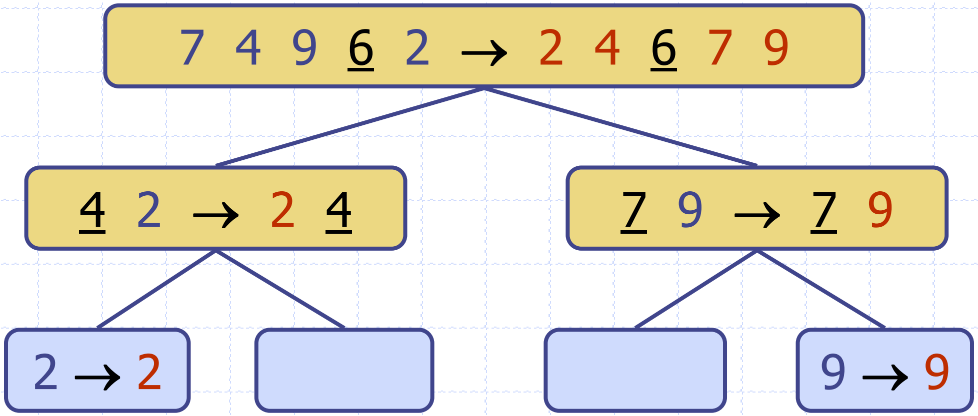
Quick-Sort

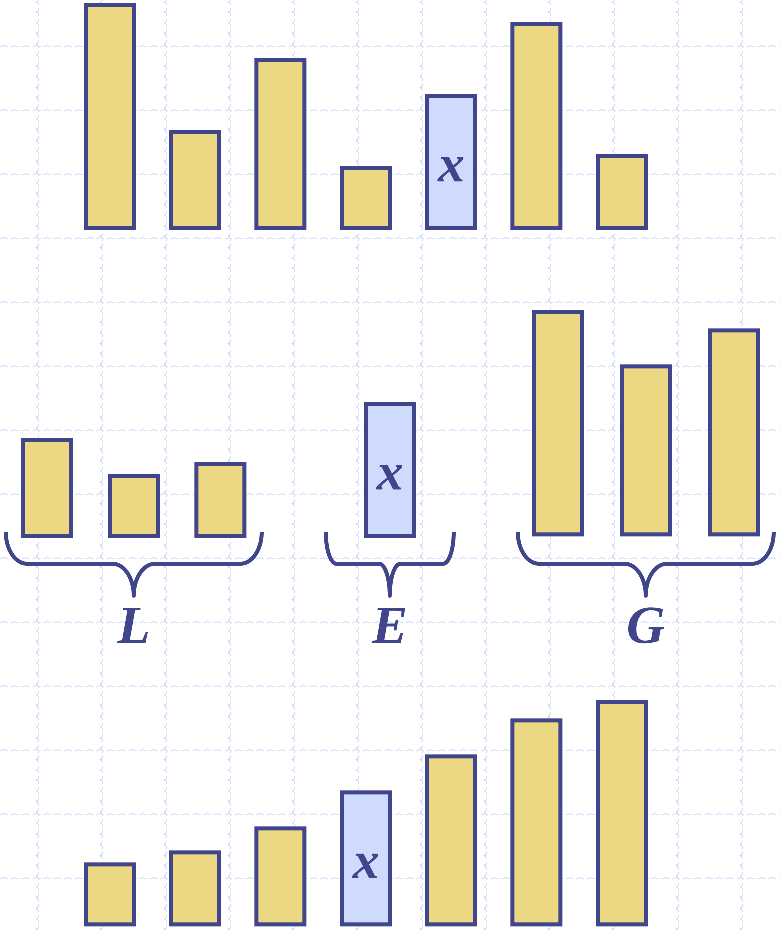


Outline and Reading

- Quick-sort
  - Algorithm
  - Partition step
  - Quick-sort tree
  - Execution example
- Analysis of quick-sort
- In-place quick-sort
- Summary of sorting algorithms

Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Isn't that Merge-Sort?

- Quick-Sort is similar to Merge-Sort but with several key differences – details later

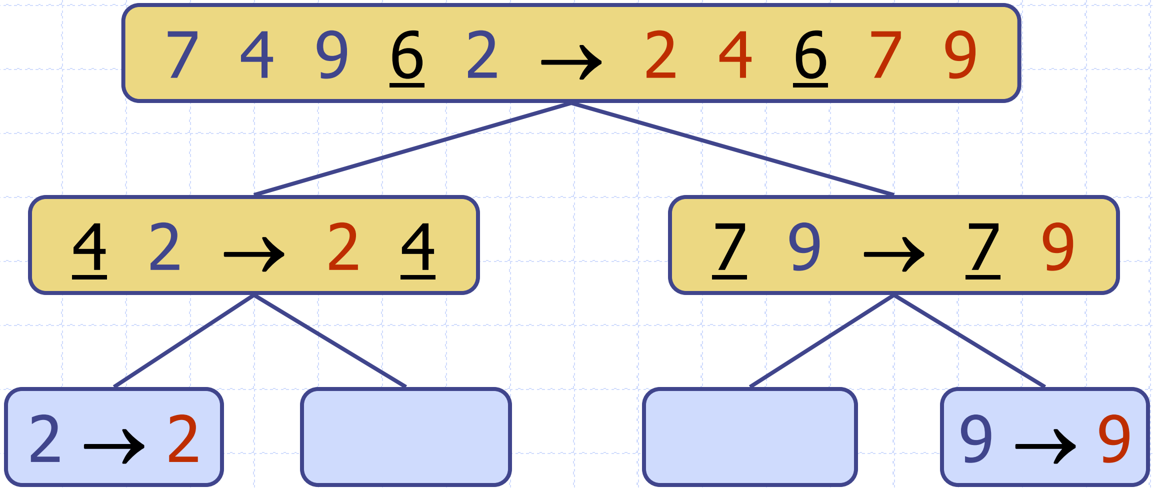
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Algorithm *partition*(S, p)
 Input sequence S, position p of pivot
 Output subsequences L, E, G of the
  elements of S less than, equal to,
  or greater than the pivot, resp.
 $L, E, G \leftarrow$ empty sequences
 $x \leftarrow S.remove(p)$
 while $\neg S.isEmpty()$
  $y \leftarrow S.remove(S.first())$
  if $y < x$
   $L.insertLast(y)$
  else if $y = x$
   $E.insertLast(y)$
  else $\{ y > x \}$
   $G.insertLast(y)$
 return L, E, G

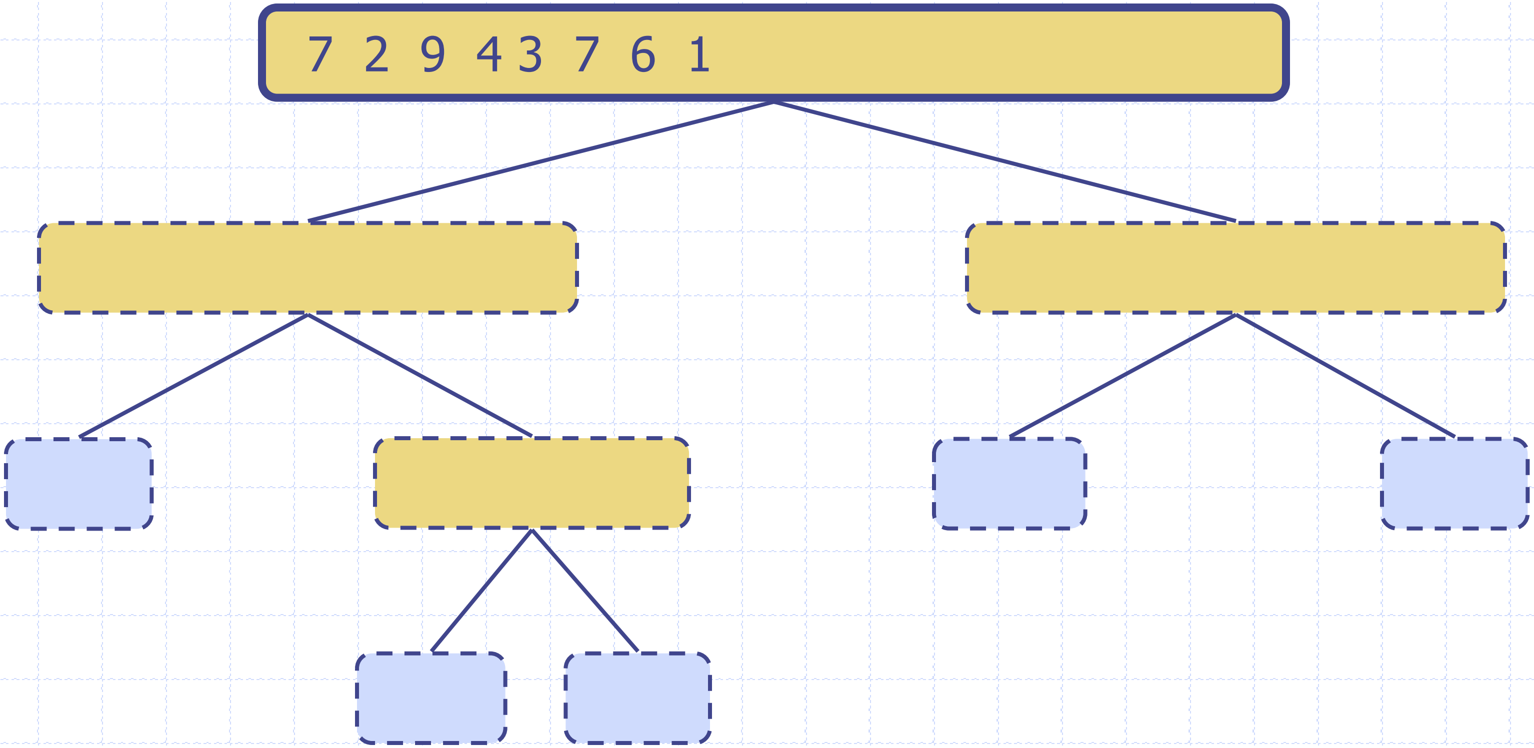
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



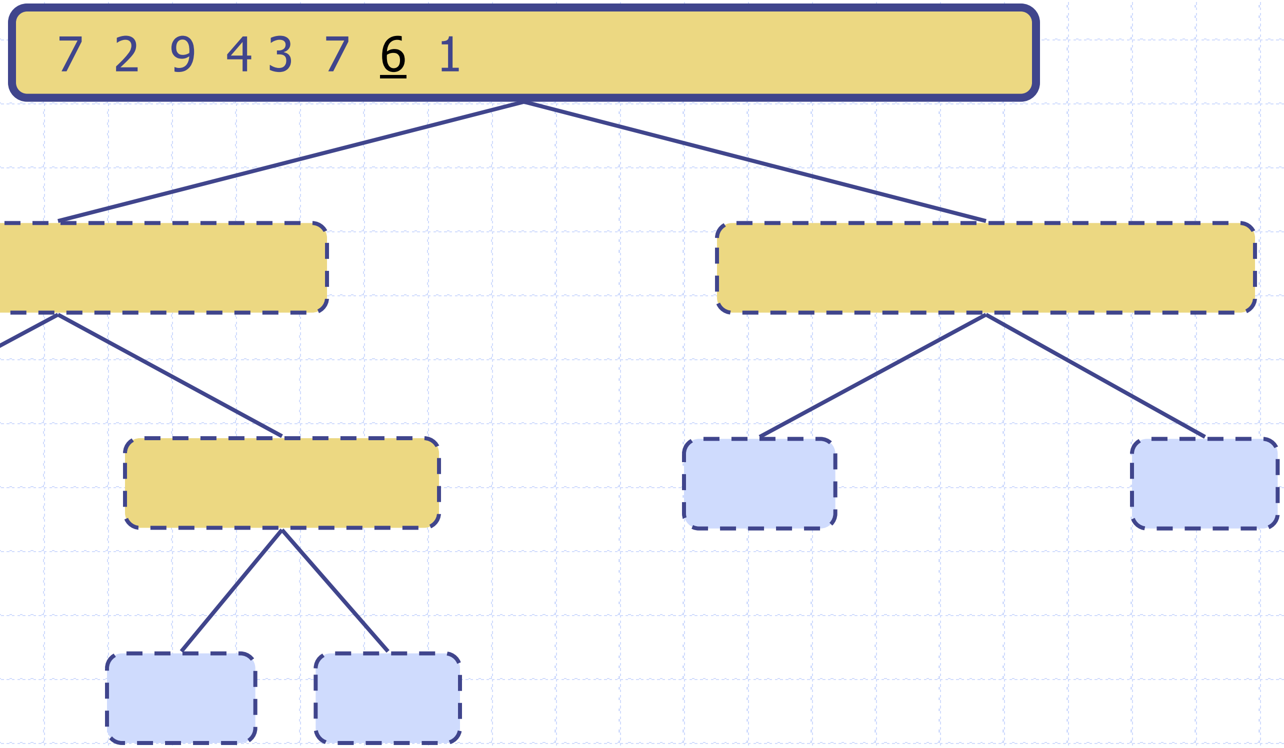
Execution Example

- Pivot selection



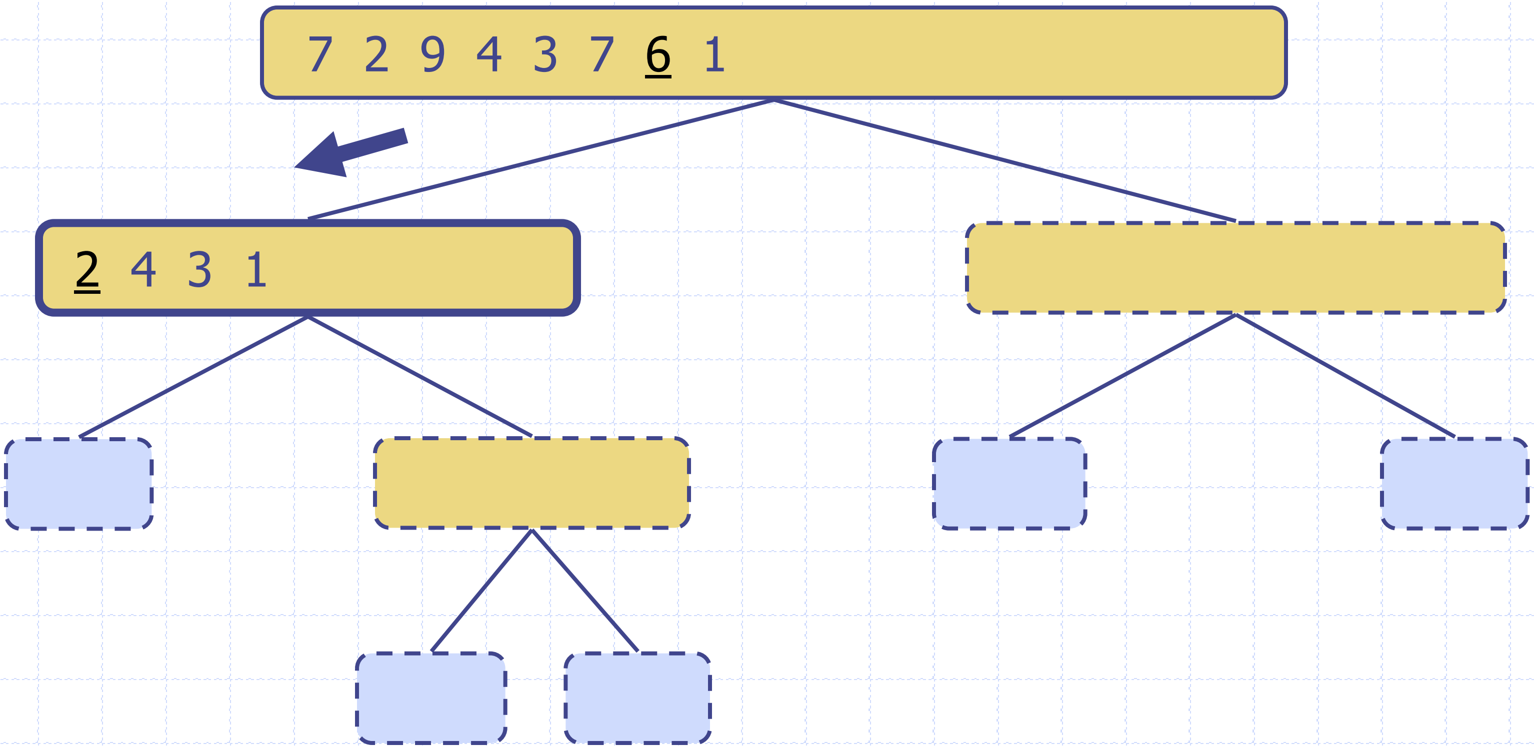
Execution Example

- Pivot selection, e.g. "6"



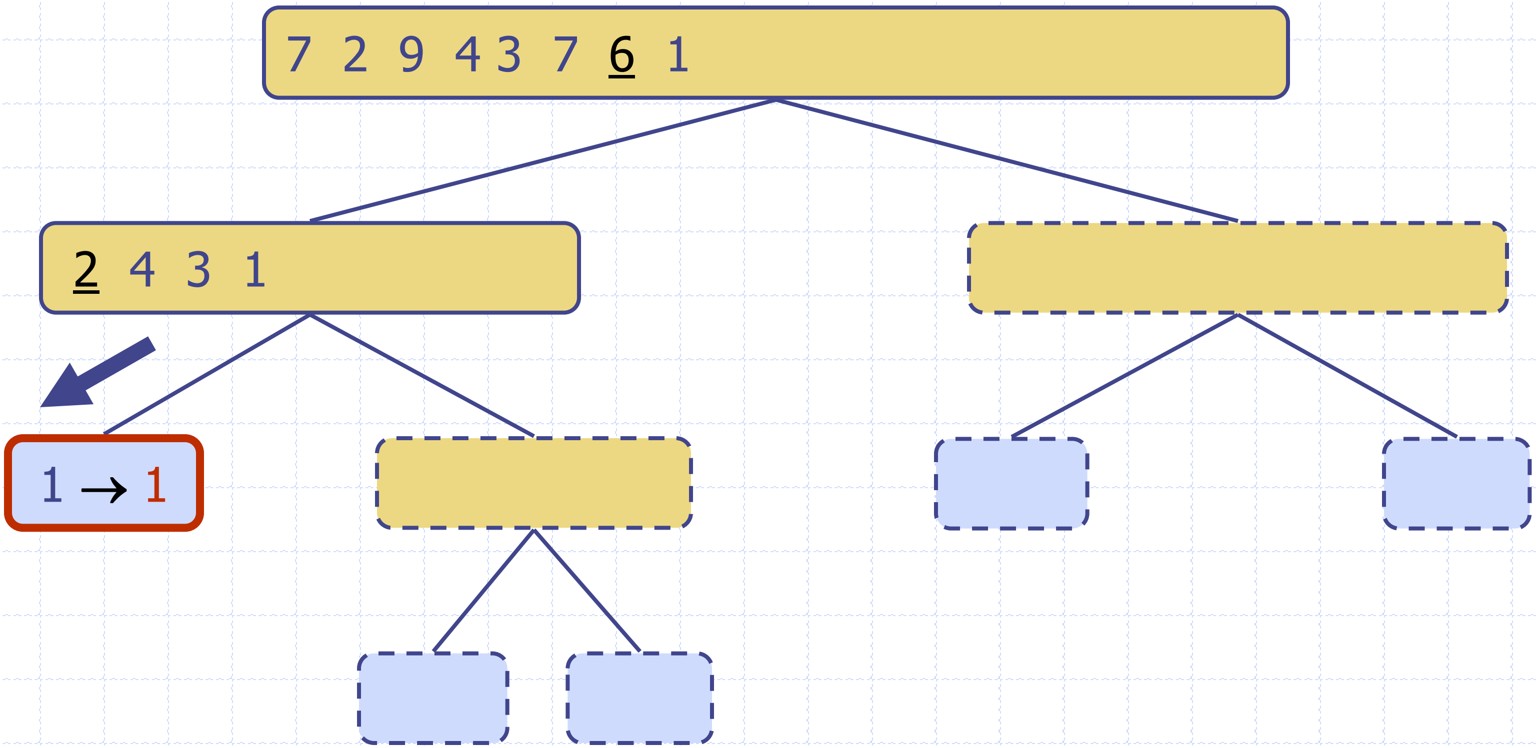
Execution Example (cont.)

- Partition, recursive call, pivot selection



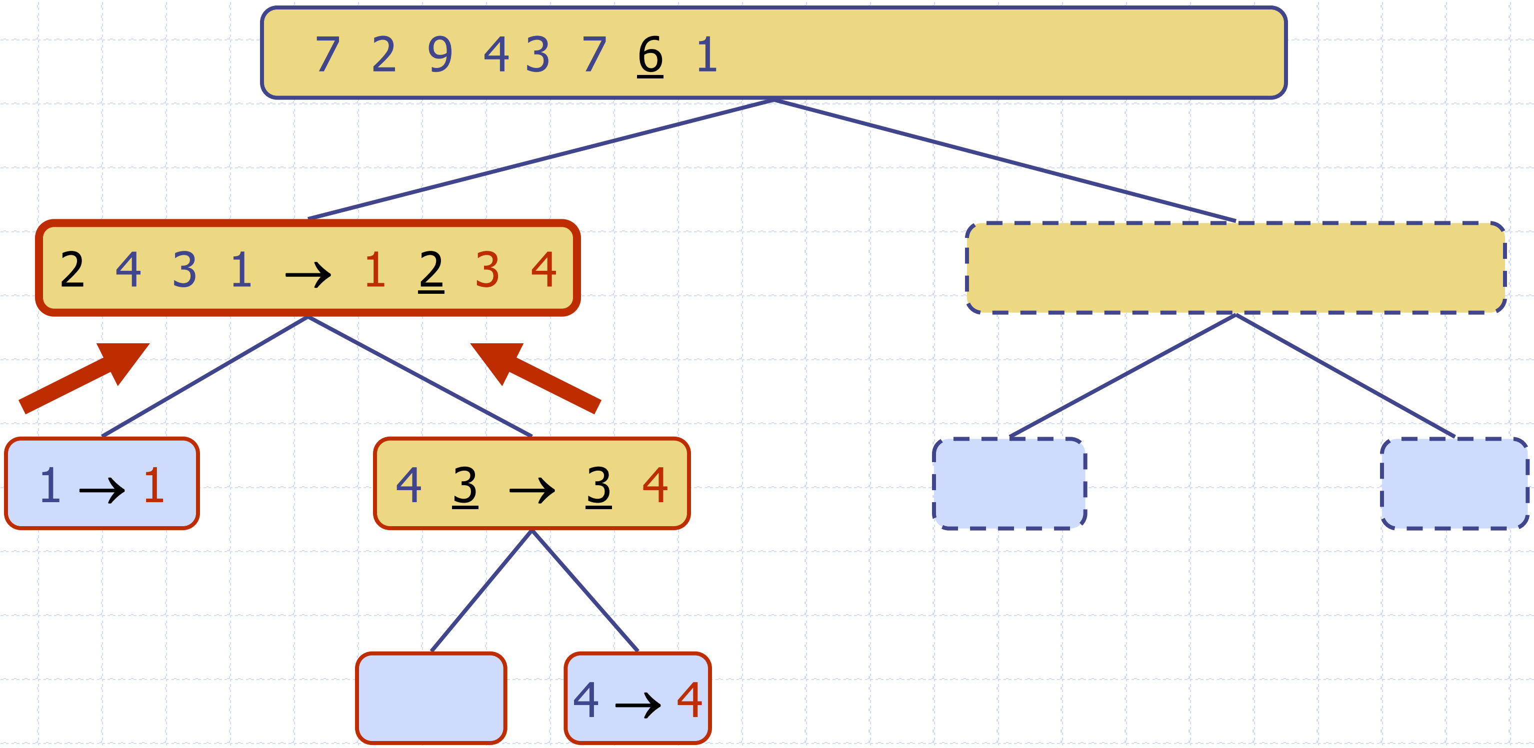
Execution Example (cont.)

- Partition, recursive call, base case



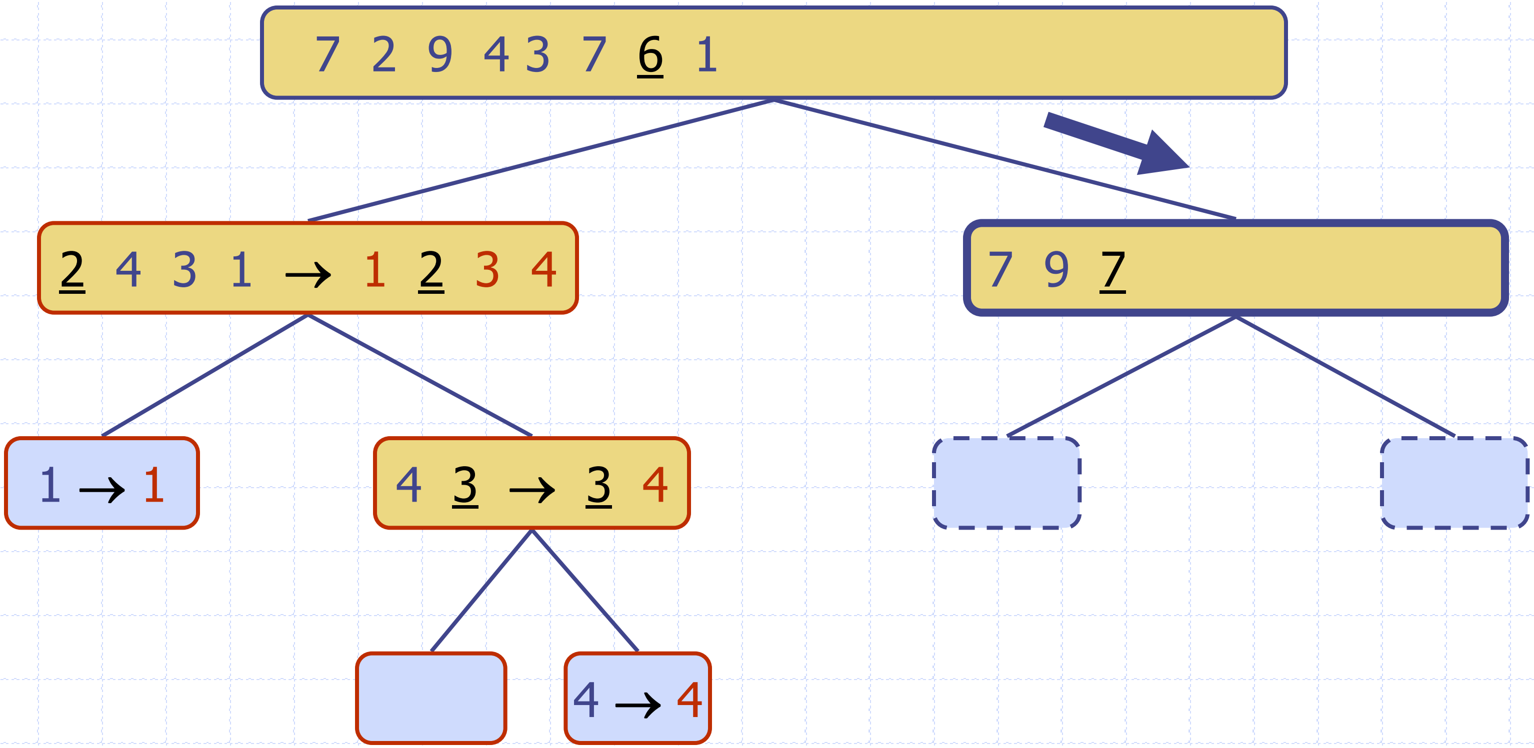
Execution Example (cont.)

- Recursive call, …, base case, join



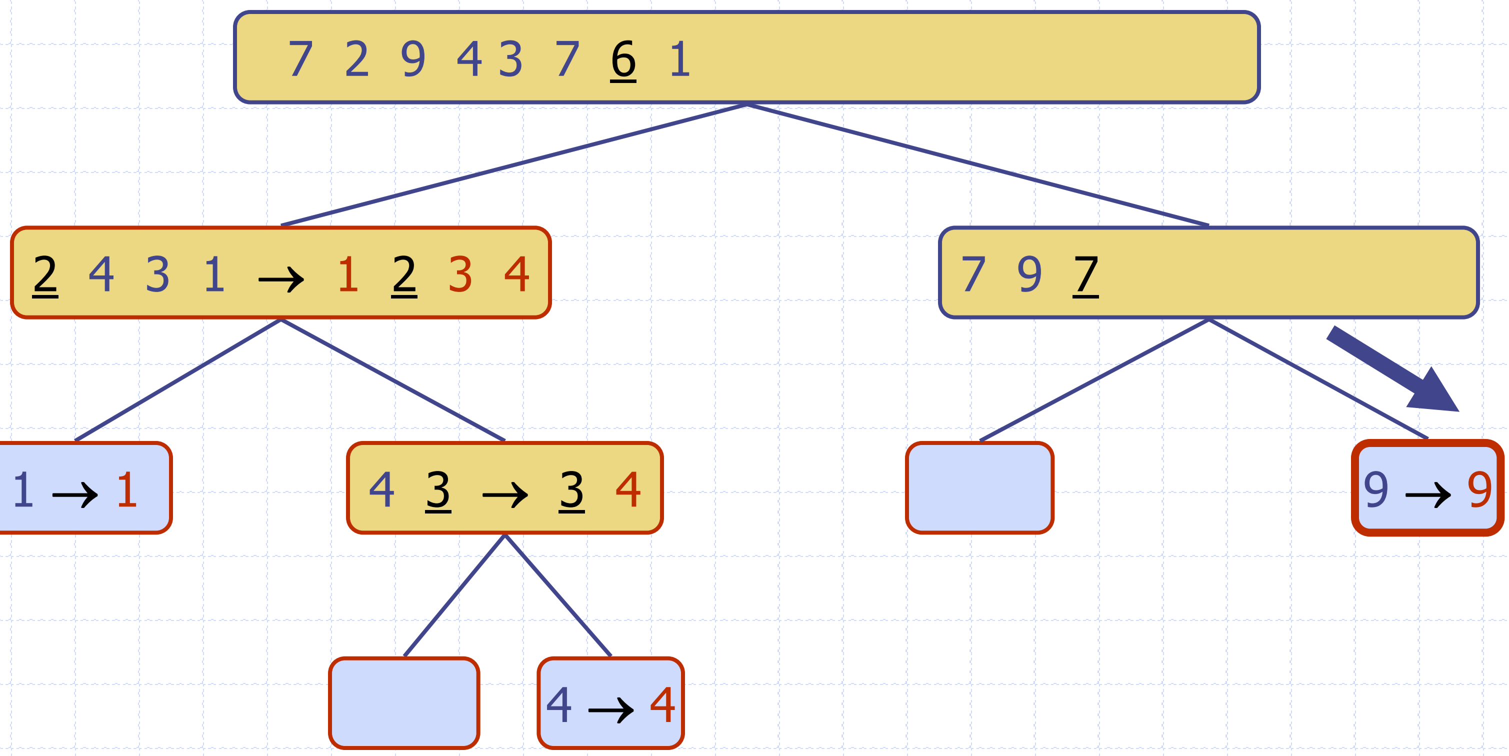
Execution Example (cont.)

- Recursive call, pivot selection



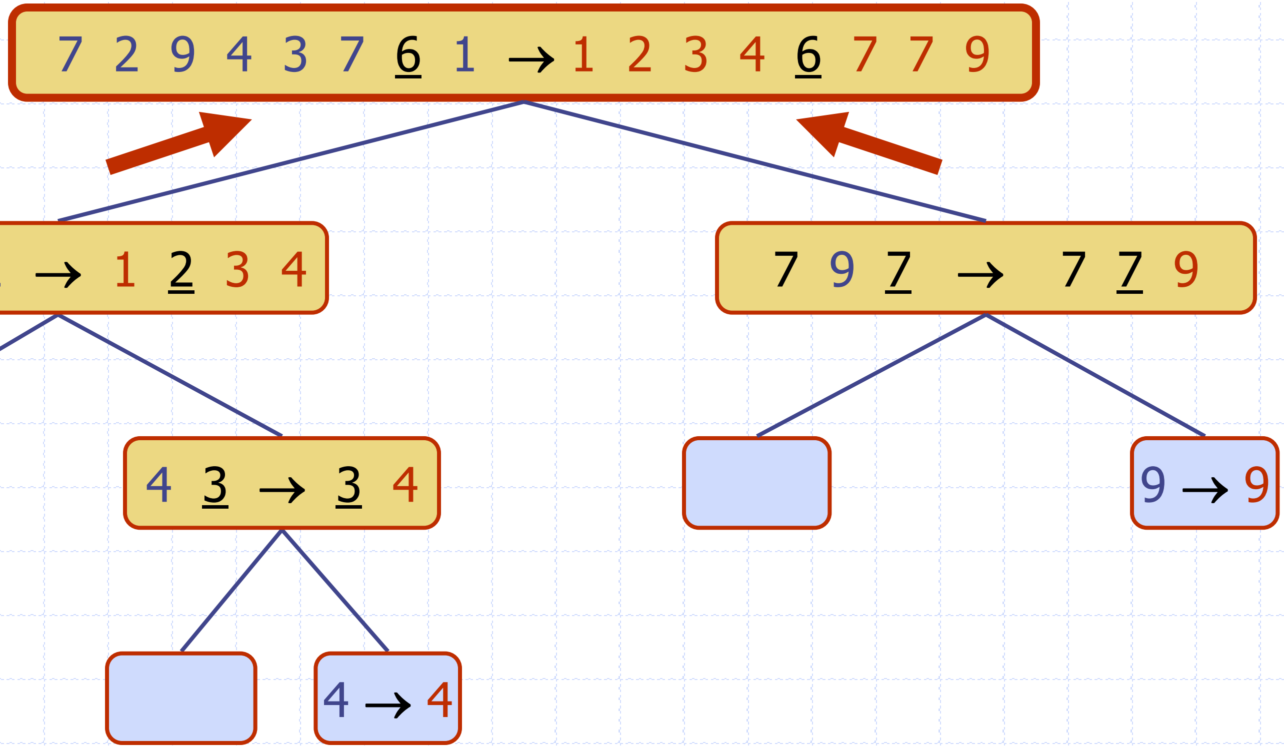
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



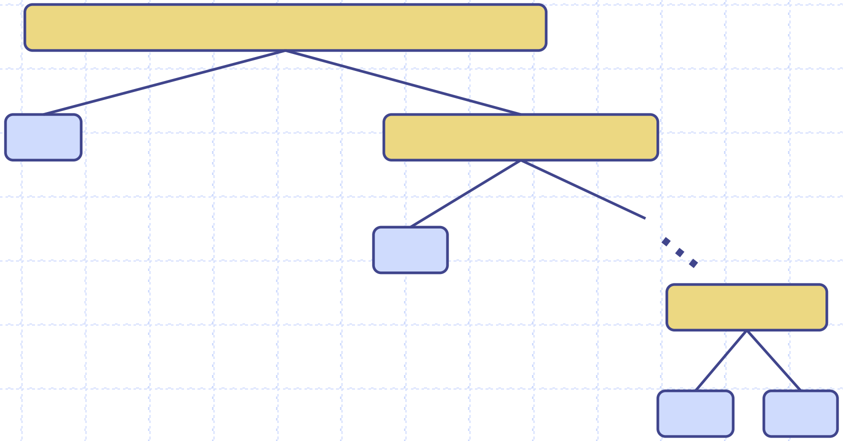
Worst-case Running Time

- When does the worst-case running time occur?
  - When the pivot is the unique minimum or maximum element
- In such cases, one of L and G has size $n - 1$ and the other size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

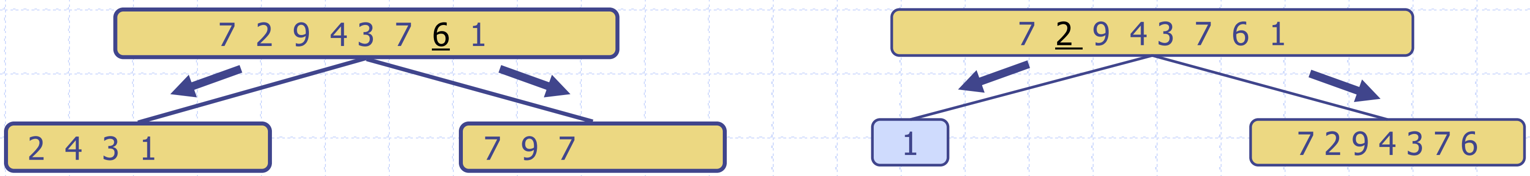
- Thus, the worst-case running time of quick-sort is
  - $O(n^2)$

depth	time
0	n
1	$n - 1$
…	…
$n - 1$	1



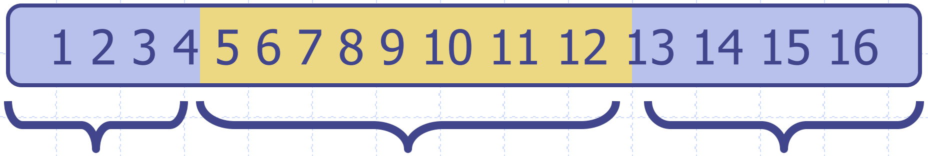
Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$



Good call **Bad call**

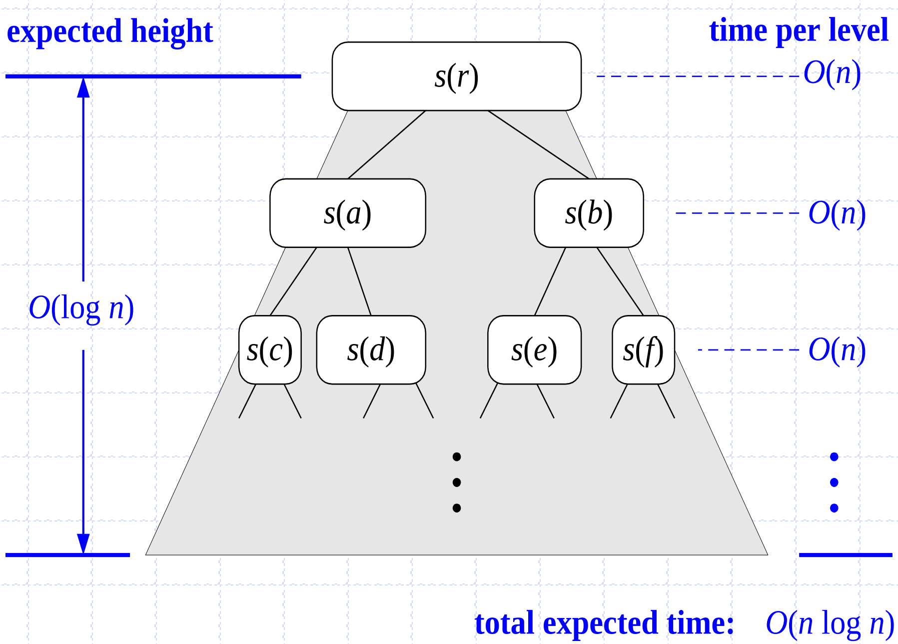
- A call is **good** with what probability?
  - 1/2 of the possible pivots cause good calls:



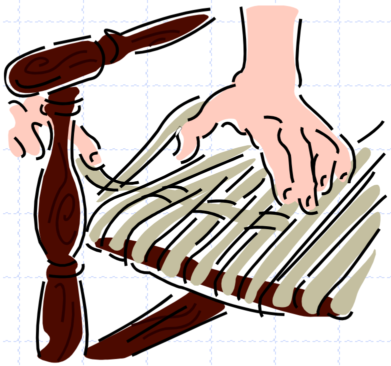
Bad pivots **Good pivots** **Bad pivots**

Expected Running Time, Part 2

- (Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$)
- For a node of depth i, we expect
  - How many of the ancestor calls to be good?
    - $i/2$ ancestors
  - The size of the input sequence for the current call is at most
    - $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - Thus, the expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Hence, the expected running time of quick-sort is $O(n \log n)$



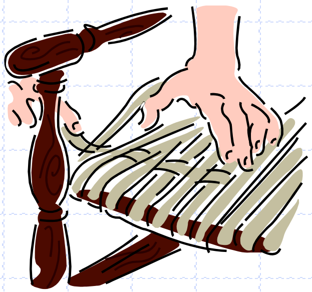
In-Place Quick-Sort



- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(S, l, r)
    Input sequence S, ranks l and r
    Output sequence S with the elements of rank between l and r rearranged in increasing order
    if $l \geq r$
        return
    $i \leftarrow$ a random integer between l and r
    $x \leftarrow S.elemAtRank(i)$
    $(h, k) \leftarrow inPlacePartition(x)$
    $inPlaceQuickSort(S, l, h - 1)$
    $inPlaceQuickSort(S, k + 1, r)$

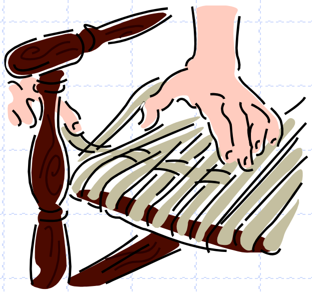
In-Place Partitioning



- Perform the partition with pivot Y using two indices to split S into L and EYG (a similar method can split EYG into E and G)

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 6 9

In-Place Partitioning



- Perform the partition with pivot Y using two indices to split S into L and EYG (a similar method can split EYG into E and G)

j k

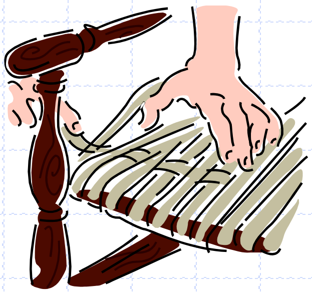
3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 ("Y" pivot = 6)

- Repeat
  - Scan j to the right until finding an element $\geq$ x
  - Scan k to the left until finding an element < x
  - Swap elements at indices j and k

j k

3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

In-Place Partitioning



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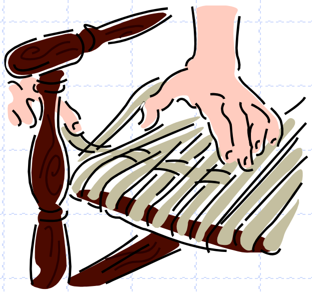
- Repeat
  - Scan j to the right until finding an element $\geq$ x
  - Scan k to the left until finding an element < x
  - Swap elements at indices j and k
- Until j and k meet

j k

3 2 5 1 0 $\boxed{2}$ 3 5 9 $\boxed{7}$ 7 9 8 9 7 **6** 9

In-Place Partitioning



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j k

3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 ("Y" pivot = 6)

- Repeat
  - Scan j to the right until finding an element $\geq$ x
  - Scan k to the left until finding an element < x
  - Swap elements at indices j and k
- Until j and k meet

jk

3 2 5 1 0 2 3 $\boxed{5}$ 9 7 7 9 8 9 7 **6** 9

(done with all partitioning operations and eventually sorting the array)

Isn't all this Merge-Sort?

- Quick-Sort is similar to Merge-Sort but with the following key differences
  - (In-place) Quick-Sort uses at most $O(\log n)$ space vs. Merge-Sort uses $O(n)$ space
  - Quick-Sort is $O(n^2)$ vs. Merge-Sort which is $O(n \log n)$
    - But a "good" Quick-Sort is $O(n \log n)$ or better
  - Quick-Sort implementation details are more friendly towards current computer architecture and thus in practice the "constant" is very small

3-Way Quicksort

- Instead of partitioning into 2 sets each time, partition into 3 sets:
  - Less than set
  - Equal than set
  - Greater than set
- Helps slightly with many repeated keys

Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	◆ in-place ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$	◆ in-place, O(n) for almost sorted ◆ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	◆ in-place, randomized ◆ fastest (good for large inputs)
heap-sort	$O(n \log n)$	◆ in-place, O(n) first results ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$	◆ sequential data access ◆ fast (good for huge inputs)

Demo

- https://www.youtube.com/watch?v=kPRA0W1kECg
- https://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html
- https://www.toptal.com/developers/sorting-algorithms