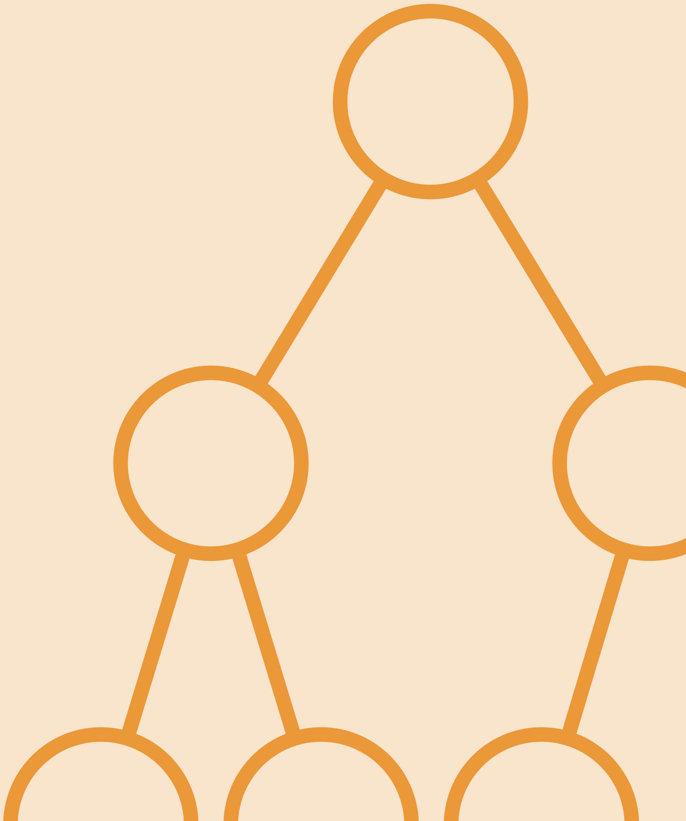
Data Structures and Analysis



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Tutorial

I

Data Structures

Lecture 1 Abstract Data Types

1.1 Abstract Data Type

Definition 1.1.1 — Abstract Data Type. An abstract data type (ADT) is a set of mathematical objects and a set of operations on those objects. An ADT describes how information can be used in a program, which is important for specification and provides modularity and reuseability.

■ **Example 1.1 — ADT for Integers.**

- Objects: $\mathbb{Z}$
- Operations: $\textsc{Add}(x,y)$ $\textsc{Substract}(x,y)$, $\textsc{Multiply}(x,y)$, $\textsc{Quotient}(x,y)$, and $\textsc{Remainder}(x,y)$.

■

■ **Example 1.2 — Stack ADT.**

- Objects: sequences
- Operations: $\textsc{Push}(S)$ $\textsc{Pop}(S)$, $\textsc{Empty}(S)$.

■

1.2 Data Structures

Definition 1.2.1 — Data Structure. A data structure is an implementation of an abstract data type.

■ **Example 1.3 — Data Structures for Stack.** A data structure for stacks is an array with a counter. Alternatively, a stack can be implemented as a singly linked list with the top of the stack at the beginning of the list. ■

An ADT specifies what kind of data you can have and what you can do with the data. A data structure specifies how the data is implemented; in other words, it specifies how the data is stored and how you actually do the operations on the data.

1.3 Algorithm Compleixty

The complexity of an algorithm tells us the amount of resources used by the algorithm, expressed by a function of the size of the input. Such resources can be time, space, number of messages, numbers of bits of communication, etc. We are interested in analyzing the complexity of algorithms because it allows us to:

- compare different algorithms
- give bound to resources needed for a given input
- determine the largest size of input for which the algorithm is still efficient

The definition of input size depends on the problem that we are interested in. Below are examples of some common definitions of input size for the type of data that we are dealing with.

- integer: number of bits
- list: number of elements
- array: dimension of the array, or number of bits
- graph: number of vertices, or number of edges, or both

1.4 Dictionary ADT and Implementations

1.4.1 Dictionary ADT

In this seciton, we will take a look at an example of ADT and some implementation of it. For the dictionary ADT, the objects are defined to be the set of elements each of which has a key drawn from a totally ordered set. And the dictionary ADT should support the following operations:

- SEARCH(S,k): search the set S for an element with the key k and return a pointer to one such element. If no such element exists, return NIL.
- INSERT(S,x): insert element pointed by the pointer x into the set S.
- DELTE(S,x): delete the element pointed to by the pointer x from the set S.

Each element x will contain a property k that stores the key.

R Some examples of totally ordered sets are: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, colors, English words. The set of complex numbers $\mathbb{C}$ is not totally ordered. A more formal definition of a total order can be found in the notes on Theory of Computation (CSC 240/CSC 236).

1.4.2 Data Structures for Dictionary

There are many ways to implement a dictionary. The simplest and most common way is to use a hash table, but there are also equally valid implementations.

Hashing

SEARCH: average complexity $O(1)$, worst-case complexity $O(n)$
INSERT: average complexity $O(1)$, worst-case complexity $O(n)$

DELETE: average complexity $O(1)$, worst-case complexity $O(n)$

Array

We can use two arrays, one with keys, the other with values in the correesponding position of the keys. In the case of unsorted arrays, the time complexities of the operations are:

SEARCH: worst-case complexity $O(n)$; since unsorted, we need to perform linear search to find the element
INSERT: worst-case complexity $O(1)$
DELETE: worst-case complexity $O(1)$

Another assumption that we need to make is that the elements and keys are placed consecutively in the array. However, empty slots might be created upon deleting elements. To solve this, we simply replace the deleted element with the last element in the array. By doing so, we ensure that the number of elements in S is at most the size of the array.

Binary Search Tree

SEARCH: $\Theta(h)$ INSERT: $\Theta(h)$
DELETE: $\Theta(h)$
where h is the height of the tree.

Sorted Array with Counter

SEARCH: $O(\log n)$ using binary search
INSERT: $\Theta(n)$
DELETE: $\Theta(n)$

Unsorted Singly Linked List

SEARCH: $\Theta(n)$
INSERT: $\Theta(1)$
DELETE: $\Theta(n)$; this is because for a singly linked list, it takes $\Theta(n)$ time to find the pointer to the previous element in order to reconnect the links after deleting

Unsorted Doubly Linked List

SEARCH: $\Theta(n)$
INSERT: $\Theta(1)$
DELETE: $\Theta(1)$

Sorted Doubly Linked List

SEARCH: $\Theta(n)$; we cannot perform binary search because we don't know the length of the array
INSERT: $\Theta(n)$ to insert into the correct position to keep the list sorted
DELETE: $\Theta(1)$

Direct Access Table

If our set of keys S is a subset of some finite universe $U = \{0, 1, \ldots, m-1\}$ where $|U| = m$, then we can use a direct access table to implement a dictionary ADT. To represent the dictionary, we use an array A with m slots indexed from 0 to m, each of which corresponds to a key in S. The value of the slot $A[i]$ is the pointer to the element in the set S with the key i. The limitations of such implementation include:

- keys have to be unique
- size of the list can get arbitrarily large
- size of the list is limited

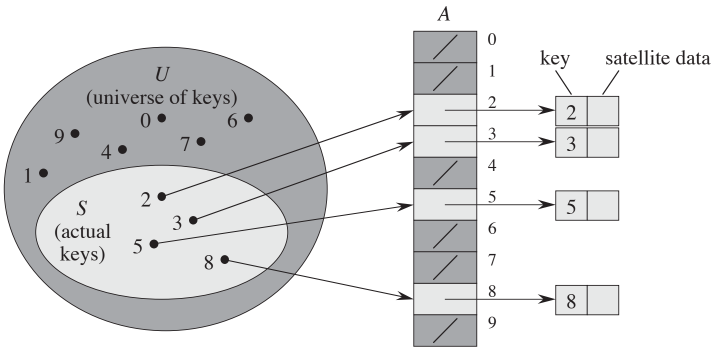


Figure 1.1: Direct access table

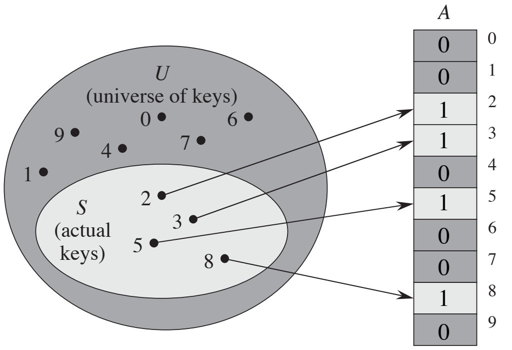


Figure 1.2: Bit vector direct access table

```
SEARCH$(A, k)$
1  return $A[k]$
```

```
INSERT$(A, x)$
    $A[x.key] = x$
```

```
DELETE$(A, x)$
    $A[x.key] =$ NIL
```

Each of the three operations above takes constant time $O(1)$.

Alternatively, we can use value 1 to indicate the presence of element at a given slot, and value 0 to denote the absence of element at a given slot. The resulting data structure called a bit vector direct access table.

Lecture 2 Binary Search Trees

2.1 Binary Search Tree

Definition 2.1.1 — Binary Search Property. If y is in the left subtree of x, then $y.key \leq x.key$. If y is in the right subtree of x, hen $y.key \geq x.key$.

Definition 2.1.2 — Binary Search Tree. A binary tree is a binary search tree if all nodes x, y of the tree satisfies the binary search property.

If we perform an in-order traversal of a binary search tree, we gets the nodes in increasing order by key.

2.2 Searching in BST

```
SEARCH($T, k$)
if $T$ = NIL
    return NIL
if $T.key == k$
    return $T$
if $T.key > k$
    return SEARCH($T.left, k$)
if $T.key < k$
return SEARCH($T.right, k$)
```

2.3 Insertion

```
INSERT($T, x$)
$X$ = SEARCH($T, x$)
if $X$ == NIL
    $X = x$
else
    // there is already an element with key $x.k$ stored in BST, we can do the following
    // 1. return without inserting $x$
    // 2. continues to search until NIL is found
    // 3. store all elements with the same key in an auxillary data structure (e.g. linked list)
    // 4. replace old element with $x$
```

2.4 Deletion

Deletion is the trickiest operation for BST.

```
DELETE$(T, x)$
$y = x.left$
while $y.right \neq$ NIL
    $y = y.right$
$x = y$
$y =$ NIL
```

Lecture 3 Balanced Search Trees

3.1 Balanced Trees

There are two different ways of defining balancedness of a binary tree: weight balance and height balance. In this chapter, we will mainly focus on height balanced trees. As it turns out, weight balance is a more strict requirement than height balance, and weight balance implies height balance. Since the runtime complexity of binaryy tree operations are height-dependent, both definitions should give us $O(\lg n)$ time on most operations.

> **Definition 3.1.1 — Height Balancedness.** A binary tree is height balanced if for every node in the tree, the height of its left and right subtrees differ by at most one.

> **Definition 3.1.2 — Weight Balancedness.** A binary tree is weight balanced if for every node in the tree, the number of nodes of its left and right subtrees differ by at most one.

> **Corollary 3.1.1** Weight-balanced binary trees are height-balanced.

In this section we will look at a few height balanced search tree including red-black trees, AVL (Adelson-Velskii and Landis) trees, 2-3 trees, and B-trees which is a more general form of 2-3 trees. The first two are binary trees while 2-3 tree and B-tree are not necessarily binary.

3.2 Red-Black Tree

3.2.1 Definition and Properties

> **Definition 3.2.1 — Red-Black Tree.** A red-black tree is a binary search tree in which every node is either red or black and satisfies the following properties:
>
> 1. The root is black
> 2. Every leaf node (NIL node) is black
> 3. If a node is red, then both its children are black
> 4. For each node, all paths from the node to descendant leaves (NIL nodes) contain the same number of black nodes
>
> Alternatively, the properties can be stated without using the NIL node.

1. The root is black
2. A red node has no red children
3. Every path from the root to a node with at most one child contains the same number of black nodes

Figure 3.1 illustrates the properties of red-black trees.

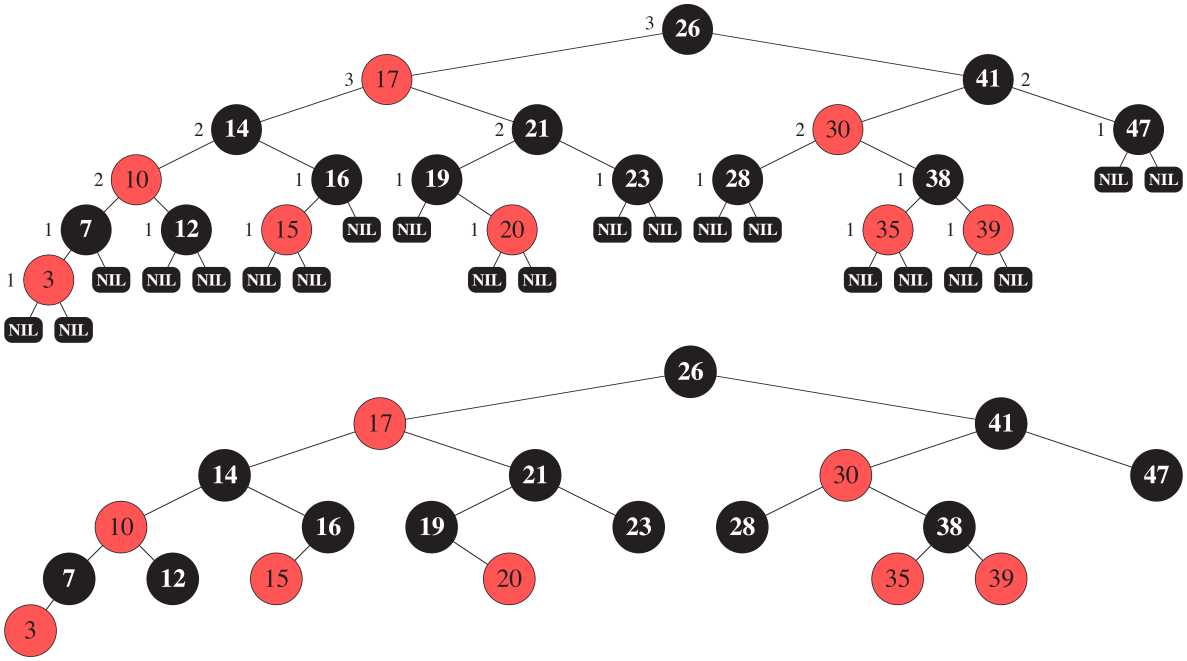


Figure 3.1: Red-black trees. The first tree is represented with the NIL sentinel node. The second tree is without the NIL node. The two trees are equivalent and both satisfies the red-black tree properties.

Lemma 3.2.1 The number of nodes in a red-black tree of height h is at least $2^{\lceil (h+1)/2 \rceil} - 1$.

Proof. A red-black tree of height h has a path of length h from the root to a leaf node. This path contains $h+1$ nodes, the first of which is black. Since it does not contain two consecutive red nodes, the path contains at least $b = \lceil (h+1)/2 \rceil$ black nodes (i.e. at least half of the nodes on that path are black). Hence, every path from the root to a node with at most one child contains at least b black nodes. It suffices to prove that there are at least $2^b - 1$ black nodes in a red-black tree of height h such that the number of black nodes in the path from the root to a leaf node is b.

BASE CASE: If the height of the tree is 0, then the number of nodes is 1.

INDUCTIVE STEP: Let $h \in \mathbb{N}$ be arbitrary. Assume that for all tree with height $h' < h$, there are $2^{b'} - 1$ black nodes where b' is the number of black nodes in the path from the root of that tree to a leaf node.

Consider an arbitrary tree T with height h with two subtrees. It follows that there are $b = \lceil (h+1)/2 \rceil$ black nodes from the root of the tree to a leaf node. If the subtree has a black root, then the path going from the root of the subtree to a leaf contains $b-1$ black nodes. If the root is red, then the path contains b black nodes. Therefore, the number of black nodes in the path from the root of the tree to a leaf node is $b-1$ or b. Then, by induction hypothesis, the number black nodes in each subtree is at least $2^{b-1}-1$.

Since the root of a red-black tree is black, the number of black nodes in T is the number of nodes in the left subtree plus the number of nodes in the right subtree plus the root node.

$$\left(2^{b-1}-1\right)+\left(2^{b-1}-1\right)+1=2^{b}-1$$

By induction, the number of black nodes in a red-black tree of height h is at least $2^{b}-1=2^{\lceil (h+1)/2\rceil}-1$.

■

Corollary 3.2.2 A red-black tree with n nodes has height $h \leq 2\log_2(n+1)-1$.

It follows immediately from this corollary that the SEARCH operation will run in $O(\lg n)$ time on a red-black tree.

3.2.2 Rotations

It is obvious that INSERT and DELETE will also run in $O(\lg n)$ time, but the resulting tree may not satisfy the red-black tree properties, meaning that the tree after insertion and deletion of nodes may not be a red-black tree. We can fix this using a technique known as rotation.

Case 1

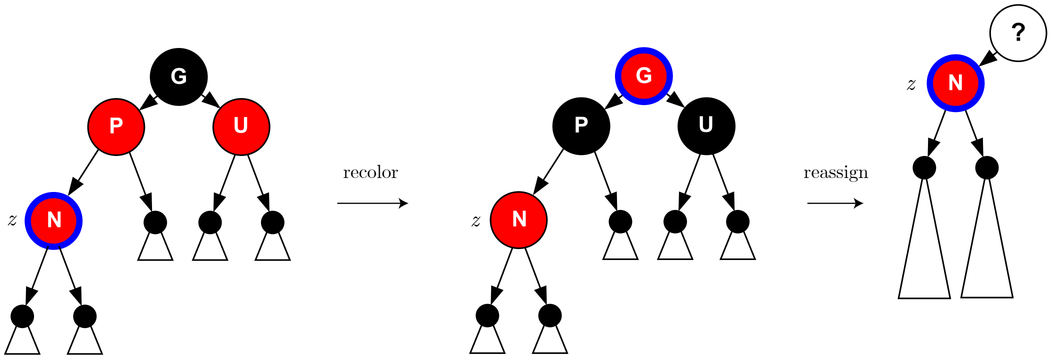


Figure 3.2: <caption>

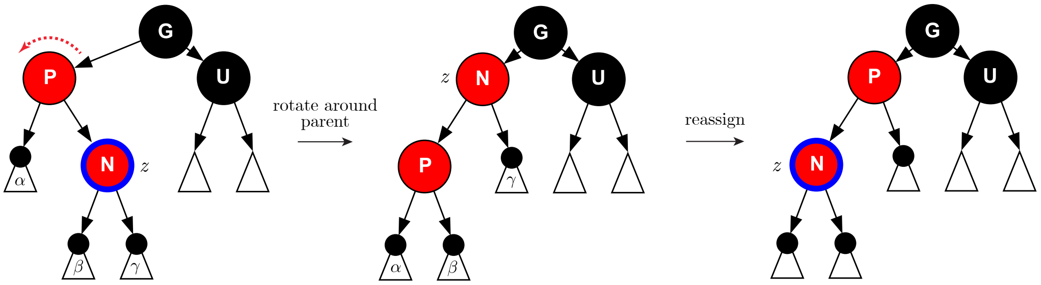


Figure 3.3: <caption>

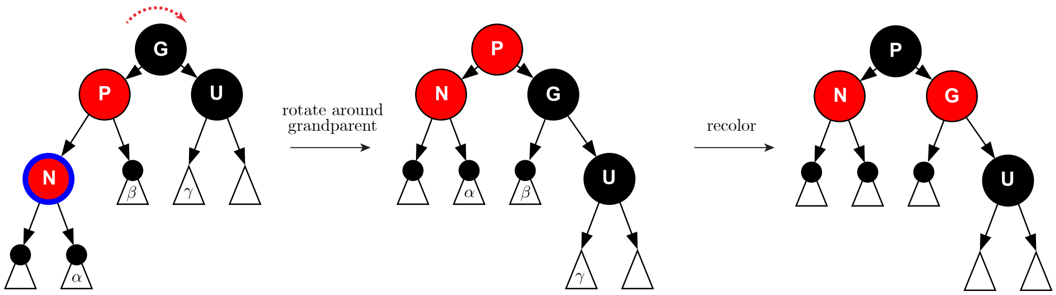


Figure 3.4: <caption>

Case 2

Case 3

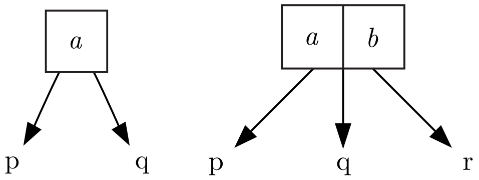
3.3 AVL Tree

3.4 B-Tree

3.4.1 2-3 Tree

Definition 3.4.1 A **search** tree is a 2-3 tree if every internal node has either two children with one element, or three children with two elements. Leaf nodes of a 2-3 tree have no children and one or two elements. More formally, a tree T is a 2-3 tree if and only if one of the following is true:

- T is empty
- T has one element a and two children p, q (p being the left child and q being the right child). p and q are 2-3 trees of the same height, and a is greater than every element in p, and a is less than every element in q
- T has two elements $a < b$ and three children p, q, r (left, middle, right children, respectively). p and q are 2-3 trees of the same height; and a is greater than every element in p and less than every element in q; and b is greater than every element in q and less than every element in r.


a
p
q
a
b
p
q
r

Lecture 4 Augmenting Data Structures

4.1 Augmenting Data Structures

For many problems, it is not enough to use only the elementary data structures such as linked list, hash table, or binary tree. But for most of those problems, we don't need to reinvent the wheel. Instead, we can augment the data structures we already have along with some additional information.

4.2 Order Statistics With Red-Black Trees

4.3 Steps To Create Augmented Data Structures

To create augmented data strucutres, we typically follow these four steps:

1. choose an underlying data structure
2. determine additional information to be maintained
3. verify that the additional information can be maintained by update operations (or basic steps of update operations, e.g. rotations)
4. develop new operations

Theorem 4.3.1 — Augmenting Red-Black Tree. Let f be a field augmenting each node of a red-black tree and suppose that $x.f$ can be computed using information in node x, $x.left$, and $x.right$, possibly including $x.left.f$ and $x.right.f$. Then, the f field can be maintained in all nodes during insertion and deletion without asymptotically affecting the $O(\lg n)$ performance of these operations.

Proof Idea. A change to $x.f$ only propagates to $y.f$ for the ancestors y of x. Since the height of a red-black tree is $O(\lg n)$, at most $O(\lg n)$ nodes have their f fields changed an each change takes $O(1)$ time.

Lecture 5 Priority Queue and Heap

5.1 Priority Queue

5.1.1 Priority Queue ADT

The priority queue ADT is a data type that stores a collection of items with priorities (keys) that supports the following opeations:

- INSERT(Q,x) inserts the element x into the priority queue Q.
- MAXIMUM(Q) returns the element of Q with the largest key.
- EXTRACT-MAX(Q) removes and returns the element of q with the largest key.
- INCREASE-KEY(S,x,k) increases the value of element x's key into the new value k, assuming that $k \geq x$.

Priority queue allows us to access the element with largest (if max-priority queue) or smallest (if min-priority queue) more efficiently. It has many applications in computer science, such as: job scheduling in operation systems, bandwith management, or finding minimum spanning tree of a graph, etc.

5.1.2 Primitive Implementation Using Linked Lists

We can have a naive implementation of a priority queue simply using a sorted linked list, which has the following time complexity:

- INSERT(Q,x): $\Theta(n)$ in the worst case. We have to linearly search the correct location of insertion.
- MAXIMUM(Q): $\Theta(1)$ by returning the head of the list.
- EXTRACT-MAX(Q): $\Theta(1)$ by removing and returning the head of the list.
- INCREASE-KEY(S,x,k): $\Theta(n)$ in the worst case. Need to move element to new location after increase.

However, we want to have something that is more efficient than $\Theta(n)$. As it turns out, by putting the elements in a specific way, we can achieve worst-case time complexity of $\Theta(\log n)$ for INSERT, EXTRACT-MAX, and INCREASE-KEY. Even better, we can show that the amortized complexity of INSERT is $\Theta(1)$ and EXTRACT-MAX is $\Theta(\log n)$.

5.2 Heap

5.2.1 Types of Binary Trees

Before starting to formally define heaps, let's review some definitions about binary trees.

Definition 5.2.1 — Full Binary Tree. A full binary tree (sometimes proper binary tree or 2-tree) is a tree in which every node other than the leaves has two children.

Definition 5.2.2 — Complete Binary Tree. A complete binary tree is a binary tree in which every level, except possibly the last, is completely filled, and all nodes are as far left as possible.

Importantly, a complete binary tree with n nodes has $\lfloor n/2 \rfloor$ internal nodes (nodes that are not leaves).

Definition 5.2.3 — Perfect Binary Tree. A perfect binary tree is a binary tree in which all interior nodes have two children and all leaves have the same depth or same level.

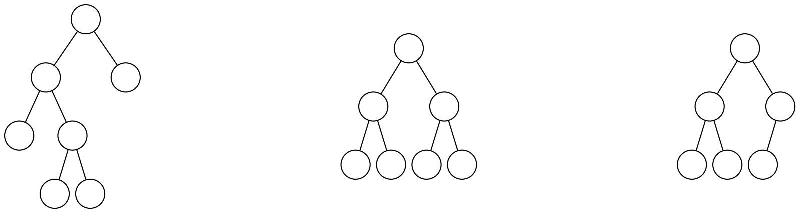


Figure 5.1: From left to right: full binary tree, complete binary tree, perfect binary tree.

Conveniently, a complete binary tree can be represented as an array.

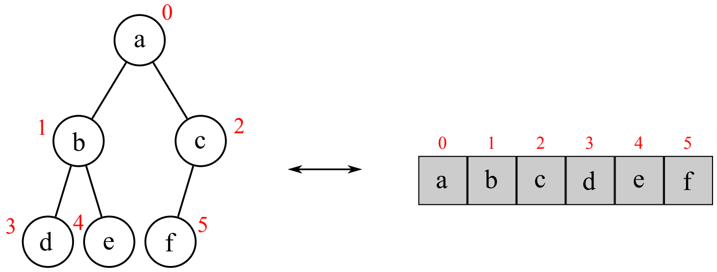


Figure 5.2: A complete binary tree and its corresponding array representation

Assuming that we index from 0, we can compute the indices of each node's parent, left, and right child.

PARENT(i) = $\lfloor (i-1)/2 \rfloor$
LEFT(i) = $(2i)+1$
RIGHT(i) = $(2i)+2$

If we index from 1, the indices are calculated as follows:

PARENT(i) = $\lfloor i/2 \rfloor$

Left(i) = $2i$
Right(i) = $2i+1$

5.2.2 Heap Property

Then, we can define a max-heap as a complete binary tree with the max-heap property.

> **Definition 5.2.4 — Max-heap Property.** In a max-heap represented by the array A , the max-heap property is that for every node i other than the root,
>
> $$A[\text{Parent}(i)] \geq A[i].$$
>
> that is, the value of a node is at most the value of its parent.

The max-heap property guarantees that the largest element in a heap is always stored at its root.

For our heap implementation, we will include the following operations: Insert, Maximum, Extract-Max, Increase-Key, Max-Heapify, Build-Max-Heap. The first few operations allow us to use heap to implement the priority queue ADT, and in addition to those, Build-Max-Heap allows us to produce a max-heap from an unordered array.

5.3 Maintaining the Heap Property

Given an array A and index i, Max-Heapify will correct a single violation of the max-heap property in the subtree with i as its root. To implement Max-Heapify, we use a technique called "trickle down".

First, assume that the trees rooted at Left(i) and Right(i) are max-heaps. If element $A[i]$ violates the max-heap property, we correct thie violation by "trickling" element $A[i]$ down the tree until it reaches the correct position. By doing so, we can make the subtree rooted at index i a max-heap. In every trickle-down step, swap $A[i]$ with its largest child.

```
Max-Heapify$(A, i)$
  $l = \text{Left}(i)$
  $r = \text{Right}(i)$
  if $l \leq A.heapsize$ and $A[l] > A[i]$
      $largest = l$
  else $largest = i$
  if $r \leq A.heapsize$ and $A[r] > A[largest]$
      $largest = r$
  if $largest \neq i$
      exchange $A[i]$ with $A[largest]$
      Max-Heapify$(A, largest)$
```

5.3.1 Correctness of MAX-HEAPIFY

5.3.2 Running Time of MAX-HEAPIFY

5.4 Inserting Into Max-Heap

To insert into a max-heap while maintaining the heap property, we use a similar technique. We first append the new element to the end of the heap. The new element will become the right-most leaf at the last level. Then, we check if the element is already at the right position. If not, we "bubble" the element up the tree, until it reaches the correct position.

5.5 Build Heap From Unsorted Array

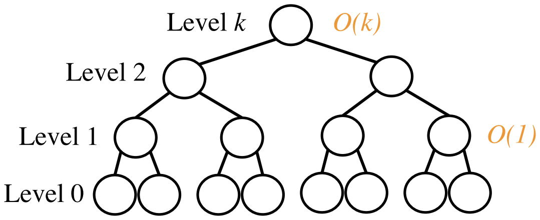
```
BUILD-MAX-HEAP($A$)
$A.heapsize = A.length$
for $i = \lfloor A.length/2 \rfloor$ downto 1
    MAX-HEAPIFY($A, i$)
```

The reason we start calling MAX-HEAPIFY at $\lfloor A.length/2 \rfloor$ is because elements beyond that $A[n/2+1, \cdots, n]$ are all leaves of the tree. Recall that for a complete binary tree with n nodes, there are only $\lfloor n/2 \rfloor$ internal nodes.

5.5.1 Running Time of BUILD-MAX-HEAP

Each call of MAX-HEAPIFY takes $O(\log n)$ time, and BUILD-MAX-HEAP calls MAX-HEAPIFY $O(n)$ times. Thus, the running time of BUILD-MAX-HEAP is $O(n \log n)$. However, this upper bound is not tight. We will prove a tighter upper bound of $O(n)$.

Note that MAX-HEAPIFY takes $O(1)$ time for nodes tthat are one level above the leaves, and in general, it takes $O(k)$ for nodes that are k levels above the leaves. We have $n/4$ nodes at level 1, $n/8$ at level 2, etc. At the root level, which is $\log_2 n$ levels above the leaves, we have only 1 node.



More generally, a heap with n nodes has a height of $\lfloor \log_2 n \rfloor$ and at most $\lceil n/2^{h+1} \rceil$ nodes at any height h. Hence, the total cost of BUILD-MAX-HEAP can be written as

$$\sum_{h=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\log_2 n} \frac{h}{2^h} \right).$$

The last summation is bounded by a constant, namely

$$\sum_{h=0}^{\log_2 n} \frac{h}{2^h} \leq \sum_{h=0}^{\infty} \frac{h}{2^h} = 2 = O(1).$$

Thus,

$$O\left(n \sum_{h=0}^{\log_2 n} \frac{h}{2^h}\right) = O(n)O(1) = O(n).$$

5.6 Heapsort

```
HEAPSORT($A$)
for $i = A.length$ downto 2
    exchange $A[1]$ with $A[i]$
    $A.heapsize = A.heapsize - 1$
    MAX-HEAPIFY($A, 1$)
```

Lecture 6 Mergeable Heaps

II

Algorithm Analysis

Lecture 7 Average Case Complexity and Randomized Algorithms

7.1 Basic Probability Theory

7.1.1 Sample Space and Events

Definition 7.1.1 — Probability Space and Sample Space. A ***probability space*** (Ω, Prob) consists of a finite or countable set Ω called the sample space, and the probability function $\mathrm{Prob}: \Omega \to \mathbb{R}$ such that for all $\omega \in \Omega$, $\mathrm{Prob}(\omega) \geq 0$ and $\sum_{\omega\in\Omega} \mathrm{Prob}(\omega) = 1$. We cann an element $\omega \in \Omega$ a sample point, or ***outcome***, or ***simple event***.

A sample space models some random experiment, where Ω contains all possible outcomes, and $\mathrm{Prob}(\omega)$ is the probability of the outcome ω. We always talk about probabilities in relation to a sample space.

Definition 7.1.2 — Event. An event A is a set of outcomes, $A \subseteq \Omega$. We define the probability of an event A to be the sum of the probabilities of its elements

$$\mathrm{Prob}(A) = \sum_{\omega\in A} \mathrm{Prob}(\omega)$$

If we have $\mathrm{Prob}(\omega) = \mathrm{Prob}(\omega')$ for all distinct $\omega, \omega' \in \Omega$, we say that the probability is uniform over Ω.

7.1.2 Properties of Probability Functions

Definition 7.1.3 — Complement. The complement of an event A is

$$\overline{A} = \Omega \setminus A$$

The complement of A is also denoted by not A.

Theorem 7.1.1

$$\begin{aligned}\mathrm{Prob}(\overline{A}) &= 1 - \mathrm{Prob}(A)\\ \mathrm{Prob}(A\cup B) &= \mathrm{Prob}(A) + \mathrm{Prob}(B) - \mathrm{Prob}(A\cap B)\\ \mathrm{Prob}(A\cup B) &\leq \mathrm{Prob}(A) + \mathrm{Prob}(B)\end{aligned}$$

It can then be proved by induction that

$$\text{Prob}(A_1 \cup A_2 \cup \cdots \cup A_k) \leq \text{Prob}(A_1) + \text{Prob}(A_2) + \cdots + \text{Prob}(A_k)$$

for any events $A_1, \cdots A_k$.

We say that two events A and B are disjoint or mutually exclusive if $A \cap B = \emptyset$. If A and B are disjoint, $\text{Prob}(A \cup B) = \text{Prob}(A) + \text{Prob}(B)$.

7.2 Conditional Probability and Independence

Conditional probability allows us to compute the probability of some event given that we already know that some other events have occurred.

Definition 7.2.1 — Conditional Probability. The probability of an event A conditional on an event B is

$$\text{Prob}(A \mid B) = \frac{\text{Prob}(A \cap B)}{\text{Prob}(B)}$$

given that $\text{Prob}(B) > 0$.

7.3 Average Case Analysis

Let A be an algorithm, and let

$t(x) =$ number of steps taken by A on input x.

We know that the worst case time complexity of A is

$$T(n) = \max\{t(x) \mid x \text{ has size } n\}$$

We can define the average case time complexity of A as

Definition 7.3.1 The average case time complexity of A is

$$T'(n) = \text{E}[t(x) \mid x \text{ has size } n]$$

If all inputs of size n are equally likely, then

$$T'(n) = \frac{\sum\{t(x) \mid size(x) = n\}}{|x \mid size(x) = n|}$$

In general, the average case time complexity is less than or equal to the worst case time complexity, that is $T'(n) \leq T(n)$ for all n.

The average case time complexity is dependent on the probability distribution of the inputs, which in turn can depends on the application where the algorithm is to be used. However, this is usually unknown, in which case, uniform distribution usually makes the analysis easier. For some applications, an algorithm with good average case behavior but bad worst case behavior is sufficient.

We follow the following steps before we can analyze the average case time complexity of an algorithm:

- define the sample space of the inputs
- define the probability distribution function
- define any necessary random variables

7.4 Average Case Analysis of Linear Search

Consider the following algorithm for linear search in an unsorted array.

```
LINEAR-SEARCH($L, k$)
$j = 1$
while $j \leq n$
    if $L[k] == k$
        return $j$
    $j = j + 1$
return 0
```

It is obvious that the worst case complexity is $O(n)$ because there will be at most n comparisons with the searching key k. Following the procedure introduced above, we can perform an average case complexity analysis.

1. **Define sample space**: This will be the set of possible inputs that we are considering in our analysis. We define our sample space to be all pairs (L,k) where L is an input of size n and k is a key. Some observations on our choice of sample space:
    - If we fix $L = [l_1, \cdots, l_n]$, then the sample space for k is $[l_1, \cdots, l_n, \text{NIL}]$ where NIL is used to indicate a value that is not in L. Hence, for a L of size n, the sample space for k contains $n+1$ sample points.
    - The number of comparisons performed by LINEAR-SEARCH(L,k) is the same for (L,k) and (L',k') if k occurs in L at the same position as k' in L'.
    - If k and k' are not in L, then LINEAR-SEARCH(L,k) will also perform the same number of comparisons.
    - The relative order between elements of L does not matter since LINEAR-SEARCH only performs equality tests, so choosing $L = [1, \cdots, n]$ is just as good as $L = [n, \cdots, 1]$.

    Hence, we can formally define our sample space as

    $$S_n = \{(L,i) \mid i \in \{\text{NIL}, 1, \cdots, n\}\}$$

    where $L = [1, \cdots, n]$.

2. **Probability distribution**: We choose uniform distribution, which means that every point in the sample space has the same probability $\dfrac{1}{n+1}$.
3. **Define random variable**: Let $t_n : S_n \to \mathbb{N}$ be such that $t_n(i)$ is the number of comparisons performed when the input is (L,i) where

$$t_n(i) = \begin{cases} i & \text{for } i = 1, \cdots, n \\ n & \text{for } i = 0 \end{cases}$$

4. **Analysis**:
   For the uniform distribution,

$$\begin{aligned} T'(n) = \mathrm{E}[t_n] &= \frac{\sum\{t_n(L,i) \mid i = \textsc{nil}, \cdots, n\}}{n+1} \\ &= \frac{n + \sum_{i=1}^{n} i}{n+1} \\ &= \frac{n + \frac{n(n+1)}{2}}{n+1} \\ &= \frac{n}{2} + \frac{n}{n+1} \\ &< \frac{n}{2} + 1 \end{aligned}$$

   Hence, $T'(n) \in O(\frac{n}{2})$.

7.5 Average Case Analysis of Quick Sort

Quicksort is ued to sort a multi-set of elements S from a totally ordered domain. The pseudocode for quicksort is shown below.

```
Quicksort($S$)
if $|S| \le 1$
    return $S$
$pivot$ = select a pivot
// partition $S$ into $L$, $E$, and $G$ such that $L$ contains all elements less than $pivot$
// $E$ contains all elements equal to $pivot$, and $G$ contains all elements greater than $pivot$
$L, E, G = \textsc{Partition}(S, pivot)$
return $\textsc{Quicksort}(L) + E + \textsc{Quicksort}(G)$
```

In practice, we often choose the first element of S as the pivot. The worst case time complexity of quicksort is $O(n^2)$ because it has to partition the input S into three parts, L, E, and G, which gives us this recurrence.

$$T(n) = T(n-1) + T(0) + \Omega(n) \in O(n^2)$$

Following the procedure introduced above, we can perform an average case complexity analysis. In Quicksort, only the relative order of the elements matter, not their actual values.

1. Define sample space: $S_n =$ all permutations of $\{1, \cdots, n\}$
2. Probability distribution: We know that $|S_n| = n!$. Assuming uniform distribution, we have

$$\text{Prob}[\pi] = \frac{1}{n!} \qquad \text{for all } \pi \in S_n$$

3. Random variables: Let $t_n : S_n \to \mathbb{N}$ be the random variable such that

$$t_n(i) = \text{number of element comparison performed by } \textsc{Quicksort}(S)$$

4. Analysis: Let $T'(n) = \text{E}[t_n]$. Let $X_{ij}\ S_n \to \{0,1\}$ be an indicator random variable such that $X_{ij}(\pi) = 1$ if and only if elements i and j are compared during $\textsc{Quicksort}(\pi)$. Since no pair of elements is compared more than once,

$$t_n(\pi) = \sum_{1 \le i < j \le n} X_{ij}(\pi),$$

and

$$\begin{aligned} T'(n) = \text{E}[t_n] &= \sum_{1 \le i < j \le n} \text{E}[X_{ij}] && \text{by linearity of expectation} \\ &= \sum_{1 \le i < j \le n} \text{Prob}[X_{ij} = 1] && \text{since } X_{ij} \text{ is an indicator variable} \end{aligned}$$

In $\textsc{Quicksort}(\pi)$, as long as no element in $\{i, \cdots, j\}$ is chosen as a pivot, these elements will stay together, either all going to L, or all going to G in recursive calls.
Eventually, one of the elements in $\{i, \cdots, j\}$ is chosen as pivot. Suppose that p is the first of the elements in $\{i, \cdots, j\}$ that is chosen as pivot. If $i < p < j$, then i and j are not compared during $\textsc{Quicksort}$. If $p = i$ or $j = j$, then i and j are compared.
There are $j - i + 1$ possibilities for p to be chosen as pivot. Among those possible choices, $X_{ij} = 1$ for two possibilities, and $X_{ij} = 0$ for the rest. Since all permutations of the inputs are equally likely, each of these possibilities for p is equally like and has probability of $\frac{1}{j-i+1}$.
Hence,

$$\text{Prob}[X_{ij} = 1] = \frac{2}{j - i + 1}$$

Using this, we can rewrite $T'(n)$ as with $k = j - i + 1$

$$\begin{aligned}
T'(n) &= \mathrm{E}[t_n] \\
&= \sum_{1 \le i < j \le n} \mathrm{Prob}[X_{ij} = 1] \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n-1} \frac{2}{j - i + 1} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n-1} \frac{2}{k + 1} && \text{substituting } k \\
&< \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{2}{k} \\
&= \sum_{i=1}^{n} O(n \lg n) && \text{harmonic series} \\
&= O(\lg n)
\end{aligned}$$

Therefore, the average case time complexity of quicksort is $O(n \lg n)$.

From this analysis, we notice that the average case time complexity for quicksort is much smaller than the worst case time complexity. If the actual distribution of inputs is close to uniform distribution, then the average case time complexity will serve as a more realistic estimate of running time. But problem arises when the actual distribution is not uniform. To solve this issue, we can modify the quicksort algorithm to use a randomized pivot selection.

7.6 Randomized Quicksort

In randomized quicksort, we randomly pick the pivot instead of using the first element of S. Pick each element of S to be the pivot with probability of $\frac{1}{|S|}$ using a random number generator.

```
RANDOMIZED-QUICKSORT($S$)
if $|S| \le 1$
    return $S$
$r = $ RANDOM$(1, |S|)$
$pivot = S[r]$
// partition $S$ into $L$, $E$, and $G$ such that $L$ contains all elements less than $pivot$
// $E$ contains all elements equal to $pivot$, and $G$ contains all elements greater than $pivot$
$L, E, G = $ PARTITION$(S, pivot)$
return RANDOMIZED-QUICKSORT$(L) + E + $ RANDOMIZED-QUICKSORT$(G)$
```

RANDOM(i, j) will return a number in $\{i, i+1, \cdots, j-1, j\}$ each equally likely, assuming $i \le j$.

Let $t(I,p)$ denote the running time of algorithm A on input I with the sequence of random choices p. Then, the expected running time of algorithm A on input I is

$$\mathrm{E}_p\left[t(T,p)\right] = \sum_e \mathrm{Prob}[p]t(I,p)$$

The worst case expected time complexity of algorithm A is

$$T(n) = \max_{I\in S_n} \mathrm{E}_p\left[t(I,p)\right]$$

where S_n is the set of all inputs of size n.

Note that the worst case complexity does not depend on any assumption about the input distribution.

Lecture 8 Hashing

8.1 Hashing and Hash Function

Let U be the universe of possible keys, and let m be the size of the hash table. Then, we say that

$$h : U \to \{0, \cdots, m-1\}$$

is a hash function.

If two keys are mapped to the same location/bucket/slot by the hash function, we say that they collide.

Furthermore, if $|U| > m$, then by the pigeonhole principle, there are at least two keys that collide. And because in virtually all cases, $|U| > m$, collision is unavoidable. A well-chosen hash function will minimize the number of collisions, but we still need some means to resolve collisions.

> R A fun fact about the etymology of the word "hash": it is said that the word "hash" originated from the french word "hache", which refers to the action of chopping something into pieces. Hashing, as we will see, involves the same notion of randomly chopping and mixing.

8.2 Resolving Collision

8.3 Universal Hashing

Let $\mathcal{H}$ be a finite collection of hash functions that map a given universe U into the range $\{0, \cdots, m-1\}$. Such a collection is said to be universal if for each pair of distinct keys, $k, l \in U$, the number of hash functions $h \in \mathcal{H}$ for which $h(k) = h(l)$ is at most $|\mathcal{H}|/m$. In other words, with a hash function randomly chosen from $\mathcal{H}$, the chance of a collision between distinct keys is not more than the chance $1/m$ of a collision if $h(k)$ and $h(l)$ were randomly and independently chosen from the set $\{0, \cdots, m-1\}$.

Lecture 9 Amortized Analysis

9.1 Amortized Cost

> **Definition 9.1.1 — Amortized Cost.** The amortized cost per operation for a sequence of n operations is the total cost of the operations divided by n.

Generally, there are three methods for performing amortized analysis. Although all methods should give us the same answer, depending on the circumstances, some methods will be easier than others.

- Aggregate analysis determines the upper bound $T(n)$ on the total cost of a sequence of n operations, then calculates the amortized cost to be $T(n)/n$.
- The accounting method determines the individual cost of each operation, combining its immediate execution time and its influence on the running time of future operations. We use the analogy of a bank account. Prior operations that may impact future operations can store some credits in the bank for uses by future operations. Usually, many short-running operations accumulate credits in small increments, while rare long-running operations use the credits.
- The potential method is like the accounting method, but the balance of the imaginary bank account at each state is given by the potential function.

As a starter, we will consider the data structure known as dynamic array. It is similar to a regular array, but its length changes according to the "fullness" of the array.

The array will be initialized with a fixed length. Upon calling INSERT, it will insert an element into the array, and whenver the current array becomes full, we create a longer array and copy everything into the new array. Similarly, after calling DELETE, we will shrink the array whenever the array becomes too empty. If we only consider the worst case, the runtime complexities are bad: $O(n)$ for both operations. However, if we look at the amortized cost, the runtime is actually smaller.

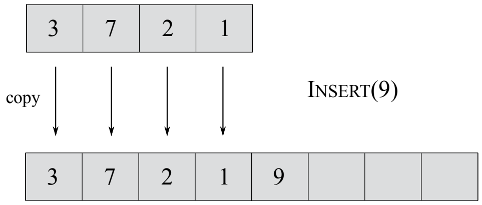


Figure 9.1 The dynamic array after the operation INSERT(9). The length of the new array is doubled, and old elements are copied to the new array.

9.2 Aggregate Method

9.3 Accounting Method

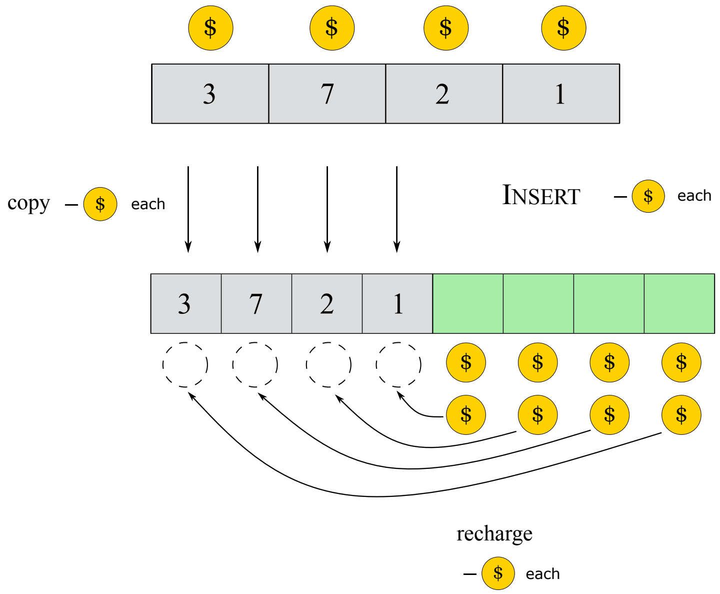


Figure 9.2: <caption>

9.4 Potential Method

Definition 9.4.1 — Potential Function. Let D be some data structure with initial state D_0. For each $i = 1,2,\cdots,n$, let c_i be the actual cost of the i-th operation and D_i be the resulting data structure after applying the i-th operation to data structure D_{i-1}.

The potential function Φ maps each data structure at state i, denoted D_i, to a real number $\Phi(D_i)$, which is the potential associated with data structure D_i.

The amortized cost $\hat{c}_i$ of the i-th operation with respect to the potential function Φ is

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

And the total amortized cost of n operations is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

III Advanced Data Structures

Lecture 10 Dynamic Arrays

Lecture 11 Fibonacci Heaps

Lecture 12 Disjoint Sets

IV

Lower Bounds

Lecture 13 Decision Trees

13.1 Lower Bounds

So far, we have been talking almost exclusively about how we can use different algorithms and data structures to solve certain problems as fast as possible. In this part, we will focus on proving certain problems cannot be solved as quickly as we might want. In other words, there is a limit to how good we can do. For example, in a comparison model, sorting can only be achieved at best in $\Omega(n \log n)$ time in the worst case.

Let $T_A(X)$ be the running time of algorithm A given input X. Then, the worst-case running time is

$$T_A(n) = \max_{|X|=n} T_A(X)$$

The worst-case complexity of a **problem** Π is the worst-case running time of the fastest algorithm for solving it.

$$T_\Pi(n) = \min_{A \text{ solves } \Pi} T_A(n) = \min_{A \text{ solves } \Pi} \max_{|X|=n} T_A(X)$$

We can prove the upper-bound of the complexity of a problem by giving a specific algorithm A that solves Π, and faster algorithms give us smaller (tighter and better) upper bounds.

However, to prove that a problem has a certain lower bound, we have to show that every algorithm that solves Π has a worst-case running time $\Omega(f(n))$, or equivalently, that there is no algorithm that solves Π that runs in $o(f(n))$ time. To be more specific, we need to specify what kinds of algorithms we want to consider, which is formally known as model of computation. For example, the comparison model is one model that is used to solve sorting and searching problems.

13.2 Comparison Model

In the comparison model, we consider all input items as black boxes, or more precisely, ADTs. The only operations allowed on the items are comparisons: $<, \leq, >, \geq, =$. Most searching and sorting algorithms we have been looking at so far use the comparison model: heap sort, merge sort, binary search and binary search tree, etc. In the comparison model, we count the number of comparisons and define it as the time cost of the algorithm.

13.3 Decision Tree

13.3.1 Intuition

Any comparison algorithm can be viewed as a tree of all possible comparisons, the outcomes of the comparisons, and the resulting answer. This tree is called a decision tree.

For any particular n,

- *internal node* corresponds to binary decision in the algorithm (in this case, binary comparisons)
- *leaf* corresponds to a possible answer of the problem
- *root-to-leaf path* corresponds to an execution of the algorithm
- *length of the root-to-leaf path* corresponds to the time cost of the execution associated with that path
- *height of the tree* (or depth of the deepest leaf) corresponds to the worst-case running time.

13.3.2 Decision Tree for Searching

In this subsection, we will look at the decision tree for binary search and use it to prove that the lower bound of searching under the comparison model is $\Omega(\log n)$.

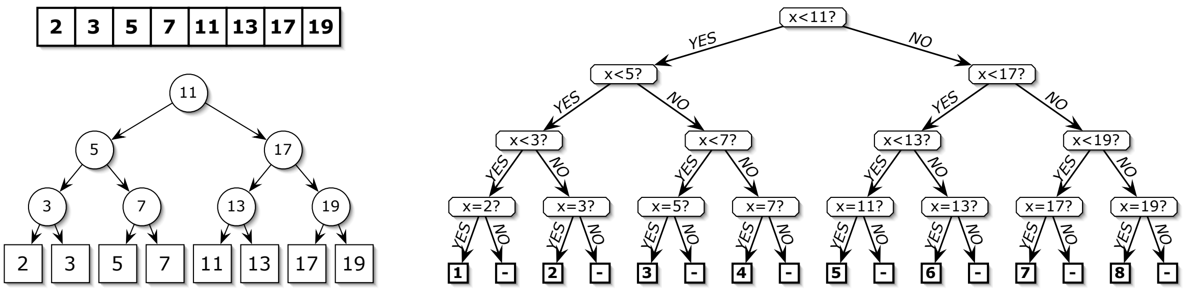


Figure 13.1: <caption>

13.3.3 Decision Tree for Sorting

13.4 Sorting in Linear Time

Lecture 14 Information Theory

Lecture 15 Adversary Arguments

Lecture 16 Reduction

V

Graphs

Lecture 17 Breadth-First Search

17.1 Definition

Definition 17.1.1 — Graphs. A graph is defined as a tuple $G = (V,E)$ where $V$is an arbitrary non-empty finite set, whose elements are called vertices or nodes; and E is a set of pairs of elements of V, which we call edges. For an undirected graph, the edges are unordered pairs u,v. In a directed graph, the edges are ordered pairs (u,v).

Definition 17.1.2 — Neighbors and Degrees. For any edge uv in an undirected graph, we call u neighbor of v and vice versa, and we say that u and v are adjacent. The degree of a node is its number of neighbors.

In directed graphs, for every edge $u \to v$, we call u a predecessor of v, and we call v a successor of u. The in-degree of a vertex is its number of predecessors; the out-degree is its number of sucessors.

Definition 17.1.3 — Subgraphs. A graph $G' = (V',E')$ is a subgraph of $G = (V,E)$ if $V' \subseteq V$ and $E' \subseteq E$. A proper subgraph of G is any subgraph that is not G itself.

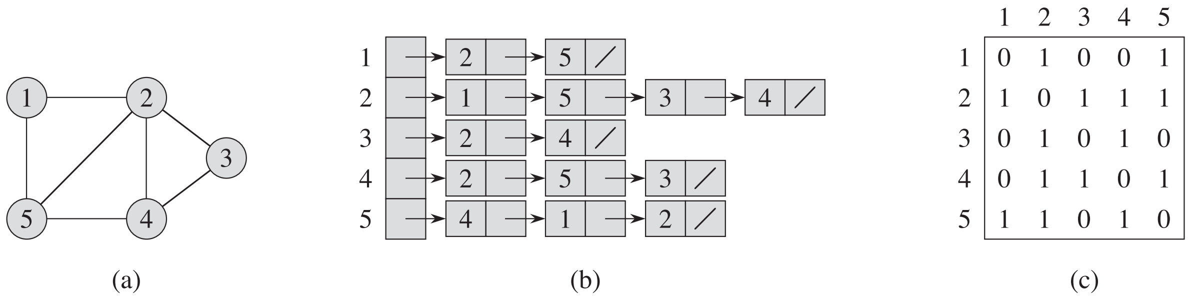
17.2 Representations of Graphs

17.2.1 Adjacency List

Definition 17.2.1 — Adjacency List. The adjacency-list representation of a graph $G = (V,E)$ consists of an arrray *Adj* of $|V|$ lists, one for each vertex in V. For each $u \in V$, the adjacency list *Adj[u]* contains all the vertices v such that there is an edge $(u,v) \in E$. That is, *Adj*[u] contains all the vertices adjacent to u in G.

17.3 Breadth-first Search

Given a graph $G = (V,E)$ and a distinguished source vertex s, breadth-first search systematically explores the edges of G to discover every vertex that is reachable from s. Breadth-first search expands the frontier between disvoered and undiscovered vertices uniformly across the breadth of the frontier. The algorithm discovers all vertices at distance k from s before discovering any vertices at distance $k+1$.



$$\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 \\
4 & 0 & 1 & 1 & 0 & 1 \\
5 & 1 & 1 & 0 & 1 & 0 \\
\hline
\end{array}$$

(a) (b) (c)

Figure 17.1: Representations of undirected graph: (a) the graph; (b) adjacency list of the graph; (c) adjacency matrix of the graph.

Lecture 18 Depth-First Search

Lecture 19 Minimum Spanning Trees

Lecture 20 Bellman-Ford’s Algorithm

Lecture 21 Dijkstra’s Algorithm

Appendix

Commonly Used Axioms & Theorems

Rules of Inference

Axiom 1 — Modus Ponens. $(P \wedge (P \text{ IMPLIES } Q)) \text{ IMPLIES } Q$

Axiom 2 — Modus Tollens. $(\neg Q \wedge (P \text{ IMPLIES } Q)) \text{ IMPLIES } \neg P$

Axiom 3 — Hypothetical Syllogism (transitivity).

$((P \text{ IMPLIES } Q) \wedge (Q \text{ IMPLIES } R)) \text{ IMPLIES } (P \text{ IMPLIES } R)$

Axiom 4 — Disjunctive Syllogism. $((P \vee Q) \wedge \neg P) \text{ IMPLIES } Q$

Axiom 5 — Addition. $P \text{ IMPLIES } (P \vee Q)$

Axiom 6 — Simplification. $(P \wedge Q) \text{ IMPLIES } P$

Axiom 7 — Conjunction. $((P) \wedge (Q)) \text{ IMPLIES } (P \wedge Q)$

Axiom 8 — Resolution. $((P \vee Q) \wedge (\neg P \vee R))$ IMPLIES $(Q \vee R)$

Laws of Logic

Axiom 9 — Implication Law. $(P \text{ IMPLIES } Q) \equiv (\neg P \vee Q)$

Axiom 10 — Distributive Law.

$$(P \wedge (Q \vee R)) \equiv ((P \wedge Q) \vee (P \wedge R))$$
$$(P \vee (Q \wedge R)) \equiv ((P \vee Q) \wedge (P \vee R))$$

Axiom 11 — De Morgan's Law.

$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$$
$$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$$

Axiom 12 — Absorption Law.

$$(P \vee (P \wedge Q)) \equiv P$$
$$(P \wedge (P \vee Q)) \equiv P$$

Axiom 13 — Commutativity of AND. $A \wedge B \equiv B \wedge A$

Axiom 14 — Associativity of AND. $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

Axiom 15 — Identity of AND. $\mathbf{T} \wedge A \equiv A$

Axiom 16 — Zero of AND. $\mathbf{F} \wedge A \equiv \mathbf{F}$

Axiom 17 — Idempotence for AND. $A \wedge A \equiv A$

Axiom 18 — Contradiction for AND. $A \wedge \neg A \equiv \mathbf{F}$

Axiom 19 — Double Negation. $\neg(\neg A) \equiv A$

Axiom 20 — Validity for OR. $A \vee \neg A \equiv \mathbf{T}$

Induction

Axiom 21 — Well Ordering Principle. Every nonempty set of nonnegative integers has a smallest element. i.e., For any $A \subset \mathbb{N}$ such that $A \neq \emptyset$, there is some $a \in A$ such that $\forall a' \in A.a \leq a'$.

Basic Prerequisite Mathematics

Basic Prerequisite Mathematics

SET THEORY

Common Sets

- $\mathbb{N} = \{0, 1, 2, \ldots\}$: the natural numbers, or non-negative integers. The convention in computer science is to include 0 in the natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$: the integers
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$: the positive integers
- $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$: the negative integers
- $\mathbb{Q}$ the rational numbers, $\mathbb{Q}^+$ the positive rationals, and $\mathbb{Q}^-$ the negative rationals.
- $\mathbb{R}$ the real numbers, $\mathbb{R}^+$ the positive reals, and $\mathbb{R}^-$ the negative reals.

Notation

For any sets A and B, we will use the following standard notation.

- $x \in A$: "x is an element of A" or "A contains x"
- $A \subseteq B$: "A is a subset of B" or "A is included in B"
- $A = B$: "A equals B" (Note that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.)
- $A \subsetneq B$: "A is a proper subset of B"
  (Note that $A \subsetneq B$ if and only if $A \subseteq B$ and $A \neq B$.)
- $A \cup B$: "A union B"
- $A \cap B$: "A intersection B"
- $A - B$: "A minus B" (*set* difference)
- $|A|$: "cardinality of A" (the number of elements of A)
- $\varnothing$ or $\{\}$: "the empty set"
- $\mathcal{P}(A)$ or 2^A: "powerset of A" (the set of all subsets of A)
  If $A = \{a, 34, \triangle\}$, then $\mathcal{P}(A) = \big\{\{\}, \{a\}, \{34\}, \{\triangle\}, \{a, 34\}, \{a, \triangle\}, \{34, \triangle\}, \{a, 34, \triangle\}\big\}$.
  $S \in \mathcal{P}(A)$ means the same as $S \subseteq A$.
- $\{x \in A \mid P(x)\}$: "the set of elements x in A for which $P(x)$ is true"
  For example, $\{x \in \mathbb{Z} \mid \cos(\pi x) > 0\}$ represents the set of integers x for which $\cos(\pi x)$ is greater than zero, *i.e.*, it is equal to $\{\ldots, -4, -2, 0, 2, 4, \ldots\} = \{x \in \mathbb{Z} \mid x \text{ is even}\}$.

- $A \times B$: "the cross product or Cartesian product of A and B"
  $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
  If $A = \{1, 2, 3\}$ and $B = \{5, 6\}$, then $A \times B = \{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}$.)

- A^n: "the cross product of n copies of A"
  This is set of all sequences of $n \geq 1$ elements, each of which is in A.

- B^A or $A \to B$: "the set of all functions from A to B."

- $f : A \to B$ or $f \in B^A$: "f is a function from A to B"
  f associates one element $f(x) \in B$ to every element $x \in A$.

NUMBER THEORY

For any two natural numbers a and b, we say that a *divides* b if there exists a natural number c such that $b = ac$. In such a case, we say that a is a *divisor* of b (*e.g.*, 3 is a divisor of 12 but 3 is not a divisor of 16). Note that any natural number is a divisor of 0 and 1 is a divisor of any natural number. A number a is *even* if 2 divides a and is *odd* if 2 does not divide a.

A natural number p is *prime* if it has exactly two positive divisors (*e.g.*, 2 is prime since its positive divisors are 1 and 2 but 1 is **not** prime since it only has one positive divisor: 1). There are an infinite number of prime numbers and any integer greater than one can be expressed in a unique way as a finite product of prime numbers (*e.g.*, $8 = 2^3$, $77 = 7 \times 11$, $3 = 3$).

Inequalities

For any integers m and n, $m < n$ if and only if $m + 1 \leq n$ and $m > n$ if and only if $m \geq n + 1$. For any real numbers w, x, y, and z, the following properties always hold (they also hold when $<$ and $\leq$ are exchanged throughout with $>$ and $\geq$, respectively).

- if $x < y$ and $w \leq z$, then $x + w < y + z$
- if $x < y$, then $\begin{cases} xz < yz & \text{if } z > 0 \\ xz = yz & \text{if } z = 0 \\ xz > yz & \text{if } z < 0 \end{cases}$
- if $x \leq y$ and $y < z$ (or if $x < y$ and $y \leq z$), then $x < z$

Functions

Here are some common number-theoretic functions together with their definitions and properties of them. (Unless noted otherwise, in this section, x and y represent arbitrary real numbers and k, m, and n represent arbitrary positive integers.)

- $\min\{x, y\}$: "minimum of x and y" (the smallest of x or y)

  Properties: $\min\{x, y\} \leq x$
  $\min\{x, y\} \leq y$

- $\max\{x,y\}$: "maximum of x and y" (the largest of x or y)

  Properties: $x \leq \max\{x,y\}$
  $y \leq \max\{x,y\}$

- $\lfloor x \rfloor$: "floor of x" (the greatest integer less than or equal to x, *e.g.*, $\lfloor 5.67 \rfloor = 5$, $\lfloor -2.01 \rfloor = -3$)

  Properties: $x - 1 < \lfloor x \rfloor \leq x$
  $\lfloor -x \rfloor = -\lceil x \rceil$
  $\lfloor x + k \rfloor = \lfloor x \rfloor + k$
  $\lfloor \lfloor k/m \rfloor /n \rfloor = \lfloor k/mn \rfloor$
  $(k - m + 1)/m \leq \lfloor k/m \rfloor$

- $\lceil x \rceil$: "ceiling of x" (the least integer greater than or equal to x, *e.g.*, $\lceil 5.67 \rceil = 6$, $\lceil -2.01 \rceil = -2$)

  Properties: $x \leq \lceil x \rceil < x + 1$
  $\lceil -x \rceil = -\lfloor x \rfloor$
  $\lceil x + k \rceil = \lceil x \rceil + k$
  $\lceil \lceil k/m \rceil /n \rceil = \lceil k/mn \rceil$
  $\lceil k/m \rceil \leq (k + m - 1)/m$

  Additional property of $\lfloor\ \rfloor$ and $\lceil\ \rceil$: $\lfloor k/2 \rfloor + \lceil k/2 \rceil = k$.

- $|x|$: "absolute value of x" ($|x| = x$ if $x \geq 0$; $-x$ if $x < 0$, *e.g.*, $|5.67| = 5.67$, $|-2.01| = 2.01$)

  BEWARE! The same notation is used to represent the cardinality $|A|$ of a set A and the absolute value $|x|$ of a number x so be sure you are aware of the context in which it is used.

- m div n: "the quotient of m divided by n" (integer division of m by n, *e.g.*, $5 \text{ div } 6 = 0$, $27 \text{ div } 4 = 6$, $-27 \text{ div } 4 = -6$)

  Properties: If $m, n > 0$, then $m \text{ div } n = \lfloor m/n \rfloor$
  $(-m) \text{ div } n = -(m \text{ div } n) = m \text{ div } (-n)$

- $m \,\text{rem}\, n$: "the remainder of m divided by n" (*e.g.*, $5 \,\text{rem}\, 6 = 5$, $27 \,\text{rem}\, 4 = 3$, $-27 \,\text{rem}\, 4 = -3$)

  Properties: $m = (m \text{ div } n) \cdot n + m \text{ rem } n$
  $(-m) \text{ rem } n = -(m \text{ rem } n) = m \text{ rem } (-n)$

- $m \bmod n$: "m modulo n" (*e.g.*, $5 \bmod 6 = 5$, $27 \bmod 4 = 3$ $-27 \bmod 4 = 1$)

  Properties: $0 \leq m \bmod n < n$
  n divides $m - (m \bmod n)$.

- $\gcd(m,n)$: "greatest common divisor of m and n" (the largest positive integer that divides both m and n)

  For example, $\gcd(3,4) = 1$, $\gcd(12,20) = 4$, $\gcd(3,6) = 3$

- $\text{lcm}(m,n)$: "least common multiple of m and n" (the smallest positive integer that m and n both divide)

  For example, $\text{lcm}(3,4) = 12$, $\text{lcm}(12,20) = 60$, $\text{lcm}(3,6) = 6$

  Properties: $\gcd(m,n) \cdot \text{lcm}(m,n) = m \cdot n$.

CALCULUS

Limits and Sums

An infinite sequence of real numbers $\{a_n\} = a_1, a_2, \ldots, a_n, \ldots$ *converges* to a limit $L \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $n_0 \geq 0$ such that $|a_n - L| < \varepsilon$ for every $n \geq n_0$. In this case, we write $\lim\limits_{n\to\infty} a_n = L$ Otherwise, we say that the sequence *diverges*.

If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = L_1$ and $\lim\limits_{n\to\infty} b_n = L_2$, then

$$\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2 \quad \text{and} \quad \lim_{n\to\infty}(a_n \cdot b_n) = L_1 \cdot L_2.$$

In particular, if c is any real number, then

$$\lim_{n\to\infty}(c \cdot a_n) = c \cdot L_1.$$

The sum $a_1 + a_2 + \cdots + a_n$ and product $a_1 \cdot a_2 \cdots a_n$ of the finite sequence $a_1, a_2, \ldots, a_n$ are denoted by

$$\sum_{i=1}^{n} a_i \text{ and } \prod_{i=1}^{n} a_i.$$

If the elements of the sequence are all different and $S = \{a_1, a_2, \ldots, a_n\}$ is the set of elements in the sequence, these can also be denoted by

$$\sum_{a\in S} a \text{ and } \prod_{a\in S} a.$$

Examples:

- For any $a \in \mathbb{R}$ such that $-1 < a < 1$, $\lim\limits_{n\to\infty} a^n = 0$.
- For any $a \in \mathbb{R}^+$, $\lim\limits_{n\to\infty} a^{1/n} = 1$.
- For any $a \in \mathbb{R}^+$, $\lim\limits_{n\to\infty} (1/n)^a = 0$.
- $\lim\limits_{n\to\infty} (1 + 1/n)^n = e = 2.71828182845904523536\ldots$

- For any $a, b \in \mathbb{R}$, the *arithmetic* sum is given by:

$$\sum_{i=0}^{n}(a + ib) = (a) + (a + b) + (a + 2b) + \cdots + (a + nb) = \frac{1}{2}(n + 1)(2a + nb).$$

- For any $a, b \in \mathbb{R}^+$, the *geometric* sum is given by:

$$\sum_{i=0}^{n}(ab^i) = a + ab + ab^2 + \cdots + ab^n = \frac{a(1 - b^{n+1})}{1 - b}.$$

EXPONENTS AND LOGARITHMS

Definition: For any $a, b, c \in \mathbb{R}^+$, $a = \log_b c$ if and only if $b^a = c$.

Notation: For any $x \in \mathbb{R}^+$, $\ln x = \log_e x$ and $\lg x = \log_2 x$.

For any $a, b, c \in \mathbb{R}^+$ and any $n \in \mathbb{Z}^+$, the following properties always hold.

- $\sqrt[n]{b} = b^{1/n}$
- $b^a b^c = b^{a+c}$
- $(b^a)^c = b^{ac}$
- $b^a / b^c = b^{a-c}$
- $b^0 = 1$
- $a^b c^b = (ac)^b$
- $b^{\log_b a} = a = \log_b b^a$
- $a^{\log_b c} = c^{\log_b a}$
- $\log_b(ac) = \log_b a + \log_b c$
- $\log_b(a^c) = c \cdot \log_b a$
- $\log_b(a/c) = \log_b a - \log_b c$
- $\log_b 1 = 0$
- $\log_b a = \log_c a / \log_c b$

BINARY NOTATION

A *binary number* is a sequence of bits $a_k \cdots a_1 a_0$ where each bit a_i is equal to 0 or 1. Every binary number represents a natural number in the following way:

$$(a_k \cdots a_1 a_0)_2 = \sum_{i=0}^{k} a_i 2^i = a_k 2^k + \cdots + a_1 2 + a_0.$$

For example, $(1001)_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 1 = 9$, $(01110)_2 = 8 + 4 + 2 = 14$.

Properties:

- If $a = (a_k \cdots a_1 a_0)_2$, then $2a = (a_k \cdots a_1 a_0 0)_2$, *e.g.*, $9 = (1001)_2$ so $18 = (10010)_2$.
- If $a = (a_k \cdots a_1 a_0)_2$, then $\lfloor a/2 \rfloor = (a_k \cdots a_1)_2$, *e.g.*, $9 = (1001)_2$ so $4 = (100)_2$.
- The smallest number of bits required to represent the positive integer n in binary is called the *length* of n and is equal to $\lceil \log_2(n+1) \rceil$.

Make sure you know how to add and multiply two binary numbers. For example, $(1111)_2 + (101)_2 = (10100)_2$ and $(1111)_2 \times 101)_2 = (1001011)_2$.

Proof Templates

Proof Outlines

LINE NUMBERS: Only lines that are referred to have labels (for example, L1) in this document. For a formal proof, all lines are numbered. Line numbers appear at the beginning of a line. You can indent line numbers together with the lines they are numbering or all line numbers can be unindented, provided you are consistent.

INDENTATION: Indent when you make an assumption or define a variable. Unindent when this assumption or variable is no longer being used.

1. **Implication**: Direct proof of A IMPLIES B.

   > L1. Assume A.
   >
   > $\vdots$
   >
   > L2. B

   A IMPLIES B; direct proof: L1, L2

2. **Implication**: Indirect proof of A IMPLIES B.

   > L1. Assume $NOT(B)$.
   >
   > $\vdots$
   >
   > L2. $NOT(A)$

   A IMPLIES B; indirect proof: L1, L2

3. **Equivalence**: Proof of A IFF B.

   > L1. Assume A.
   >
   > $\vdots$
   >
   > L2. B

   L3. A IMPLIES B; direct proof: L1, L2

   > L4. Assume B.
   >
   > $\vdots$
   >
   > L5. A

   L6. B IMPLIES A; direct proof: L4, L5

   A IFF B; equivalence: L3, L6

4. **Proof by contradiction** of A.

   > L1. To obtain a contradiction, assume $NOT(A)$.
   >
   > $\vdots$
   >
   > > L2. B
   >
   > $\vdots$
   >
   > > L3. $NOT(B)$
   >
   > L4. This is a contradiction: L2, L3

   Therefore A; proof by contradiction: L1, L4

5. **Modus Ponens**.

   ⋮
   L1. A
   ⋮
   L2. A IMPLIES B
   B; modus ponens: L1, L2

6. **Conjunction:** Proof of A AND B:

   ⋮
   L1. A
   ⋮
   L2. B
   A AND B; proof of conjunction; L1, 2

7. **Use of Conjunction**:

   ⋮
   L1. A AND B
   A; use of conjunction: L1
   B; use of conjunction: L1

8. **Implication with Conjunction**: Proof of (A_1 AND A_2) IMPLIES B.

      L1. Assume A_1 AND A_2.
      A_1; use of conjunction, L1
      A_2; use of conjunction, L1
      ⋮
      L2. B

   (A_1 AND A_2) IMPLIES B; direct proof, L1, L2

9. **Implication with Conjunction**: Proof of A IMPLIES (B_1 AND B_2).

      L1. Assume A.
      ⋮
      L2. B_1
      ⋮
      L3. B_2
      L4. B_1 AND B_2; proof of conjunction: L2, L3

   A IMPLIES (B_1 AND B_2); direct proof: L1, L4

10. **Disjunction**: Proof of A OR B and B OR A.

    ⋮
    L1. A
    A OR B; proof of disjunction: L1
    B OR A; proof of disjunction: L1

11. **Proof by cases**.

L1. C OR $NOT(C)$ tautology
L2. Case 1: Assume C.
$\vdots$
L3. A
L4. C IMPLIES A; direct proof: L2, L3
L5. Case 2: Assume $NOT(C)$.
$\vdots$
L6. A
L7. $NOT(C)$ IMPLIES A; direct proof: L5, L6
A proof by cases: L1, L4, L7

12. **Proof by cases** of A OR B.

L1. C OR $NOT(C)$ tautology
L2. Case 1: Assume C.
$\vdots$
L3. A
L4. A OR B; proof of disjunction, L3
L5. C IMPLIES (A OR B); direct proof, L2, L4
L6. Case 2: Assume $NOT(C)$.
$\vdots$
L7. B
L8. A OR B; proof of disjunction, L7
L9. $NOT(C)$ IMPLIES (A OR B); direct proof: L6, L8
A OR B; proof by cases: L1, L5, L9

13. **Implication with Disjunction**: Proof by cases of (A_1 OR A_2) IMPLIES B.

L1. Case 1: Assume A_1.
$\vdots$
L2. B
L3. A_1 IMPLIES B; direct proof: L1,L2
L4. Case 2: Assume A_2.
$\vdots$
L5. B
L6. A_2 IMPLIES B; direct proof: L4, L5
(A_1 OR A_2) IMPLIES B; proof by cases: L3, L6

14. **Implication with Disjunction**: Proof by cases of A IMPLIES (B_1 OR B_2).

    L1. Assume A.
    L2. C OR NOT(C) tautology
    L3. Case 1: Assume C.
    $\vdots$
    L4. B_1
    L5. B_1 OR B_2; disjunction: L4
    L6. C IMPLIES (B_1 OR B_2); direct proof: L3, L5
    L7. Case 2: AssumeNOT(C).
    $\vdots$
    L8. B_2
    L9. B_1 OR B_2; disjunction: L8
    L10. NOT(C) IMPLIES (B_1 OR B_2); direct proof: L7, L9
    L11. B_1 OR B_2; proof by cases: L2, L6, L10
    A IMPLIES (B_1 OR B_2): direct proof. L1, L11

15. **Substitution of a Variable in a Tautology**:
    Suppose P is a propositional variable, Q is a formula, and R' is obtained from R by replacing *every* occurrence of P by (Q).

    L1. R tautology
    R'; substitution of all P by Q: L1

16. **Substitution of a Formula by a Logically Equivalent Formula**:
    Suppose S is a subformula of R and R' is obtained from R by replacing *some* occurrence of S by S'.

    L1. R
    L2. S IFF S'
    L3. R'; substitution of an occurrence of S by S': L1, L2

17. **Specialization**:

    L1. $c \in D$
    L2. $\forall x \in D.P(x)$
    $P(c)$; specialization: L1, L2

18. **Generalization**: Proof of $\forall x \in D.P(x)$.

    L1. Let x be an arbitrary element of D.
    $\vdots$
    L2. $P(x)$
    Since x is an arbitrary element of D,
    $\forall x \in D.P(x)$; generalization: L1, L2

19. **Universal Quantification with Implication**: Proof of $\forall x \in D.(P(x) \text{ IMPLIES } Q(x))$.

    L1. Let x be an arbitrary element of D.
        L2. Assume $P(x)$
        $\vdots$
        L3. $Q(x)$
    L4. $P(x)$ IMPLIES $Q(x)$; direct proof: L2, L3
    Since x is an arbitrary element of D,
    $\forall x \in D.(P(x) \text{ IMPLIES } Q(x))$; generalization: L1, L4

20. **Implication with Universal Quantification**: Proof of $(\forall x \in D.P(x))$ IMPLIES A.

    L1. Assume $\forall x \in D.P(x)$.
    $\vdots$
    L2. $a \in D$
    $P(a)$; specialization: L1, L2
    $\vdots$
    L3. A
    Therefore $(\forall x \in D.P(x))$ IMPLIES A; direct proof: L1, L3

21. **Implication with Universal Quantification**: Proof of A IMPLIES $(\forall x \in D.P(x))$.

    L1. Assume A.
        L2. Let x be an arbitrary element of D.
        $\vdots$
        L3. $P(x)$
    Since x is an arbitrary element of D,
    L4. $\forall x \in D.P(x)$; generalization, L2, L3
    A IMPLIES $(\forall x \in D.P(x))$; direct proof: L1, L4

22. **Instantiation**:

    L1. $\exists x \in D.P(x)$
    Let $c \in D$ be such that $P(c)$; instantiation: L1
    $\vdots$

23. **Construction**: Proof of $\exists x \in D.P(x)$.

    L1. Let $a = \cdots$
    $\vdots$
    L2. $a \in D$
    $\vdots$
    L3. $P(a)$
    $\exists x \in D.P(x)$; construction: L1, L2, L3

24. **Existential Quantification with Implication**: Proof of $\exists x \in D.(P(x)$ IMPLIES $Q(x))$.

> L1. Let $a = \cdots$
>
> $\vdots$
>
> L2. $a \in D$
>
>> L3. Suppose $P(a)$.
>>
>> $\vdots$
>>
>> L4. $Q(a)$
>
> L5. $P(a)$ IMPLIES $Q(a)$; direct proof: L3, L4

$\exists x \in D.(P(x)$ IMPLIES $Q(x))$; construction: L1, L2, L5

25. **Implication with Existential Quantification**: Proof of $(\exists x \in D.P(x))$ IMPLIES A.

> L1. Assume $\exists x \in D.P(x)$.
>
>> Let $a \in D$ be such that $P(a)$; instantiation: L1
>>
>> $\vdots$
>>
>> L2. A

$(\exists x \in D.P(x))$ IMPLIES A; direct proof: L1, L2

26. **Implication with Existential Quantification**: Proof of A IMPLIES $(\exists x \in D.P(x))$.

> L1. Assume A.
>
>> L2. Let $a = \cdots$
>>
>> $\vdots$
>>
>> L3. $a \in D$
>>
>> $\vdots$
>>
>> L4. $P(a)$
>
> L5. $\exists x \in D.P(x)$; construction: L2, L3, L4

A IMPLIES $(\exists x \in D.P(x))$; direct proof: L1, L5

27. **Subset**: Proof of $A \subseteq B$.

> L1. Let $x \in A$ be arbitrary.
>
> $\vdots$
>
> L2. $x \in B$
>
> *The following line is optional:*
>
> L3. $x \in A$ IMPLIES $x \in B$; direct proof: L1, L2

$A \subseteq B$; definition of subset: L3 (or L1, L2, if the optional line is missing)

28. **Weak Induction**: Proof of $\forall n \in N.P(n)$

```
Base Case:
$\vdots$
L1. $P(0)$
    L2. Let $n \in N$ be arbitrary.
        L3. Assume $P(n)$.
        $\vdots$
        L4. $P(n+1)$
    *The following two lines are optional:*
    L5. $P(n)$ IMPLIES $(P(n+1)$; direct proof of implication: L3, L4
L6. $\forall n \in N.(P(n)$ IMPLIES $P(n+1))$; generalization L2, L5
$\forall n \in N.P(n)$ induction; L1, L6 (or L1, L2, L3, L4, if the optional lines are missing)
```

29. **Strong Induction**: Proof of $\forall n \in N.P(n)$

```
    L1. Let $n \in N$ be arbitrary.
        L2. Assume $\forall j \in N.(j < n$ IMPLIES $P(j))$
        $\vdots$
        L3. $P(n)$
    *The following two lines are optional:*
    L4. $\forall j \in N.(j < n$ IMPLIES $P(j))$ IMPLIES $P(n)$; direct proof of implication: L2, L3
L5. $\forall n \in N.[\forall j \in N.(j < n$ IMPLIES $P(j))$ IMPLIES $P(n)]$; generalization: L1, L4
$\forall n \in N.P(n)$; strong induction: L5 (or L1, L2, L3, if the optional lines are missing)
```

30. **Structural Induction**: Proof of $\forall e \in S.P(e)$, where S is a recursively defined set

```
Base case(s):
    L1. For each base case $e$ in the definition of $S$
    L2. $P(e)$.
Constructor case(s):
    L3. For each constructor case $e$ of the definition of $S$,
        L4. assume $P(e')$ for all components $e'$ of $e$.
        $\vdots$
        L5. $P(e)$
$\forall e \in S.P(e)$; structural induction: L1, L2, L3, L4, L5
```

31. **Well Ordering Principle**: Proof of $\forall e \in S.P(e)$, where S is a well ordered set, i.e. every nonempty subset of S has a smallest element.

    - L1. To obtain a contradiction, suppose that $\forall e \in S.P(e)$ is false.
    - L2. Let $C = \{e \in S \mid P(e) \text{ is false}\}$ be the set of counterexamples to P.
    - L3. $C \neq \phi$; definition: L1, L2
        - L4. Let e be the smallest element of C; well ordering principle: L2, L3
            - Let $e' = \cdots$
            - $\vdots$
            - L5. $e' \in C$
            - $\vdots$
            - L6. $e' < e$.
    - L7. This is a contradiction: L4, L5, L6

    $\forall e \in S.P(e)$; proof by contradiction: L1, L7

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Tutorial