MAT344 Intro to Combinatorics

Lecture 6: The Pigeonhole Principle

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6.1 The Pigeonhole Principle

Theorem 6.1 (The Pigeonhole Principle)**.** Let X be a set of n objects. Suppose $\{X_1, \ldots, X_k\}$ from a partition of X (i.e. a family of disjoint sets whose union is X). If $k < n$, then $\exists i \in \{1, \ldots, k\}$ such that $|X_i| \geq 2$.

Proof. Suppose for contradiction that for all $i \in \{1, \ldots, k\}$, $|X_i| = 1$. Since $\{X_1, \ldots, X_k\}$ is a partition,

$$n = |X| = \sum_{i=1}^{k} |X_i| = k.$$

But this is contradiction since $k < n$. ■

Alternatively, we can state the Pigeonhole Principle in terms of a mapping function.

Theorem 6.2 (The Pigeonhole Principle)**.** Let X be a set of n objects. Let $s : X \to \{1, \ldots, k\}$. If $k < n$, then s cannot be injective. That is, there exist distinct $x, y \in X$ such that $s(x) = s(y)$.

Proof. For $i \in \{1, \ldots, k\}$, let $X_i = \{x \in X \mid s(x) = i\}$. Suppose for contradiction that s is injective. Then, it follows that X_i are disjoint and

$$\bigcup_{i=1}^{k} X_i = X.$$

But again, since s is injective $|X_i| \leq 1$ for all $i \in \{1, \ldots, k\}$ and this is a contradiction since $k < n$. ■

6.2 Applications of the Pigeonhole Principle