CSC2420 - Algorithm Design, Analysis and Theory **Fall 2022**

Lecture 6: Online Bipartite Matching and Secretary Problem

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6.1 Online Bipartite Matching in ROM

In the random order model (ROM), the nodes V of G are presented to $\mathcal{A}$ uniformly at random, so the matching $\mathcal{A}(G)$ returned by $\mathcal{A}$ is random.

Performance of $\mathcal{A}$, namely $\mathbb{E}[w(\mathcal{A}(G))]$, is averaged over π. The competitive ratio of $\mathcal{A}$ in the random order model is then

$$\inf_G \frac{\mathbb{E}[w(\mathcal{A}(G))]}{OPT(G)}$$

Huang et al. 2018 introduced a generalization of Weighted-Ranking, defined using a function $g(x,y)$ where $g : [0,1]^2 \to [0,1]$.

The price of $u \in U$ is then $w_u \cdot g(X_u, Y_V)$ where $X_U \sim \mathcal{U}[0,1]$ is the rank of $u \in U$ and $Y_V \sim \mathcal{U}[0,1]$ is the arrival time of vertex $v \in V$.

For fixed $x \in [0,1]$, if $y_1 < y_2$, then $g(x,y_2) < g(x,y_1)$. Thus, the later a vertex v arrives, the larger Y_V is, the lower the prices for v.

Theorem 6.1. *Generalized-Ranking achieves a competitive ratio of* 0.6534 *in the vertex-weighted ROM setting.*

6.2 Online Matching with Edge Weight

We have so far only considered the case when $G = (U,V,E)$ is vertex weighted. All the arrival models and corresponding competitive ratios generalize to the edge weighted setting.

However, in the adversarial arrival model, no algorithm attains a constant competitive ratio. This is true even when $|U| = 1$ and $|V| = 2$. Consider the case when $w_{e_1} << w_{e_2}$.

6.2.1 The Secretary Problem

When $|U| = 1$, $E = \{u\} \times V$, observe that $OPT(G) = \max_{e \in E} w_e$. This is the secretary problem. Moreover, the algorithm can select at most one edge, and the edges arrive uniformly at random. The secretary algorithm is simple:

SECRETARY

1 pass on the first n/e arriving edges
2 accept the first edge whose weight is at least as large as the first n/e edges

If the max edge is in the first n/e edges, we don't pick anything. The probability of this happening is $1/e$.

Problem 6.2.

Input: Let there be n applicants, say $1, \ldots, n$, where w_i is the "value/weight" of applicant i.

Goal: Output $I \in \{1, \ldots, n\}$ such that

$$\mathbb{E}[w_I] \geq \alpha \cdot \max_{1 \leq i \leq n} w_i$$

In the following pseudocode, π is a random permutation of the applicants.

Threshold-Secretary

for $t = 1$ **to** $c \cdot n - 1$
    pass on applicant $\pi(t)$
for $t = c \cdot n$ **to** n
    if $w_{\pi(t)} \geq \max\{w_{\pi(1), \ldots, w_{\pi(cn-1)}}\}$
        return $w_{\pi(t)}$

Theorem 6.3. *If* $j = \arg\max_{1 \leq i \leq n} w_i$,

$$\Pr[I = i] \geq \alpha$$

where $\alpha = 1/e$.

Proof: Let I be the output of the cn threshold. We would like to find a lower bound on $\Pr[w_I = \max\{w_1, \ldots, w_n\}]$. This probability can be expressed as

$$\Pr[w_I = \max\{w_1, \ldots, w_n\}] = \sum_{t=1}^{n} \Pr[I = \pi(t) \wedge w_{\pi(t)} = \max\{w_1, \ldots, w_n\}] \tag{6.1}$$

Observation: $\pi(t)$ is selected if and only if $\pi(t)$ is the best amongst the first cn candidates and all candidates from $\pi(cn)$ to $\pi(t-1)$ are rejected. Equivalently, this is equal to the probability that $\pi(t)$ is the best applicant AND $\pi(cn), \ldots, \pi(t-1)$ are rejected.

$$\begin{aligned}\Pr[I = \pi(t) \wedge \pi(t) \text{ is the best}] &= \Pr[\pi(cn), \ldots, \pi(t-1) \text{ rejected} \mid \pi(t) \text{ is the best}] \cdot \Pr[\pi(t) \text{ is the best}] \\ &= \Pr[\pi(cn), \ldots, \pi(t-1) \text{ rejected} \mid \pi(t) \text{ is the best}] \cdot \frac{1}{n}\end{aligned}$$

For every job $\pi(j)$ between $\pi(cn)$ and $\pi(t-1)$, it is rejected if and only if $w_{\pi(j)} < \max\{w_{\pi(1) \ldots w_{\pi(cn-1)}}\}$. This transforms the event of "all jobs rejected" into "all jobs have value smaller than the max of the first $cn - 1$ jobs", which is something that we can calculate.

$$\begin{aligned}\Pr\left[\bigcap_{j=cn}^{t-1} \left(w_{\pi(j)} < \max\left\{w_{\pi(1), \ldots, w_{\pi(cn-1)}}\right\}\right)\right] &= \Pr\left[\begin{matrix}\text{best applicant amongst the first} \\ t-1 \text{ arrivals in the first } cn-1\end{matrix}\right] \\ &= \frac{cn-1}{t-1}\end{aligned}$$

Substitute this result back to (5.1).

$$\Pr[w_I = \max\{w_1, \ldots, w_n\}] = \sum_{t=cn}^{n} \frac{cn-1}{t-1} \cdot \frac{1}{n} \tag{6.2}$$

As $n \to \infty$, (5.2) can be upper bounded as follows.

$$\begin{aligned}\sum_{t=cn}^{n} \frac{cn-1}{t-1} \cdot \frac{1}{n} &= (1+o(1))c \cdot \sum_{t=cn}^{n} \frac{1}{t-1} \\ &\leq (1+o(1))c \int_{s=cn}^{n} \frac{1}{s} ds \\ &= (1+o(1))c \cdot [\ln(n) - \ln(cn)] \\ &= c \ln c\end{aligned}$$

This attains maximum for $c = \frac{1}{e}$. ■

It turns out this ratio can be achieved in the general case where we have more than one offline vertices as well. The algorithm is as follows.

Secretary-Matching

$\mathcal{M} = \emptyset$
$G_0 = (U, \emptyset, \emptyset)$
for $t = 1, \ldots, n$
    input v_t and compute G_t by updating G_{t-1} to contain v_t
    if $t < \lfloor n/e \rfloor$
        pass on v_t
    else
        compute an optimal matching $\mathcal{M}_t$ of G_t
        $e_t =$ the edge matched to v_t via $\mathcal{M}_t$
        if $e_t = (u_t, v_t)$ exists and u_t is unmatched
            $\mathcal{M} = \mathcal{M} \cup \{e_t\}$
return $\mathcal{M}$

Theorem 6.4 (Kesselheim 2013, ESA)**.** *Secretary-Matching attains an asymptotic as* $|V| \to \infty$ *competitive ratio of* $1/e$ *for the case* $|U| > 1$.