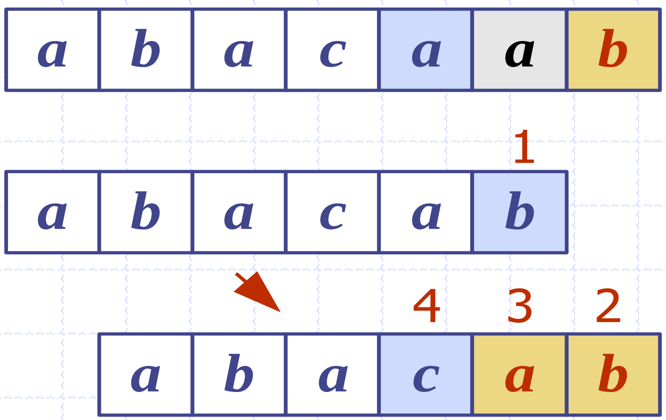
Strings and Pattern Matching



Outline

- Strings
- Pattern matching algorithms
  - Brute-force algorithm
  - Boyer-Moore algorithm
  - Knuth-Morris-Pratt algorithm

Strings

- A string is a sequence of characters
- Examples of strings:
  - C++ program
  - HTML document
  - DNA sequence
  - Digitized image
- An alphabet Σ is the set of possible characters for a family of strings
- Example of alphabets:
  - ASCII (used by C and C++)
  - Unicode (used by Java)
  - {0, 1}
  - {A, C, G, T}
- P is a string of size m
  - A substring $P[i\,..\,j]$ of P is the subsequence of P consisting of the characters with ranks between i and j
  - A prefix of P is a substring of the type $P[0\,..\,i]$
  - A suffix of P is a substring of the type $P[i\,..m-1]$

Alphabets

name	R()	lgR()	characters
BINARY	2	1	01
OCTAL	8	3	01234567
DECIMAL	10	4	0123456789
HEXADECIMAL	16	4	0123456789ABCDEF
DNA	4	2	ACTG
LOWERCASE	26	5	abcdefghijklmnopqrstuvwxyz
UPPERCASE	26	5	ABCDEFGHIJKLMNOPQRSTUVWXYZ
PROTEIN	20	5	ACDEFGHIKLMNPQRSTVWY
BASE64	64	6	ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef ghijklmnopqrstuvwxyz0123456789+/
ASCII	128	7	*ASCII characters*
EXTENDED_ASCII	256	8	*extended ASCII characters*
UNICODE16	65536	16	*Unicode characters*

Standard alphabets

Interesting Fact

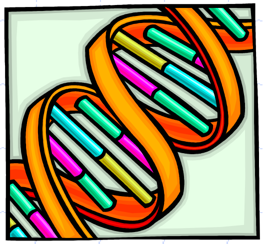
- IBM System/360 defined 8 bit byte (1964)
  - Instead of 4 or 6 bits which was cheaper
  - Enabled the extended ASCII set, both upper and lowercase, as well as symbols
  - Huge ramifications!
  - Decided by Fred Brooks
    - His most important decision:

"The most important single decision I ever made was to change the IBM 360 series from a 6-bit byte to an 8-bit byte, thereby enabling the use of lowercase letters. That change propagated everywhere."

Pattern Matching: Problem Statement

- Given strings T (text) and P (pattern), the pattern matching problem consists of finding a substring of T equal to P
- Applications:
  - Text editors
  - Search engines
  - Biological research
  - And many others…

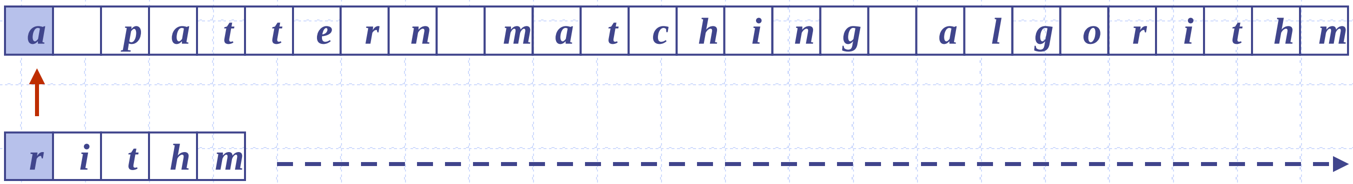
Brute-Force Algorithm



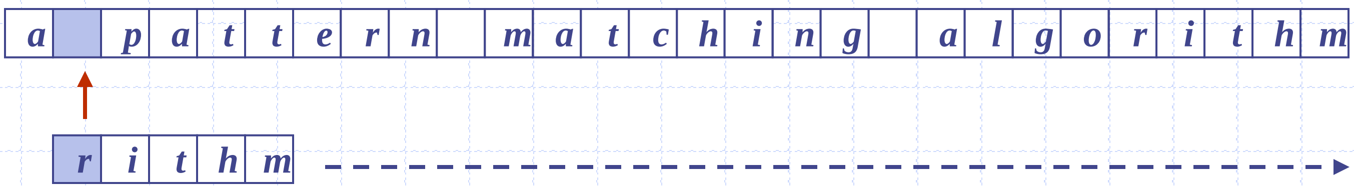
- Compares the pattern P (of length **m**) with the text T (of length **n**) for each possible shift of P relative to T, until
  - a match is found, or
  - all placements of the pattern have been tried
- Brute-force pattern matching runs in what time?
  - $O(nm)$
- What is an example worst case?
  - $T = aaa \ldots ah$
  - $P = aaah$
  - may occur in images and DNA sequences
  - unlikely in English text

Algorithm *BruteForceMatch*(T, P)
  Input text T of size n and pattern P of size m
  Output starting index of a substring of T equal to P or -1 if no such substring exists
  for $i \leftarrow 0$ **to** $n - m$
    { test shift i of the pattern }
    $j \leftarrow 0$
    while $j < m \wedge T[i + j] = P[j]$
      $j \leftarrow j + 1$
    if $j = m$
      return i {match at i}
    else
      break while loop {mismatch}
  return -1 {no match anywhere}

Brute Force Algorithm Example



Brute Force Algorithm Example



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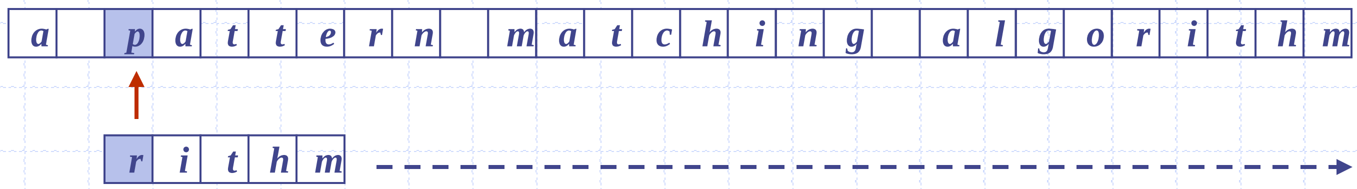

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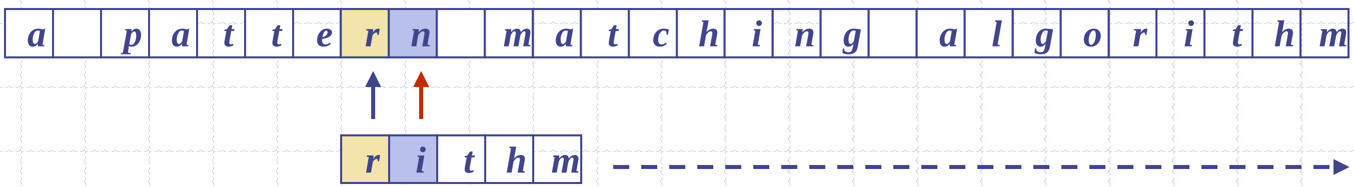

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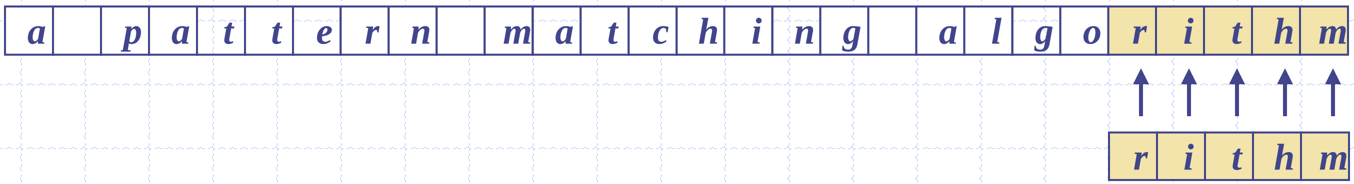
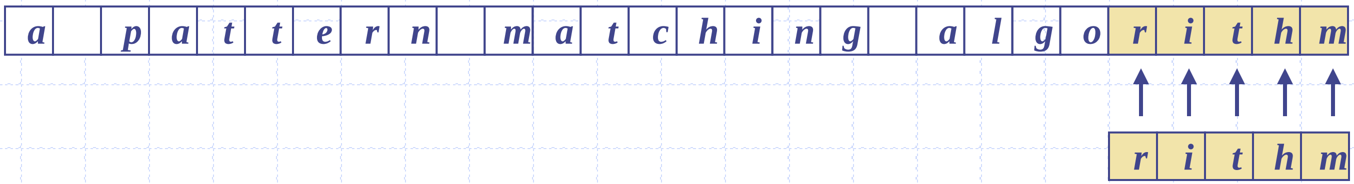


How can we do better? Ideas?



Boyer-Moore Heuristics

- The Boyer-Moore's pattern matching algorithm is based on two heuristics

  Looking-glass heuristic: Compare P with a subsequence of T moving backwards

  Character-jump heuristic: When a mismatch occurs at $T[i] = c$
  - If P contains c, shift P to align the last occurrence of c in P with $T[i]$
  - Else, shift P to align $P[0]$ with $T[i+1]$
- Example

a		*p*	*a*	*t*	*t*	*e*	*r*	*n*		*m*	*a*	*t*	*c*	*h*	*i*	*n*	*g*		*a*	*l*	*g*	*o*	*r*	*i*	*t*	*h*	*m*

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r	*i*	*t*	*h*	*m*

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1

r	*i*	*t*	*h*	*m*

2

r	*i*	*t*	*h*	*m*

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1

r	*i*	*t*	*h*	*m*

2

r	*i*	*t*	*h*	*m*

3

r	*i*	*t*	*h*	*m*

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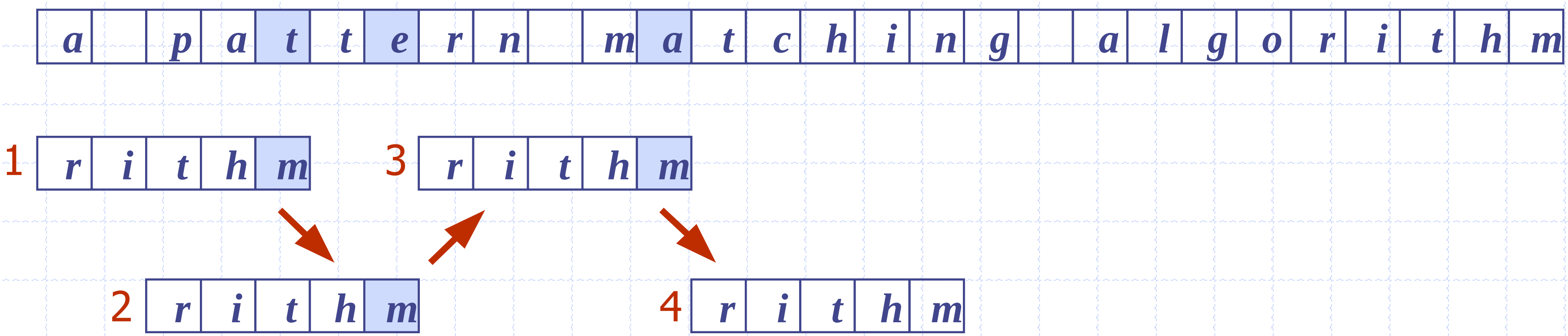

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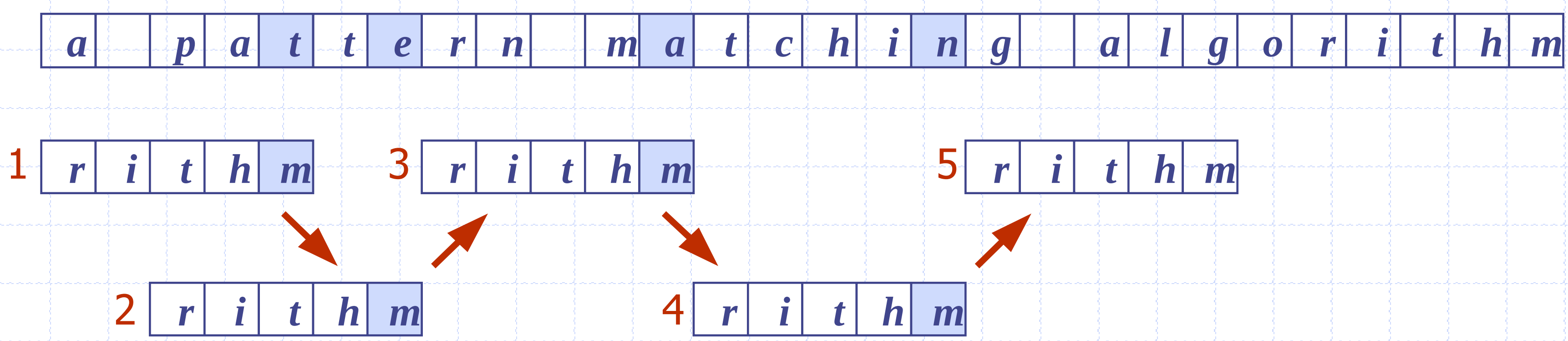

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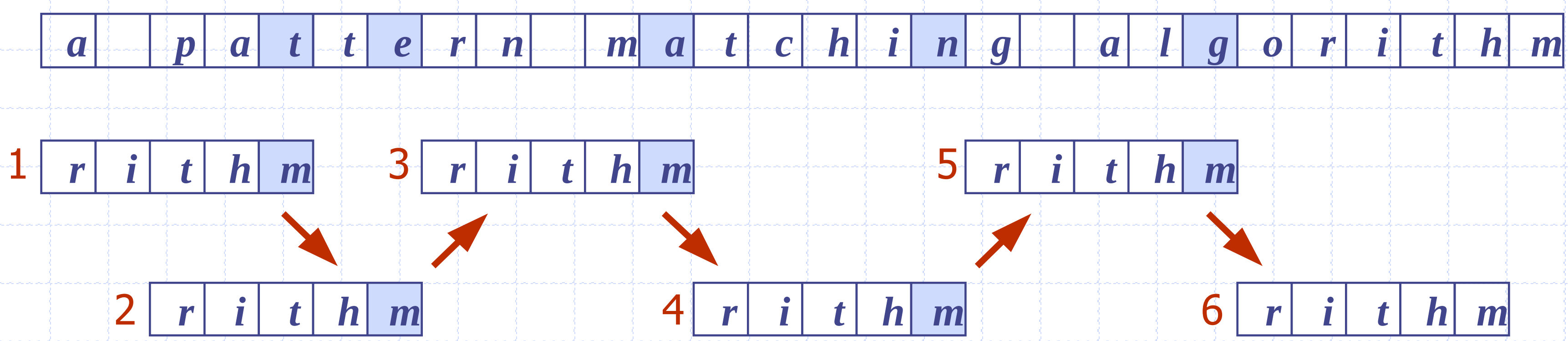

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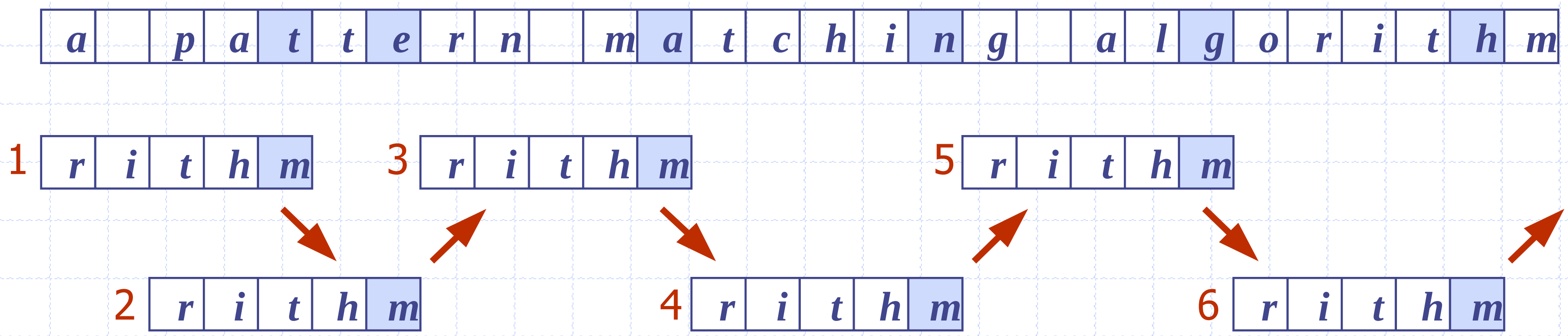

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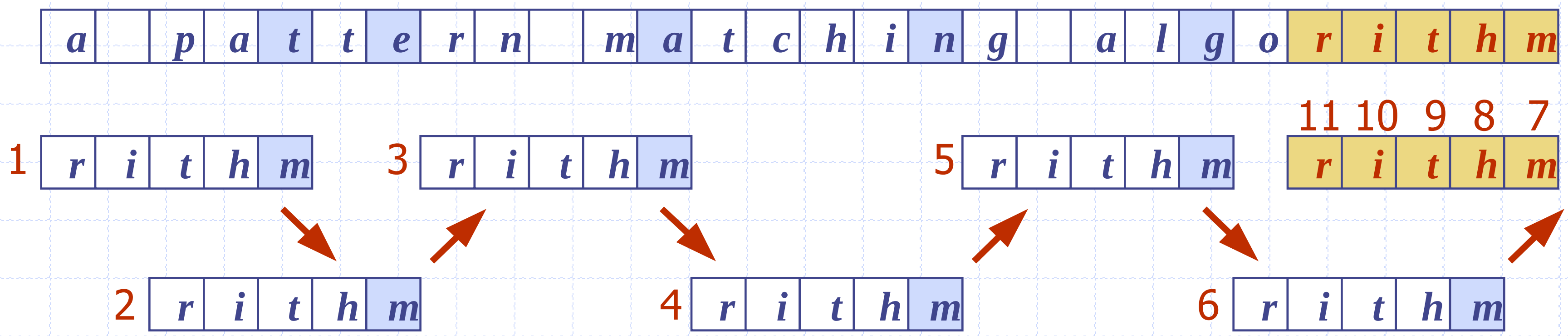


Last-Occurrence Function

- Boyer-Moore's algorithm preprocesses the pattern P and the alphabet Σ to build the last-occurrence function L mapping Σ to integers, where $L(c)$ is defined as
  - the largest index i such that $P[i] = c$ or
  - -1 if no such index exists
- Example:
  - $\Sigma = \{a, b, c, d\}$
  - $P = abacab$

c	a	b	c	d
$L(c)$	4	5	3	-1

- The last-occurrence function can be represented by an array indexed by the numeric codes of the characters
- The last-occurrence function can be computed in time $O(m + s)$, where m is the size of P and s is the size of Σ

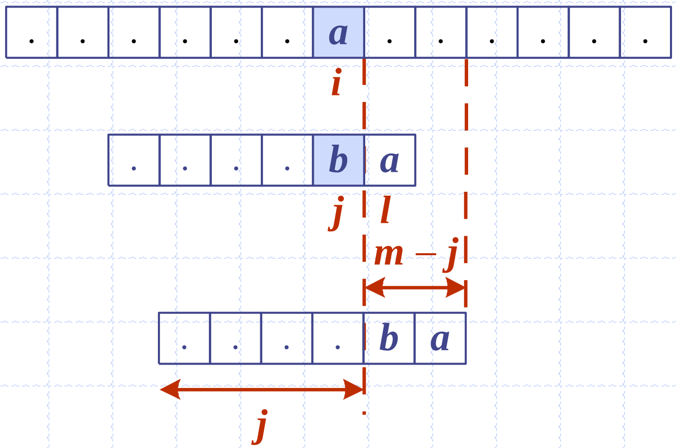
Additional examples

- P=ab, S={a,b,c,d}
  - L=[a,0], [b,1], [c,-1], [d,-1]
- P=abab
  - L=[a,2], [b,3], [c,-1], [d,-1]
- P=dcba
  - L=[a,3], [b,2], [c,1], [d,0]

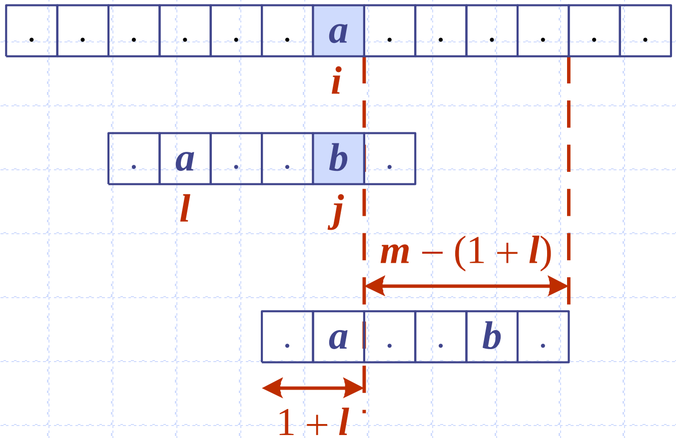
The Boyer-Moore Algorithm

```
Algorithm BoyerMooreMatch(T, P, Σ)
    L ← lastOccurenceFunction(P, Σ )
    i ← m − 1
    j ← m − 1
    repeat
        if T[i] = P[j]
            if j = 0
                return i  { match at i }
            else
                i ← i − 1
                j ← j − 1
        else
            { character-jump }
            l ← L[T[i]]
            i ← i + m – min(j, 1 + l)
            j ← m − 1
    until i > n − 1
    return −1 { no match }
```

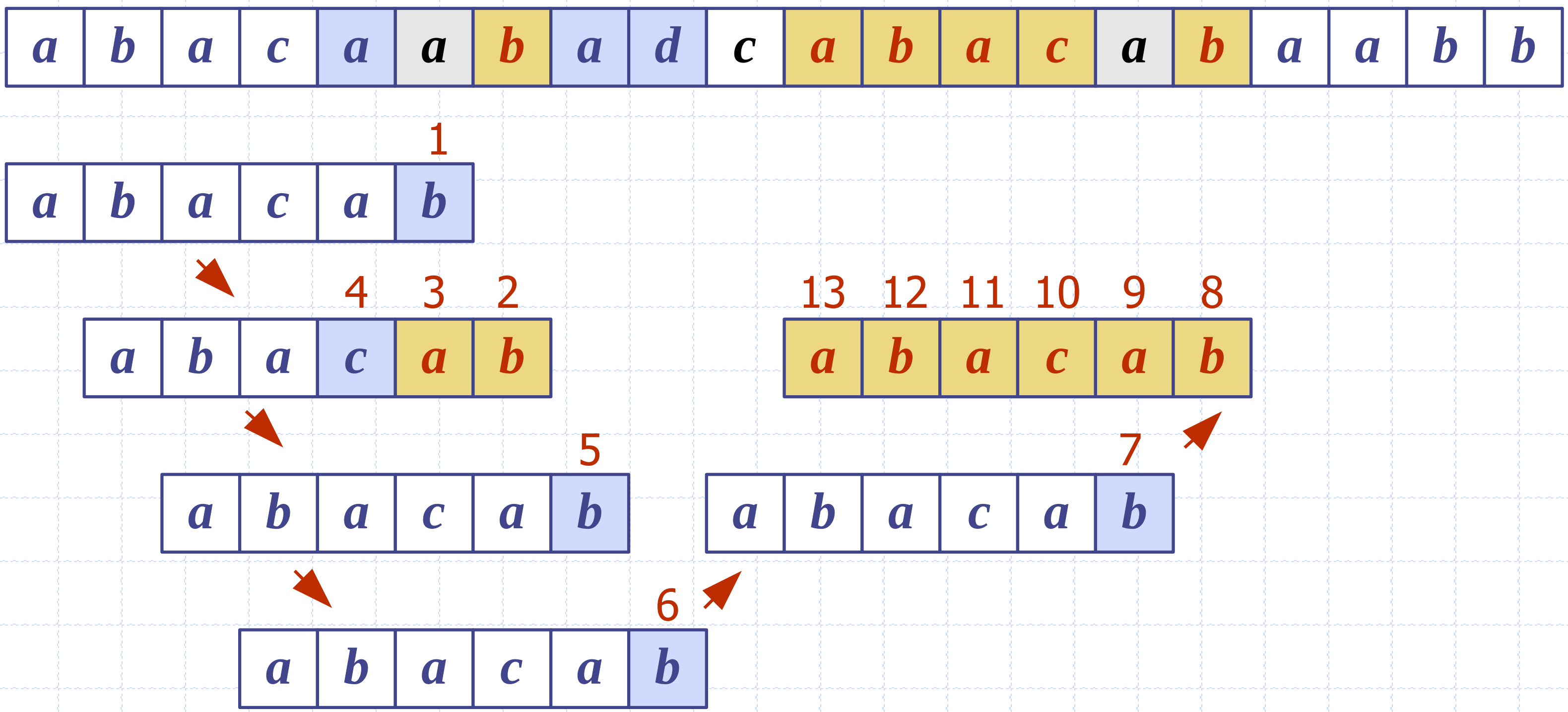
Case 1: $j \leq 1 + l$



Case 2: $1 + l \leq j$

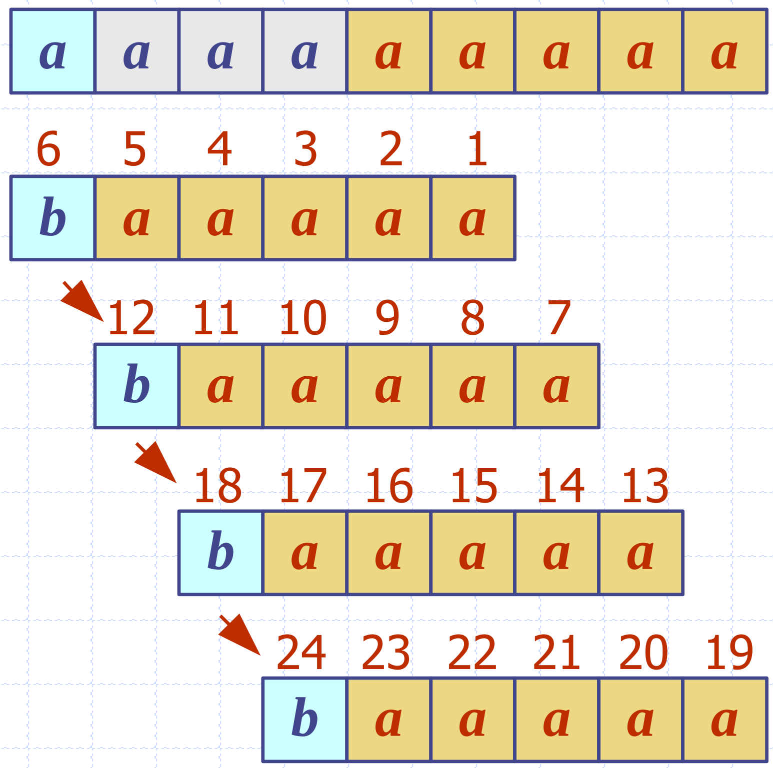


Example


a b a c a a b a d c a b a c a b a a b b
1
a b a c a b
4 3 2
a b a c a b
13 12 11 10 9 8
a b a c a b
5
a b a c a b
7
a b a c a b
6
a b a c a b

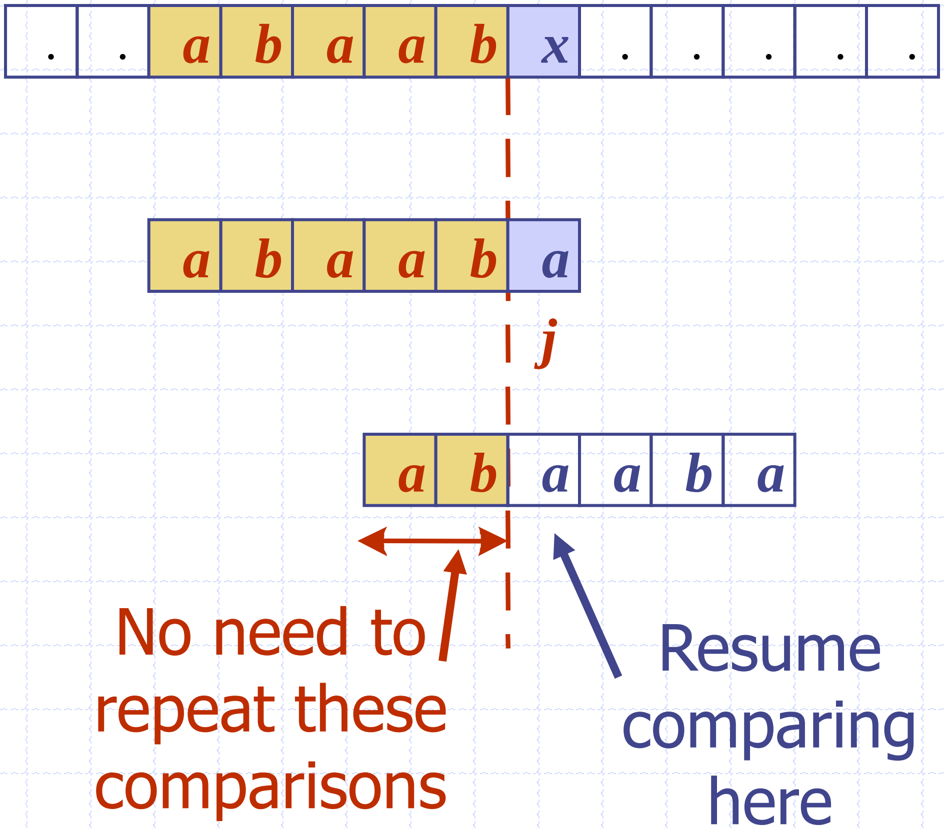
Analysis

- Boyer-Moore's algorithm runs in time $O(nm + s)$
- Example of worst case:
  - $T = aaa \ldots a$
  - $P = baaa$
- The worst case may occur in images and DNA sequences but is unlikely in English text
- Boyer-Moore's algorithm is **significantly** faster in practice than the brute-force algorithm on English text



The KMP Algorithm - Motivation

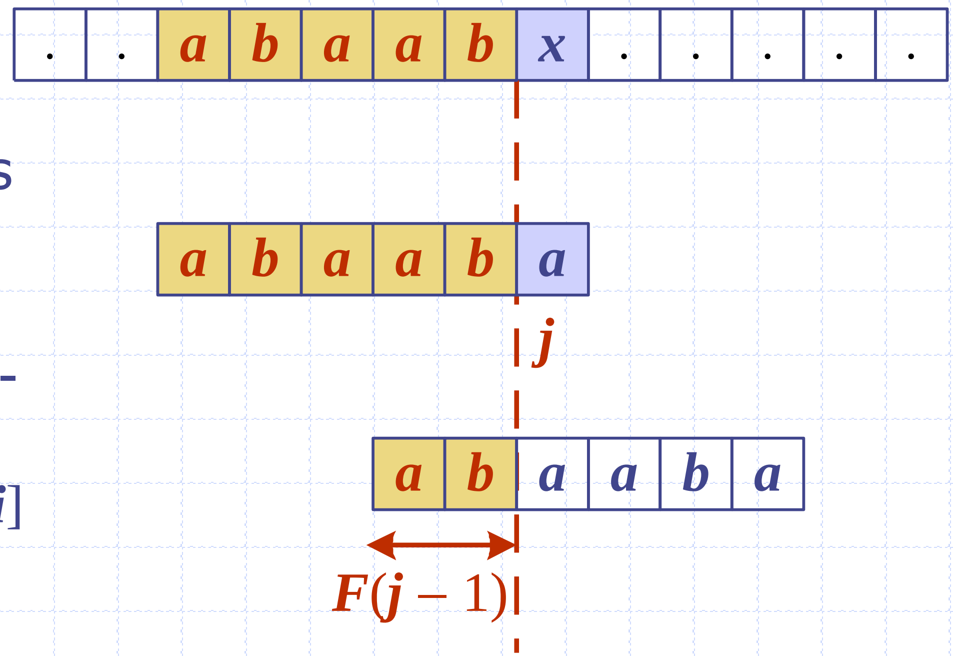
- Knuth-Morris-Pratt's algorithm compares the pattern to the text in **left-to-right**, but shifts the pattern more intelligently than the brute-force algorithm.
- When a mismatch occurs, what is the **most** we can shift the pattern so as to avoid redundant comparisons?
- Answer: the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$



KMP Failure Function

- Knuth-Morris-Pratt's algorithm preprocesses the pattern to find matches of prefixes of the pattern with the pattern itself
- The **failure function** $F(j)$ is defined as the size of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- Knuth-Morris-Pratt's algorithm modifies the brute-force algorithm so that if a mismatch occurs at $P[j] \neq T[i]$ we set $j \leftarrow F(j-1)$

j	0	1	2	3	4	5
$P[j]$	a	b	a	a	b	a
$F(j)$	0	0	1	1	2	3

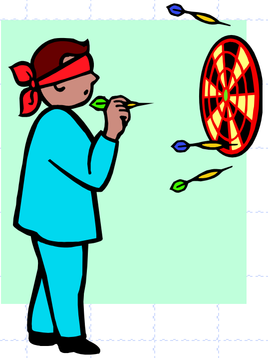


The KMP Algorithm

- The failure function can be represented by an array and can be computed in $O(m)$ time
- At each iteration of the while-loop, either
  - i increases by one, or
  - the shift amount $i - j$ increases by at least one (observe that $F(j - 1) < j$)
- Hence, there are no more than $2n$ iterations of the while-loop
- Thus, KMP's algorithm runs in time
  - $O(m + n)$!!

Algorithm *KMPMatch*(T, P)
$F \leftarrow failureFunction(P)$
$i \leftarrow 0$
$j \leftarrow 0$
while $i < n$
  if $T[i] = P[j]$
    if $j = m - 1$
      return $i - j$ { match }
    else
      $i \leftarrow i + 1$
      $j \leftarrow j + 1$
  else
    if $j > 0$
      $j \leftarrow F[j - 1]$
    else
      $i \leftarrow i + 1$
return -1 { no match }

Computing the Failure Function



- The failure function can be represented by an array and can be computed in $O(m)$ time
- The construction is similar to the KMP algorithm itself
- At each iteration of the while-loop, either
  - i increases by one, or
  - the shift amount $i - j$ increases by at least one (observe that $F(j - 1) < j$)
- Hence, there are no more than $2m$ iterations of the while-loop

Algorithm *failureFunction*(P)
  $F[0] \leftarrow 0$
  $i \leftarrow 1$
  $j \leftarrow 0$
  while $i < m$
    if $P[i] = P[j]$
      {we have matched $j + 1$ chars}
      $F[i] \leftarrow j + 1$
      $i \leftarrow i + 1$
      $j \leftarrow j + 1$
    else if $j > 0$ **then**
      {use failure function to shift P}
      $j \leftarrow F[j - 1]$
    else
      $F[i] \leftarrow 0$ { no match }
      $i \leftarrow i + 1$

Additional Example

Pattern:

aba

123

Failure function f:

0 0 1

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

??

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(a) = 0 (always = 0 for one letter)

f(aa) = 1 ('a' is both a prefix and suffix)

f(aab) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(a) = 0 (always = 0 for one letter)

f(aa) = 1 ('a' is both a prefix and suffix)

f(aab) = 0 (no same suffixes and prefixes: a != b, aa != ab)

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aaba) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aaba) = 1 ('a' is the same in the beginning and the end, but if you take 2 letters, they won't be equal: aa != ba)

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaa) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaa) = 2 (you can take 'aa' but no more: aab != baa)

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = 4 (you can take 'aaba')

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = 4 (you can take 'aaba')

f(aabaabab) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = 4 (you can take 'aaba')

f(aabaabab) = 0 ('a' != 'b', 'aa' != 'ab`, etc & can't be = 5, 'aabaa' != 'aabab')

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = 4 (you can take 'aaba')

f(aabaabab) = 0

f(aabaababb) = ?

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

f(aabaab) = 3 (you can take 'aab')

f(aabaaba) = 4 (you can take 'aaba')

f(aabaabab) = 0

f(aabaababb) = 0

Additional Example

Pattern:

a a b a a b a b b

1 2 3 4 5 6 7 8 9

Failure function f:

0 1 0 1 2 3 4 0 0

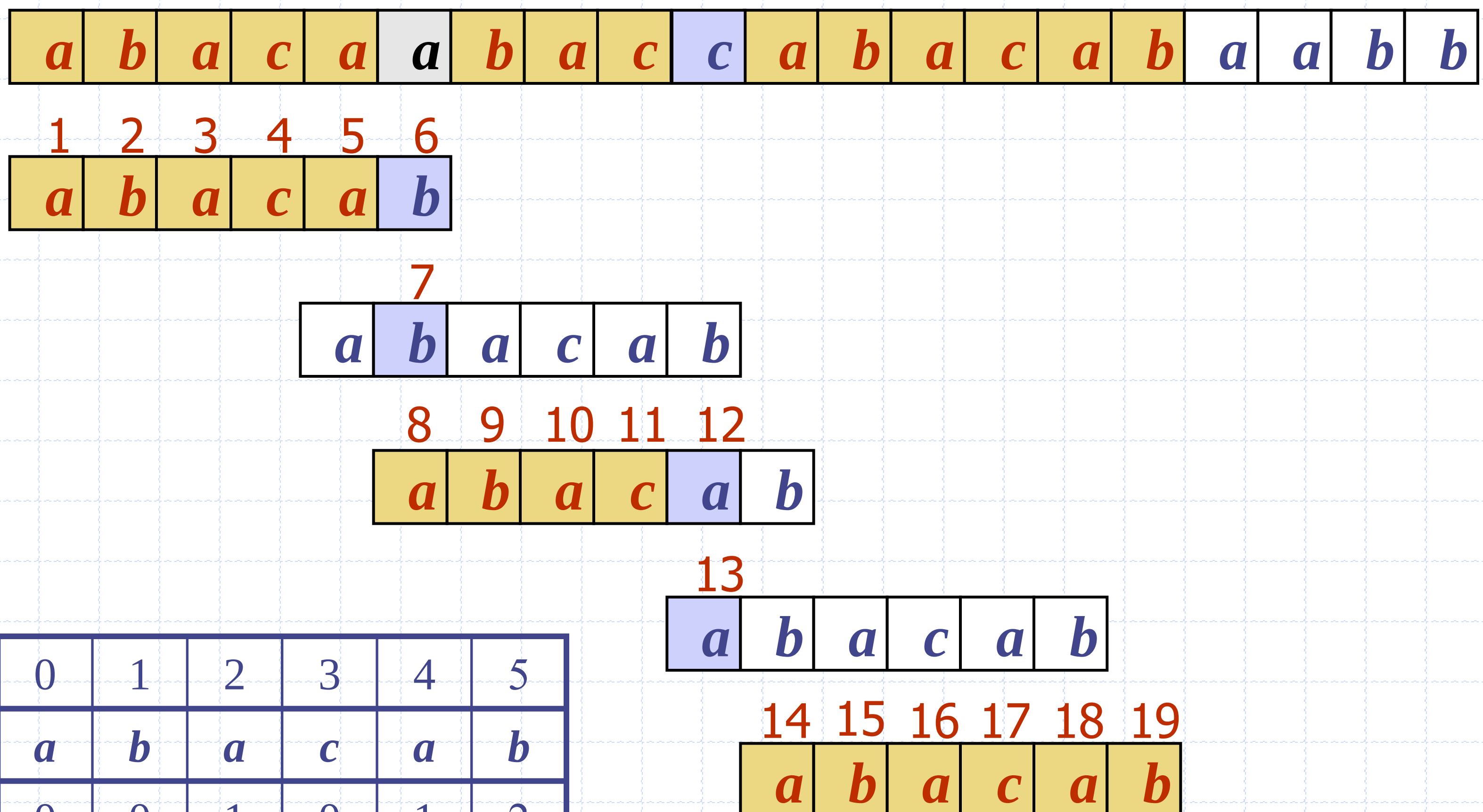
Example

a	*b*	*a*	*c*	*a*	*a*	*b*	*a*	*c*	*c*	*a*	*b*	*a*	*c*	*a*	*b*	*a*	*a*	*b*	*b*

a	*b*	*a*	*c*	*a*	*b*

j	0	1	2	3	4	5
$P[j]$	*a*	*b*	*a*	*c*	*a*	*b*
$F(j)$	0	0	1	0	1	2

Example



j	0	1	2	3	4	5
P[*j*]	*a*	*b*	*a*	*c*	*a*	*b*
F(*j*)	0	0	1	0	1	2

Rabin-Karp

- Calculates a **hash value** for the pattern, and for each M-character subsequence of text.
- If hash values unequal, then calculate the hash value for next M-character sequence.
- If hash values equal, then do Brute Force comparison.

Rabin-Karp: Analysis

- For "good" hash functions, the hashed values of two different patterns will usually be distinct.
- Thus, average case O(N), where N is size of text.
- Worst case complexity O(MN) but rare for good hash functions.