Perturbation theory on Yukawa matrices

Suppose the Yukawa matrices have the following form.

$$Y^{(-\frac{1}{3})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A & b \\ c & 0 & d \end{pmatrix} \tag{1}$$

$$Y^{(-1)} = \begin{pmatrix} 0 & 0 & c \\ 0 & -3A & 0 \\ 0 & b & d \end{pmatrix} \tag{2}$$

where A is the **45** coupling and others are **5** couplings. We assume that $d \gg b, c, A$. Then we can normalize the matrices by d and take $d = 1$. Then,

$$Y^{-(\frac{1}{3})} Y^{-(\frac{1}{3})\intercal} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A^2 + b^2 & bd \\ 0 & bd & c^2 + d^2 \end{pmatrix} \tag{3}$$

$$Y^{(-1)} Y^{(-1)\intercal} = \begin{pmatrix} c^2 & 0 & cd \\ 0 & 9A^2 & -3Ab \\ cd & -3Ab & b^2 + d^2 \end{pmatrix} \tag{4}$$

Eigenvalues of $Y^{(-\frac{1}{3})}\ Y^{(-\frac{1}{3})\intercal}$ are

$$m_d^2 = 0; \tag{5}$$

$$m_s^2 = \frac{1}{2}\left[1 + A^2 + b^2 + c^2 - \sqrt{\left(1 - A^2 - b^2 + c^2\right)^2 + 4b^2}\right] \tag{6}$$

$$m_b^2 = \frac{1}{2}\left[1 + A^2 + b^2 + c^2 + \sqrt{\left(1 - A^2 - b^2 + c^2\right)^2 + 4b^2}\right] \tag{7}$$

Eigenvalues of $Y^{(-1)}\ Y^{(-1)\intercal}$ are obtained by replacing $A \to -3A$.

$$m_e^2 = 0 \tag{8}$$

$$m_\mu^2 = \frac{1}{2}\left[1 + 9A^2 + b^2 + c^2 - \sqrt{\left(1 - 9A^2 - b^2 + c^2\right)^2 + 4b^2}\right] \tag{9}$$

$$m_\tau^2 = \frac{1}{2}\left[1 + 9A^2 + b^2 + c^2 + \sqrt{\left(1 - 9A^2 - b^2 + c^2\right)^2 + 4b^2}\right] \tag{10}$$

Writing

$$\alpha = A^2 + b^2 + c^2 \tag{11}$$
$$\beta = A^2 + b^2 - c^2 \tag{12}$$
$$\alpha' = 9A^2 + b^2 + c^2 \tag{13}$$
$$\beta' = 9A^2 + b^2 - c^2 \tag{14}$$

we can rewrite the eigenvalues as

$$m_s^2 = \frac{1+\alpha}{2} - \sqrt{(1-\beta)^2 + 4b^2} \approx \frac{1}{2}(\alpha+\beta) - b^2 = A^2 \tag{15}$$
$$m_b^2 = \frac{1+\alpha}{2} + \sqrt{(1-\beta)^2 + 4b^2} \approx 1 + \frac{1}{2}(\alpha-\beta) + b^2 = 1 + c^2 + b^2 \tag{16}$$
$$m_\mu^2 = \frac{1+\alpha'}{2} - \sqrt{(1-\beta')^2 + 4b^2} \approx \frac{1}{2}(\alpha'+\beta') - b^2 = 9A^2 \tag{17}$$
$$m_\tau^2 = \frac{1+\alpha'}{2} + \sqrt{(1-\beta')^2 + 4b^2} \approx 1 + \frac{1}{2}(\alpha'-\beta') + b^2 = 1 + c^2 + b^2 \tag{18}$$

where $\alpha, \beta, \alpha', \beta' \ll 1$. Then, the expected mass relations

$$m_b = m_\tau \tag{19}$$
$$3m_s = m_\mu \tag{20}$$

are automatically satisfied for the heavier two generations.

Our first objective is to check the PMNS angles for this setup.

Suppose the eigenvectors of $Y^{-1}Y^{(-1)\intercal}$ have the form $(X, Y, Z)^\intercal$.

For eigenvalue $m_e^2 = 0$:

$$c^2X + cZ = 0$$
$$9A^2Y - 3AbZ = 0$$

Taking $X = 1$, $Z = -c$ and $Y = \dfrac{bc}{-3A}$. Thus the eigenvector is

$$|\psi_e\rangle = \frac{1}{\sqrt{N_e}}\left(1, \frac{bc}{-3A}, -c\right)^\intercal \tag{21}$$

where N_e is a normalizing factor.

For eigenvalue m_μ^2:

$$c^2X + cZ = m_\mu^2 X$$
$$9A^2Y - 3AbZ = m_\mu^2 Y$$

Taking $Y = 1$, $Z = \dfrac{9A^2 - m_\mu^2}{3Ab}$ and $X = \dfrac{(9A^2 - m_\mu^2)c}{3Ab(m_\mu^2 - c^2)}$. Thus the eigenvector is

$$|\psi_\mu\rangle = \frac{1}{\sqrt{N_\mu}}\left(\frac{(9A^2 - m_\mu^2)c}{3Ab(m_\mu^2 - c^2)}, 1, \frac{9A^2 - m_\mu^2}{3Ab}\right)^\intercal \tag{22}$$

where N_μ is a normalizing factor.

For eigenvalue m_τ^2, the eigenvector is same as for m_μ^2 with $m_\mu \to m_\tau$. However, we will represent in a different form.

$$|\psi_\tau\rangle = \frac{1}{\sqrt{N_\tau}} \left(\frac{c}{m_\tau^2 - c^2}, \frac{3Ab}{9A^2 - m_\tau^2}, 1 \right)^\intercal \tag{23}$$

where N_τ is a normalizing factor.

Then, the diagonalizing matrix

$$U^{(-1)} = \begin{pmatrix} \frac{1}{\sqrt{N_e}} & \frac{1}{\sqrt{N_\mu}} \frac{(9A^2 - m_\mu^2)c}{3Ab(m_\mu^2 - c^2)} & \frac{1}{\sqrt{N_\tau}} \frac{c}{(m_\tau^2 - c^2)} \\ -\frac{1}{\sqrt{N_e}} \frac{bc}{3A} & \frac{1}{\sqrt{N_\mu}} & \frac{1}{\sqrt{N_\tau}} \frac{3Ab}{(9A^2 - m_\tau^2)} \\ -c\frac{1}{\sqrt{N_e}} & \frac{1}{\sqrt{N_\mu}} \frac{(9A^2 - m_\mu^2)}{3Ab} & \frac{1}{\sqrt{N_\tau}} \end{pmatrix} \tag{24}$$

For a real symmetric matrix $Y^{(-1)}Y^{(-1)\intercal}$, the eigenvectors are orthogonal. [1] Since we have normalized the eigenvectors, they are orthonormal. Therefore $U^{(-1)}$ has orthonormal column vectors. This implies that the row vectors of $U^{(-1)}$ are orthonormal. [2] Hence $U^{(-1)}$ is a orthogonal matrix, $U^{(-1)}U^{(-1)\intercal} = U^{(-1)\intercal}U^{(-1)} = I$.

Comparing (24) with the standard form

$$U^{(-1)} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix} \tag{25}$$

we get

$$\begin{aligned} \frac{U^{(-1)}(2,3)}{U^{(-1)}(3,3)} &= t_{23} = \frac{3Ab}{9A^2 - m_\tau^2} \\ \theta_{23}^{(-1)} &= \tan^{-1} \left(\frac{3Ab}{9A^2 - m_\tau^2} \right) \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{U^{(-1)}(2,3)}{U^{(-1)}(1,3)} &= \frac{s_{23}}{t_{13}} = \frac{3Ab\,(m_\tau^2 - c^2)}{c\,(9A^2 - m_\tau^2)} \\ \theta_{13}^{(-1)} &= \tan^{-1} \left(\frac{\sin\theta_{23}^{(-1)} c\,(9A^2 - m_\tau^2)}{3Ab\,(m_\tau^2 - c^2)} \right) \end{aligned} \tag{27}$$

[1] For any real matrix A and any vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T \mathbf{y} \rangle.$$

Now assume that A is symmetric, and $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors of A corresponding to distinct eigenvalues λ and μ. Then

$$\lambda\langle \mathbf{x}, \mathbf{y} \rangle = \langle \lambda\mathbf{x}, \mathbf{y} \rangle = \langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^T\mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mu\mathbf{y} \rangle = \mu\langle \mathbf{x}, \mathbf{y} \rangle.$$

Therefore, $(\lambda - \mu)\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Since $\lambda - \mu \neq 0$, then $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, i.e., $\mathbf{x} \perp \mathbf{y}$

[2] For any real matrix A with orthonormal column vectors, $A^\intercal A = I$. A is invertible by the 'Invertible Matrix Theorem'. Then, $A^\intercal = A^{-1}$. Now, $AA^\intercal = AA^{-1} = I = (A^\intercal)^\intercal(A^\intercal)$. Thus the column vectors of $A^\intercal$, which are the row vectors of A, are orthonormal.

And

$$\frac{U^{(-1)}(1,3)}{U^{(-1)}(1,1)} = \frac{t_{12}s_{23}}{c_{13}} - c_{23}t_{13} = -c$$
$$\theta_{12}^{(-1)} = \tan^{-1}\left(\frac{(-c+c_{23}t_{13})c_{13}}{s_{23}}\right) \tag{28}$$

Eqns (26), (27), (28) have been derived without any new assumption. Now, assume that the angles are very small. Hence we can approximate $\theta \approx \sin^{-1}\theta \approx \tan^{-1}\theta$ and $\cos\theta \approx 1$. Using the approximate relation $m_\tau^2 \approx 1 + b^2 + c^2$, (26) gives

$$\theta_{23}^{(-1)} \approx \frac{3Ab}{9A^2 - 1 - b^2 - c^2} \tag{29}$$

and (27) gives

$$\theta_{13}^{(-1)} \approx \frac{c}{1+b^2} \tag{30}$$

Finally (28) gives

$$\theta_{12}^{(-1)} \approx \frac{-b^2c(9A^2 - 1 - b^2 - c^2)}{3Ab(1+b^2)} \tag{31}$$

Using Bin's values after normalizing each entry by $d = 0.9925$, we get from (26), (27) and (28)

$$\theta_{23}^{(-1)} = -0.0022$$
$$\theta_{13}^{(-1)} = -0.122$$
$$\theta_{12}^{(-1)} = -0.102 \tag{32}$$

If we use the approximate relations in (29), (30) and (31), we get $\theta_{23}^{(-1)} \approx -0.0022, \theta_{13}^{(-1)} \approx -0.123$ and $\theta_{12}^{(-1)} \approx -0.104$. These are in excellent agreement with the exact values, justifying our approximation. All of these angles are small indeed, so this is expected.

Now

$$U_{PMNS} = U^{(-1)\intercal} U_{TBM}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{N_e}} & \frac{1}{\sqrt{N_\mu}} \frac{(9A^2 - m_\mu^2)c}{3Ab(m_\mu^2 - c^2)} & \frac{1}{\sqrt{N_\tau}} \frac{c}{(m_\tau^2 - c^2)} \\ -\frac{1}{\sqrt{N_e}} \frac{bc}{3A} & \frac{1}{\sqrt{N_\mu}} & \frac{1}{\sqrt{N_\tau}} \frac{3Ab}{(9A^2 - m_\tau^2)} \\ -c\frac{1}{\sqrt{N_e}} & \frac{1}{\sqrt{N_\mu}} \frac{(9A^2 - m_\mu^2)}{3Ab} & \frac{1}{\sqrt{N_\tau}} \end{pmatrix}^\intercal \cdot \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{N_e}} \left(\sqrt{\frac{2}{3}} + \frac{bc}{3\sqrt{6}A} - \frac{c}{\sqrt{6}} \right) & \frac{1}{\sqrt{N_e}} \left(\frac{1}{\sqrt{3}} - \frac{bc}{3\sqrt{3}A} + \frac{c}{\sqrt{3}} \right) & \frac{1}{\sqrt{N_e}} \left(-\frac{bc}{3\sqrt{2}A} - \frac{c}{\sqrt{2}} \right) \\ & & \frac{1}{\sqrt{N_\mu}} \left(\frac{1}{\sqrt{2}} + \frac{9A^2 - m_\mu^2}{3\sqrt{2}Ab} \right) \\ & & \frac{1}{\sqrt{N_\tau}} \left(\frac{-3Ab}{\sqrt{2}(m_\tau^2 - 9A^2)} + \frac{1}{\sqrt{2}} \right) \end{pmatrix} \tag{33}$$

The standard form of U_{PMNS} is same as (25). Comparing the above matrix to the standard form, we get

$$U_{PMNS}(1,3) = \sin\theta_{13} = \frac{1}{\sqrt{N_e}} \left(-\frac{bc}{3\sqrt{2}A} - \frac{c}{\sqrt{2}} \right) = -\frac{c}{\sqrt{2N_e}} \left(\frac{b}{3A} + 1 \right) \tag{34}$$

$$\frac{U_{PMNS}(1,2)}{U_{PMNS}(1,1)} = \tan\theta_{12} = \frac{\frac{1}{\sqrt{3}} - \frac{bc}{3\sqrt{3}A} + \frac{c}{\sqrt{3}}}{\sqrt{\frac{2}{3}} + \frac{bc}{3\sqrt{6}A} - \frac{c}{\sqrt{6}}}$$
$$= \frac{\sqrt{2}\left(1 + c\left(1 - \frac{b}{3A}\right)\right)}{2 - c\left(1 - \frac{b}{3A}\right)} \tag{35}$$

$$\frac{U_{PMNS}(2,3)}{U_{PMNS}(3,3)} = \tan\theta_{23} = \frac{\frac{1}{\sqrt{N_\mu}} \left(\frac{1}{\sqrt{2}} + \frac{9A^2 - m_\mu^2}{3\sqrt{2}Ab} \right)}{\frac{1}{\sqrt{N_\tau}} \left(\frac{-3Ab}{\sqrt{2}(m_\tau^2 - 9A^2)} + \frac{1}{\sqrt{2}} \right)}$$
$$= \frac{\sqrt{N_\tau}\left(1 + \frac{9A^2 - m_\mu^2}{3Ab}\right)}{\sqrt{N_\mu}\left(1 + \frac{3Ab}{9A^2 - m_\tau^2}\right)} \tag{36}$$

We can use the approximation $m_\mu^2 \approx 9A^2$ and $m_\tau^2 \approx 1 + b^2 + c^2$ to simplify this expression.

$$\tan\theta_{23} \approx \frac{\sqrt{N_\tau}}{\sqrt{N_\mu}\left(1+\frac{3Ab}{9A^2-1-b^2-c^2}\right)}$$
$$= \frac{\sqrt{N_\tau}}{\sqrt{N_\mu}\left(1+\frac{\frac{b}{3A}}{1-\frac{1+c^2}{9A^2}-\left(\frac{b}{3A}\right)^2}\right)} \tag{37}$$

Interestingly, all the PMNS angles can be expressed in terms of $\frac{b}{3A}$. Also, using (29), we see that

$$\tan\theta_{23} \approx \frac{\sqrt{N_\tau}}{\sqrt{N_\mu}\left(1+\theta_{23}^{(-1)}\right)} \tag{38}$$

which shows how the PMNS angle θ_{23} is modified from the TBM angle by the angle $\theta_{23}^{(-1)}$ from $U^{(-1)}$.

Using Bin's values (after normalizing all matrix elements by $d = 0.9925$) we get the following results from (34), (35) and (36)

$$\theta_{13} = 0.158 \approx 9.07^\circ \tag{39}$$
$$\theta_{12} = 0.601 \approx 34.42^\circ \tag{40}$$
$$\theta_{23} = 0.791 \approx 45.33^\circ \tag{41}$$

Using the approximate formula (37) gives $\theta_{23} \approx 0.790 \approx 45.27^\circ$.

A Appendix: Bin's values after normalization

Bin's original values were

$$b = 0.043$$
$$c = -0.122$$
$$d = 0.9925$$
$$A = 0.0171$$

We normalize all values by d so that we can write $d = 1$. Then the values are

$$b = 0.0426$$
$$c = -0.123$$
$$d = 1$$
$$A = 0.0172$$

These are the values we have used in this article.