MAT344 Intro to Combinatorics

Lecture 10: Prüfer Code and Counting Trees, Poset

Lecturer: Keegan Dasilva Barbosa *Scribe: Kevin Gao*

10.1 Counting Trees

Recall that a tree is a connected graph with the property that any two vertices are connected by a unique path. Equivalently, a tree is a connected graph with $n-1$ edges and n vertices. A tree with at least $n \geq 2$ vertices has at least two vertices of degree one (the leaves).

Definition 10.1 (Labeled Tree)**.** A ***labeled tree*** is a tree T whose vertex set is $\{1, \ldots, n\}$.

We would like to know how many possible E exist such that $([n], E)$ forms a **labeled tree**. We are not counting isomorphisms here so the labels matter. To answer this question, we introduce **Prüfer's cod**e, which transforms trees into strings that are easy to count.

10.1.1 Prüfer Code

Fix $n \geq 2$. Let $s : [n-2] \to \{1, \ldots, n\}$ be a sequence (string over $\{1, \ldots, n\}$ of length $n-1$). Consider the following algorithm that generates a set E of size $n-2$.

Prufer-Decode(s)

```
L = {1, ..., n}
while |s| > 0
    find the smallest j that is not in s, add {j, s[1]} to E
    set L = L \ {j}
    delete s[1]
set E = E ∪ {L}
return E
```

The algorithm above assigns to each sequence s, a set E of size $n-1$ such that $([n], E)$ is connected.

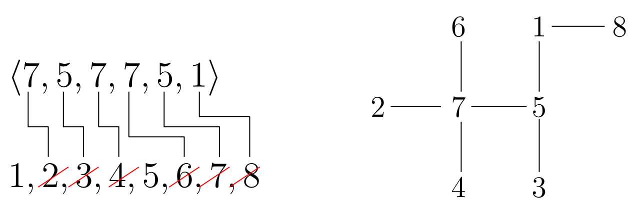


Figure 10.1: Example of a tree and its corresponding Prufer code

We can reverse the algorithm above to encode a tree into its Prüfer code.

PRUFER-ENCODE(E)

for $i = 1$ **to** $n - 2$

  $s[i]$ = the unique vertex adjacent to the leaf with the smallest value

  remove the leaf with the smallest value

return s

10.1.2 Back to Counting Trees

Each labeled tree with n vertices has a unique Prüfer's code of length $n - 2$. Then, counting the number of labeled trees is equivalent to counting the number of Prüfer codes for all the trees.

Theorem 10.1 (Cayley's Formula). For each $n \geq 2$, there are exactly n^{n-2} labeled trees on $\{1, \ldots, n\}$.

Proof. To each tree, find the unique associated Prüfer code $s : [n-2] \to [n]$. This creates a bijective correspondence between labeled trees and sequences $s : [n-2] \to [n]$.

There are exactly n^{n-2} such sequences and hence that many labeled trees on $\{1, \ldots, n\}$. ∎

10.2 Poset

Definition 10.2 (Poset (partially ordered set)). A ***poset*** $\mathbb{P}$ is a pair $\mathbb{P} = (X, P)$ where X is a set and $P \subseteq X \times X$ is a relation that is

- reflexive: $\forall a \in X. (a, a) \in P$
- anti-symmetric: $a \neq b \wedge (a, b) \in P \implies (b, a) \notin P$
- transitive: $(a, b) \in P \wedge (b, c) \in P \implies (a, c) \in P$

Instead of writing $(a, b) \in P$, we use the notation $a \leq_{\mathbb{P}} b$.

Definition 10.3 (Embedding, Isomorphism, Automorphism). Given a poset $\mathbb{P} = (X, P)$ and $\mathbb{Q} = (Y, Q)$, an **embedding** from $\mathbb{P}$ into $\mathbb{Q}$ is an injective map $f : X \to Y$ with the property that $a \leq_{\mathbb{P}} b$ if and only if $f(a) \leq_{\mathbb{Q}} f(b)$. If the embedding is surjective, we call it an **isomorphism**. If $\mathbb{Q} = \mathbb{P}$, we call it an **automorphism**.

For example, consider $X = \{\star, \circ, \diamond\}$ with $P = \{(\star, \star), (\circ, \circ), (\diamond, \diamond), (\diamond, \star)\}$, and $Y = \{\star, \circ, \diamond, \Box\}$ with $Q = \{(\star, \star), (\circ, \circ), (\diamond, \diamond), (\Box, \Box), (\Box, \star)\}$. Let $\mathbb{P} = (X, P)$ and $\mathbb{Q} = (Y, Q)$. Consider the function $f : X \to Y$ such that $f(\star) = \star$, $f(\circ) = \circ$, and $f(\diamond) = \Box$. f is an embedding because $\diamond \leq_{\mathbb{P}} \star$ if and only if $f(\diamond) = \Box \leq_{\mathbb{Q}} \star = f(\star)$. However, f is not an isomorphism because $\diamond$ is not in the range of f so f is not surjective.

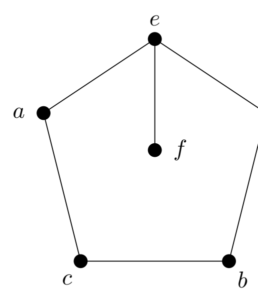
Definition 10.4 (Dual). Given a poset $\mathbb{P} = (X, P)$, we call the poset $\mathbb{P}^d = (X, P^d)$ where $a \leq_{\mathbb{P}^d} b \iff b \leq_{\mathbb{P}_d} a$ the **dual** of $\mathbb{P}$. We say a poset is self-dual if it is isomorphic to its dual.

Definition 10.5 (Cover). Given a poset $\mathbb{P} = (X, P)$ and a point $a \in X$, we say a is **covered by** a point $b \in X$ if $a <_{\mathbb{P}} b$ and there is no c such that $a <_{\mathbb{P}} c <_{\mathbb{P}} b$.

Definition 10.6 (Cover Graph)**.** Given a poset $\mathbb{P} = (X, P)$, we call the graph $G = (X, E)$ given by $\{x, y\} \in E$ if and only if x covers y or y covers x, the **cover graph** associated to $\mathbb{P}$.

If we draw the cover graph in an oriented fashion where lower vertices correspond to the $\leq_{\mathbb{P}}$-smaller elements, we have a special kind of cover graph known as **Hasse diagram**.

Let $\mathbb{P} = (X, P)$ be a poset where $X = \{a, b, c, d, e, f\}$ and $P = \{(a, c), (b, c), (b, d), (d, e), (a, e), (e, f)\}$. One possible cover graph and the Hasse diagram is shown below.



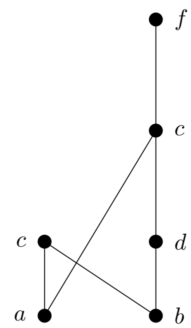


Figure 10.2: Cover graph and Hasse diagram for the poset described above.

10.2.1 Linear Orders

Definition 10.7 (Comparability)**.** Given a poset $\mathbb{P} = (X, P)$, we say two points $a, b \in X$ are ***comparable*** if either $a <_{\mathbb{P}} b$ or $b <_{\mathbb{P}} b$ If two points are not comparable, we call them ***incomparable***.

Definition 10.8 (Total/Linear Order)**.** Given a poset $\mathbb{P} = (X, P)$, we say $\mathbb{P}$ is ***linearly ordered*** or ***totally ordered*** if no two distinct points are incomparable.

10.2.2 Height and Width

Definition 10.9 (Antichain and Chain)**.** Given a poset $\mathbb{P} = (X, P)$, we call $A \subseteq X$ an ***antichain*** if every pair of distinct elements in A are **incomparable**. We call a subset $C \subseteq X$ a ***chain*** is every pair of distinct elements is **comparable**.

Definition 10.10 (Height and Width)**.** Given a poset $\mathbb{P} = (X, P)$, we define the parameters $\mathsf{width}(\mathbb{P})$ and $\mathsf{height}(\mathbb{P})$ to denote the size of the largest antichain and chain of $\mathbb{P}$, respectively.

10.2.3 Subset Lattice and Sperner's Theorem

Theorem 10.2 (Sperner's Theorem)**.** Consider the poset $\mathbb{P} = (\mathcal{P}([n]), \subseteq)$. Then, $\mathsf{width}(\mathbb{P}) = \binom{n}{\lfloor n/2 \rfloor}$.

Proof. One can easily verify that $A = \{S \subseteq [n] \mid |S| = \lfloor \frac{n}{2} \rfloor\}$ is an **antichain**. Two sets of the same size are comparable if and only if they are equal. There are $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ such subsets of $[n]$ of size $\lfloor \frac{n}{2} \rfloor$, so $\mathsf{width}(\mathbb{P}) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. This shows that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is a lower bound.

Now, we proceed to show that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is also an upper bound. Let $A = \{S_1, \ldots, S_w\}$ be a **maximal antichain** of $\mathbb{P}$. It suffices to show that $w \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. For each $S_i \in A$, let $\mathcal{C}_i$ be the set of all **maximal chains** that contains S_i. We note that a maximal chain but be an $\subseteq$-increasing sequences that differs by at most one element per successive cover. If the next subset in the chain **differs** from the previous subset **by more than one element**, then the chain **would not be maximal**. In other words, starting from S_i, we remove 1 element until we reaches the empty set, and add 1 element until we reaches $[n]$. There are $|S_i|!$ ways to remove points successively from S_i. Similarly, there are $(k - |S_i|)!$ ways to add points successively to S_i. Therefore, for any i, $|\mathcal{C}_i| = |S_i|! \cdot (k - |S_i|)!$.

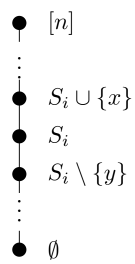


Figure 10.3: Each subset must differ from the previous and next subset by exactly one element in a maximal chain. Otherwise, we form a chain of longer length by inserting a new subset in between.

We also observe that there are exactly $n!$ many maximal chains, each corresponding to one ordering (permutation) to remove elements from $[n]$.

For any $i \neq j$, $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ because otherwise there would be a chain C with $S_i, S_j \in C$ and hence S_i and S_j are comparable, which is a contradiction to our assumption that S_i and S_j are members of a maximal **antichain**. It follows that

$$\left| \bigcup_{i=1}^{w} \mathcal{C}_i \right| = \sum_{i=1}^{w} |\mathcal{C}_i| \leq n!$$

Since $|\mathcal{C}_i| = |S_i|! \cdot (k - |S_i|)!$, it follows that

$$\sum_{i=1}^{w} |\mathcal{C}_i| = \sum_{i=1}^{w} (|S_i|! \cdot (k - |S_i|)!) \leq n!$$

and hence

$$\sum_{i=1}^{w} \frac{|S_i|! \cdot (k - |S_i|)!}{n!} = \sum_{i=1}^{w} \frac{1}{\binom{n}{|S_i|}} \leq 1$$

by definition of combination. It follows from $\binom{n}{|S_i|} \leq \binom{n}{\lfloor n/2 \rfloor}$ that

$$\sum_{i=1}^{w} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{i=1}^{w} \frac{1}{\binom{n}{|S_i|}} \leq 1$$

and $w \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Therefore, $\mathsf{width}(\mathbb{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and because $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is also a lower bound, $\mathsf{width}(\mathbb{P}) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. ■