MAT344 Intro to Combinatorics

Lecture 5: Induction and Recursion

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5.1 Induction

Definition 5.1. A mathematical statement or a ***statement*** is a sentence or expression which can be true or false.

$1 + 1 = 2$ is a statement, but $1 + 1$ is not a statement. An open statement is a statement that depends on some variables.

5.1.1 Principle of Induction

Note that in this course, $\mathbb{N} = \{1, 2, \ldots\}$.

Principle of Weak Induction: Suppose we have a family of statements $P(n)$ for all $n \in \mathbb{N}$, and suppose the following holds:

1. $P(1)$ holds, and
2. if $P(k)$ holds, then $P(k+1)$ holds.

Then, we can conclude $P(n)$ is true for all $n \in \mathbb{N}$.

Remark. The base case does not have to be for $n = 1$ if we are not proving the statement for all $n \in \mathbb{N}$. Say, for example, that we want to show $P(n)$ holds for all $n \in \mathbb{N}$ such that $n \geq 5$. Then, we can use $P(5)$ as the base case.

For example, suppose we want to show $\forall n \in \mathbb{N}.\ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Proof. Let $P(n)$ be the statement

$$\text{“}\sum_{i=0}^{n} 2^i = 2^{n+1} - 1\text{”}$$

1. Base Case: When $n = 1$, LHS $= \sum_{i=0}^{n} 2^i = 2^0 + 2^1 = 3$. RHS $= 2^{i+1} - 1 = 3$. LHS = RHS, so $P(1)$ holds.

2. Inductive Case: Assume $P(k)$ holds (inductive hypothesis). WTS $P(k+1)$ holds. The LHS of $P(k+1)$ is

$$\text{LHS} = \sum_{i=0}^{k+1} 2^i = \underbrace{\sum_{i=0}^{k} 2^i}_{\substack{\text{LHS of } P(k) \\ \text{by IH, equals} \\ \text{RHS of } P(k)}} + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$

which is equal to RHS of $P(k+1)$.

Since we have (1) and (2), by the Principle of Induction, we can conclude that $P(n)$ holds for all $n \in \mathbb{N}$. ■

5.1.2 Strong Induction

Principle of Strong Induction: Suppose we have a family of statements $P(n)$ for all $n \in \mathbb{N}$. Suppose the following are true:

1. $P(1)$ is true, and
2. if $P(m)$ is true for all $1 \leq m \leq k$, then $P(k+1)$ is true.

Then, we can conclude $P(n)$ is true for all $n \in \mathbb{N}$.

Suppose we want to show the Fundamental Theorem of Arithmetics (without the uniqueness part), which states that for all $n \in \mathbb{N}$ such that $n \geq 2$, n is a product of primes.

Proof. Let $P(n)$ be that $P(n) =$ "n is a product of primes".

1. Base Case: $P(2)$. Since 2 is a prime, it is a product of primes.

2. Inductive Case: Suppose that for all m such that $2 \leq m \leq k$, $P(m)$ holds. WTS $P(k+1)$ holds. If $k+1$ is prime, then $P(k+1)$ trivially holds. Otherwise, $k+1$ is composite, so there exists $a, b \in \mathbb{N}$ with $2 \leq a, b < k+1$ such that $k+1 = ab$. So since $P(a)$ and $P(b)$ hold by inductive hypothesis, a and b are both products of primes. So $k+1 = ab$ is a product of primes, so $P(k+1)$ holds. ■

5.2 Recursion

As an example, consider the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. Note that $C_0 = 1$ because of the empty path and $C_1 = 1$. Prove the recursive identity

$$C_{n+1} = \sum_{i=1}^{n+1} C_{i-1} C_{n+1-i}$$

for $n \geq 0$.

Proof. Let $p = p_1 p_2 \dots p_{2n}$ be a Dyck path of length $2n$. Let $p_1 \dots p_{2i}$ be the segment of p up until the first time it touches $x = y$ (but not crosses). So, $p_1 =$ E and $p_{2i} =$ N. p_i cannot go North because otherwise we would cross the diagonal. For a similar reason, p_{2i} must go North.

Then, $p_2 \dots p_{2i-1}$ is a Dyck path of length $2(i-1)$. We can split into cases based on the first time after $(0,0)$ where the path touches the diagonal. It can be at $i = 1, \dots, n+1$. At a given i, we break into two Dyck paths: $p_2 \dots p_{2i-1}$ of length $2(i-1)$ and $p_{2i+1} \dots p_{2(n+1)}$ of length $2(n+1-i)$. The first path have C_{i-1} possibilities, and the second path have C_{n+1-i} possibilities.

In total, we have $\sum_{i=i}^{n+1} C_{i-1} C_{n+1-i}$ number of Dyck paths of length $n+1$. ■

Definition 5.2. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. We say that this sequence satisfies a ***recurrence relation*** if we can define the nth term based on the terms that come before it.

What we proved just now is a recurrence formula for the Catalan numbers. Namely,

$$C_{n+1} = \sum_{i=1}^{n+1} C_{i-1} C_{n+1-i}$$

is a recurrence relation on $\{C_n\}_{n\geq 0}$ with $C_0 = 1$.

5.2.1 Proving Recurrences

Consider the Fibonacci numbers with $F_1 = 1$ and $F_2 = 1$, given by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 3$$

Induction is a method of proving statements. Recursion is a way of defining/giving a formula for some quantities.

Example. We have a closed form expression of F_n. Let $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6$, and let $\psi = \frac{1-\sqrt{5}}{2}$ be the conjugate of φ. These are the roots of $x^2 = x + 1$. We would like to show that

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

for all $n \geq 1$.

We prove this closed form formula using induction.

Proof. Let $P(n)$ be the statement we want to prove.

1. Base Case: $P(1)$. LHS $= F_1 = 1$. RHS $= \frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} = \frac{2\frac{\sqrt{5}}{2}}{\sqrt{5}} = 1$.

2. Base Case: $P(2)$. LHS $= F_2 = 1$. RHS $= \frac{\varphi^2 - \psi^2}{\sqrt{5}} = \frac{\varphi - \psi}{\sqrt{5}} = 1$.

3. Inductive Case: Assume $P(n)$ holds for all $1 \leq m \leq k$. WTS $P(k+1)$ holds. Consider the LHS of $P(k+1)$. We have

$$F_{k+1} = F_k + F_{k-1} = \frac{\varphi^k - \psi^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \psi^{k-1}}{\sqrt{5}}$$

by IH. Collect like terms

$$\text{LHS} = \frac{\varphi^{k-1}(\varphi + 1) - \psi^{k-1}(\psi + 1)}{\sqrt{5}} = \frac{\varphi^{k-1}(\varphi^2) - \psi^{k-1}(\psi^2)}{\sqrt{5}}$$

since φ and ψ are the solutions to $x^2 = x + 1$. This implies that

$$\text{LHS} = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}} = \text{RHS}$$

So $P(k+1)$ holds. ∎

Remark. If instead, we prove the statement

$$P(n) = \text{“}F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \wedge F_{n+1} = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}\text{”}$$

We can, in fact, use weak induction.

Let a_n be the max number of intersections of n lines in the plane. Find a recurrence relation for a_n. $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, and $a_3 = 3$. Note that two lines intersect unless parallel. Any two parallel lines can be nudged slightly to not be parallel. Moreover, if 3 lines intersect at a point, we can nudge them so they intersect at distinct pairwise intersections. If we know a_n, we can have $a_{n+1} = a_n + n$.