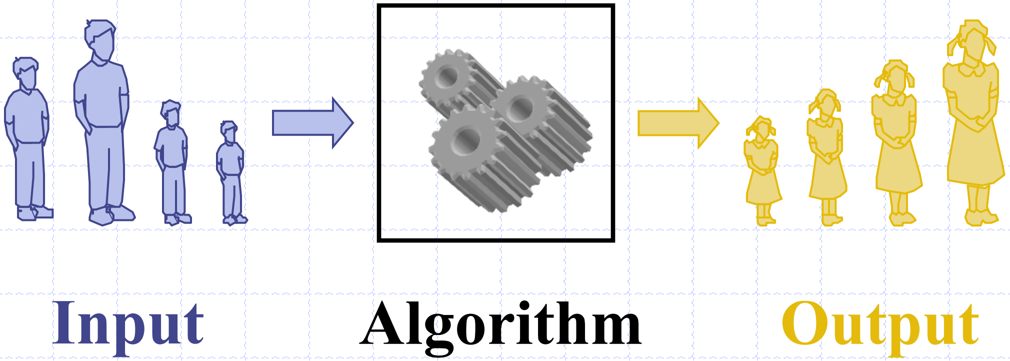
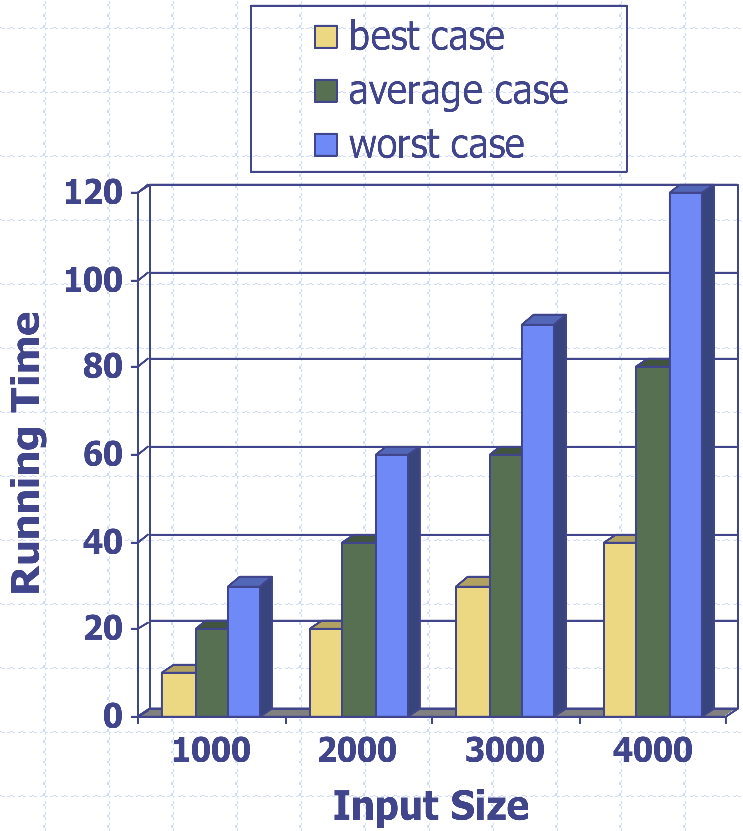
# Analysis of Algorithms




An **algorithm** is a step-by-step procedure for solving a problem in a finite amount of time.

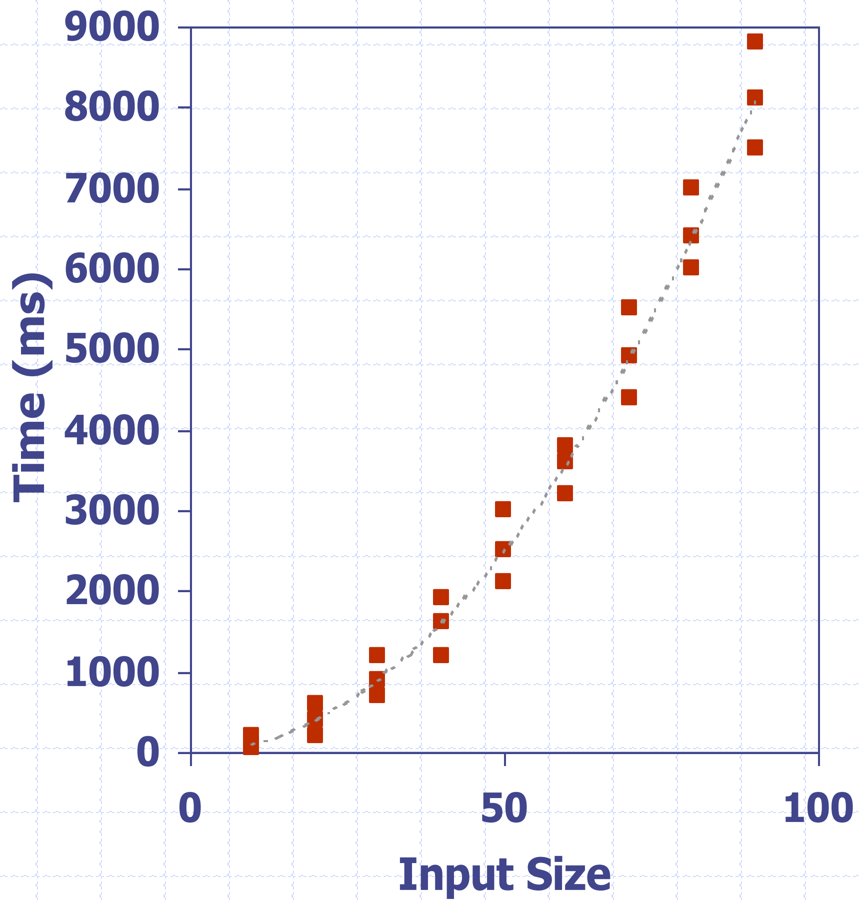
# Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



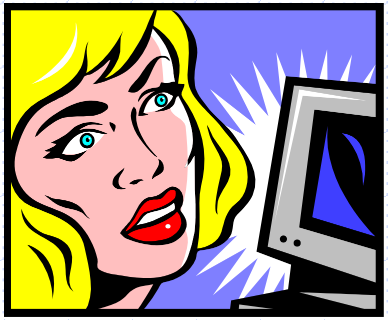
# Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a function, like the built-in clock() function, to get an accurate measure of the actual running time
- Plot the results

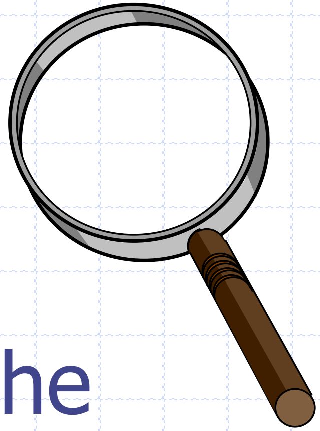


# Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used



# Theoretical Analysis



- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, *n*.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

# Pseudocode

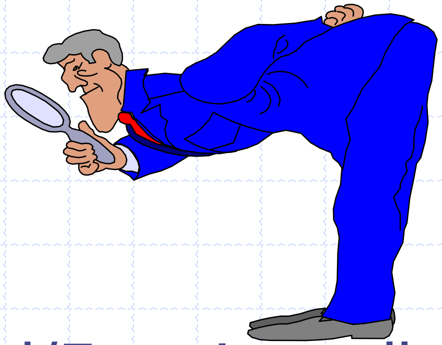
- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

**Algorithm** ***arrayMax*(*A*, *n*)**
  **Input** array ***A*** of ***n*** integers
  **Output** maximum element of ***A***

  ***currentMax*** $\leftarrow$ ***A***[0]
  **for** ***i*** $\leftarrow$ 1 **to** ***n*** $-$ 1 **do**
    **if** ***A***[*i*] $>$ ***currentMax*** **then**
      ***currentMax*** $\leftarrow$ ***A***[***i***]
  **return** ***currentMax***

# Pseudocode Details



- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  **Algorithm** *method* (*arg* [, *arg*…])

  **Input** …

  **Output** …
- Method/Function call

  *var.method* (*arg* [, *arg*…])
- Return value

  **return** *expression*
- Expressions

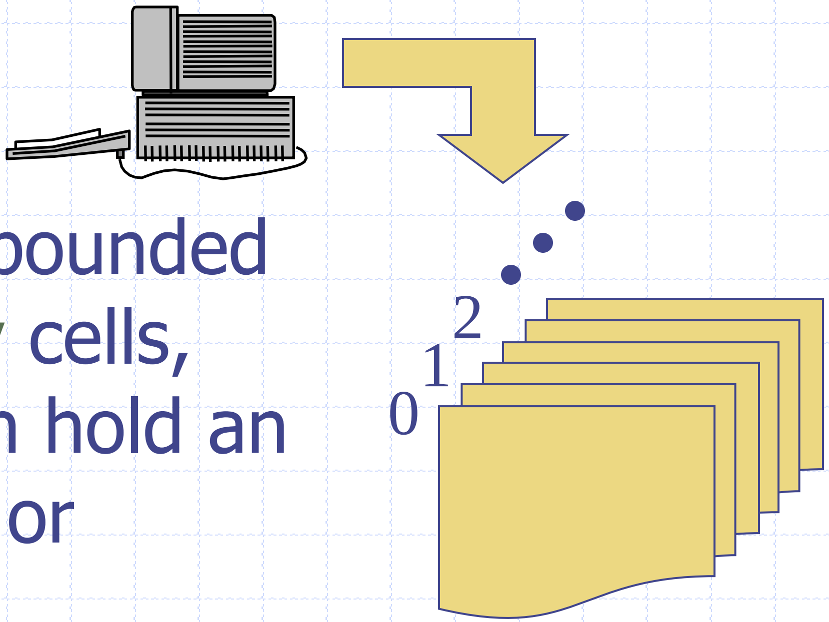
  $\leftarrow$ Assignment (like $=$ in C++)

  $=$ Equality testing (like $==$ in C++)

  $n^2$ Superscripts and other mathematical formatting allowed

# The Random Access Machine (RAM) Model

- A **CPU**




- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character

- Memory cells are numbered and accessing any cell in memory takes unit time.

# Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

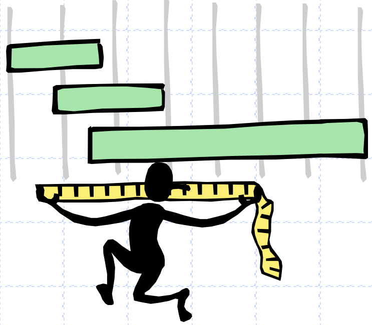
- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

# Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

| **Algorithm** *arrayMax*($A$, $n$) | # operations |
|---|---|
| $currentMax \leftarrow A[0]$ | $2$ |
| **for** $i \leftarrow 1$ **to** $n - 1$ **do** | $2 + n$ |
|     **if** $A[i] > currentMax$ **then** | $2(n - 1)$ |
|         $currentMax \leftarrow A[i]$ | $2(n - 1)$ |
| { increment counter $i$ } | $2(n - 1)$ |
| **return** $currentMax$ | $1$ |
| Total | $7n - 1$ |

# Estimating Running Time



- Algorithm ***arrayMax*** executes $7n - 1$ primitive operations in the worst case.
- Define:

  $a$ = Time taken by the fastest primitive operation

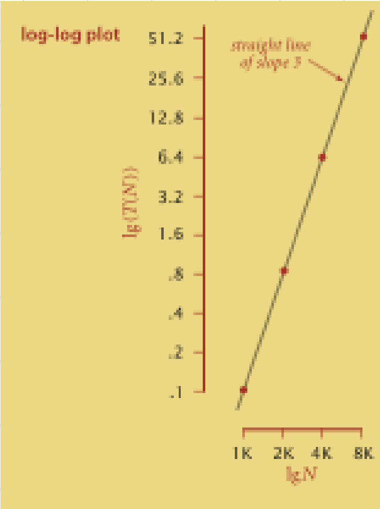
  $b$ = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of ***arrayMax***. Then

$$a(7n - 1) \leq T(n) \leq b(7n - 1)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

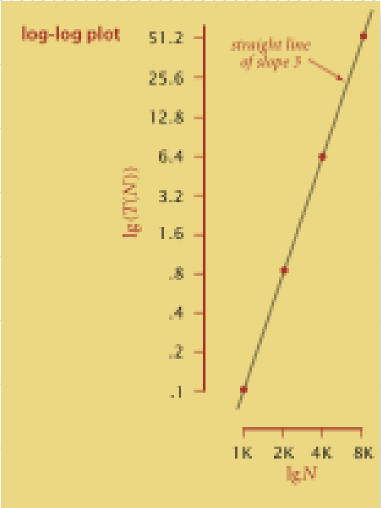
# Power Law

- Log-log plot. Plot running time $T(N)$ vs. input size $N$ using log-log scale.



# Power Law

- **Log-log plot.** Plot running time $T(N)$ vs. input size $N$ using log-log scale.




*Let* $T(N) = a N^{b}$, *where* $a = 2^{c}$

*Thus* $\lg(T(N)) = b \lg N + c$

*After regression:*

$b = 2.999$

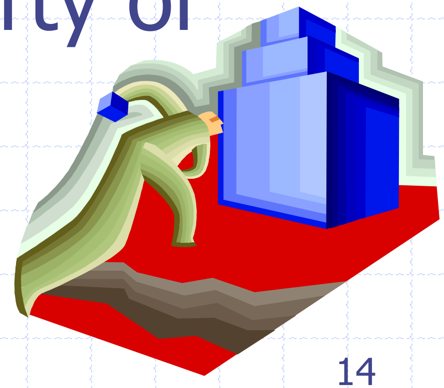
$c = -33.2103$

power law

- **Regression.** Fit straight line through data points: $a N^{b}$.
- **Hypothesis.** The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

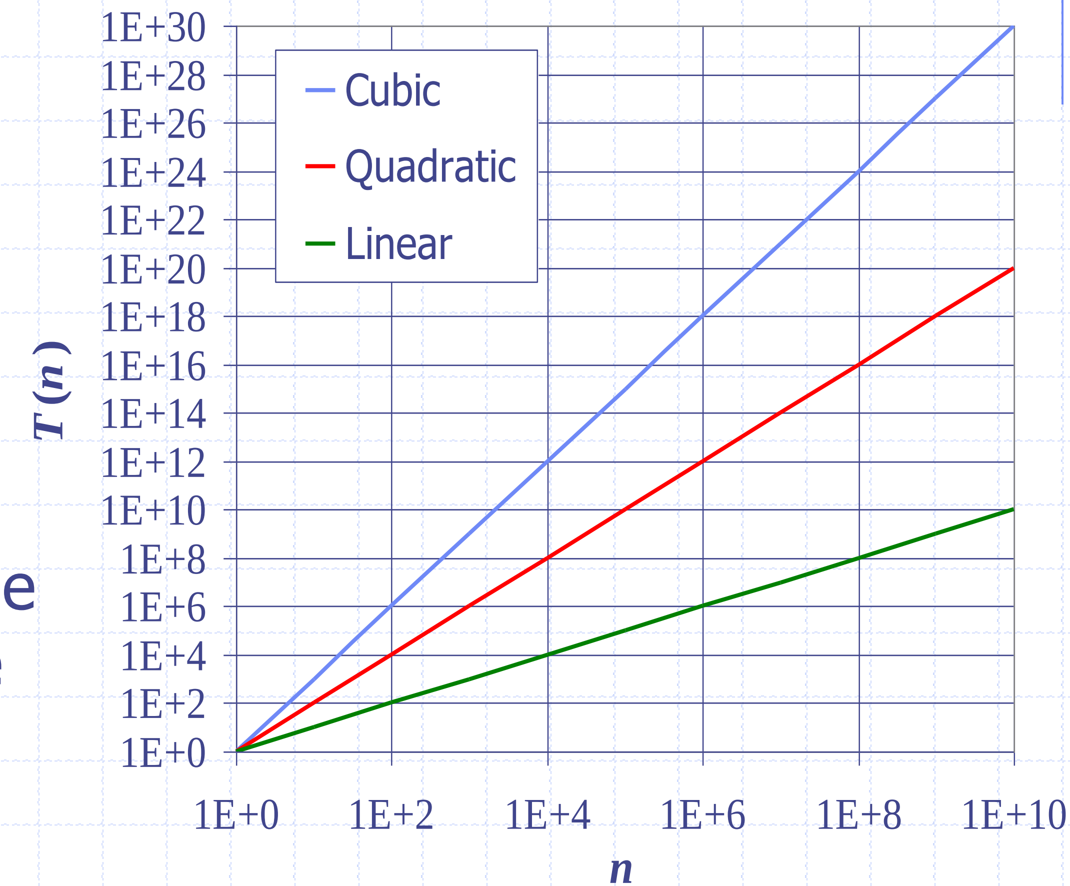
# Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor
  - But does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***



# Growth Rates

- Growth rates of functions:
  - Linear $\approx n$
  - Quadratic $\approx n^2$
  - Cubic $\approx n^3$
- In a log-log chart, the slope of the line corresponds to the growth rate of the function



# Practical implications of order-of-growth

| growth rate | problem size solvable in minutes | | | |
|---|---|---|---|---|
| | 1970s | 1980s | 1990s | 2000s |
| 1 | any | any | any | any |
| log N | any | any | any | any |
| N | millions | tens of millions | hundreds of millions | billions |
| N log N | hundreds of thousands | millions | millions | hundreds of millions |
| $N^2$ | hundreds | thousand | thousands | tens of thousands |
| $N^3$ | hundred | hundreds | thousand | thousands |
| $2^N$ | | | | |

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| $N^2$ | hundreds | thousand | thousands | tens of thousands |
| $N^3$ | hundred | hundreds | thousand | thousands |
| $2^N$ | 20 | 20s | 20s | 30 |

# Practical implications of order-of-growth

| growth rate | name | description | effect on a program that runs for a few seconds | |
|---|---|---|---|---|
| | | | time for 100x more data | size for 100x faster computer |
| 1 | constant | independent of input size | – | – |
| log N | logarithmic | nearly independent of input size | – | – |
| N | linear | optimal for N inputs | a few minutes | 100x |
| N log N | linearithmic | nearly optimal for N inputs | a few minutes | 100x |
| $N^2$ | quadratic | not practical for large problems | several hours | 10x |
| $N^3$ | cubic | not practical for medium problems | several weeks | 4–5x |
| $2^N$ | exponential | | | |

# Practical implications of order-of-growth

| growth rate | name | description | effect on a program that runs for a few seconds | |
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| $N^2$ | quadratic | not practical for large problems | several hours | 10x |
| $N^3$ | cubic | not practical for medium problems | several weeks | 4–5x |
| $2^N$ | exponential | useful only for tiny problems | forever | 1x |

# Practical implications of order-of-growth

| growth rate | problem size solvable in minutes | | | | time to process millions of inputs | | | |
|---|---|---|---|---|---|---|---|---|
| | 1970s | 1980s | 1990s | 2000s | 1970s | 1980s | 1990s | 2000s |
| 1 | any | any | any | any | instant | instant | instant | instant |
| log N | any | any | any | any | instant | instant | instant | instant |
| N | millions | tens of millions | hundreds of millions | billions | minutes | seconds | second | instant |
| N log N | hundreds of thousands | millions | millions | hundreds of millions | hour | minutes | tens of seconds | seconds |
| $N^2$ | hundreds | thousand | thousands | tens of thousands | decades | years | months | weeks |
| $N^3$ | | | | | | | | |

# Practical implications of order-of-growth

| growth rate | problem size solvable in minutes | | | | time to process millions of inputs | | | |
|---|---|---|---|---|---|---|---|---|
| | 1970s | 1980s | 1990s | 2000s | 1970s | 1980s | 1990s | 2000s |
| 1 | any | any | any | any | instant | instant | instant | instant |
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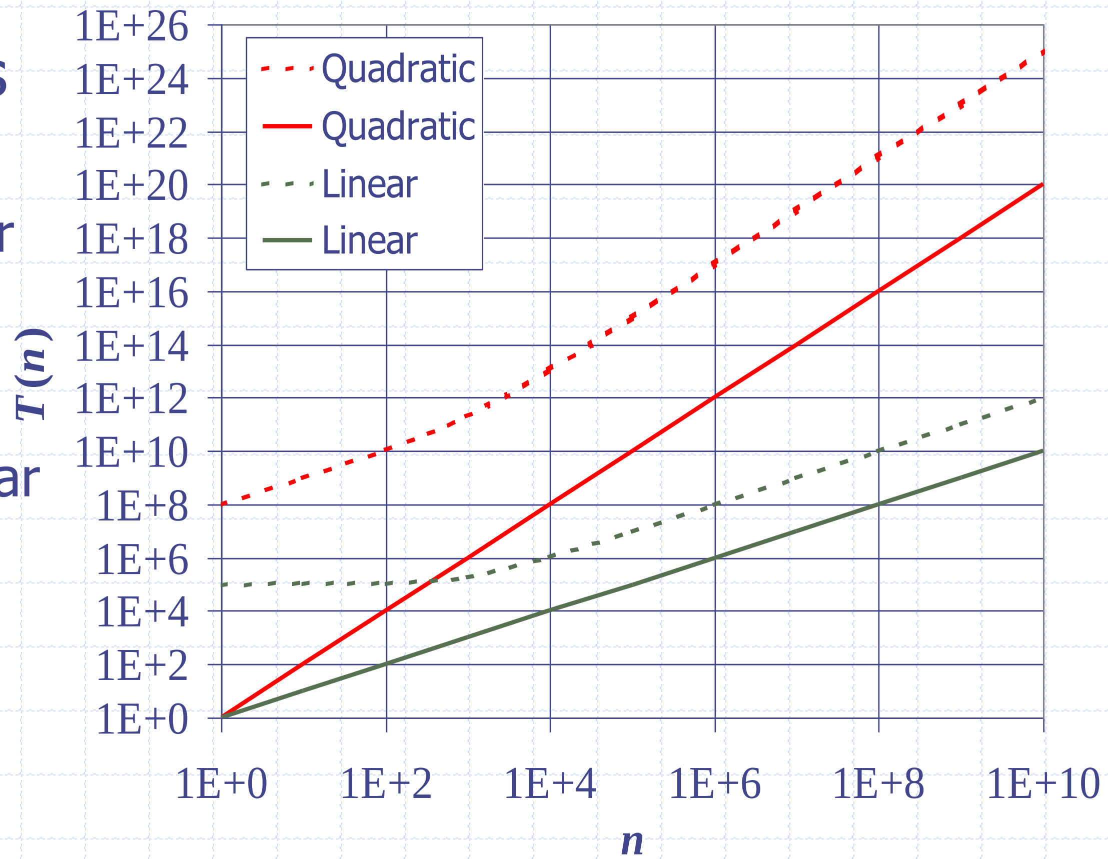
| N log N | hundreds of thousands | millions | millions | hundreds of millions | hour | minutes | tens of seconds | seconds |
| $N^2$ | hundreds | thousand | thousands | tens of thousands | decades | years | months | weeks |
| $N^3$ | hundred | hundreds | thousand | thousands | | | | |

# Practical implications of order-of-growth

| growth rate | problem size solvable in minutes | | | | time to process millions of inputs | | | |
|---|---|---|---|---|---|---|---|---|
| | 1970s | 1980s | 1990s | 2000s | 1970s | 1980s | 1990s | 2000s |
| 1 | any | any | any | any | instant | instant | instant | instant |
| log N | any | any | any | any | instant | instant | instant | instant |
| N | millions | tens of millions | hundreds of millions | billions | minutes | seconds | second | instant |
| N log N | hundreds of thousands | millions | millions | hundreds of millions | hour | minutes | tens of seconds | seconds |
| $N^2$ | hundreds | thousand | thousands | tens of thousands | decades | years | months | weeks |
| $N^3$ | hundred | hundreds | thousand | thousands | never | never | never | millennia |

# Constant Factors

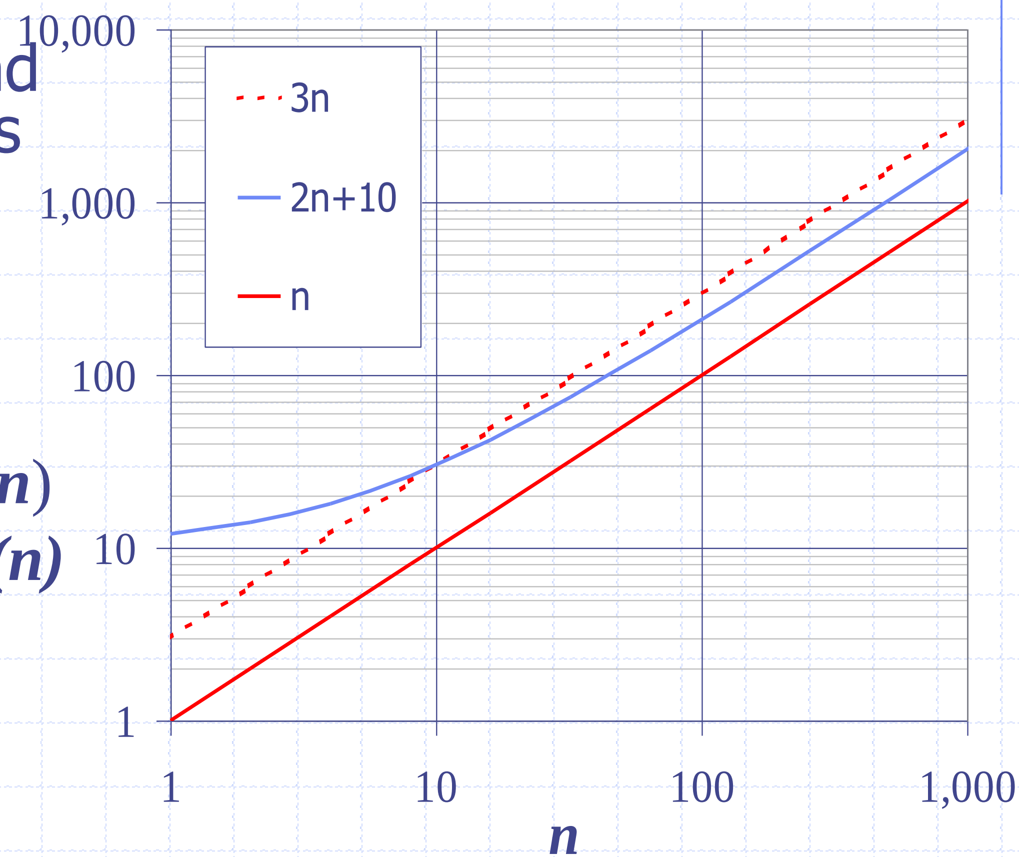
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



# Big-Oh Notation

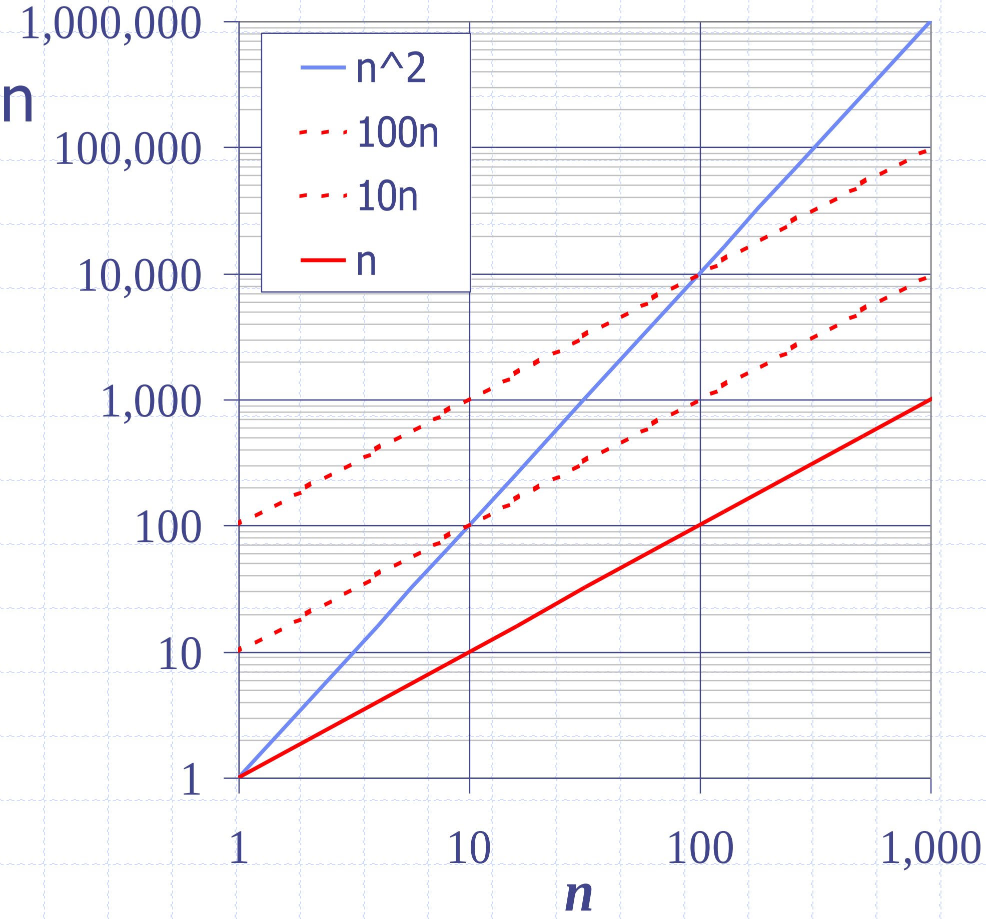
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants $c$ and $n_0$ such that

  $f(n) \leq cg(n)$ for $n \geq n_0$
- Example: $2n + 10$ is $O(n)$
- For what $c$ and $n$ is $cg(n) \geq f(n)$?
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$

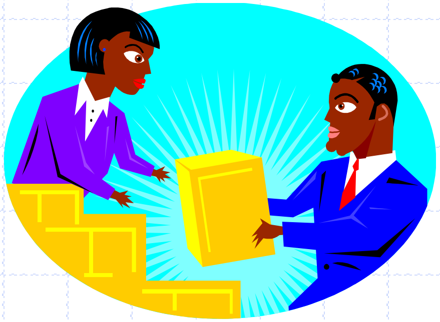


# Big-Oh Example

- Example: the function $n^2$ is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since $c$ must be a constant



# More Big-Oh Examples



- 7n-2

  e.g. Is there a c and $n_0$ such that $c \bullet n \geq 7n-2$?

What is the big-Oh?

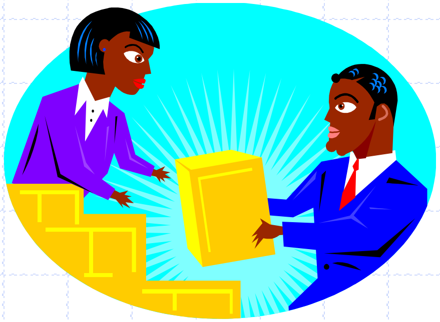
7n-2 is **O(n)**

[TEAMS] What are the values for c and $n_0$ ?

Need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

This is true for $c = 7$ and $n_0 = 1$

# More Big-Oh Examples



- $2n^2+16$

  $2n^2+16$ is $\mathbf{O(n^2)}$

  need $c > 0$ and $n_0 \geq 1$ such that $2n^2+16 \leq c \bullet n^2$ for $n \geq n_0$

  this is true for $c = 3$ and $n_0 = 4$

- $3n^3 + 2n^2 + 9$

  $3n^3 + 2n^2 + 9$ is $\mathbf{O(n^3)}$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 2n^2 + 9 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 3$

# Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

| | $f(n)$ is $O(g(n))$ | $g(n)$ is $O(f(n))$ |
|---|---|---|
| $g(n)$ grows more | Yes | No |
| $f(n)$ grows more | No | Yes |
| Same growth | Yes | Yes |

# Big-Oh Rules

- If is $f(n)$ a polynomial of degree $d$, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

# Asymptotic Algorithm Analysis

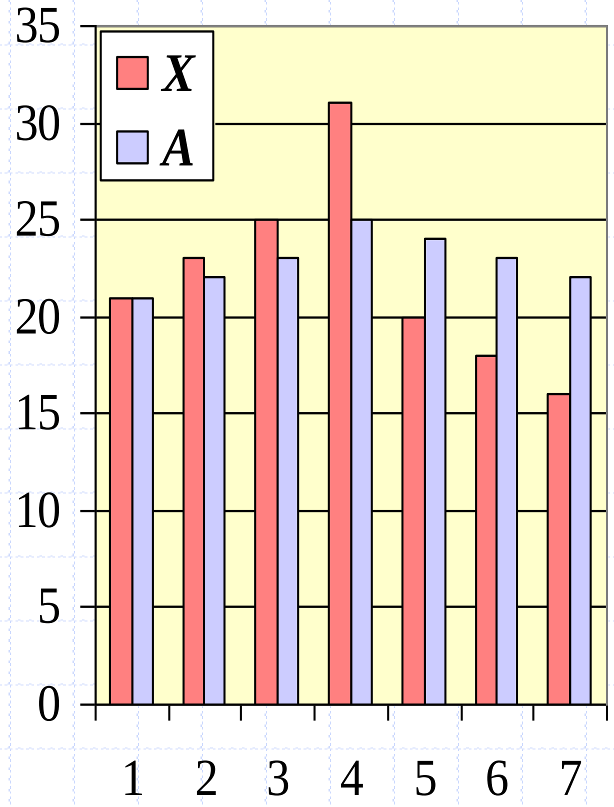
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $7n - 1$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

# Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The $i$-th prefix average of an array $X$ is average of the first $(i + 1)$ elements of $X$:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array $A$ of prefix averages of another array $X$ has applications to financial analysis



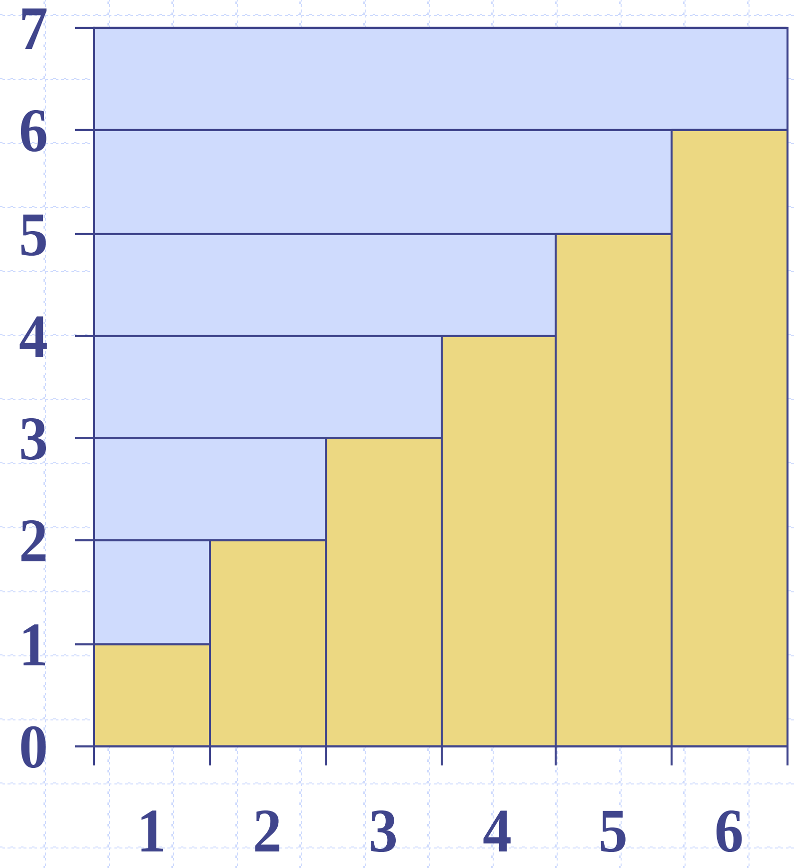
# Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

| | |
|---|---|
| **Algorithm** *prefixAverages1*($X$, $n$) | |
| **Input** array $X$ of $n$ integers | |
| **Output** array $A$ of prefix averages of $X$ | #operations |
| $A \leftarrow$ new array of $n$ integers | $n$ |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $s \leftarrow X[0]$ | $n$ |
| **for** $j \leftarrow 1$ **to** $i$ **do** | $1 + 2 + \ldots + (n - 1)$ |
| $s \leftarrow s + X[j]$ | $1 + 2 + \ldots + (n - 1)$ |
| $A[i] \leftarrow s \,/\, (i + 1)$ | $n$ |
| **return** $A$ | $1$ |

# Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first $n$ integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $O(n^2)$ time



# Prefix Averages

- Asymptotic analysis tells us the complexity of the original algorithm.
- [TEAMS] Given such a tool, how can we make the algorithm more efficient?

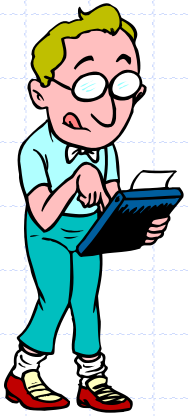
# Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

| | |
|---|---|
| **Algorithm** *prefixAverages2*($X$, $n$) | |
| **Input** array $X$ of $n$ integers | |
| **Output** array $A$ of prefix averages of $X$ | #operations |
| $A \leftarrow$ new array of $n$ integers | $n$ |
| $s \leftarrow 0$ | 1 |
| **for** $i \leftarrow 0$ **to** $n - 1$ **do** | $n$ |
| $\quad s \leftarrow s + X[i]$ | $n$ |
| $\quad A[i] \leftarrow s / (i + 1)$ | $n$ |
| **return** $A$ | 1 |

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

# Math you need to Review

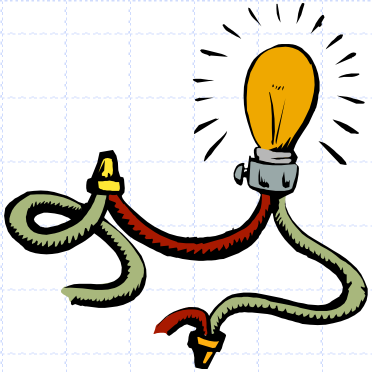


- Summations
- Logarithms and Exponents
  - **properties of logarithms:**
    - $\log_b(xy) = \log_b x + \log_b y$
    - $\log_b (x/y) = \log_b x - \log_b y$
    - $\log_b x^a = a\log_b x$
    - $\log_b a = \log_x a/\log_x b$
  - **properties of exponentials:**
    - $a^{(b+c)} = a^b a^c$
    - $a^{bc} = (a^b)^c$
    - $a^b / a^c = a^{(b-c)}$
    - $b = a^{\log_a b}$
    - $b^c = a^{c*\log_a b}$
- Proof techniques
- Basic probability

# Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$
- **little-oh**
  - f(n) is $o(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) < c \bullet g(n)$ for $n \geq n_0$
- **little-omega**
  - f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) > c \bullet g(n)$ for $n \geq n_0$

# Intuition for Asymptotic Notation



**Big-Oh**

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

**big-Omega**

- f(n) is $\Omega$(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)

**big-Theta**

- f(n) is $\Theta$(g(n)) if f(n) is asymptotically **equal** to g(n)

**little-oh**

- f(n) is o(g(n)) if f(n) is asymptotically **strictly less** than g(n)

**little-omega**

- f(n) is $\omega$(g(n)) if is asymptotically **strictly greater** than g(n)

# Example Uses of the Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  [TEAMS] What are values for c and $n_0$?

  Let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  Let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\omega(n)$**

  f(n) is $\omega(g(n))$ if, for any constant $c > 0$, there is an integer constant $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

  Need $5n_0^2 > c \cdot n_0 \rightarrow$ given c, the $n_0$ that satisfies this is $n_0 > c/5 > 0$

# Euclid's Algorithm

- ***An algorithm for computing the greatest common divisor (GCD) of two numbers $M \geq N$:***

```
Algorithm GCD(M, N)
    while (N != 0)
        rem ← M mod N
        M ← N
        N ← rem
    endwhile
    return M
```

- [TEAMS] What is the ***Big-Oh***?

# Euclid's Algorithm

- ***An algorithm for computing the greatest common divisor (GCD) of two numbers $M \geq N$:***

```
Algorithm GCD(M, N)
    while (N != 0)
        rem ← M mod N
        M ← N
        N ← rem
    endwhile
    return M
```

- The algorithm GCD runs is O(***logn***)

# Euclid's Algorithm

- Why?
  - If $M \geq N$, then $(M \bmod N) < M/2$
  - Thus, each iteration at least halves the value of M

# Exponentiation

- ***A recursive algorithm to compute X to the power N:***

```
Algorithm pow(X, N)
    if (N == 0) return 1
    if (N == 1) return X
    if (N is even)
        return pow(X*X, N/2)
    else
        return pow(X*X, N/2) * X
```

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- [TEAMS] What is the ***Big-Oh***?

# Exponentiation

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- The algorithm *pow* is O(***logn***)