MAT344 Intro to Combinatorics

Lecture 9: Planarity and Euler's Formula

Lecturer: Keegan Dasilva Barbosa *Scribe: Kevin Gao*

9.1 Planarity

Definition 9.1 (Planarity). We say a graph G is ***planar*** if there **exists** a depiction of it on a plane without any edge crossing.

To put in simple terms, a graph is planar if one can draw the graph on a piece of paper without any edge crossing each other. The definition as it is presented here is quite vague and lack mathematical rigor. Thankfully, this is formalized with a theorem by Kuratowski that we will introduce later that presents an equivalence to our definition of planarity.

There are many practical applications of graph planarity. In computer science, there is a notion of thickness that correlates planarity to how difficult it is to embed a network. Planarity is also related to integrated circuit design in cases when lines or wires are not allowed to cross.

Theorem 9.1 (Four Color Theorem). Every planar graph has chromatic number of at most 4.

The proof of the Four Color theorem relies on brute-force computation methods and was only completed 1976. It is complex and not very readable, so we will not provide the proof here but instead treat it as a proven fact.

The Four Color Theorem implies that if G is planar, $\chi(G) \leq 4$, so any graph with $\chi(G) \geq 5$ cannot be planar.

Corollary 9.2. Any graph containing a copy of K_5 cannot be planar.

9.1.1 Face

Roughly speaking, a ***face*** (in context of graph theory) is a region not containing other vertices or edges, bounded by edges of a graph in its planar depiction. Note that when discussing the **number of faces**, it is important to only consider the **planar depiction** of a graph because otherwise we can create arbitrary many faces.

Is there any pattern between the number of vertices, edges, and faces in a planar graph? Consider the graph shown in Figure 9.1. We have 4 edges, 6 vertices, and 4 faces.

The triangle graph K_3 (complete graph of 3 vertices) has 3 vertices, 3 edges, and 2 faces. The path graph of n vertices P_n have n vertices, $n-1$ edges, and one face. The cycle graph of n vertices C_n has n vertices, n edges, and 2 faces. See Figure 9.2

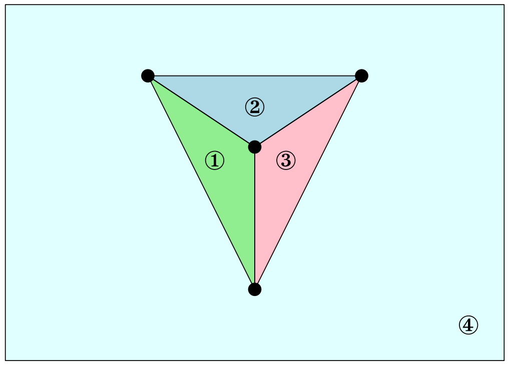

Figure 9.1: A planar graph and its 4 faces.

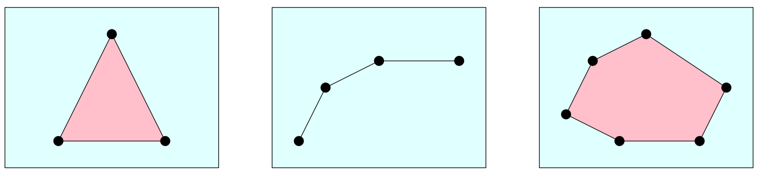

Figure 9.2: The triangle graph K_3, path graph P_n, and cycle graph C_n.

9.2 Euler's Formula

From the previous examples (Figure 9.1 and 9.2), we know there is some relationship between the number of vertices, edges, and faces in a planar depiction of a graph. This pattern is characterized by Euler's formula:

Theorem 9.3 (Euler's Formula for Planar Graphs)**.** Let G be a connected planar graph with n vertices, m edges, and f faces. Then,

$$n - m + f = 2$$

Proof. By induction on m.

Base Case: When $m = 0$. G is connected, so graph is a singleton vertex. It has one face. And clearly, $1 - 0 + 1 = 2$.

Inductive Step: Let $m \in \mathbb{N} \cup \{0\}$ be arbitrary. Assume that Euler's formula holds for graphs with m edges. Let G be a graph with $m + 1$ edges.

Case 1: If we can remove an edge so that the graph is still connected, then the edge being removed must divide some face into two. Let G' be the graph after removing the edge. Removing the edge does not change the number of vertices, so G' has n vertices, m edges, and f. By inductive hypothesis,

$$n - m + f = 2$$

After adding the edge back, we have $m + 1$ edges and $f + 1$ since the edge divides a face into two so adding it back adds one more face. Hence, in G, we have

$$n - (m + 1) + (f + 1) = m - m + f = 2$$

In this case, Euler's formula holds for G with $m + 1$.

Case 2: If removing any edge leaves the graph disconnected, the graph must be a tree. Tree has 1 face, and $n = m + 1$. So, for G,

$$n - m + f = m + 1 - m + f = 2$$

Euler's formula also holds for this case.

By induction, Euler's formula holds for all graphs. ■

9.2.1 A Useful Corollary

A useful corollary about planar graphs follows from Euler's formula.

Corollary 9.4. Let G be a planar graph with $n \geq 3$ vertices and m edges. Then,

$$m \leq 3n - 6$$

Proof. We consider two cases: G is connected and G is disconnected.

Case 1: G is connected. We can assume that G is not a tree since the case where G is a tree is trivial. Let S be the set of pairs (e, F) where e is an edge and F is a face such that e **is an edge on the face** F. For a fixed e,

$$|\{F \mid (e, F) \in S\}| \leq 2$$

as an edge borders at most two faces.

For a fixed F,

$$|\{e \mid (e, F) \in S\}| \geq 3$$

since an enclosed area must have at least three boundaries. Then it follows that $3f \leq |S| \leq 2m$. By Euler's formula,

$$m = n + f - 2 \leq n + \frac{2}{3}m - 2 \implies m \leq 3n - 6$$

Case 2: G is disconnected. Consider each connected component of G. Use the same argument as Case 1 to conclude that the inequality holds for each connected components and thus must hold for the entire graph. ■

9.3 Graph Minor and Contraction

Definition 9.2 (Contraction)**.** Let $G = (V, E)$ be a graph and let $e \in E$ be an edge. A ***contraction*** along e is a graph $G' = ((V \setminus e) \cup \{e\}, E')$ where E' is defined such that for all $u, v \in (V \setminus e)$,

$$\{u, v\} \in E \iff \{u, v\} \in E'$$

and

$$\{u, e\} \in E' \iff \exists x \in e. \{u, x\} \in E.$$

Definition 9.3 (Graph Minor)**.** A ***graph minor*** of a graph G is a graph H obtained via a process of taking subgraphs and contractions.

We are interested in graph minor because it gives us an important equivalent characterization for planar graphs.

Theorem 9.5 (Kuratowski, 1930)**.** A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

Theorem 9.6 (Wagner, 1937)**.** A graph is planar if and only if it has no minor isomorphic to K_3 or $K_{3,3}$.

9.3.1 Application of Kuratowski's Theorem

We use Kuratowski's theorem to show that Petersen's graph is not planar.

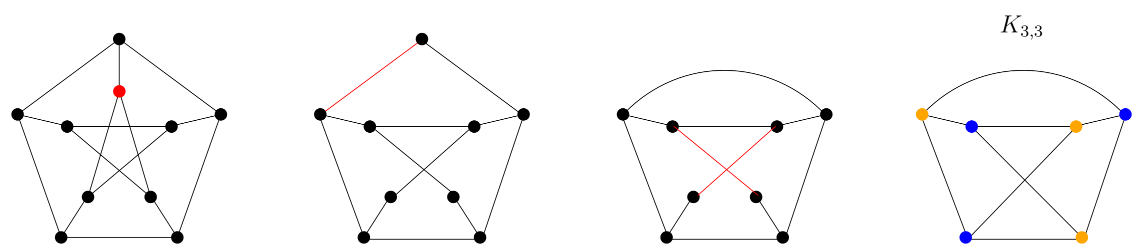

Figure 9.3: First, take the subgraph by removing the red vertex and edges connected to it. Then, contract the red edges. We notice that after these steps, we get a graph minor that is isomorphic to $K_{3,3}$.

9.4 Formalizing Planrity Using Topology (not covered in class)