MAT344 Intro to Combinatorics

Lecture 8: Intro to Graph Theory

Lecturer: Keegan Dasilva Barbosa *Scribe: Kevin Gao*

8.1 Eulerian Graphs and Circuit

Definition 8.1 (Eulerian Circuit). Given a graph $G = (V, E)$, an ***Eulerian circuit*** is a sequence of vertices $x_0, \dots, x_t$ such that

- $x_0 = x_t$
- $\forall i \in [t].\, \{x_i, x_{i+1}\} \in E$
- $\forall e \in E.\, \exists$ unique $i \in [t].\, e = \{x_i, x_{i+1}\}$ (i.e. every edge appears exactly once in an Eulerian circuit)

We say a graph is ***Eulerian*** if and only if it has an **Eulerian circuit**. An Eulerian circuit is also referred to as an **Euler tour** in some texts. The notion of an Eulerian circuit and graph appears in the famous problem of the **bridges of Königsberg**.

In 1736, Euler gave his famous characterization of an Eulerian graph, stated as follows

Theorem 8.1 (Euler, 1736). A connected graph $G = (V, E)$ is Eulerian if and only if all its vertices have even degree.

Proof. The forward direction of the proof is quite straightforward whereas the reverse direction requires a slightly more involved proof by induction.

($\Longrightarrow$): Let G be a connected graph. Assume that G is Eulerian so it must have an Eulerian circuit. Note that in an Eulerian circuit, every time we enter a vertex, we must also leave the vertex. This is the case for all vertices because otherwise we would have an infinite graph. Hence, all vertices in G must have even degree.

($\Longleftarrow$): Let G be a graph. We proceed by strong induction on the number of edges.

Base Case: G is a graph with $m = 0$ edge. The result trivially holds.

Inductive Step: Let $m \in \mathbb{N}$ be arbitrary. Assume that for all $k \in \mathbb{N}$ such that $0 \leq k < m$, the implication holds. Let $G = (V, E)$ be a connected graph with m edges. Further, assume that $\deg_G(v)$ is even for all $v \in V$. Since the graph is connected and every vertex has even degree, it follows immediately that $\deg_G(v) \geq 2$ for all $v \in V$. This also implies that G contains a cycle. Let $c = v_1 \dots v_k$ be such cycle of maximal length and E' be the edges contained in this cycle.

If c contains all edges exactly once, we are done. Hence, suppose $E' \neq E$ and consider the graph $G' = (V, E \setminus E')$. It has connected components $S_1, \dots, S_l$. Since $E' \neq E$, each of the connected components S_i contains strictly fewer edges than $|E| = m$. For every $v \in G$, an even number of edges of G at v are in the cycle c, so we we remove these edges, each vertex in the remaining graph should still have even degree. Apply the induction hypothesis to the components, which asserts that each of the components $S_1, \dots, S_l$

possess an Eulerian circuit. Further, since c is a cycle, $C = (\{v_1, \ldots, v_k\}, E')$ itself is also Eulerian. Now, we recursively construct an Eulerian circuit, say x, in the original graph G. Start from v_1, find the component S_i containing v_1, and concatenate the Eulerian tour in S_i to x. Next, move from v_1 to v_2 along $\{v_1, v_2\} \in E'$. If $\{v_1, v_2\} \notin E$, then they must have been in the same connected component, in which case we skip v_2 and move to v_3. Repeat this until we have walked through every edge in each one of the l connected components and the edges in E' connecting each component. It is clear that x is Eulerian since it visits every edge exactly once.

By induction, the implication holds. ■

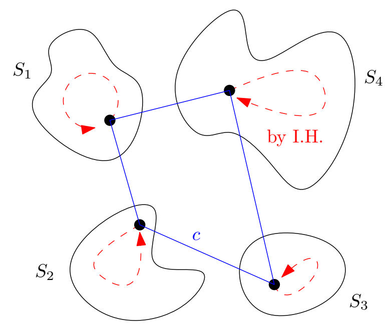

Figure 8.1: Construct an Eulerian circuit in the original graph G by first finding a cycle c, remove the cycle, find an Eulerian circuit within each component $S_1, \ldots, S_l$, and concatenate these circuits via the cycle c.

8.2 Hamiltonian Graphs

Definition 8.2 (Circuits)**.** A ***circuit*** in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, \ldots, v_k$ such that for all $i \in \{1, \ldots, k-1\}$, $\{v_i, v_{i+1}\} \in E$ and $\{v_k, v_1\} \in E$. Note, a circuit induces a copy of a subgraph isomorphic to C_k for $k \geq 3$. Specially, we define a **single vertex** a circuit as well.

Definition 8.3 (Hamiltonian Graph)**.** We call a graph with n vertices ***Hamiltonian*** if it admits a circuit of length n, which is to say that the graph has a spanning subgraph that is isomorphic to C_n.

A sufficient condition for Hamiltonian graphs.

Theorem 8.2 (Dirac, 1952)**.** Let $G = (V, E)$ be a graph with n vertices where $n \geq 3$. Suppose that for every $v \in V$, $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$. Then, G is Hamiltonian.

Proof. Let $G = (V, E)$ be a graph with $n \geq 3$ vertices. Assume that $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ for all $v \in V$. Let $\delta(G)$ denote the minimum degree. That is, $\delta(G) = \min\{\deg_G(v) \mid v \in V\}$. Then, the assumption is equivalent to that $\delta(G) \geq \lceil \frac{n}{2} \rceil$.

We claim that G is connected. We prove the claim by contradiction. So suppose not, consider the component $G' = (V_{G'}, E_{G'})$ of G with the fewest number of vertices. Since $\delta(G) \geq \lceil \frac{n}{2} \rceil$, each vertex is connected to at least $\lceil \frac{n}{2} \rceil$ other vertices. Since C is a component that is not connected to vertices in other components, $|V_{G'}| \leq \lceil \frac{n}{2} \rceil$. But then, $\delta(G') < |V_C| \leq \lceil \frac{n}{2} \rceil$, which contradicts the assumption that $\deg_G(v) \geq \lceil \frac{n}{2} \rceil$ for all $v \in V$, including those in G'.

Since G is connected, we can find the longest path in G. Let $P = v_0 \dots v_k$ be a longest path in G of length k (with k edges and $k+1$ vertices). We claim that there exists some $0 \le i \le k-1$ such that $\{v_0, v_{i+1}\} \in E$, $\{v_i, v_k\} \in E$, and $\{v_i, v_{i+1}\} \in E$ as shown in Figure 8.2.

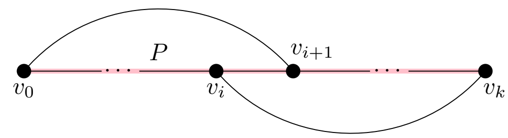

Figure 8.2: The longest path $P = v_0 \dots v_k$ with $k+1$ vertices is colored in red. There exists some $0 \le i \le k$ such that $\{v_0, v_{i+1}\} \in E$ and $\{v_i, v_k\} \in E$.

Such adjacent vertices v_i and v_{i+1} such that v_i is adjacent to v_k and v_{i+1} is adjacent to v_0 must exists. By way of contradiction, suppose v_i and v_{i+1} do not exist. Then, for every vertex adjacent to v_0, there must exists some vertex adjacent to it that is NOT adjacent to v_k. Similarly, for every vertex adjacent to v_k, there must exists some adjacent vertex that is NOT adjacent to v_0. Note that these two sets of vertices are disjoint and do not include v_k. This implies that

$$\deg_G(v_0) + \deg_G(v_k) + 1 \le k + 1$$

since we are not overcounting and the number of vertices being counted is at most the length of path P. The additional 1 on the LHS of the inequality came from couting v_k as it is not included in either $\deg_G(v_0)$ or $\deg_G(v_k)$. Now, since $\deg_G(v) \le \lceil \frac{n}{2} \rceil$ for all $v \in V$,

$$n + 1 \le \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + 1 \le \deg_G(v_0) + \deg_G(v_k) + 1 \le k + 1$$

so $n+1 \le k+1$. This implies that $n < k+1$ since both n and k are integers. But this leads to a contradiction because the number of vertices on the path P cannot be more than the total number of vertices in the entire graph.

The existence of such v_i and v_{i+1} allows us to construct a cycle $C = v_0 \to v_{i+1} \rightsquigarrow_P v_k \to v_i \rightsquigarrow_P v_0$. We claim this cycle is Hamiltonian.

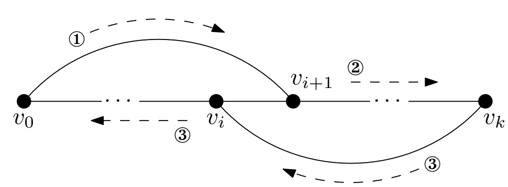

Figure 8.3: The Hamiltonian path is $C = v_0 \to v_{i+1} \rightsquigarrow_P v_k \to v_i \rightsquigarrow_P v_0$.

To see why C is Hamiltonian, again we use a contradiction proof. Suppose C is not Hamiltonian. Then, by definition, there must be some vertex $w \in V$ such that w is not on C. But since G is connected, w must be adjacent to some vertices, say $v_w \in V$. Without loss of generality, suppose that this v_w is on the cycle C. There must also be a v_{w+1} immediately adjacent to v_w. By construction, the cycle C contains $k+1$ edges (that's all edges on P along with $\{v_0, v_{i+1}\}, \{v_i, v_k\}$ and without $\{v_i, v_{i+1}\}$). We then consider the path from v_w to v_{w+1} by following the edges on the cycle. This leads to a path of length k, namely $p = v_w \rightsquigarrow v_0 \to v_{i+1} \rightsquigarrow v_k \to v_i \rightsquigarrow v_{w+1}$. Now, we extend the left end of this path to w since w is adjacent to v_w and still get back a valid path. The new path $P' = w \to v_w \rightsquigarrow v_0 \to v_{i+1} \rightsquigarrow v_k \to v_i \rightsquigarrow v_{w+1}$ is one edge longer than p. However, this contradicts the maximality assumption for P since now we would have a path, P', that contains more vertices than P. Therefore, C is indeed a Hamiltonian path.

It follows immediately that G is Hamiltonian by definition. ■

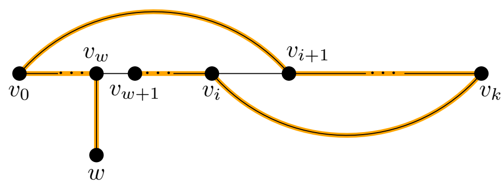

Figure 8.4: With the existence of a vertex w outside of the previously constructed cycle, we would have a longer path (colored in orange), contradicting the maximality of P.

8.3 Graph Coloring

8.3.1 Independent Sets and Bipartite Graphs

Definition 8.4 (Independent Set). Given a graph $G = (V, E)$, we say $A \subseteq V$ is ***independent*** if and only if no no vertices in A are adjacent.

The only independent sets in K_n are singletons. We can prove this using a contradiction and the definition of a complete graph and independent set.

Definition 8.5 (Bipartite Graph). We say a graph $G = (V, E)$ is ***bipartite*** if and only if we can partition V into two **disjoint independent** sets.

A bipartite graph $(V_1 \cup V_2, E)$ is a complete bipartite graph iff every $v_1 \in V_1$ is connected to every vertex in V_2 and vice versa. We denote a complete bipartite graph by $K_{|V_1|,|V_2|}$ (K with a subscript denoting the size of the left and right partition, respectively).

Theorem 8.3. A graph is bipartite if and only it does not contain a circuit of odd length.

Proof.

($\implies$): Let $G = (V, E)$ be a bipartite graph. In particular, $V = A \cup B$ for some $A, B \subseteq V$ such that $A \cap B = \emptyset$ and for all $\{a, b\} \in E$, $a \in A$ and $b \in B$. Suppose, for contradiction, that G contains an odd-length cycle $C = v_1 v_2 \ldots v_n v_1$ of length n. Without loss of generality, suppose that v_i and v_{i+1} alternates between A and B. So, $v_1 \in A$, $v_2 \in B$, $v_3 \in A$, and so on. If the cycle is not in that particular order, we can reindex the vertices and still have the same cycle.

Then, for $k \in \{1, 2, 3, \ldots, n\}$,

$$v_k \in \begin{cases} A & k \text{ is odd} \\ B & k \text{ is even} \end{cases}$$

Since C is a cycle of odd length, n is odd. It follows that $v_n \in A$. But then, since $v_1 \in A$ and $\{v_n, v_1\} \in E$, this is a contradiction to the assumption that G is bipartite.

($\impliedby$): Let $G = (V, E)$ be a graph. Without loss of generality, assume that G is connected. Otherwise, we can consider the connected components individually. Assume that G contains no odd cycle. Let $w \in V$ be a vertex in G.

Let A be the set of vertices whose shortest distance from w is even, and let B be the set of vertices whose shortest distance from w is odd. That is,

$$A = \{v \in V \mid d(v, w) \equiv 0 \mod 2\}$$
$$B = \{v \in V \mid d(v, w) \equiv 1 \mod 2\}$$

Since G is connected, every vertex is either at an even distance or odd distance from $w \in V$. A vertex cannot be both at an even distance and an odd distance from w at the same time. Hence, $A \cup B = V$ and $A \cap B = \emptyset$. This implies that A and B are a valid partition of V.

Now, we would like to show that G is bipartite. It suffices to show for all vertices $a_1, a_2 \in V$ and $b_1, b_2 \in B$, $\{a_1, a_2\} \notin E$ and $\{b_1, b_2\} \notin E$. To prove this fact, we suppose the contrary and derive a contradiction. So, suppose that there does exist such $x, y \in A$ or $x, y \in B$ such that $\{x, y\} \in E$. Fix such x, y. We can assume that $x \neq y \neq w$. Otherwise, we have $w = x$ and $d(x, w) = 0$. Since x and y are in the same partition, $d(y, w)$ is even and $d(y, x) = 0$. However, this is not possible since $d(y, x) = 1$, which is odd. By a similar argument, we can show that $w \neq y$ either.

To obtain a contradiction, we consider the shortest path from x to w and the shortest path from y to w. Let p be the shortest path from x to w, and let q be the shortest path from y to w. Let z be the last common vertex of p and q. Note that z may be w. We also note that $|p|$ and $|q|$ have the same parity since we assumed that y, x are in the same partition.

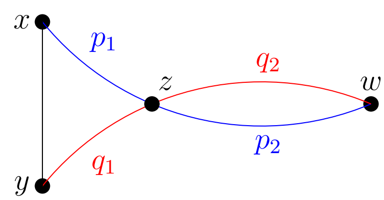

Figure 8.5: p is the shortest path from x to w. q is the shortest path from y to w. p_1 is the part of p from x to the last common vertex of p and q. Similarly, q_1 is the part of q from y to the last common vertex of p and q.

Let $p_1 = x \rightsquigarrow_p z$ be the part of the path p from x to z. Similarly, let $p_2 = z \rightsquigarrow_p w$, $q_1 = y \rightsquigarrow_q z$, and $q_2 = z \rightsquigarrow_q w$. We claim that $|p_2| = |q_2|$ since otherwise we can obtain a shorter path from x to w or from y to w. Further, we claim that $|p_1|$ and $|q_1|$ have the same parity becuase $|p|$ and $|q|$ have the same parity and the second part of both paths, p_2 and q_2, are of the same length. Recall that $\{x, y\} \in E$. Then, $C = x \rightsquigarrow_{p_1} z \rightsquigarrow_{q_1} y \rightarrow x$. Since $|p_1|$ and $|q_1|$ have the parity, $|C| = |p_1| + |q_1| + 1$ is odd. This is becuase $|p_1| + |q_1|$ can be expressed as $2k$ for some $k \in \mathbb{Z}$. This is an odd-length cycle, which is a contradiction to our initial assumption that G has no odd cycle. The only additional assumption leading to this contradiction is that G is not bipartite. Hence, G must be bipartite. ■

8.3.2 Coloring

Definition 8.6 (Proper Coloring)**.** Let $G = (V, E)$ be a graph. A (proper) ***coloring*** of G is a function $\phi : V \to [k]$ such that for all $i \in [k]$, $\phi^{-1}(i)$ is independent. Equivalently, ϕ is a (proper) coloring of G iff $\forall \{a, b\} \in E. \, \phi(a) \neq \phi(b)$. We call ϕ a k-coloring.

Definition 8.7 (Chromatic Number). The ***chromatic number*** of a graph G, is the smallest k such that there is a proper k coloring of G. The chromatic number of G is denoted by $\chi(G)$.

Theorem 8.4. A graph is 2-colorable if and only if it does not contain an odd-length cycle.

Proof. Theorem 8.3 states that a graph is bipartite iff there is no odd-length cycle. To prove this theorem, it suffices to prove that a graph is 2-colorable if and only if the graph is bipartite.

($\Longrightarrow$): Let $G = (V, E)$ be a 2-colorable graph. Take the coloring. Assign vertices with one color to V_1 and vertices with another color to V_2. V_1 and V_2 is a partition of V. By definition of a 2-color, for all $x, y \in V_1$, $\{x, y\} \notin E$ and for all $x, y \in V_2$, $\{x, y\} \notin E$.

($\Longleftarrow$): Let $G = (V, E)$ be a bipartite graph where $V = V_1 \cup V_2$ is a partition. Assign one color to all vertices in V_1 and assign another color to all vertices in V_2. It is easy to prove that this is a valid 2-coloring directly from the definition of a bipartite graph. ■

To demonstrate the concept of graph coloring and chromatic number, we consider the Petersen graph.

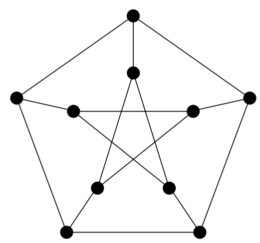

Figure 8.6: The Petersen graph.