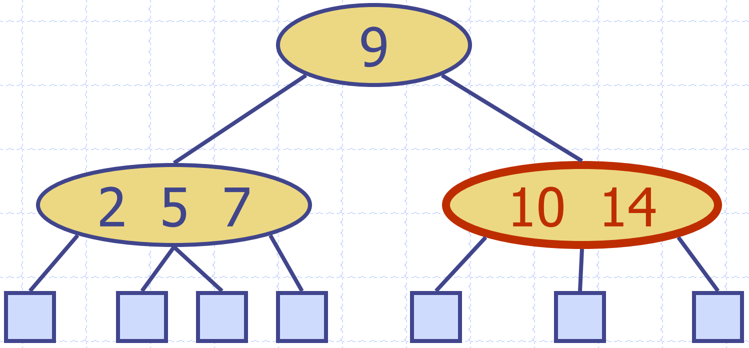
(2,4) Trees

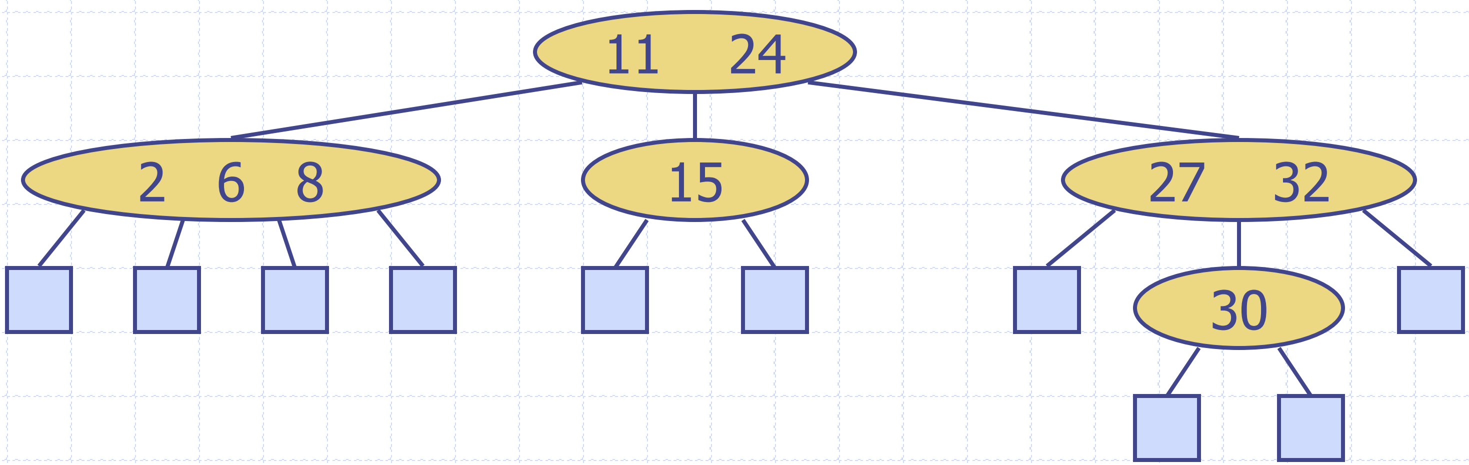


Outline and Reading

- Multi-way search tree
  - Definition
  - Search
- (2,4) tree
  - Definition
  - Search
  - Insertion
  - Deletion
- Comparison of dictionary implementations

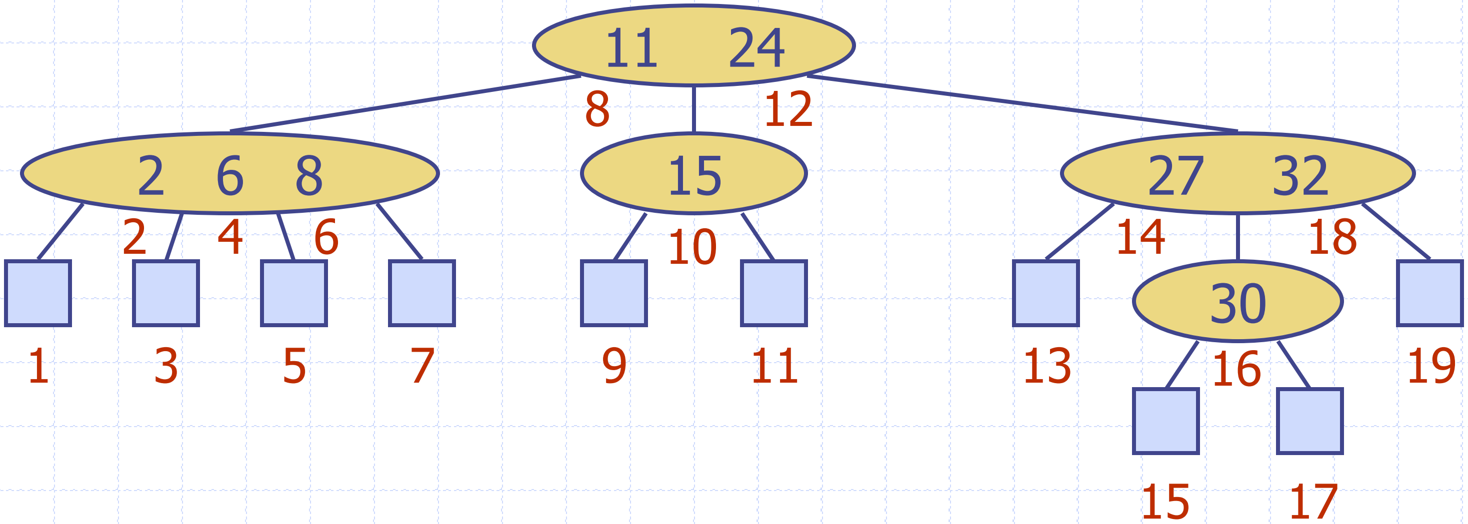
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\, v_2\, \ldots\, v_d$ storing keys $k_1\, k_2\, \ldots\, k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



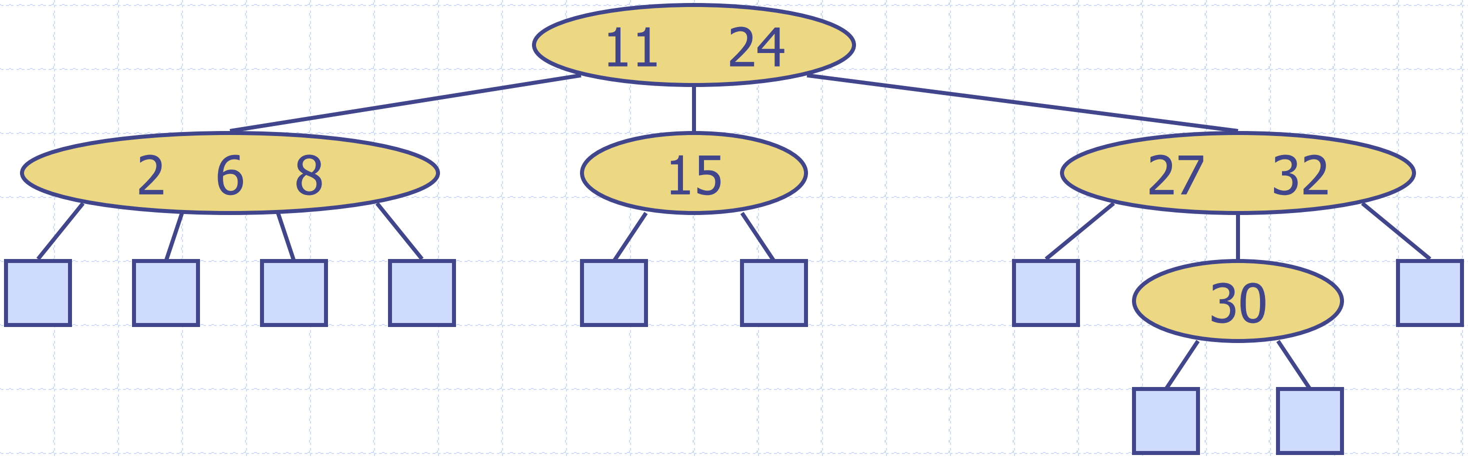
Multi-Way Inorder Traversal

- We can extend the notion of inorder traversal from binary trees to multi-way search trees
- Namely, we visit item (k_i, o_i) of node v between the recursive traversals of the subtrees of v rooted at children v_i and v_{i+1}
- An inorder traversal of a multi-way search tree visits the keys in increasing order



Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\, v_2\, \ldots\, v_d$ and keys $k_1\, k_2\, \ldots\, k_{d-1}$
  - $k = k_i$ ($i = 1, \ldots, d-1$): the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ ($i = 2, \ldots, d-1$): we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



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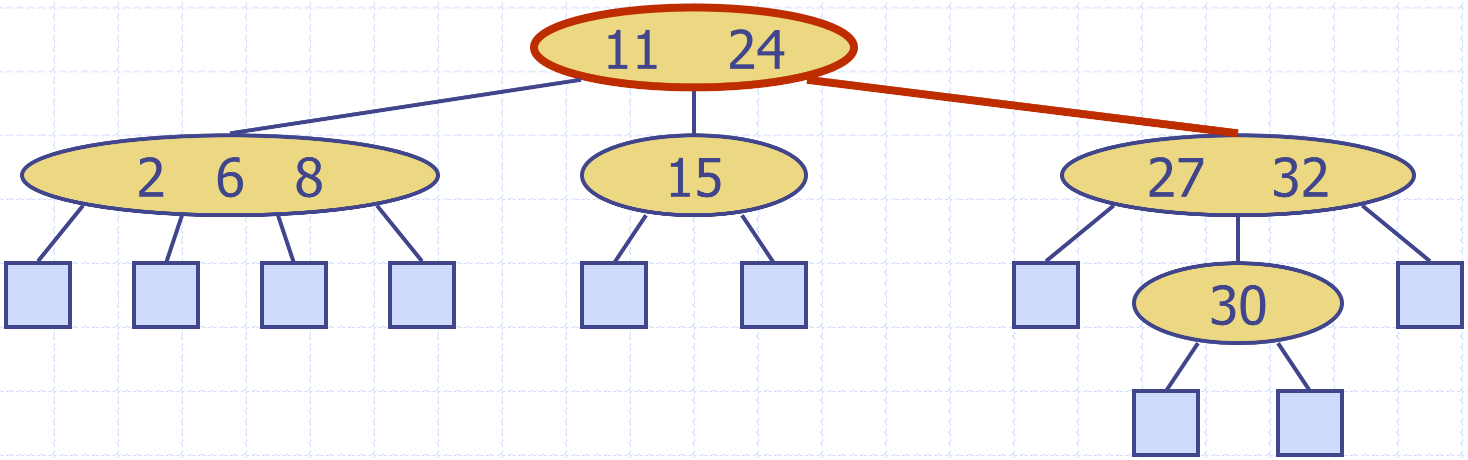

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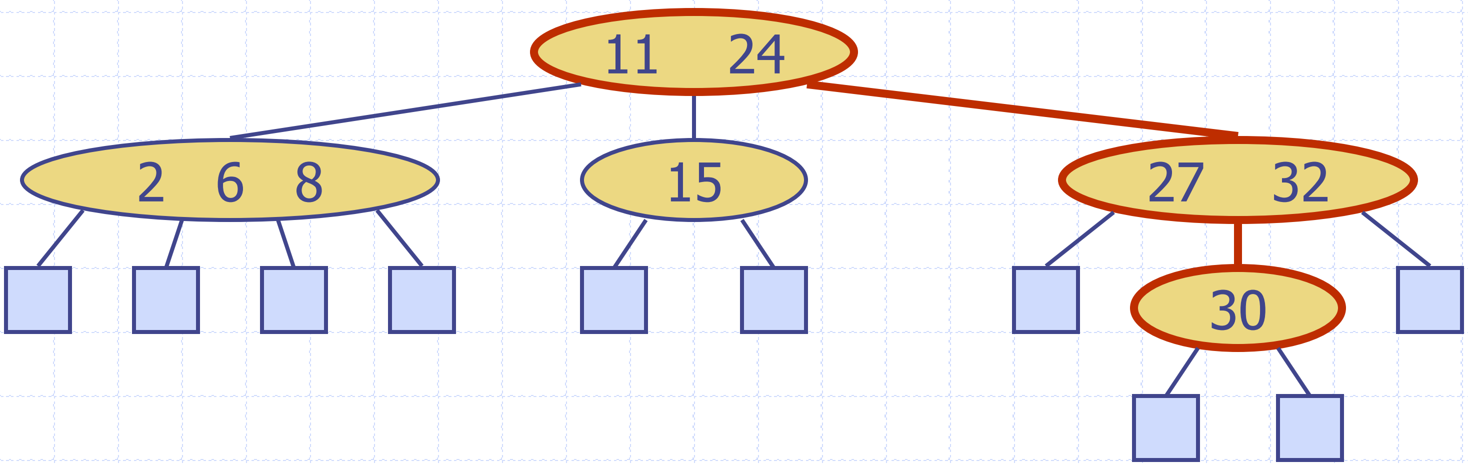
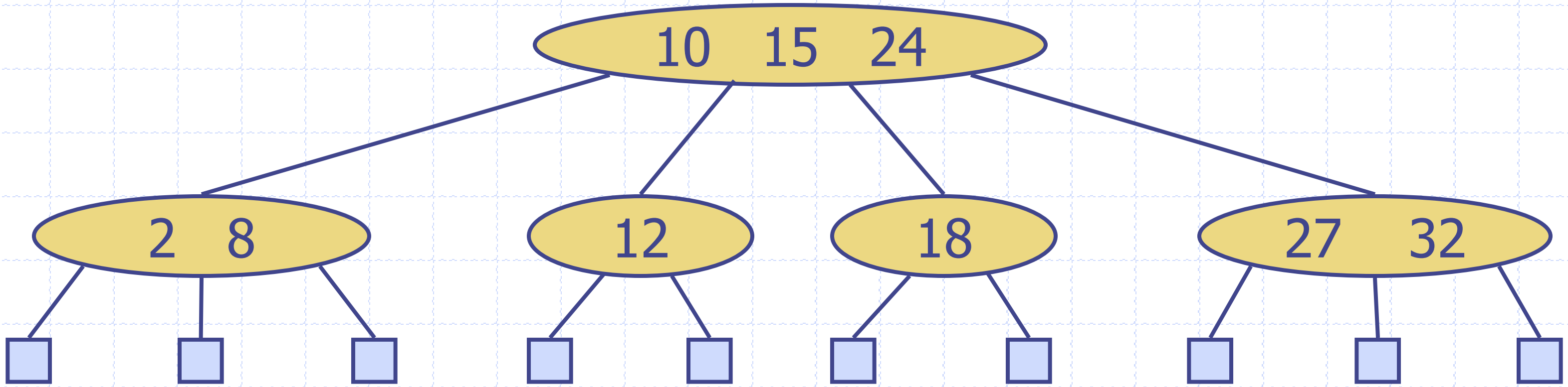


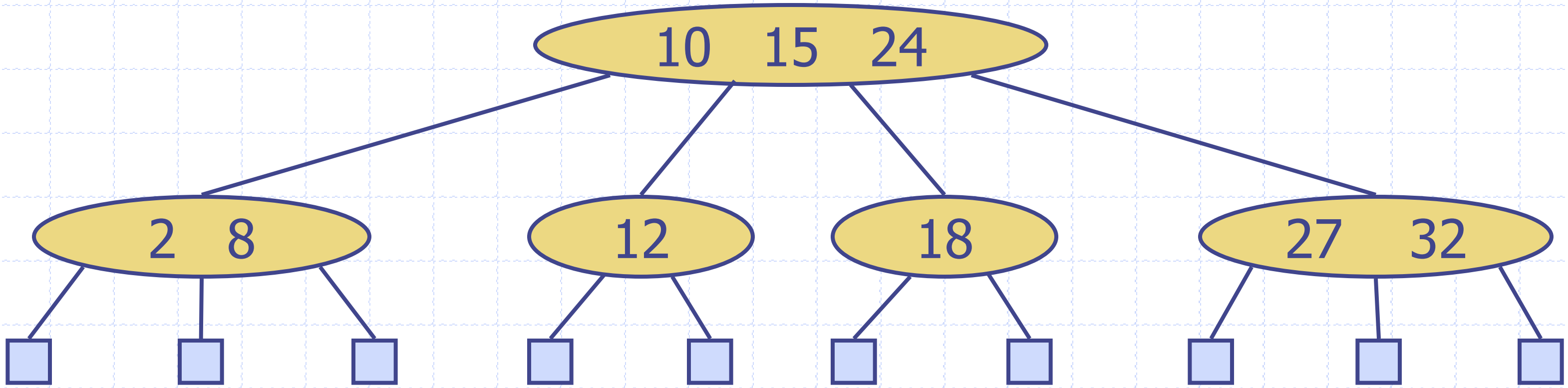
(2,4) Tree

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children, an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node



Height of a (2,4) Tree

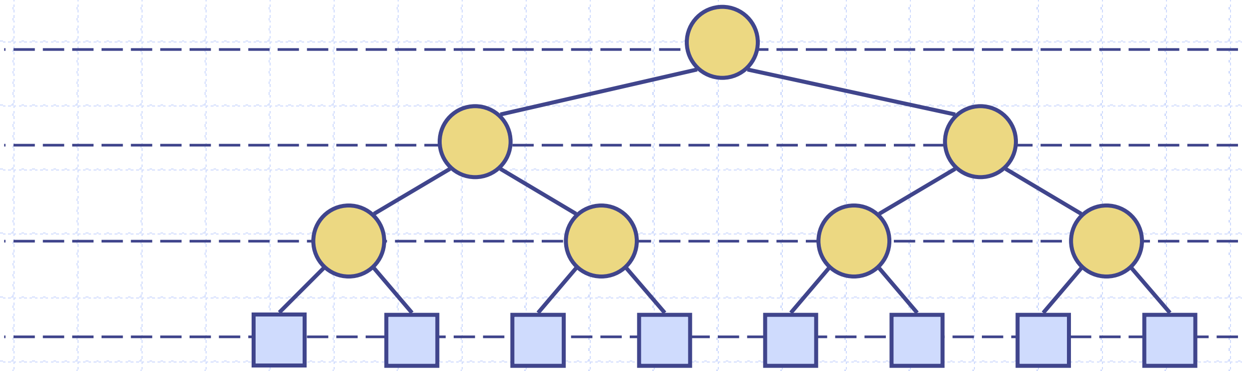
- As opposed to a binary tree, a (2,4) tree has internal nodes with 2, 3, and 4 children
- What is the height of the tree of n items?
- What is the big-Oh of the height of the tree of n items?



Height of a (2,4) Tree

- Theorem: A (2,4) tree storing n items has height $O(\log n)$
  Proof:
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \dots, h - 1$ and no items at depth h, we have
    $$n \geq 1 + 2 + 4 + \dots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n + 1)$

depth	items
0	1
1	2
$h-1$	2^{h-1}
h	0

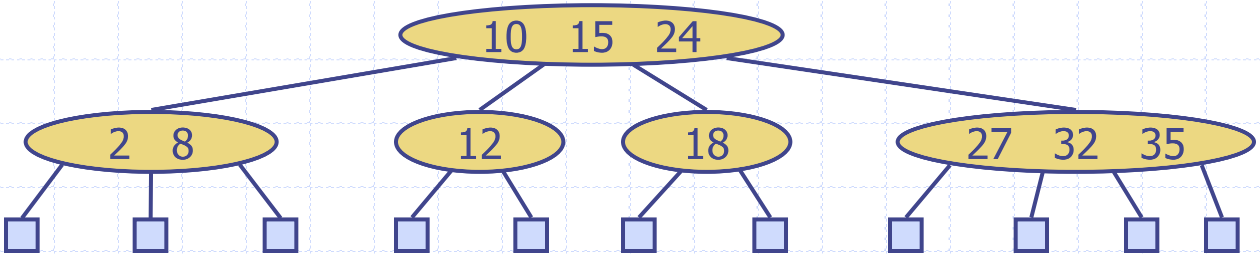


(2,4) Tree Operations

- Search
  - Depends on height of tree, thus searching in a (2,4) tree with n items takes $O(\log n)$
- Insert
  - Coming up next…
- Delete
  - Coming up next next…

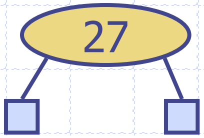
Insertion

- How do you insert an item into an existing tree? Ideas?
- Recall the (2,4) tree properties:
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
    - THIS IS CRUCIAL TO KEEP O(logN) SEARCH TIME - WHY?
- How do you maintain these properties? Ideas?



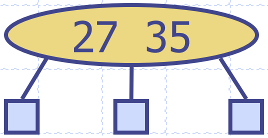
Insertion

- Let’s start at the beginning
  - Insert 27



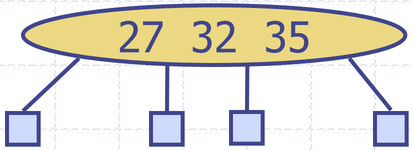
Insertion

- Let’s start at the beginning
  - Insert 27
  - Insert 35



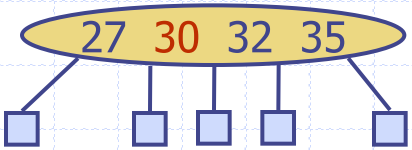
Insertion

- Let's start at the beginning
  - Insert 27
  - Insert 35
  - Insert 32
  - Insert 30?



Insertion

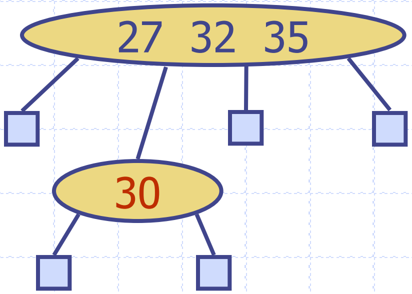
- Let's start at the beginning
  - Add 30 to the node?
    - Makes 5 children = overflow…



Node-size property is broken

Insertion

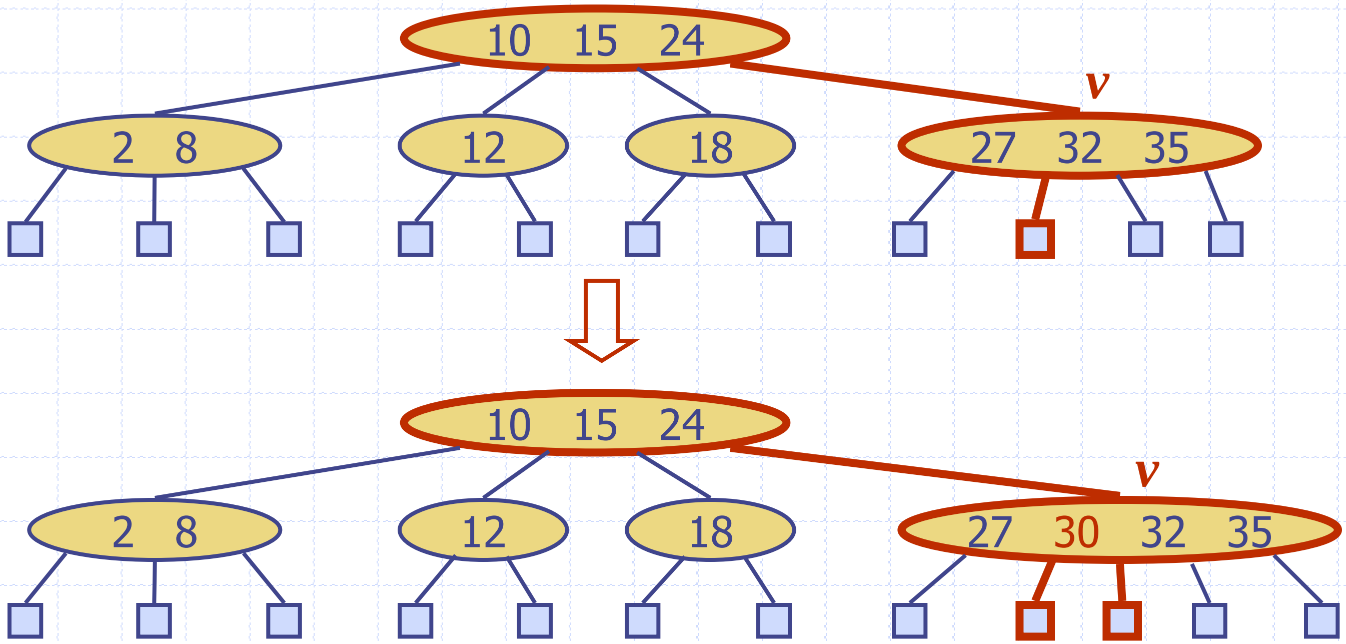
- Let’s start at the beginning
  - We make 30 a child of the node (27,32,35)?
    - External node at different depths…



Depth property is broken!

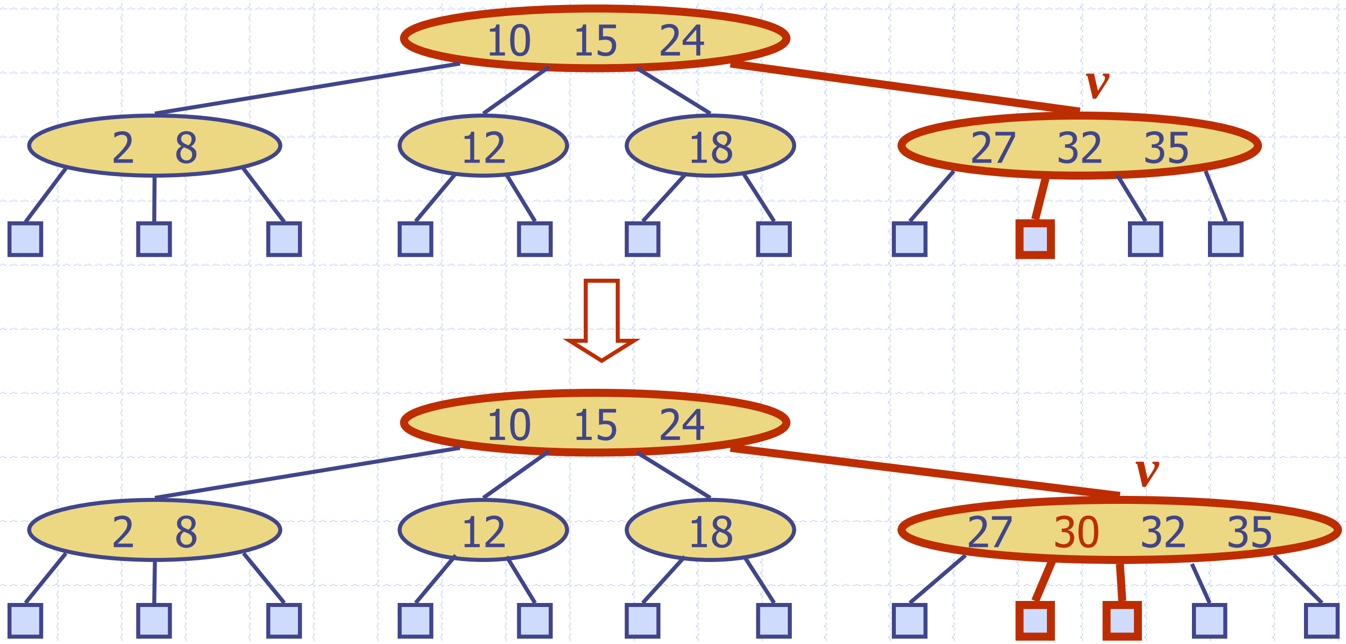
Insertion

- Another example: insert 30 into a larger tree
- We insert the new item (***k=30***, ***o***) at the parent ***v*** of the leaf reached by searching for ***k***
  - We preserve the depth property but
  - We cause an overflow (i.e., node ***v*** becomes a 5-node)



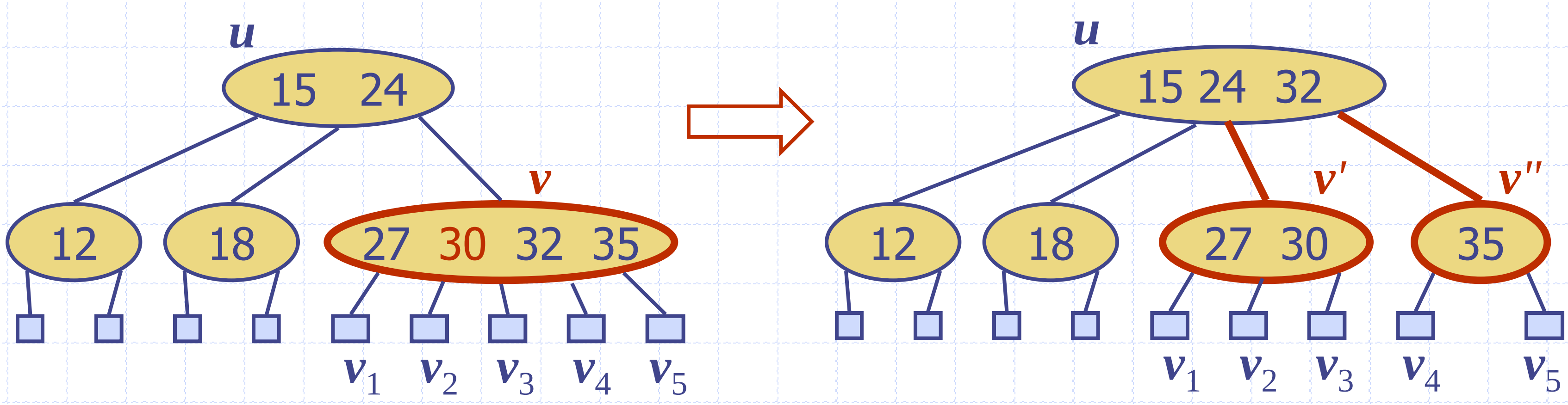
Insertion

- What can we do?



Handling Overflows

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \ldots v_5$ be the children of v and $k_1 \ldots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- **The overflow may propagate to the parent node u**



Analysis of Insertion

Algorithm *insertItem*(k, o)

1. We search for key $\boldsymbol{k}$ to locate the insertion node $\boldsymbol{v}$

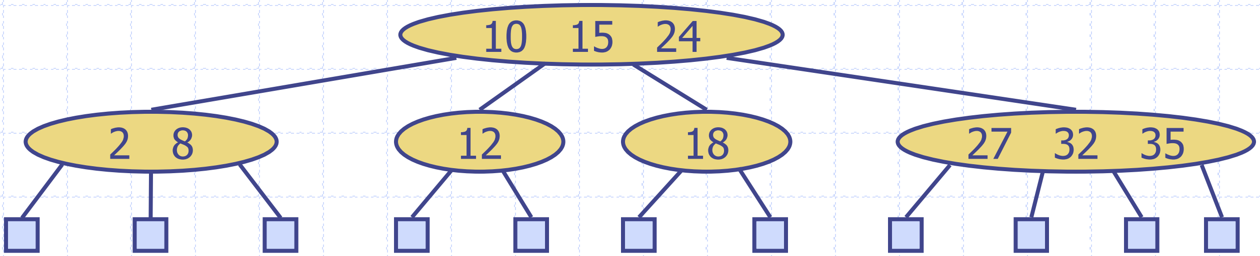
2. We add the new item ($\boldsymbol{k}$, $\boldsymbol{o}$) at node $\boldsymbol{v}$

3. **while** *overflow*(v)
   - **if** *isRoot*(v)
     - create a new empty root above $\boldsymbol{v}$
   - $v \leftarrow$ *split*(v)

- Let $\boldsymbol{T}$ be a (2,4) tree with $\boldsymbol{n}$ items
  - Tree $\boldsymbol{T}$ has $\boldsymbol{O}(\log \boldsymbol{n})$ height
  - Step 1 takes
    - $\boldsymbol{O}(\log \boldsymbol{n})$ time because we visit $\boldsymbol{O}(\log \boldsymbol{n})$ nodes
  - Step 2 takes
    - $\boldsymbol{O}(1)$ time
  - Step 3 takes
    - $\boldsymbol{O}(\log \boldsymbol{n})$ time because each split takes $\boldsymbol{O}(1)$ time and we perform $\boldsymbol{O}(\log \boldsymbol{n})$ splits
- Thus, an insertion in a (2,4) tree takes
  - $\boldsymbol{O}(\log \boldsymbol{n})$ time

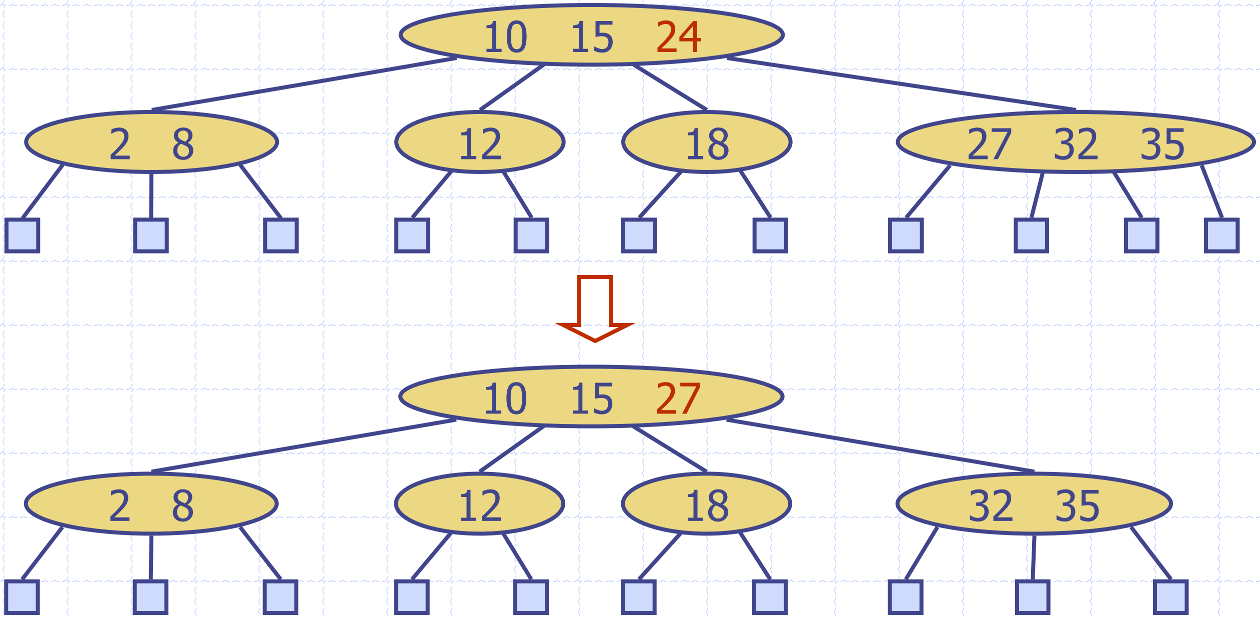
Deletion

- How do you delete an item?
- What problems can occur?



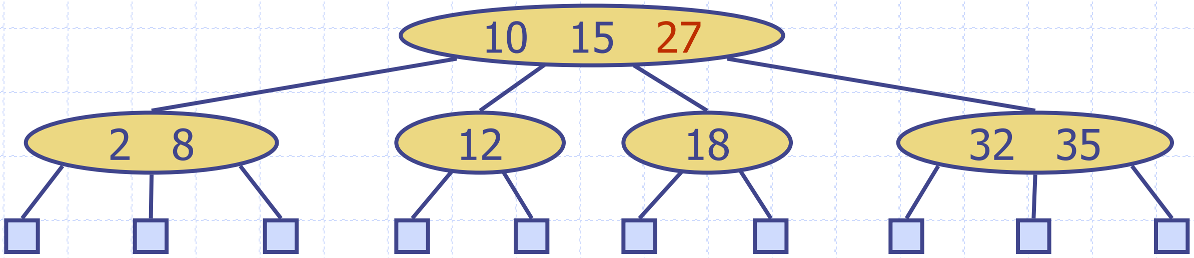
Deletion

- We reduce deletion of an item to the case where the item is at the node with leaf children
- Otherwise, we replace the item with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter item
- Example: to delete key 24, we replace it with 27 (inorder successor)



Deletion

- What happens if I delete 12? 18?
- Simply removing the node will break the depth property…

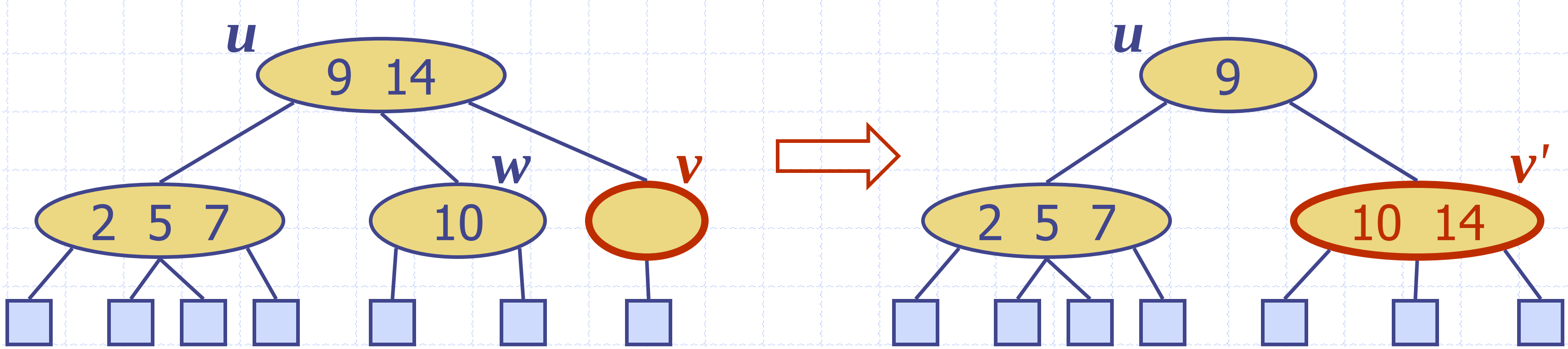


Underflow, Fusion, and Transfer

- Deleting an item from a node v may cause an underflow, where node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases in the next slides

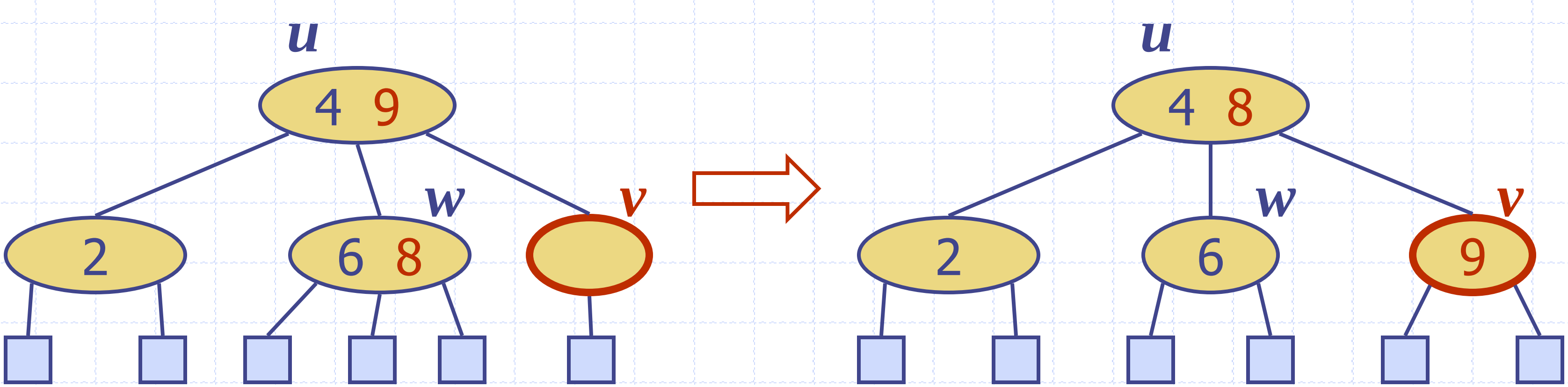
Underflow and Fusion

- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: since there is “space” in the siblings, we merge v with an adjacent sibling w and move an item from u to the merged node v'
  - **After a fusion, the underflow may propagate to the parent u**



Underflow and Transfer

- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - **After a transfer, no underflow occurs**



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the item
  - We handle an underflow with a series of $O(\log n)$ fusions, followed by at most one transfer
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Summary

- Comparison of data structures and algorithms

	Search	Insert	Delete	Notes
Hash Table	1 expected	1 expected	1 expected	no ordered dictionary methods simple to implement
Skip List	log n high prob.	log n high prob.	log n high prob.	randomized insertion simple to implement
(2,4) Tree	log n worst-case	log n worst-case	log n worst-case	complex to implement