x+2, 2, 1), lty=2)
text(2,0.34, expression(mu==2))
lines(x-2, dnorm(x-2, -2, 1), lty=4)
text(-2,0.34, expression(mu==-2))
```

Exponential Distribution $\mathrm{Exp}(\lambda)$: $X \sim \mathrm{Exp}(\lambda)$ if X has the PDF

$$f(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a parameter.

$$E(X) = \lambda, \quad \mathrm{Var}(X) = \lambda^2, \quad \psi_X(t) = 1/(1 - t\lambda).$$

Background. $\mathrm{Exp}(\lambda)$ is used to model the lifetime of electronic components and the waiting times between rare events.

Gamma Distribution $\mathrm{Gamma}(\alpha, \beta)$: $X \sim \mathrm{Gamma}(\alpha, \beta)$ if X has the PDF

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha, \beta > 0$ are two parameters, and $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$.

$$E(X) = \alpha\beta, \quad \mathrm{Var}(X) = \alpha\beta^2, \quad \psi_X(t) = (1 - t\beta)^{-\alpha}.$$

Note. Gamma$(1, \beta)$ = Exp(β).

Cauchy Distribution: the PDF of the Cauchy distribution is

$$f(x) = \frac{1}{\pi(1 + x^2)}, \qquad x \in (-\infty, \infty).$$

As $E(|X|) = \infty$, the mean and variance of the Cauchy distribution do not exist. Cauchy Distribution is particularly useful to model the data with excessively large, or negatively large outliers.