MAT344 Intro to Combinatorics

Lecture 11: Dilworth's Theorem

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11.1 Poset

11.1.1 Dilworth's Theorem

Theorem 11.1 (Dilworth's Theorem)**.** Let $\mathbb{P} = (X, P)$ be a poset with width w. There is a partition of X into w chains $C_1, \dots, C_w$. Moreover, this partition is optimal and it is not possible to cover $\mathbb{P}$ with fewer than w chains.

There is also a dual version of Dilworth's theorem.

Theorem 11.2 (Dilworth's Theorem (dual))**.** Let $\mathbb{P} = (X, P)$ be a poset with height h. There is a partition of X into h antichains $A_1, \dots, A_h$. Moreover, this partition is optimal and it is not possible to cover h with feweer than h antichains.

Assuming the existence of such partition as described in Dilworth's theorem, we first prove optimality.

Proof. (of optimality)

Let $\mathbb{P} = (X, P)$ be a poset with width w. By way of contradiction, suppose that X can be partitioned into k disjoint chains $C_1, \dots, C_k$ such that $k < w$. Take an antichaim A of size w and consider a map $f : A \to \{1, \dots, k\}$ such that $f(x) = i \iff x \in C_i$ (f outputs the index of the chain containing x). However, by Pigeonhole Principle, there is some C_i containing at least two distinct members of A and hence are comparable. This is a contradiction to the assertion that A is an antichain. ■

Next, we prove the existence of such optimal partition using induction on $|X|$.

Proof. (of existence)

Base Case: If $|X| = 1$, the poset contains exactly one element is the one point linear (total) order, so X is a chain itself and we are done.

Inductive Step: Take a poset $\mathbb{P} = (X, P)$ with width w and height h. Assume that Dilworth's theorem holds for all posets with cardinality $k < |X|$.

Let $C = \{x_1, \dots, x_h\}$ be a maximal chain in $\mathbb{P}$ and consider the subposet $\mathbb{Q} = (X \setminus C,\ P \cap (X \setminus C)^2)$. Without loss of generality, suppose $x_1 <_{\mathbb{P}} <_{\mathbb{P}} \cdots <_{\mathbb{P}} x_h$. Note that $\mathsf{width}(\mathbb{Q}) \leq \mathsf{width}(\mathbb{P})$.

Case 1: $\mathsf{width}(\mathbb{Q}) < \mathsf{width}(\mathbb{P})$. Then, $\mathsf{width}(\mathbb{Q}) = w - 1$ because we only removed one chain to get from $\mathbb{P}$ to $\mathbb{Q}$ and if the width of $\mathbb{Q}$ is smaller than $w - 1$, this would mean that some pair of elements in C is not comparable, which would lead to a contradiction. Apply the induction hypothesis to $\mathbb{Q}$ to cover it with $w - 1$ chains $C_1, \dots, C_{w-1}$. Then, adding back C gives us $C_1, \dots, C_{w-1}, C$, which is a cover of $\mathbb{P}$ with w chains.

Case 2: $\mathsf{width}(\mathbb{Q}) = \mathsf{width}(\mathbb{P})$. Then, there exists an antichain $A = \{a_1, \ldots, a_w\}$ of size w in $\mathbb{Q}$. Let

$$P^- = \{x \in P \mid \exists i.\, x \leq_{\mathbb{P}} a_i\}$$

$$P^+ = \{x \in P \mid \exists i.\, x \geq_{\mathbb{P}} a_i\}$$

Pictorially, P^+ is the set of all elements larger than some a_i under $\leq_{\mathbb{P}}$ and P^- is the set of all elements smaller than some a_i under $\leq_{\mathbb{P}}$.

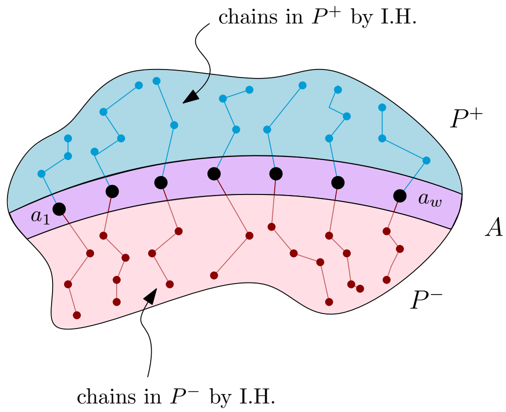

Figure 11.1: We partition P into P^+ and P^- such that $P^+ \cap P^- = A$ and $P^+ \cup P^- = P$. Then, we take the w chains in P^+ and concatenate with the w chains in P^- to form a set of w disjoint chains in P.

We note that

1. $P = P^- \cup P^+$. Otherwise, there would be an element $x \in P$ that is incomparable with every elements of A and A would not have been a max-size antichain.
2. $P^- \cap P^+ = A$. Otherwise, there would exist x, i, j such that $a_i <_{\mathbb{P}} x <_{\mathbb{P}} a_j$ so A is not antichain.
3. $x_h \notin P^-$. Otherwise, $x <_{\mathbb{P}} a_i$ for some i and the chain C is not a maximal chain.

By the third observation, $|P^-| < |P|$. Apply the inductive hypothesis, and we see that P^- can be partitioned into w chains $C_1^-, \ldots, C_w^-$. a_i would be the **minimal element** in each C_i^- because otherwise, $C^- \in P^- \cup P^+ \setminus A$, which contradicts the second observation. Using the same argument, we show that P^+ can be partitioned into w chains $C_1^+, \ldots, C_w^+$ with a_i as the minimal element of C_i^+.

Now we construct the chains in P. Let $C_i = C_i^- \cup C_i^+$ so $P = \bigcup_{i=1}^{w} C_i$. From the second observation, it is clear that C_i is a chain for $i = \{1, \ldots, w\}$.

By induction, Dilworth's theorem holds for all posets. ■

11.1.2 Erdös-Szekeres

> **Theorem 11.3** (Erdös-Szekeres). Let $\{x_i \mid i \in [rs+1]\}$ be a family of $rs+1$ distinct real numbers. There is a strictly increasing subsequence of length $r+1$ or a strictly decreasing subsequence of length $s+1$.

Erdös-Szekeres's theorem can be proved using the Pigeonhole Principle. Here, we consider an alternative proof using Dilworth's theorem.

Proof. Consider the poset $\mathbb{P} = (X, P)$ where

$$X = \{(i, x_i) \mid i \in [rs + 1]\}$$

and $(i, x_i) \leq_{\mathbb{P}} (j, x_j)$ if and only if $i \leq j$ and $x_i \leq x_j$.

If $\mathsf{height}(\mathbb{P}) \geq r + 1$, we are done since a maximal chain contains strictly increasing subsequence x_{n_k} of size at least $r + 1$.

Hence, suppose $\mathsf{height}(\mathbb{P}) \leq r$. By Dilworth's theorem, there exists a partition of P into w chains $C_1, \ldots, C_w$. Note that $|C_i| \leq r$, so $w \geq s + 1$. Otherwise, we can cover X of size $rs + 1$ with $\bigcup_{i=1}^{w} C_i$ which has at most rs points, which is a contradiction.

It follows that there is an antichain $A = \{(n_1, x_{n_1}), \ldots, (n_{r+1}, x_{n_{r+1}})\}$ where we may assume that $i < j$ implies $n_i < n_j$. By the way we defined $\mathbb{P}$, if $n_i < n_j$, then $x_{n_i} > x_{n_j}$ or else A would not be an antichain. It follows then that x_{n_i} is a decreasing subsequence of size at least $s + 1$. ■

11.2 Linear Extensions

Definition 11.1 (Linear Extension)**.** Let $\mathbb{P} = (X, P)$ be a poset. A linear extension of $\mathbb{P}$ is a linear (total) order $\mathbb{L} = (X, L)$ where $P \subseteq L$ (such that for all $a, b \in X$, $a \leq_{\mathbb{P}} b \iff a \leq_{\mathbb{L}} b$).

Questions pertaining to extending posets to linear (total) orders relate to sorting algorithms. The following theorem states that every poset has a linear extension.

Theorem 11.4. Let $\mathbb{P} = (X, P)$ be a poset. There is a linear extension $\mathbb{L} = (X, L)$ of $\mathbb{P}$.

Proof. We prove this by recursively creating a sequence of extensions $\mathbb{P}_i = (X, P_i)$ of $\mathbb{P}$ starting from $\mathbb{P}$, and in the end we get a linear order.

For some index i, we have a poset $\mathbb{P}_i$. If $\mathbb{P}_i$ is not a linear order, there must be two points $a, b \in X$ that are not $\mathbb{P}_i$-comparable. We construct $\mathbb{P}_{i+1}$ to be (X, P_{i+1}) where P_{i+1} is

$$P_{i+1} = P_i \cup \{(a, b)\} \cup \{(x, y) \in X^2 \mid x \leq_{\mathbb{P}_i} a \wedge b \leq_{\mathbb{P}_i} y\}.$$

One can verify that $\mathbb{P}_{i+1}$ satisfies all the required properties of a poset, in particular, transitivity.

We perform this process starting from the original poset $\mathbb{P}$ and eventually the process must half. Otherwise, we would have an infinite sequence of pairs $(a_i, b_i) \in X^2$ that are incomparable, contradicting that X is finite. ■