

Dynamic Programming



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- ◆ It is a general algorithm design paradigm (different than divide & conquer)
 - Rather than give the general structure, let us first give a motivating example:

Fibonacci Sequence



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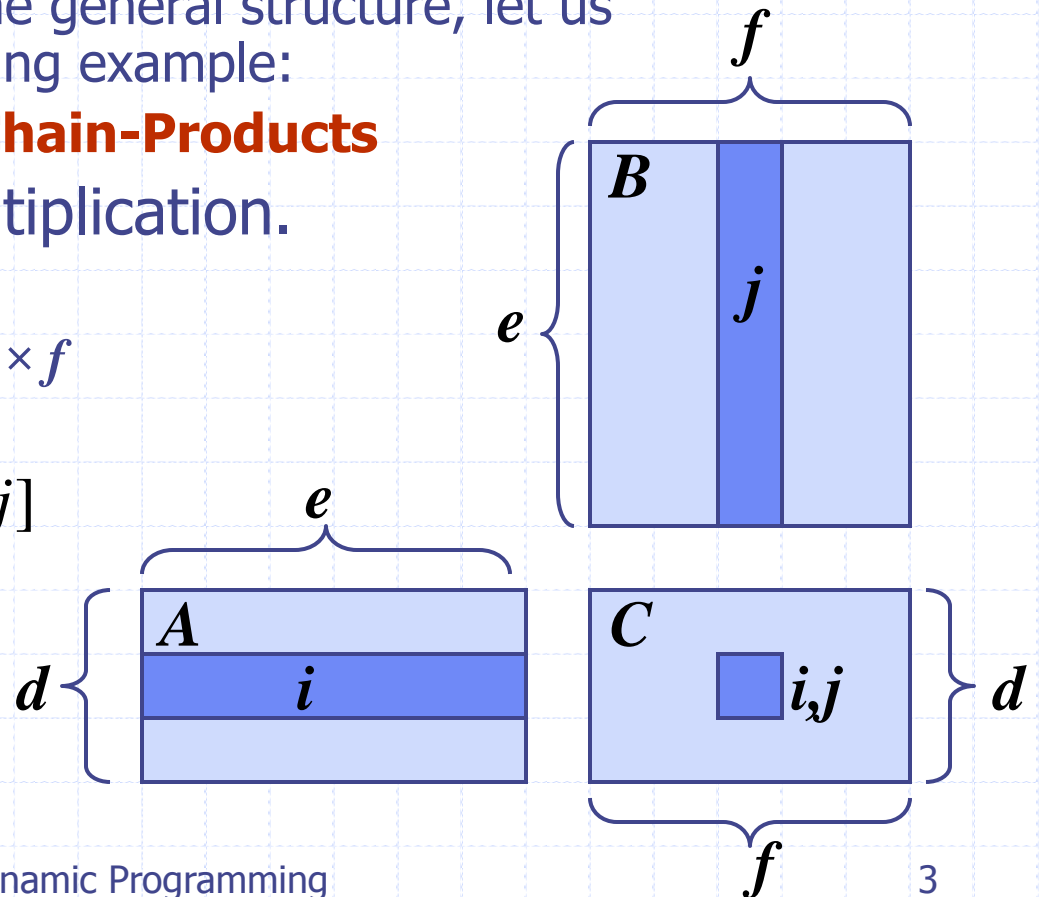
Matrix Chain-Products

- ◆ Review: Matrix Multiplication.

- $C = A * B$
- A is $d \times e$ and B is $e \times f$

$$C[i, j] = \sum_{k=0}^{e-1} A[i, k] * B[k, j]$$

- $O(def)$ time



Matrix Chain-Products



◆ Matrix Chain-Product:

- Compute $A = A_0 * A_1 * \dots * A_{n-1}$
- A_i is $d_i \times d_{i+1}$
- Problem: How to parenthesize?

◆ Example

- B is 3×100
- C is 100×5
- D is 5×5
- $(B * C) * D$ takes $1500 + 75 = 1575$ ops
- $B * (C * D)$ takes $1500 + 2500 = 4000$ ops

An Enumeration Approach



◆ Algorithm:

- Try all possible ways to parenthesize $A = A_0 * A_1 * \dots * A_{n-1}$
- Calculate number of ops for each one
- Pick the one that is best

◆ Running time:

- The number of parenthesizations is equal to the number of binary trees with n nodes
- This is **exponential**!
- It is called the Catalan number, and it is almost 4^n .
- **This is a terrible algorithm!**



A Greedy Approach

- ◆ Idea #1: repeatedly select the product that uses (up) the most operations.
- ◆ Counter-example:
 - A is 10×5
 - B is 5×10
 - C is 10×5
 - D is 5×10
 - Greedy idea #1 gives $(A*B)*(C*D)$, which takes $500+1000+500 = 2000$ ops
 - $A*((B*C)*D)$ takes $500+250+250 = 1000$ ops



Another Greedy Approach

- ◆ Idea #2: repeatedly select the product that uses the fewest operations.
- ◆ Counter-example:
 - A is 101×11
 - B is 11×9
 - C is 9×100
 - D is 100×99
 - Greedy idea #2 gives $A*((B*C)*D)$, which takes $109989 + 9900 + 108900 = 228789$ ops
 - $(A*B)*(C*D)$ takes $9999 + 89991 + 89100 = 189090$ ops
- ◆ The greedy approach is not giving us the optimal value.

A “Recursive” Approach



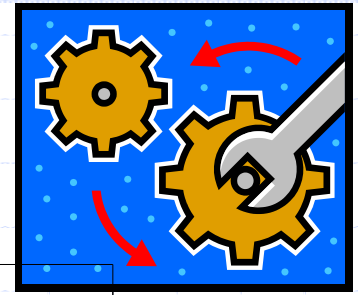
◆ Define **subproblems**:

- Find the best parenthesization of $A_i * A_{i+1} * \dots * A_j$.
- Let $N_{i,j}$ denote the number of operations done by this subproblem.
- The optimal solution for the whole problem is $N_{0,n-1}$.

◆ **Subproblem optimality**: The optimal solution can be defined in terms of optimal subproblems

- There has to be a final multiplication (root of the expression tree) for the optimal solution.
- Say, the final multiply is at index i : $(A_0 * \dots * A_i) * (A_{i+1} * \dots * A_{n-1})$.
- Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply.
- If the global optimum did not have these optimal subproblems, we could define an even better “optimal” solution.

A Characterizing Equation

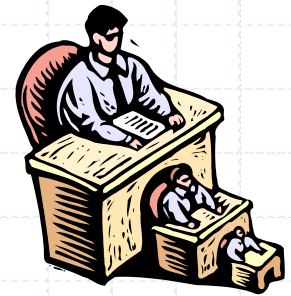


- ◆ The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- ◆ Let us consider all possible places for that final multiply:
 - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
 - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \leq k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- ◆ Note that subproblems are not independent--the **subproblems overlap**.

A Dynamic Programming Algorithm



- ◆ Since subproblems overlap, we don't use recursion.
- ◆ Instead, we construct optimal subproblems "bottom-up."
- ◆ $N_{i,i}$'s are easy, so start with them
- ◆ Then do length 2,3,... subproblems, and so on.
- ◆ The running time is $O(n^3)$

Algorithm *matrixChain*(S):

Input: sequence S of n matrices to be multiplied

Output: number of operations in an optimal parenthization of S

for $i \leftarrow 1$ **to** $n-1$ **do**

$N_{i,i} \leftarrow 0$

for $b \leftarrow 1$ **to** $n-1$ **do**

for $i \leftarrow 0$ **to** $n-b-1$ **do**

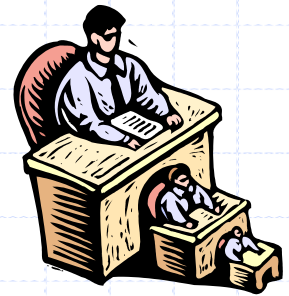
$j \leftarrow i+b$

$N_{i,j} \leftarrow +\text{infinity}$

for $k \leftarrow i$ **to** $j-1$ **do**

$N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$

A Dynamic Programming Algorithm Visualization

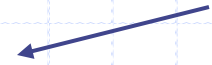


$$N_{i,j} = \min_{i \leq k < j} \{N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$$

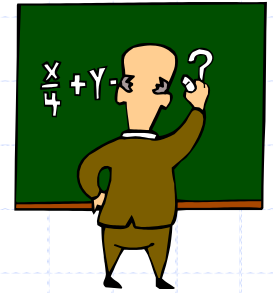
- ◆ The bottom-up construction fills in the N array by diagonals
- ◆ $N_{i,j}$ gets values from previous entries in i-th row and j-th column
- ◆ Filling in each entry in the N table takes $O(n)$ time.
- ◆ Total run time: $O(n^3)$
- ◆ Getting actual parenthesization can be done by remembering "k" for each N entry

N	0	1	2				j	...	n-1
0									
1									
...									
i									
n-1									

answer



The General Dynamic Programming Technique



- ◆ Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
 - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j , k , l , m , and so on.
 - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
 - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

The Longest Common Subsequence (LCS) Problem

- ◆ Given two strings X and Y , the longest common subsequence (LCS) problem is to find a longest subsequence common to both X and Y

Subsequences

- ◆ A ***subsequence*** of a character string $x_0x_1x_2\dots x_{n-1}$ is a string of the form $x_{i_1}x_{i_2}\dots x_{i_k}$, where $i_j < i_{j+1}$.
- ◆ Not the same as substring!
- ◆ Example String: ABCDEFGHIJK
 - Subsequence: ACEGIJK
 - Subsequence: DFGHK
 - Not subsequence: DAGH

The Longest Common Subsequence (LCS) Problem

- ◆ Given two strings X and Y , the longest common subsequence (LCS) problem is to find a longest subsequence common to both X and Y
- ◆ Has applications to DNA similarity testing (alphabet is $\{A, C, G, T\}$)
- ◆ Example: ABCDEFG and XZACKDFWGH have ACDFG as a longest common subsequence

A Poor Approach to the LCS Problem

◆ A Brute-force solution:

- Enumerate all subsequences of X
- Test which ones are also subsequences of Y
- Pick the longest one.

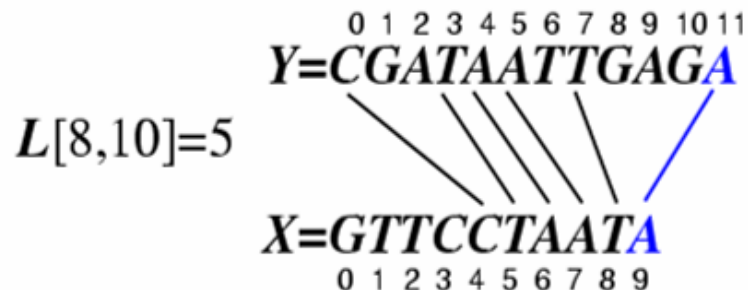
◆ Analysis:

- If X is of length n , then it has 2^n subsequences
- This is an exponential-time algorithm!

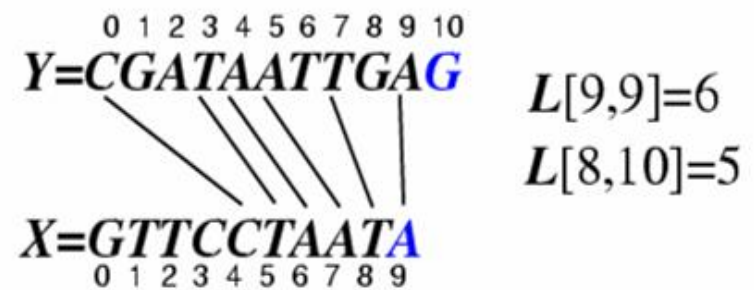
A Dynamic-Programming Approach to the LCS Problem

- ◆ Define $L[i,j]$ to be the length of the longest common subsequence of $X[0..i]$ and $Y[0..j]$.
- ◆ Allow for -1 as an index, so $L[-1,k] = 0$ and $L[k,-1]=0$, to indicate that the null part of X or Y has no match with the other.
- ◆ Then we can define $L[i,j]$ in the general case as follows:
 1. If $x_i = y_j$, then $L[i,j] = L[i-1,j-1] + 1$ (we can add this match)
 2. If $x_i \neq y_j$, then $L[i,j] = \max\{L[i-1,j], L[i,j-1]\}$ (we have no match here)

Case 1:



Case 2:



An LCS Algorithm

Algorithm LCS(X,Y):

Input: Strings X and Y with n and m elements, respectively

Output: For $i = 0, \dots, n-1$, $j = 0, \dots, m-1$, the length $L[i, j]$ of a longest string that is a subsequence of both the string $X[0..i] = x_0x_1x_2\dots x_i$ and the string $Y[0..j] = y_0y_1y_2\dots y_j$

for $i = 1$ to $n-1$ **do**

$L[i, -1] = 0$

for $j = 0$ to $m-1$ **do**

$L[-1, j] = 0$

for $i = 0$ to $n-1$ **do**

for $j = 0$ to $m-1$ **do**

if $x_i = y_j$ **then**

$L[i, j] = L[i-1, j-1] + 1$

else

$L[i, j] = \max\{L[i-1, j], L[i, j-1]\}$

return array L

Visualizing the LCS Algorithm

<https://www.cs.usfca.edu/~galles/visualization/DPLCS.html>

<i>L</i>	-1	0	1	2	3	4	5	6	7	8	9	10	11
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	2	2	2	2	2	2	2	2	2
2	0	0	1	1	2	2	2	3	3	3	3	3	3
3	0	1	1	1	2	2	2	3	3	3	3	3	3
4	0	1	1	1	2	2	2	3	3	3	3	3	3
5	0	1	1	1	2	2	2	3	4	4	4	4	4
6	0	1	1	2	2	3	3	3	4	4	5	5	5
7	0	1	1	2	2	3	4	4	4	4	5	5	6
8	0	1	1	2	3	3	4	5	5	5	5	5	6
9	0	1	1	2	3	4	4	5	5	5	6	6	6

0 1 2 3 4 5 6 7 8 9 10 11
Y=CGATAATTGAGA
 0 1 2 3 4 5 6 7 8 9
X=GTTCCTAATA

Analysis of LCS Algorithm

- ◆ We have two nested loops
 - The outer one iterates n times
 - The inner one iterates m times
 - A constant amount of work is done inside each iteration of the inner loop
 - Thus, the total running time is $O(nm)$
- ◆ Answer is contained in $L[n,m]$ (and the subsequence can be recovered from the L table).