Assume there is one Dirac Phase-δ in the PMNS Matrix:

$U_{\text{PMNS}} = R_{23} * U_{13} * R_{12}$, where

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}; \; U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} * e^{-I*\delta} \\ 0 & 1 & 0 \\ -s_{13} * e^{I*\delta} & 0 & c_{13} \end{pmatrix}; \; R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Using central values from PDG: $s_{23}{}^2 = 0.50$, $s_{13}{}^2 = 0.021$, $s_{12}{}^2 = 0.307$, δ varies from 0 to 2π.

Decompose $U^{(-1)} = U_{\text{TBM}} * U_{\text{PMNS}}{}^{\dagger} = P_L * R_{23} * U_{13} * R_{12} * P_R$, where P_L & P_R are diagonal phase matrices.

Again, take $V^{\left(-\frac{1}{3}\right)} = U^{(-1)*}$, $V^{(-1)} = U_{\text{CKM}}{}^*$, we have

$Y^{\left(-\frac{1}{3}\right)} = U_{\text{CKM}}\, D^{\left(-\frac{1}{3}\right)}\, U^{(-1)\,\mathsf{T}} = Y_5 + Y_{45}$,

$Y^{(-1)} = U^{(-1)}\, D^{(-1)}\, U_{\text{CKM}}{}^{\mathsf{T}} = Y_5{}^{\mathsf{T}} - 3\, Y_{45}{}^{\mathsf{T}}$.

Assume P_R is cancelled by the phases in U_{CKM}, and P_L is absorbed by redefining the fields, we can simply parametrize $U^{(-1)}$ as

$U^{(-1)} = R_{23}[\phi_{23}] * U_{13}[\phi_{13}, \gamma] * R_{12}[\phi_{12}]$

Some Numerical Studies:

1) δ=0 (no phase)

$\boldsymbol{\phi_{23}} = -0.003 \approx -\lambda^4$, $\boldsymbol{\phi_{13}} = -0.123 \approx -\lambda/2$, $\boldsymbol{\phi_{12}} = -0.083 \approx -\lambda/3$, γ=0

$$U^{(-1)} = \begin{pmatrix} 1 - \frac{13\lambda^2}{72} & -\frac{\lambda}{3} & -\frac{\lambda}{2} \\ \frac{\lambda}{3} & 1 - \frac{\lambda^2}{18} & -\lambda^4 \\ \frac{\lambda}{2} & -\frac{\lambda^2}{6} + \lambda^4 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O\left(\lambda^5\right)$$

$$Y_5 = \begin{pmatrix} \frac{2\lambda^4}{9} & -\frac{A\lambda^3}{2} & -\frac{\lambda}{2} \\ 0 & 0 & -\lambda^4 \\ A\,(\rho - i\eta)\,\lambda^3 & A\lambda^2 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O\left(\lambda^5\right)$$

$$Y_{45} = \begin{pmatrix} 0 & -\frac{\lambda^3}{9} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{27} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O\left(\lambda^5\right).$$

2) $\delta=\pi$ (sign different)

$\phi_{23} = 0.007 \approx \frac{5}{8}\lambda^3$, $\phi_{13} = 0.083 \approx \frac{\lambda}{3}$, $\phi_{12} = 0.123 \approx \frac{\lambda}{2}$, $\gamma = \pi$.

$$U^{(-1)} = \begin{pmatrix} 1 - \frac{13\lambda^2}{72} & \frac{\lambda}{2} & -\frac{\lambda}{3} \\ -\frac{\lambda}{2} & 1 - \frac{\lambda^2}{8} & \frac{5\lambda^3}{8} \\ \frac{\lambda}{3} - \frac{\lambda^3}{24} + \frac{5\lambda^4}{16} & \frac{\lambda^2}{6} - \frac{5\lambda^3}{8} & 1 - \frac{\lambda^2}{18} \end{pmatrix} + O\left(\lambda^5\right)$$

$$Y_5 = \begin{pmatrix} \frac{2\lambda^4}{9} & -\frac{A\lambda^3}{3} & -\frac{\lambda}{3} \\ 0 & 0 & 0.627\,\lambda^3 \\ A\,(\rho - i\eta)\,\lambda^3 & A\lambda^2 & 1 - \frac{\lambda^2}{18} \end{pmatrix} + O\left(\lambda^5\right)$$

$$Y_{45} = \begin{pmatrix} \frac{5\lambda^4}{18} & \frac{\lambda^3}{6} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{24} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O\left(\lambda^5\right).$$

3) $\delta = \pi/2$

$\phi_{23} = 0.0055 \approx \lambda^3/2$, $\phi_{13} = 0.1045 \approx \frac{\lambda}{2}$, $\phi_{12} = 0.1051 \approx \frac{\lambda}{2}$,

$\gamma = 0.377$, $e^{-i\gamma} = 0.9298 - 0.3681\, i \approx 2\,(2 - A * i)\,\lambda$

$$U^{(-1)} = \begin{pmatrix} 1 - \frac{\lambda^2}{4} & \frac{\lambda}{2} - \frac{\lambda^3}{16} & 2\lambda^2 - i\,A\,\lambda^2 \\ -\frac{\lambda}{2} & 1 - \frac{\lambda^2}{8} & \frac{\lambda^3}{2} \\ -2\lambda^2 + i\,A\,\lambda^2 + \frac{\lambda^4}{2} & -\frac{3\lambda^3}{2} + \frac{1}{2} i\,A\,\lambda^3 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O(\lambda^5)$$

$$Y_5 = \begin{pmatrix} \frac{2\lambda^4}{9} & 2A\lambda^4 - i A^2 \lambda^4 & 2\lambda^2 - i A \lambda^2 \\ 0 & 0 & \frac{\lambda^3}{2} \\ A(\rho - i\eta)\lambda^3 & A\lambda^2 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O(\lambda^5)$$

$$Y_{45} = \begin{pmatrix} \frac{5\lambda^4}{18} & \frac{\lambda^3}{6} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{24} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O(\lambda^5).$$

4) $\delta = \pi/4$

$\phi_{23} = 0.0039 \approx \lambda^3/3$, $\phi_{13} = 0.1177 \approx \frac{\lambda}{2}$, $\phi_{12} = 0.0902 \approx \frac{2\lambda}{5}$,

$\gamma = 1.705$, $e^{-i\gamma} = -0.1342 - 0.9909\, i \approx 0.6\,\lambda - i$

$$U^{(-1)} = \begin{pmatrix} 1 - \frac{41\lambda^2}{200} & \frac{2\lambda}{5} & -\frac{i\lambda}{2} + \frac{3\lambda^2}{10} \\ -\frac{2\lambda}{5} & 1 - \frac{2\lambda^2}{25} & \frac{\lambda^3}{3} \\ \frac{i\lambda}{2} - \frac{3\lambda^2}{10} & \frac{i\lambda^2}{5} - \frac{34\lambda^3}{75} & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O(\lambda^5)$$

$$Y_5 = \begin{pmatrix} \frac{2\lambda^4}{9} & -\frac{1}{2} i A \lambda^3 & -\frac{i\lambda}{2} + \frac{3\lambda^2}{10} \\ 0 & 0 & \frac{\lambda^3}{3} \\ A(\rho - i\eta)\lambda^3 & A\lambda^2 & 1 - \frac{\lambda^2}{8} \end{pmatrix} + O(\lambda^5)$$

$$Y_{45} = \begin{pmatrix} \frac{5\lambda^4}{18} & \frac{\lambda^3}{6} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{24} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O(\lambda^5).$$

5) $\delta = 1.32\pi$ (central value via PDG)

$\phi_{23} = 0.0064 \approx \frac{4}{7}\lambda^3$, $\phi_{13} = 0.0934 \approx \frac{5}{12}\lambda$, $\phi_{12} = 0.1151 \approx \frac{\lambda}{2}$, $\gamma = 0.7200$

$$U^{(-1)} = \begin{pmatrix} 1 - \frac{2\lambda^2}{9} & \frac{\lambda}{2} & \left(\frac{5}{16} - \frac{5i}{18}\right)\lambda \\ -\frac{\lambda}{2} & 1 - \frac{\lambda^2}{8} & \frac{4\lambda^3}{7} \\ \left(-\frac{5}{16} + \frac{5i}{18}\right)\lambda + \frac{2\lambda^4}{7} & \left(-\frac{5}{32} + \frac{5i}{36}\right)\lambda^2 - \frac{4\lambda^3}{7} & 1 - \frac{\lambda^2}{12} \end{pmatrix} + O(\lambda^5)$$

$$Y_5 = \begin{pmatrix} \frac{2\lambda^4}{9} & \left(\frac{5}{16} - \frac{5i}{18}\right) A\lambda^3 & \left(\frac{5}{16} - \frac{5i}{18}\right)\lambda \\ 0 & 0 & \frac{4\lambda^3}{7} \\ A\eta\lambda^3 & A\lambda^2 & 1 - \frac{\lambda^2}{12} \end{pmatrix} + O(\lambda^5)$$

$$Y_{45} = \begin{pmatrix} \frac{5\lambda^4}{18} & \frac{\lambda^3}{6} & 0 \\ \frac{\lambda^3}{3} & \frac{\lambda^2}{3} - \frac{5\lambda^4}{24} & -\frac{A\lambda^4}{3} \\ 0 & 0 & 0 \end{pmatrix} + O(\lambda^5).$$

Some Conclusions:

1) Zero at (1,1) position of Y_{45} is not guaranteed, other zeros are still there.

2) Phases only appear in Y_5. Y_{45} is always real, and $Y_{45} \sim o\left(\lambda^2\right)$.

3) $\phi_{13} \approx \phi_{12} \approx \frac{\lambda}{2}$, $\phi_{23} \sim O\left(\lambda^3\right)$

12/07/2017