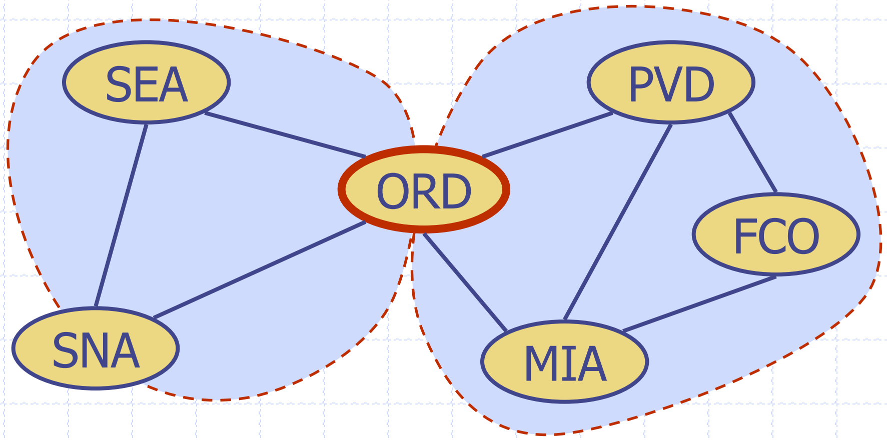
Biconnectivity

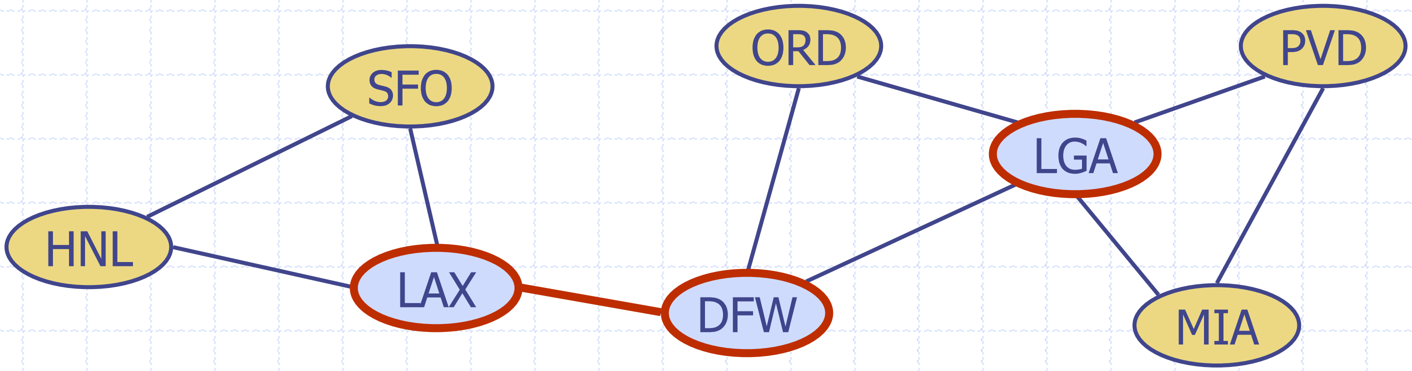


Outline and Reading

- Definitions
  - Separation vertices and edges
  - Biconnected graph
  - Biconnected components
  - Equivalence classes
  - Linked edges and link components
- Algorithms
  - Auxiliary graph
  - Proxy graph

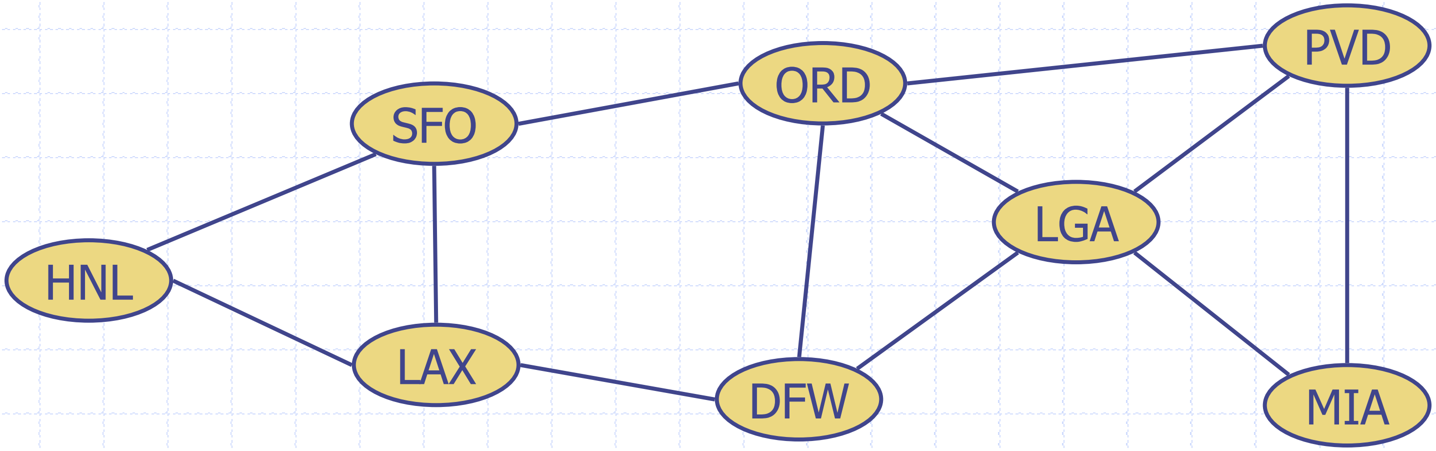
Separation Edges and Vertices

- Definitions
  - Let G be a connected graph
  - A separation edge of G is an edge whose removal disconnects G
  - A separation vertex of G is a vertex whose removal disconnects G
- Applications?
  - e.g., separation edges and vertices represent single points of failure in a network and are critical to the operation of the network
- Example
  - Separation vertices?
    - DFW, LGA and LAX
  - Separation edges?
    - (DFW,LAX)



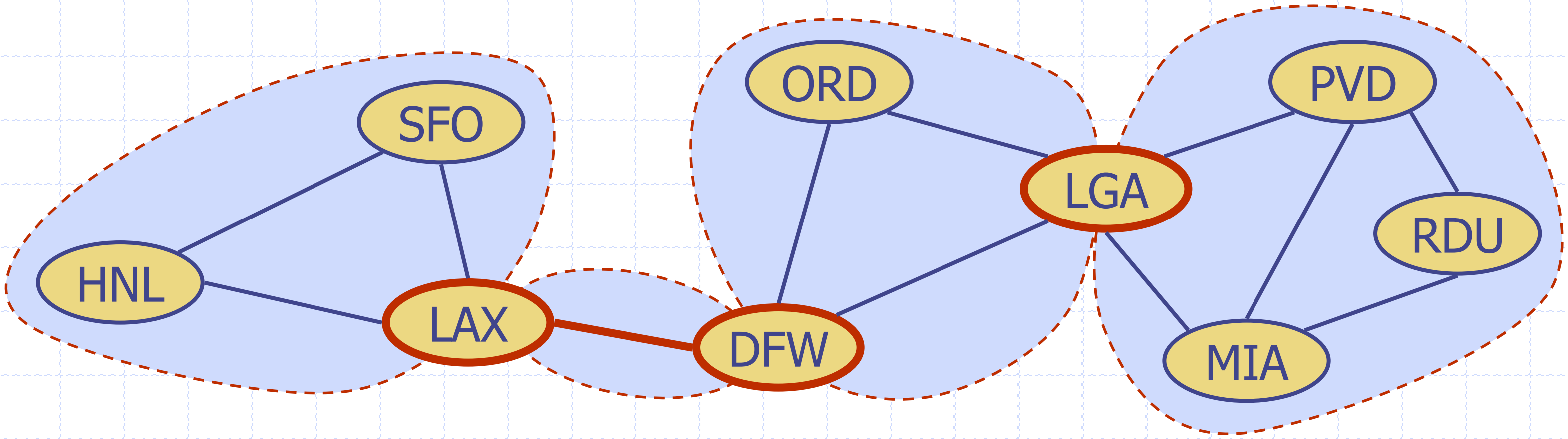
Biconnected Graph

- Equivalent definitions of a biconnected graph G
  - Graph G has no separation edges and no separation vertices
  - For any two vertices u and v of G, there are two disjoint simple paths between u and v (i.e., two simple paths between u and v that share no other vertices or edges)
  - For any two vertices u and v of G, there is a simple cycle containing u and v



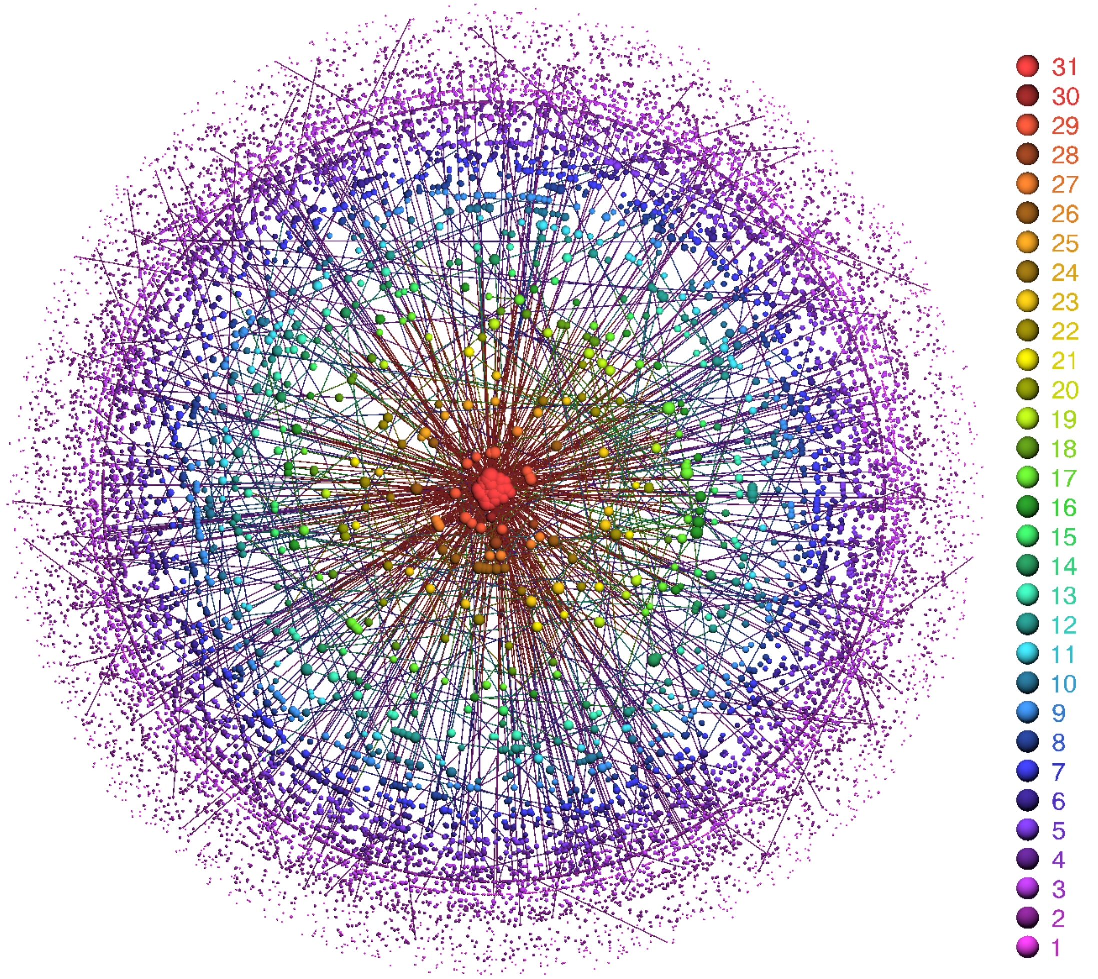
Biconnected Components

- Biconnected component of a graph ***G***
  - A maximal biconnected subgraph of ***G***, or
  - A subgraph consisting of a separation edge of ***G*** and its end vertices
- Interaction of biconnected components
  - An edge belongs to exactly one biconnected component
  - A nonseparation vertex belongs to exactly one biconnected component
  - A separation vertex belongs to two or more biconnected components
- Example of a graph with four biconnected components



More Applications

- Internet



More Applications

- Is this subset of the Internet biconnected?
- How many biconnected subgraphs are there?
- Which are the most critical nodes?
- At least how many links do I have to follow between node A and node B?
- Is node A connected to node B?

Biconnected Components

- Another way of looking at it…

Equivalence Classes

- Given a set S, a relation R on S is a set of ordered pairs of elements of S, i.e., R is a subset of $S\times S$
- An equivalence relation R on S satisfies the following properties

  Reflexive: $(x,x) \in R$

  Symmetric: $(x,y) \in R \Rightarrow (y,x) \in R$

  Transitive: $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$
- An equivalence relation R on S induces a partition of the elements of S into equivalence classes
- Example (connectivity relation among the vertices of a graph):
  - Let V be the set of vertices of a graph G
  - Define the relation

    $C = \{(v,w) \in V\times V$ such that G has a path from v to $w\}$
  - Relation C is an equivalence relation
  - The equivalence classes of relation C are the vertices in each connected component of graph G

Link Relation

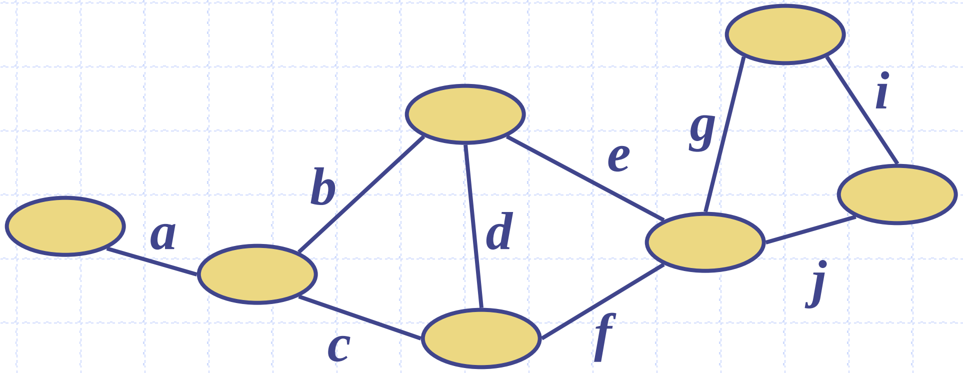
- Edges e and f of connected graph G are linked if
  - $e = f$, or
  - G has a simple cycle containing e and f
    - e.g., is biconnected

Theorem:

The link relation on the edges of a graph is an equivalence relation

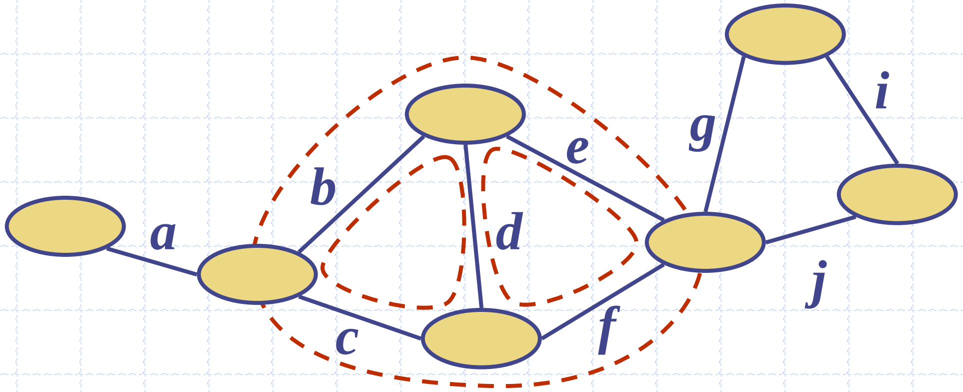
Proof Sketch:

- The reflexive and symmetric properties follow from the definition
- For the transitive property, consider two simple cycles sharing an edge



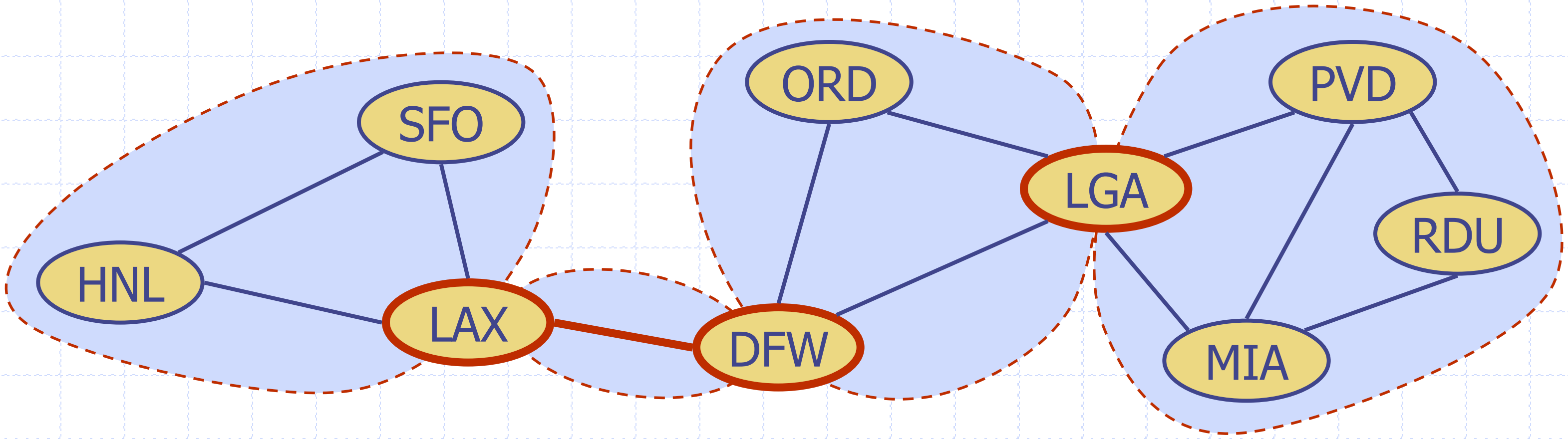
Equivalence classes of linked edges:

$\{a\}$ $\{b, c, d, e, f\}$ $\{g, i, j\}$



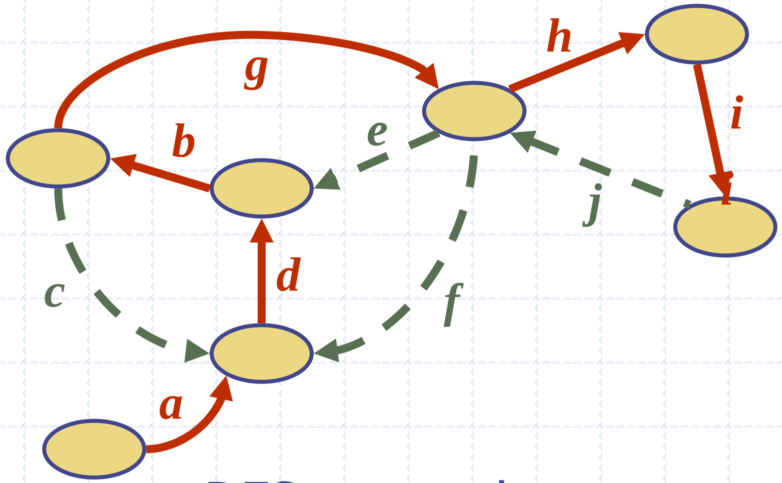
Link Components

- The link components of a connected graph G are the equivalence classes of edges with respect to the link relation
- A biconnected component of G is the subgraph of G induced by an equivalence class of linked edges
- A separation edge is a single-element equivalence class of linked edges
- A separation vertex has incident edges in at least two distinct equivalence classes of linked edge

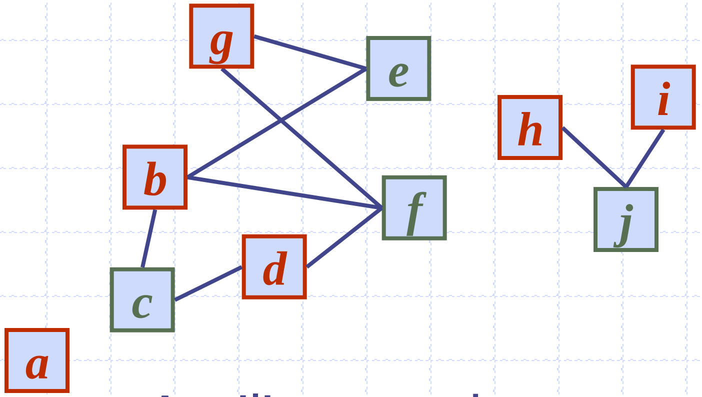


Auxiliary Graph

- Auxiliary graph B for a connected graph G
  - Associated with a DFS traversal of G
  - The vertices of B are the edges of G
  - For each back edge e of G, B has edges $(e,f_1), (e,f_2), \ldots, (e,f_k)$, where $f_1, f_2, \ldots, f_k$ are the discovery edges of G that form a simple cycle with e
  - Its connected components correspond to the the link components of G



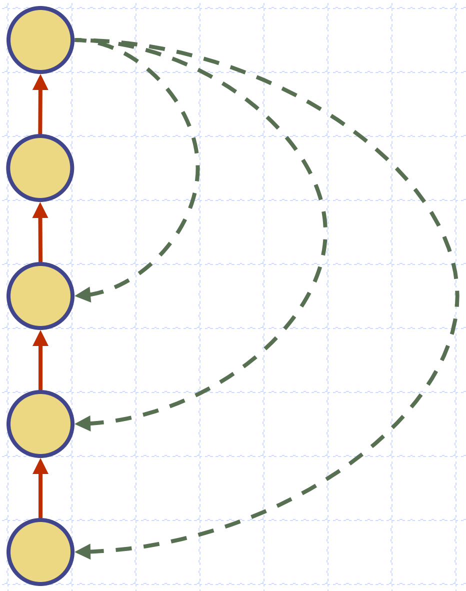
DFS on graph G



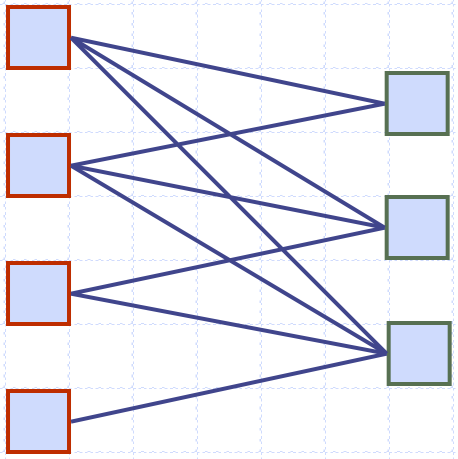
Auxiliary graph B

Auxiliary Graph (cont.)

- In the worst case, the number of edges of the auxiliary graph is proportional to nm



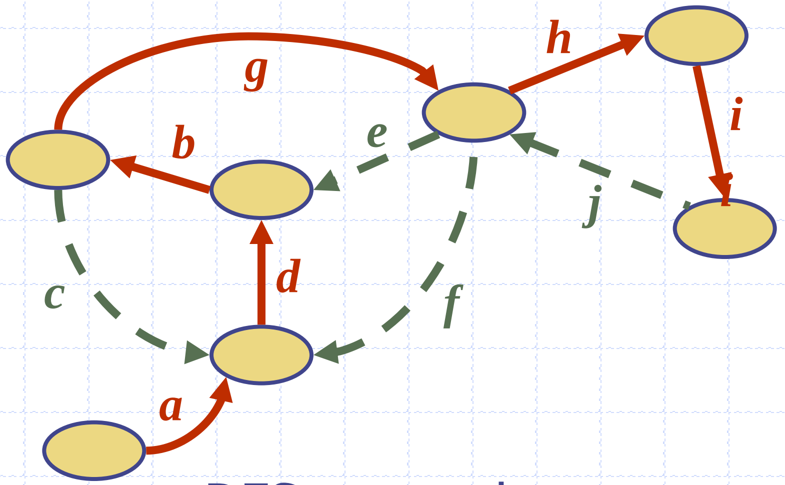
DFS on graph G



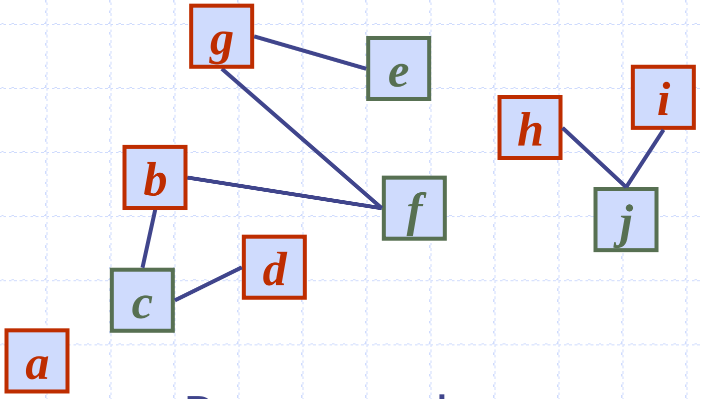
Auxiliary graph B

Proxy Graph

```
Algorithm proxyGraph(G)
    Input connected graph G
    Output proxy graph F for G
    F ← empty graph
    DFS(G, s) { s is any vertex of G}
    for all discovery edges e of G
        F.insertVertex(e)
        setLabel(e, UNLINKED)
    for all vertices v of G in DFS visit order
        for all back edges e = (u,v)
            F.insertVertex(e)
            repeat
                f ← discovery edge with dest. u
                F.insertEdge(e,f,∅)
                if f getLabel(f) = UNLINKED
                    setLabel(f, LINKED)
                    u ← origin of edge f
                else
                    u ← v { ends the loop }
            until u = v
    return F
```



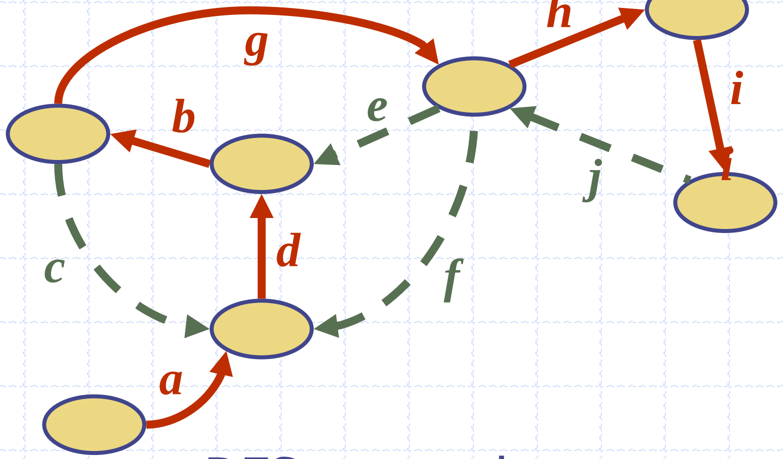
DFS on graph *G*



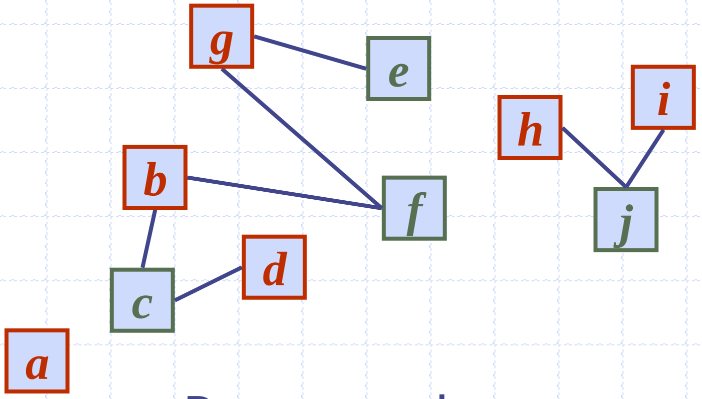
Proxy graph *F*

Proxy Graph (cont.)

- Proxy graph F for a connected graph G
  - Spanning forest of the auxiliary graph B
  - Has m vertices and $O(m)$ edges
  - Can be constructed in $O(n + m)$ time
  - Its connected components (trees) correspond to the the link components of G
- Given a graph G with n vertices and m edges, we can compute the following in $O(n + m)$ time
  - The biconnected components of G
  - The separation vertices of G
  - The separation edges of G



DFS on graph G



Proxy graph F