Recall that the ranking algorithm independently assigns a ranking for each offline vertex and match the online vertices based on the ranking of the available offline vertex as the online vertices arrive.

An intuitive interpretation of the matching problem is as follows. One side U are the sellers, and the other side V are the buyers. Each seller draws a price x_u uniformly and indepdently from $[0, 1]$. We as the "government", set an adjusted price $g(x_u)$ that the buyers end up seeing as the final pricetag. Then, when the buyer $v \in V$ arrives, the buyer greedily choose the cheapest item that it can buy (in $N(v)$), contributing to the overall utility by $(1 - g(x_u))$. In the analysis, we adjust each both the adjusted price and the utility gain by a factor of $1/F$ where $F = 1 - 1/e$. Setting $\alpha_u = g(x_u)/F$ and for $v \in N(u)$, $\beta_v = (1 - g(u))/F$, we wish to maximize

$$|\mathcal{M}| = F \cdot \left[\sum_{u \in U} \alpha_u + \sum_{v \in V} \beta_v \right]. \tag{3.1}$$

Theorem 3.1. *Ranking returns a matching $\mathcal{M}$ of $G = (U, V, E)$ such that*

$$\mathbb{E}[|\mathcal{M}|] \geq F \cdot \mathrm{OPT}(G)$$

where $F = 1 - 1/e \approx 0.62$.

Proof: Consider the convex relaxation of the integral matching LP where X_e is the indicator that is equal to 1 when e is included in the matching.

$$
\begin{array}{lll}
\text{maximize} & \sum_{e \in E} X_e & \\
\text{subject to} & \sum_{u \in N(v)} X_{uv} \leq 1 & \forall v \in V \\
& \sum_{v \in N(u)} X_{uv} \leq 1 & \forall u \in U \\
& X_e \geq 0 & \forall e \in E
\end{array}
$$

The LP has $|U| + |V|$ variables and $|E|$ constraints. Take the dual of the LP. The dual is the vertex cover LP.

$$
\begin{array}{lll}
\text{minimize} & \sum_{u \in U} \alpha_u^* + \sum_{v \in V} \beta_v^* & \\
\text{subject to} & \alpha_u^* + \beta_v^* \geq 1 & \forall uv \in E \\
& \alpha_u^*, \beta_v^* \geq 0 & \forall u \in U,\, v \in V
\end{array}
$$

α_u^* is the indicator variable that is 1 when u is in the vertex cover. By weak duality of LP,

$$\mathrm{LPOPT}(G) \leq \mathrm{DualLPOPT}(G)$$

Further, for our relaxation,

$$\mathrm{OPT}(G) \leq \mathrm{LPOPT}(G)$$

Proof Idea: As the raning algorithm executes, we will construct a primal LP solution and a dual LP solution such that:

1. primal: is just the integral matching

2. dual: fix $u \in U$, $v \in V$; if the ranking algorithm matches v to u as the online vertex v arrives, then we set $\alpha_u = g(X_u)/F$ and $\beta_u = (1 - g(X_u))/F$ where $F = 1 - 1/e$ and $g(x) = -\exp(x - 1)$; otherwise, set $\alpha_u = 0$ and $\beta_v = 0$ if u and v are unmatched.

When one more edge is admitted to $\mathcal{M}$, the objective function of the dual increased by $1/F > 1$.

$(\alpha_u)_{u \in U}$ and $(\beta_v)_{v \in V}$ are random variables dependent on $(X_u)_{u \in U}$. Similarly, $\mathcal{M}$ is a random matching of G. The goal of the proof is to show that

$$\mathbb{E}[|\mathcal{M}|] \geq (1 - 1/e) \cdot \mathrm{OPT}(G).$$

Taking expectation of on both sides of , we have

$$\mathbb{E}[|\mathcal{M}|] = F \cdot \left[\sum_{u \in U} \mathbb{E}[\alpha_u] + \sum_{v \in V} \mathbb{E}[\beta_v] \right]$$

by linearity of expectation. Further, recall that $\mathrm{OPT}(G) \leq \mathrm{LPOPT}(G) \leq \mathrm{DualLPOPT}(G)$. Then for α_u^* and β_v^* or any other feasible solution,

$$\mathrm{OPT}(G) \leq \mathrm{LPOPT}(G) \leq \mathrm{DualLPOPT}(G) \leq \sum_{u \in U} \alpha_u^* + \sum_{v \in V} \beta_v^*$$

because the dual is a minimizing LP. Hence, it suffices to prove that for $\alpha_u^* = \mathbb{E}[\alpha_u]$ and $\beta_v^* = \mathbb{E}[\beta_v]$, $(\alpha_u^*)_{u \in U}$ and $(\beta_v^*)_{v \in V}$ are feasible solution for the dual LP. We now proceed with the proof.

Fix an arbitrary edge $e_0 = (u_0, v_0) \in E$. We want to show that $\alpha_{u_0}^*$ and $\beta_{v_0}^*$ are feasible, which we can argue by showing $\alpha_{u_0}^* + \beta_{v_0}^* \geq 1$. Let $\tilde{G} = G \setminus \{u_0\}$. Fix x_u for $u \neq u_0$. Consdier an execution of the ranking algorithm on $\tilde{G}$. Define X^c to be the "critical rank" such that

$$x^c = \begin{cases} X_u & \text{if } v_0 \text{ matched to some } u \in U \setminus \{u_0\} \\ 1 & \text{otherwise} \end{cases}$$

Lemma 3.2 (Dominance Lemma). *If all ranks $(X_u)_{u \neq u_0}$ are fixed, then u_0 will be matched when the ranking algorithm executes on G, provided $X_{u_0} < X^c$.*

Lemma 3.3 (Monotonicity Lemma). *Set $\beta_{v_0}^c = (1 - g(x^c))/F$. Think of this as the "worst possible utility v_0 can gain". If all r.v. $(X_u)_{u \neq u_0}$ are fixed, then when the ranking algorithm executes on G, we get that*

$$\beta_{v_0} \geq \beta_{v_0}^c.$$

Now, we prove the main claim. Consider β_{v_0} first. By the Monotonicity Lemma, $\beta_{v_0} \geq \beta_{v_0}^c = (1 - g(x^c))/F$. Now consider α_{u_0} conditioned on $(X_u)_{u \neq u_0}$. Recall that X_{u_0} is uniformly chosen from $[0, 1]$ so $\mathcal{U}[0, 1]$.

$$\mathbb{E}\left[\alpha_{u_0} \mid (X_u)_{u \neq u_0}\right] \geq \int_0^{x^c} g(s)ds \cdot \frac{1}{F}$$

With this, we have the following bound

$$\mathbb{E}\left[\alpha_{u_0} \mid (X_u)_{u \neq u_0}\right] + \mathbb{E}\left[\beta_{v_0} \mid (X_u)_{u \neq u_0}\right] \geq \int_0^{x^c} g(s)ds \cdot \frac{1}{F} + \frac{1 - g(x^c)}{F}$$

For any real $\theta \in [0, 1]$,

$$\int_0^\theta g(s)ds + (1 - g(\theta)) \geq F = 1 - \frac{1}{e}$$

which gives a lower bound on $\mathbb{E}\left[\alpha_{u_0} \mid (X_u)_{u \neq u_0}\right] + \mathbb{E}\left[\beta_{v_0} \mid (X_u)_{u \neq u_0}\right]$ that is

$$\mathbb{E}\left[\alpha_{u_0} \mid (X_u)_{u \neq u_0}\right] + \mathbb{E}\left[\beta_{v_0} \mid (X_u)_{u \neq u_0}\right] \geq \int_0^{x^c} g(s)ds \cdot \frac{1}{F} + \frac{1 - g(x^c)}{F} \geq 1.$$

Taking expectation on both sides again,

$$\alpha_{u_0}^* + \beta_{v_0}^* = \mathbb{E}\left[\mathbb{E}\left[\alpha_{u_0} \mid (X_u)_{u \neq u_0}\right] + \mathbb{E}\left[\beta_{v_0} \mid (X_u)_{u \neq u_0}\right]\right] \geq 1.$$

$\blacksquare$