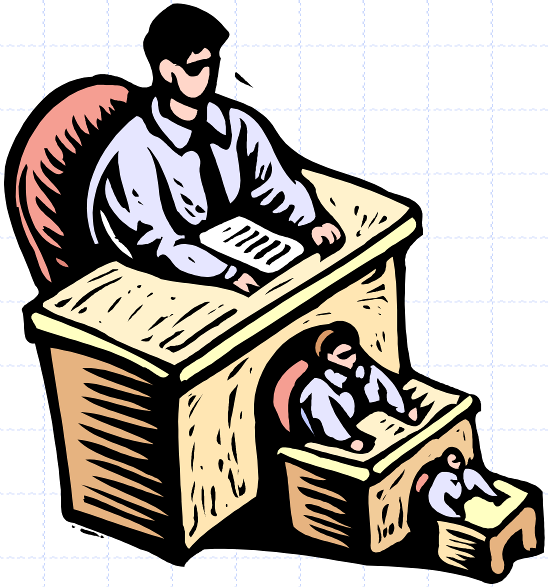
Recursion and Fibonacci Sequence



The Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example--the factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & \textit{else} \end{cases}$$

- As a C++ method:

```
// recursive factorial function
int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;      // basis case
    else return  n  *  recursiveFactorial(n- 1);  // recursive case
}
```

Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

Algorithm LinearSum(*A, n*):

Input:

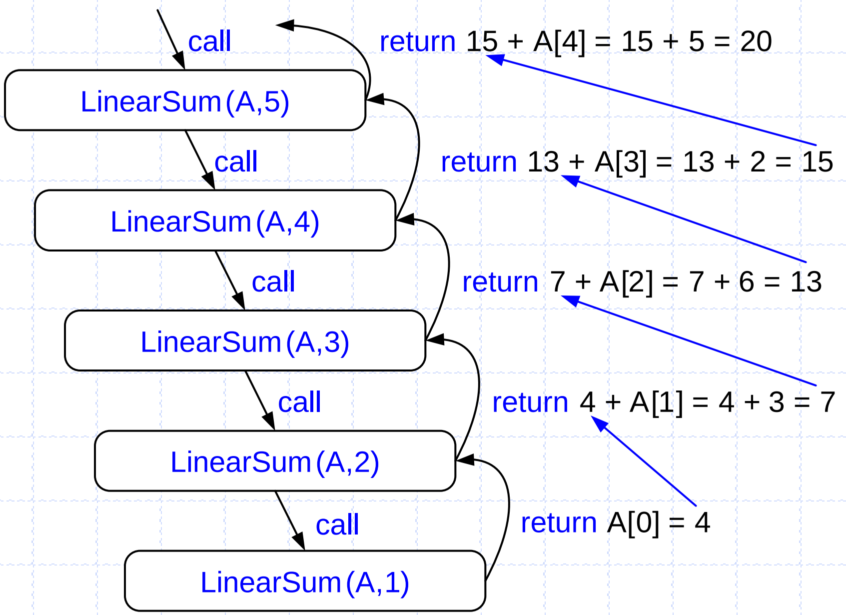
A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements

Output:

The sum of the first *n* integers in *A*

if *n* = 1 **then**
  return *A*[0]
else
  return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

return

Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional parameters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(A, i, j), not ReverseArray(A).

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

$$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

$$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$$

$$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$$

Recursive Squaring Method

Algorithm Power(x, n):

  Input: A number x and integer $n = 0$

  Output: The value x^n

  if $n = 0$ **then**

    return 1

  if n is odd **then**

    $y = $ Power(x, $(n - 1)/2$)

    return $x \cdot y \cdot y$

  else

    $y = $ Power(x, $n/2$)

    return $y \cdot y$

Analysis

Algorithm Power(x, n):
    Input: A number x and integer $n = 0$
    Output: The value x^n
   if $n = 0$ **then**
       return 1
   if n is odd **then**
       y = Power(x, $(n - 1)/2$)
       return $x \cdot y \cdot y$
   else
       y = Power(x, $n/2$)
       return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).

Reversing an Array

Algorithm ReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

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if $i < j$ **then**

Swap $A[i]$ and $A[j]$

ReverseArray(A, $i + 1$, $j - 1$)

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Tail Recursion

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- Example:

Algorithm IterativeReverseArray(A, i, j):

Input: An array A and nonnegative integer indices i and j

Output: The reversal of the elements in A starting at index i and ending at j

while $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.

```
---- 0          ----- 0         --- 0
-               -               -
--              --              --
-               -               -
---             ---             --- 1
-               -               -
--              --              --
-               -               -
---- 1          ----            --- 2
-               -               -
--              --              --
-               -               -
---             ---             --- 3
-               -
--              --
-               -
---- 2          ----- 1
```

A Binary Recursive Method for Drawing Ticks

```
    // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
    // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
   for (int i = 0; i < tickLength; i++)
       System.out.print("-");
   if (tickLabel >= 0) System.out.print(" " + tickLabel);
   System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
   if (tickLength > 0) {                        // stop when length drops to 0
       drawTicks(tickLength- 1);      // recursively draw left ticks
       drawOneTick(tickLength);       // draw center tick
       drawTicks(tickLength- 1);      // recursively draw right ticks
   }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
   drawOneTick(majorLength, 0);  // draw tick 0 and its label
   for (int i = 1; i <= nInches; i++)              {
       drawTicks(majorLength- 1);  // draw ticks for this inch
       drawOneTick(majorLength, i);               // draw tick i and its label
   }
}
```

Note the two recursive calls

Fibonacci Numbers

- Useful for
  - Stock market
  - **Search**
  - And more…

Fibonacci Search (Kiefer et al. 1953)

- Similar to binary search, but
  - Instead of dividing an array by the midpoint during search,
  - You use the largest $F_N \leq$ midpoint
  - Since

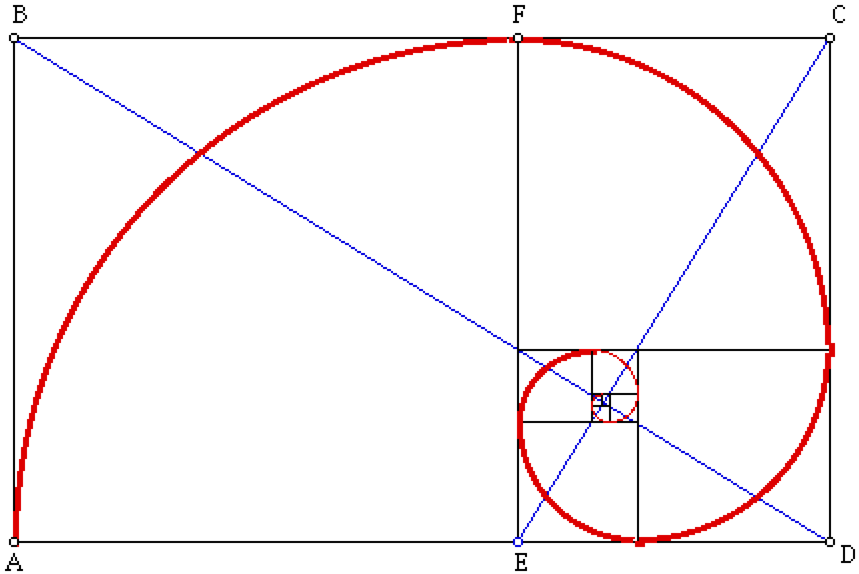
A	B	B / A
2	3	1.5
3	5	1.666666666...
5	8	1.6
8	13	1.625
13	21	1.615384615...
...	...	...
144	233	1.618055556...
233	377	1.618025751...
...	...	...

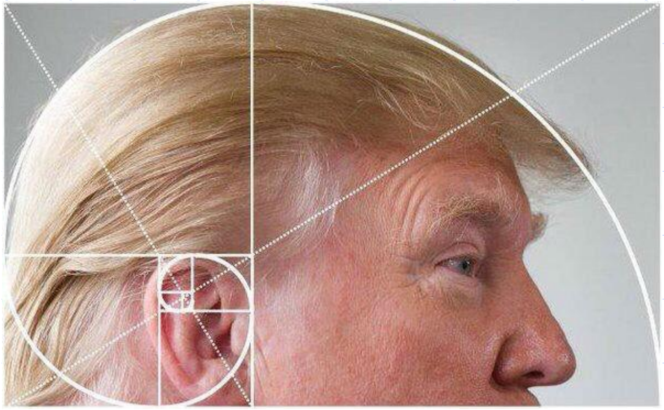
Fibonacci Search (Kiefer et al. 1953)

- Similar to binary search, but
  - Instead of dividing an array by the midpoint during search,
  - You use the largest $F_n \leq$ midpoint
  - This results in dividing the area roughly by the Golden Ratio (e.g., 62% and 38%)
  - In practice has slightly better average time performance (still O(logn))

Golden Ratio

- 1.61803398875





Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \quad \text{for } i > 1.$$

- Recursive algorithm (first attempt):

Algorithm BinaryFib(k):
Input: Nonnegative integer k
Output: The kth Fibonacci number F_k
if $k = 1$ **then**
return k
else
return BinaryFib(k - 1) + BinaryFib(k - 2)

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion in this case

```
Algorithm LinearFibonacci(k):
   Input: A nonnegative integer k
   Output: Pair of Fibonacci numbers (F_k , F_{k−1})
   if k = 1 then
      return (k, 0)
   else
      (i,  j)  =  LinearFibonacci(k − 1)
      return (i +j, i)
```

- LinearFibonacci makes k−1 recursive calls
- This is also a form of "dynamic programming"

Even Better Fibonacci Algorithm

- Binet's Fibonacci number formula:

$u_n = u_{n-1} + u_{n-2}$ for $n > 1$

where

$u_0 = 0,$

$u_1 = 1,$

$$u_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n\sqrt{5}}$$