Note that $R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$, $R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}$, where $c_{ij} = \cos(\theta_{ij})$, $s_{ij} = \sin(\theta_{ij})$

By definition

$$U_{TBM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} * \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/3} & \sqrt{2/3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Assuming there is no phase in the PMNS matrix,
and using central values from PDG : $\sin^2(\theta_{12}) = 0.307$, $\sin^2(\theta_{23}) = 0.50$, $\sin^2(\theta_{13}) = 0.021$, we get

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} * \begin{pmatrix} \sqrt{.979} & 0 & \sqrt{.021} \\ 0 & 1 & 0 \\ -\sqrt{.021} & 0 & \sqrt{.979} \end{pmatrix} * \begin{pmatrix} \sqrt{.693} & \sqrt{.307} & 0 \\ -\sqrt{.307} & \sqrt{.693} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0.824 & 0.548 & 0.145 \\ -0.477 & 0.532 & 0.700 \\ 0.306 & -0.645 & 0.700 \end{pmatrix}$$

Now we can calculate $U^{(-1)} = U_{TBM} * U_{PMNS}{}^{\dagger} = \begin{pmatrix} 0.989 & -0.082 & -0.122 \\ 0.083 & 0.997 & -0.003 \\ 0.122 & -0.007 & 0.992 \end{pmatrix}$

Decompose $U^{(-1)} = R_{23} * R_{13} * R_{12}$, we get $\boldsymbol{\phi_{12}} = -0.083 \approx -\lambda/3$,
$\boldsymbol{\phi_{13}} = -0.123 \approx -\lambda/2$, $\boldsymbol{\phi_{23}} = -0.003 \approx -\lambda^4$
We can write $U^{(-1)}$ in terms of λ

$$U^{(-1)} = R_{23}[-\lambda^4] * R_{13}[-\tfrac{\lambda}{2}] * R_{12}[-\tfrac{\lambda}{3}] = \begin{pmatrix} 1 - \frac{13\lambda^2}{72} & -\frac{\lambda}{3} & -\frac{\lambda}{2} \\ \frac{\lambda}{3} & 1 - \frac{\lambda^2}{18} & -\lambda^4 \\ \frac{\lambda}{2} & -\frac{\lambda^2}{6} + \lambda^4 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O(\lambda^5)$$

The CKM Matrix is given by

$$U_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda - \frac{\lambda^3}{6} & A * \lambda^3 * (\rho - i * \eta) \\ -\lambda + \frac{\lambda^3}{6} & 1 - \frac{\lambda^2}{2} & A * \lambda^2 \\ A * \lambda^3 (1 - \rho - i * \eta) & -A * \lambda^2 & 1 \end{pmatrix} + O(\lambda^5).$$

Assume A~O(1), ρ~O(λ), η~O(λ), and it is true since A= 0.811 ± 0.026, $\rho = 0.124^{+0.019}_{-0.018}$, $\eta = 0.356 \pm 0.011$(via PDG).

The Yukawa mass matrix of $-\frac{1}{3}$ quarks and charged leptons can be decomposed as

$\mathbf{Y}^{\left(-\frac{1}{3}\right)} = \mathbf{U}^{\left(-\frac{1}{3}\right)}\,\mathbf{D}^{\left(-\frac{1}{3}\right)}\,\mathbf{V}^{\left(-\frac{1}{3}\right)\dagger}$,

$\mathbf{Y}^{(-1)} = \mathbf{U}^{(-1)}\,\mathbf{D}^{(-1)}\,\mathbf{V}^{(-1)\dagger}$.

In the basis where $\mathbf{Y}^{\left(\frac{2}{3}\right)}$ is diagonal, $U^{\left(-\frac{1}{3}\right)} = U_{\text{CKM}}$.

$\mathbf{D}^{\left(-\frac{1}{3}\right)} = \begin{pmatrix} \frac{\lambda^4}{3} & 0 & 0 \\ 0 & \frac{\lambda^2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{D}^{(-1)} = \begin{pmatrix} \frac{\lambda^4}{9} & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are the mass eigenvalues with arbitrary choices of sign.

Based on the SO(10) GUT, $\mathbf{Y}^{\left(-\frac{1}{3}\right)}$ and $\mathbf{Y}^{(-1)}$ are related by

$\mathbf{Y}^{\left(-\frac{1}{3}\right)} = Y_5 + Y_{45}$,

$\mathbf{Y}^{(-1)} = Y_5^{\mathsf{T}} - 3\,Y_{45}^{\mathsf{T}}$.

where Y_5 is from the 5-Higgs coupling and Y_{45} is from the 45-Higgs coupling.

If we choose $\mathbf{V}^{\left(-\frac{1}{3}\right)} = \mathbf{U}^{(-1)*}$, $\mathbf{V}^{(-1)} = \mathbf{U}_{\text{CKM}}{}^{*}$, and $\mathbf{D}^{\left(-\frac{1}{3}\right)} = \begin{pmatrix} \frac{\lambda^4}{3} & 0 & 0 \\ 0 & \frac{\lambda^2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D}^{(-1)} = \begin{pmatrix} -\frac{\lambda^4}{9} & 0 & 0 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, there is a very simple and interesting pattern:

$$Y_5 = \left(3 * \mathbf{Y}^{\left(-\frac{1}{3}\right)} + \mathbf{Y}^{(-1)\,\mathsf{T}}\right) \Big/ 4$$

$$= \left[3 * \mathbf{U}_{\text{CKM}}\,\mathbf{D}^{\left(-\frac{1}{3}\right)}\,\mathbf{U}^{(-1)\,\mathsf{T}} + \left(\mathbf{U}^{(-1)}\,\mathbf{D}^{(-1)}\,\mathbf{U}_{\text{CKM}}{}^{\mathsf{T}}\right)^{\mathsf{T}}\right] \Big/ 4$$

$$= \left[\mathbf{U}_{\text{CKM}} * 3\,\mathbf{D}^{\left(-\frac{1}{3}\right)}\,\mathbf{U}^{(-1)\,\mathsf{T}} + \mathbf{U}_{\text{CKM}}\,\mathbf{D}^{(-1)}\,\mathbf{U}^{(-1)\,\mathsf{T}}\right] \Big/ 4$$

$$= U_{\text{CKM}} * \frac{3\,D^{\left(-\frac{1}{3}\right)} + D^{(-1)}}{4} * U^{(-1)\mathsf{T}}$$

$$= \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda - \frac{\lambda^3}{6} & A * \lambda^3 * (\rho - i * \eta) \\ -\lambda + \frac{\lambda^3}{6} & 1 - \frac{\lambda^2}{2} & A * \lambda^2 \\ A * \lambda^3 (1 - \rho - i * \eta) & -A * \lambda^2 & 1 \end{pmatrix} * \begin{pmatrix} \frac{2\lambda^4}{9} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 - \frac{13\lambda^2}{72} & \frac{\lambda}{3} & \frac{\lambda}{2} \\ -\frac{\lambda}{3} & 1 - \frac{\lambda^2}{18} & -\frac{\lambda^2}{6} + \lambda^4 \\ -\frac{\lambda}{2} & -\lambda^4 & 1 - \frac{\lambda^2}{8} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\lambda^4}{9} & -\frac{A\lambda^3}{2} & -\frac{\lambda}{2} \\ 0 & 0 & -\lambda^4 \\ A\,(\rho - i\eta)\,\lambda^3 & A\,\lambda^2 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O\left(\lambda^5\right),$$

$$Y_{45} = (\mathbf{Y}^{\left(-\frac{1}{3}\right)} - \mathbf{Y}^{(-1)\,\mathsf{T}})/4 = U_{\text{CKM}} * \left(\frac{D^{\left(-\frac{1}{3}\right)} - D^{(-1)}}{4}\right) * U^{(-1)\mathsf{T}} = U_{\text{CKM}} * \begin{pmatrix} \frac{\lambda^4}{9} & 0 & 0 \\ 0 & \frac{\lambda^2}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} * U^{(-1)\mathsf{T}}$$

$$= \begin{pmatrix} 0 & -\frac{\lambda^3}{9} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{27} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O\left(\lambda^5\right).$$

Note that λ=0.23≈$\frac{1}{5}$, terms like $\frac{\lambda^4}{5}$, $\frac{\lambda^3}{25}$(or even larger numerators) are treated as $O\left(\lambda^5\right)$.

11 / 30 / 2017