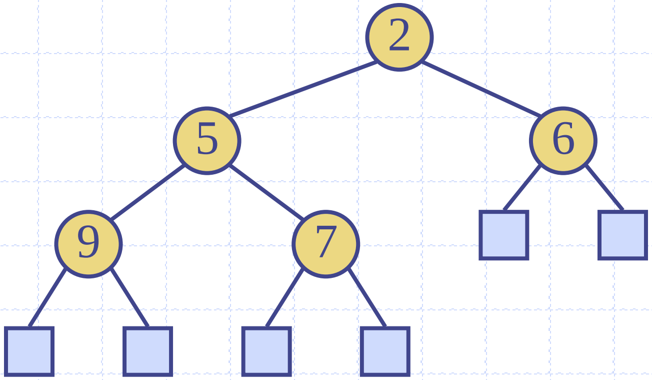
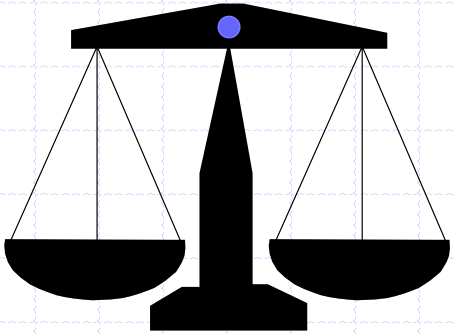
Priority Queues and Heaps



Priority Queue ADT

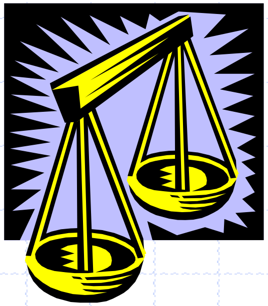
- A priority queue stores a collection of items
- An item is a pair (key, element)
- Main methods of the Priority Queue ADT
  - insertItem(k, o) inserts an item with key k and element o
  - removeMin() removes the item with the smallest key
- Additional methods
  - minKey(k, o) returns, but does not remove, the smallest key of an item
  - minElement() returns, but does not remove, the element of an item with smallest key
  - size(), isEmpty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Total Order Relation



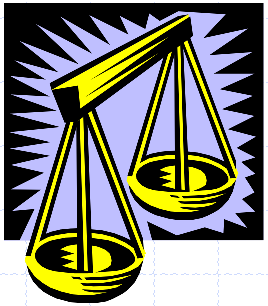
- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct items in a priority queue can have the same key
- Mathematical concept of total order relation $\leq$
  - **Reflexive** property: $x \leq x$
  - **Antisymmetric** property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - **Transitive** property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

Comparator ADT



- A *comparator* encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses a comparator as a template argument, to define the comparison function (<,=,>)
- The comparator is external to the keys being compared. Thus, the same objects can be sorted in different ways by using different comparators.
- When the priority queue needs to compare two keys, it uses its comparator

Using Comparators in C++



- A comparator class overloads the "()" operator with a comparison function.
- Example: Compare two points in the plane lexicographically.

```
class LexCompare {
public:
   int operator()(Point a, Point b) {
      if       (a.x < b.x) return –1
      else if (a.x > b.x) return +1
      else if (a.y < b.y) return  –1
      else if (a.y > b.y) return +1
      else return 0;
   }
};
```

- To use the comparator, define an object of this type, and invoke it using its "()" operator:
- Example of usage:

```
Point p(2.3, 4.5);
Point q(1.7, 7.3);
LexCompare lexCompare;

if (lexCompare(p, q) < 0)
   cout << "p less than q";
else if (lexCompare(p, q) == 0)
   cout << "p equals q";
else if (lexCompare(p, q) > 0)
   cout << "p greater than q";
```

Sorting with a Priority Queue

- We can use a priority queue to sort a set of comparable elements
  - Insert the elements one by one with a series of insertItem(e, e) operations
  - Remove the elements in sorted order with a series of removeMin() operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm *PQ-Sort*(*S*, *C*)
    Input sequence *S*, comparator *C* for the elements of *S*
    Output sequence *S* sorted in increasing order according to *C*
    P ← priority queue with comparator *C*
    while !*S.isEmpty* ()
        e ← *S.remove* (*S. first* ())
        P.insertItem(*e*, *e*)
    while !*P.isEmpty*()
        e ← *P.minElement*()
        P.removeMin()
        S.insertLast(*e*)

Sequence-based Priority Queue

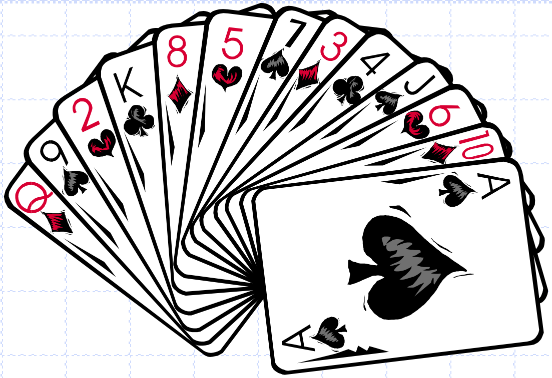
- Implementation with an unsorted list

- Performance:
  - insertItem
    - takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin, minKey and minElement
    - take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list

- Performance:
  - insertItem
    - takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin, minKey and minElement
    - take $O(1)$ time since the smallest key is at the beginning of the sequence

Selection-Sort



- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence

  4 — 5 — 2 — 3 — 1

- Running time of Selection-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes $O(n)$ time
  - Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence

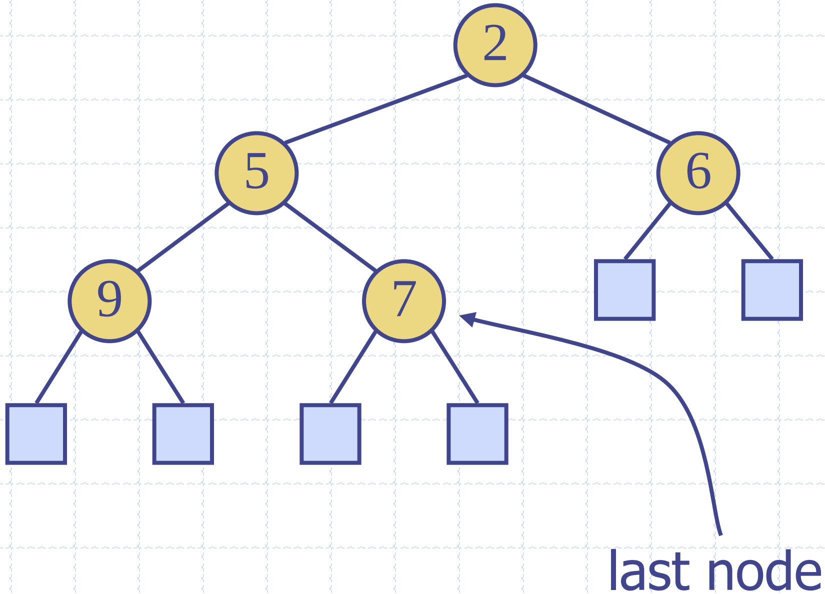
- Running time of Insertion-sort:
  - Inserting the elements into the priority queue with n insertItem operations takes time proportional to

    $$1 + 2 + \ldots + n$$
  - Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

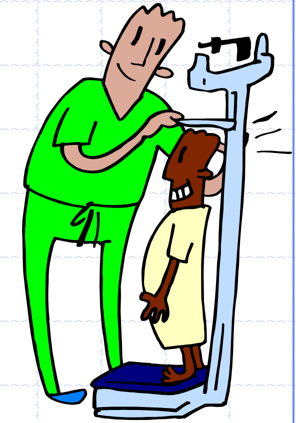
What is a heap?

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order:
    - for every internal node v other than the root, $key(v) \geq key(parent(v))$
  - Complete Binary Tree:
    - let h be the height of the heap
    - for $i = 0, \ldots, h - 1$, there are 2^i nodes of depth i
    - at depth $h - 1$, the internal nodes are to the left of the external nodes

- The last node of a heap is the rightmost internal node of depth $h - 1$

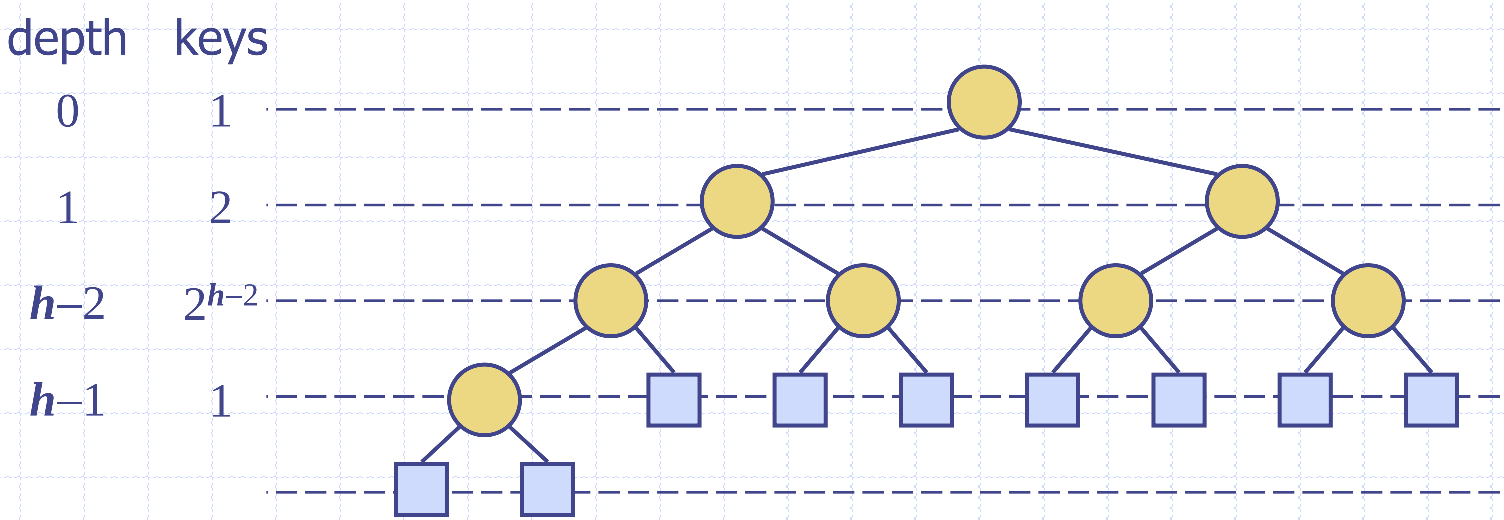


Height of a Heap



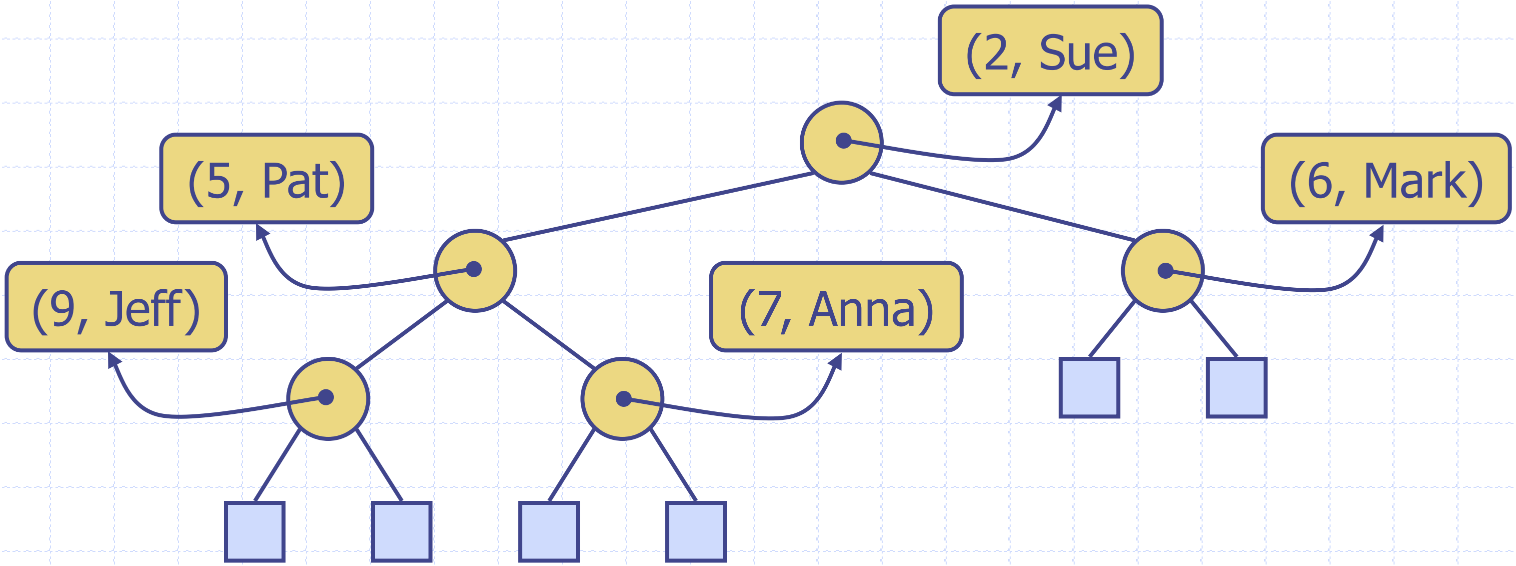
- **Theorem:** A heap storing n keys has height $O(\log n)$

  Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-2$ and at least one key at depth $h-1$, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-2} + 1$
  - Thus, $n \geq 2^{h-1}$, i.e., $h \leq \log n + 1$



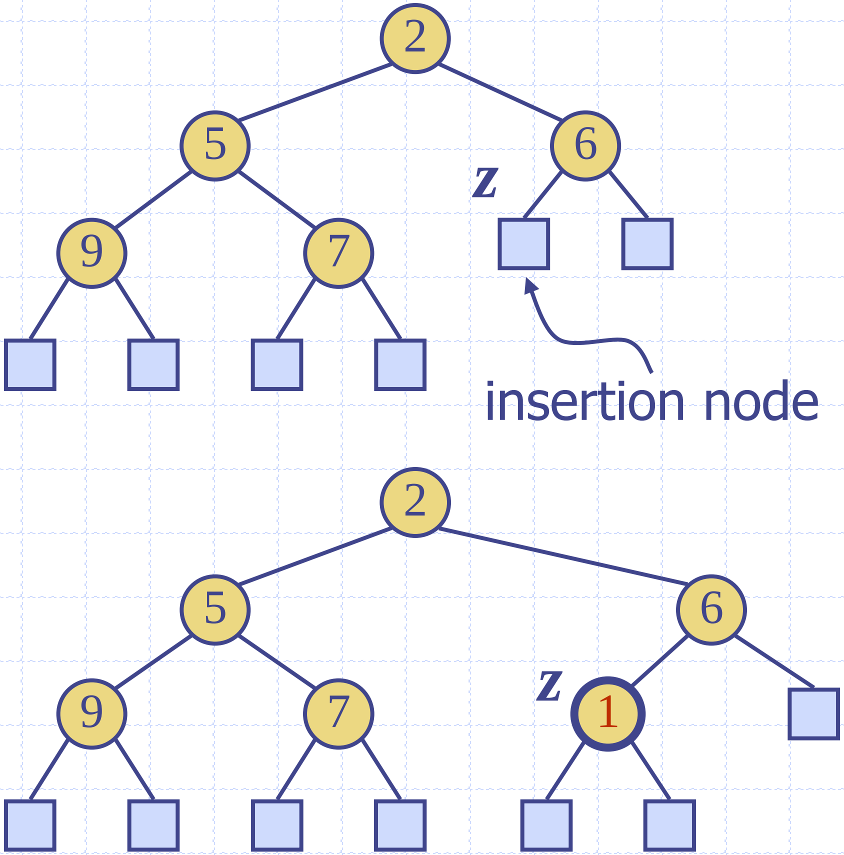
Heaps and Priority Queues

- We can use a heap to implement a priority queue
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
- For simplicity, we show only the keys in the pictures



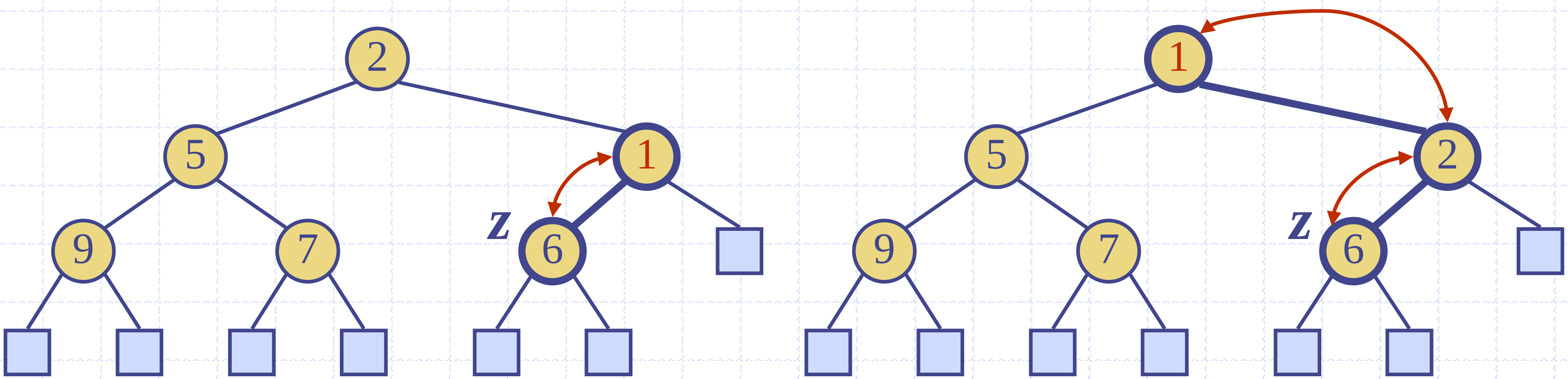
Insertion into a Heap

- Method insertItem of the priority queue ADT corresponds to the insertion of a key k to the heap
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
  - Store k at z and expand z into an internal node
  - Restore the heap-order property (discussed next)



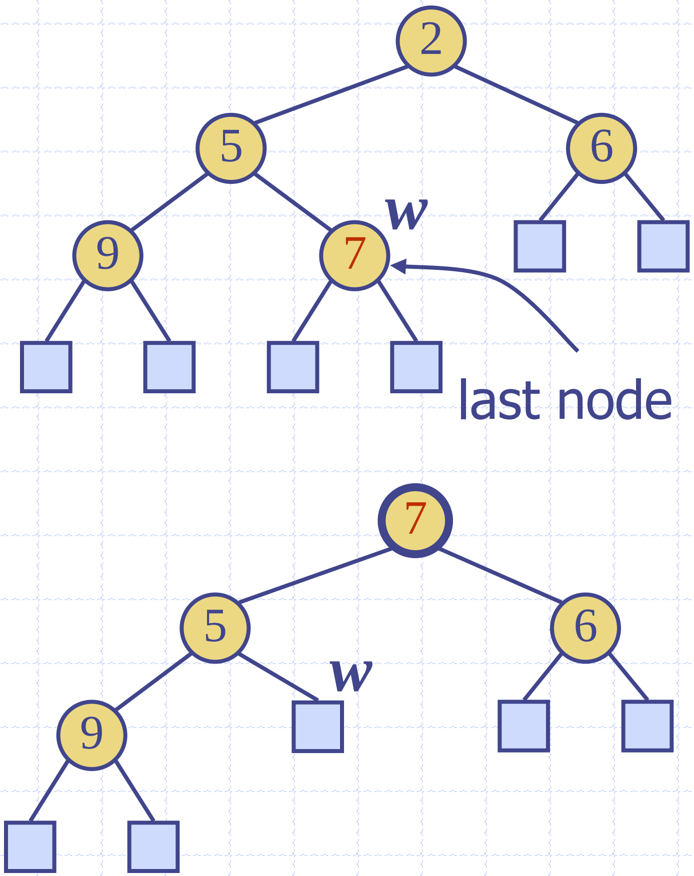
Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Performance
  - Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



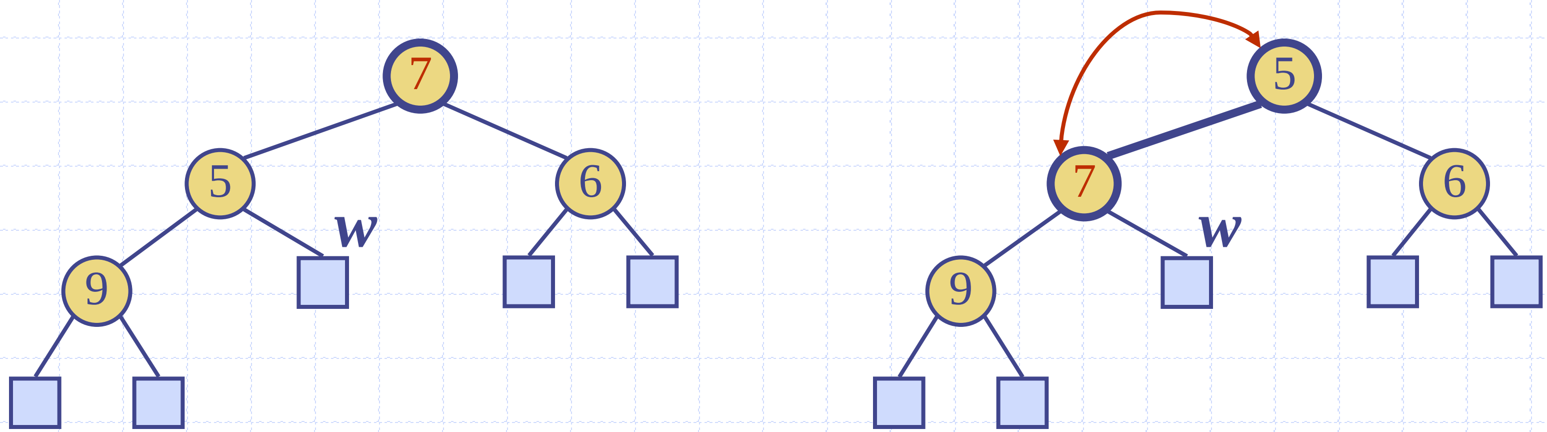
Removal from a Heap

- Method removeMin of the priority queue ADT corresponds to the removal of the root key from the heap
- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node w
  - Compress w and its children into a leaf
  - Restore the heap-order property (discussed next)

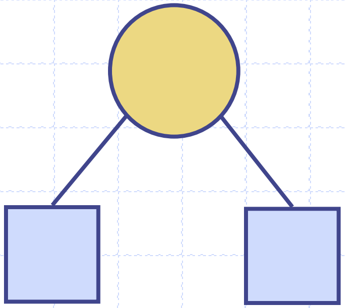


Downheap

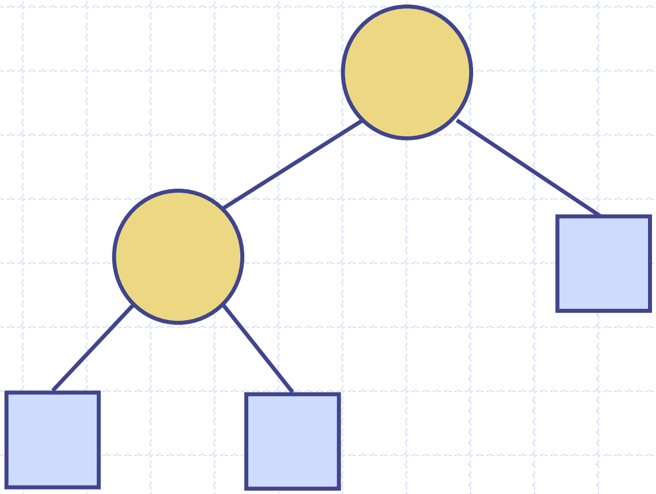
- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root
- Upheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Performance
  - Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



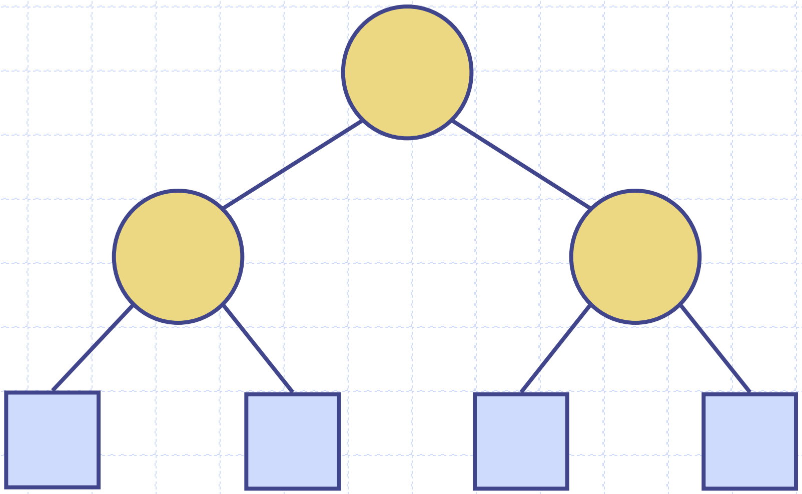
Example



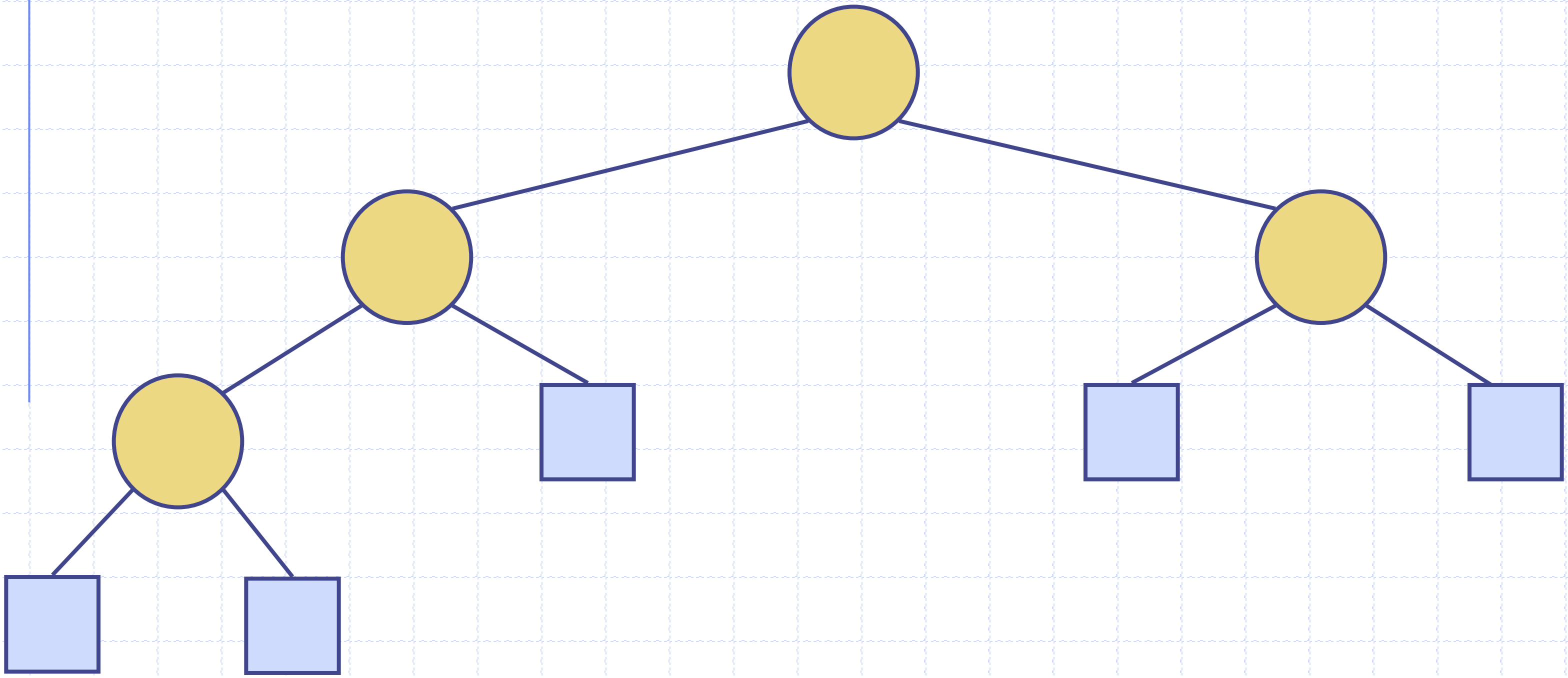
Example



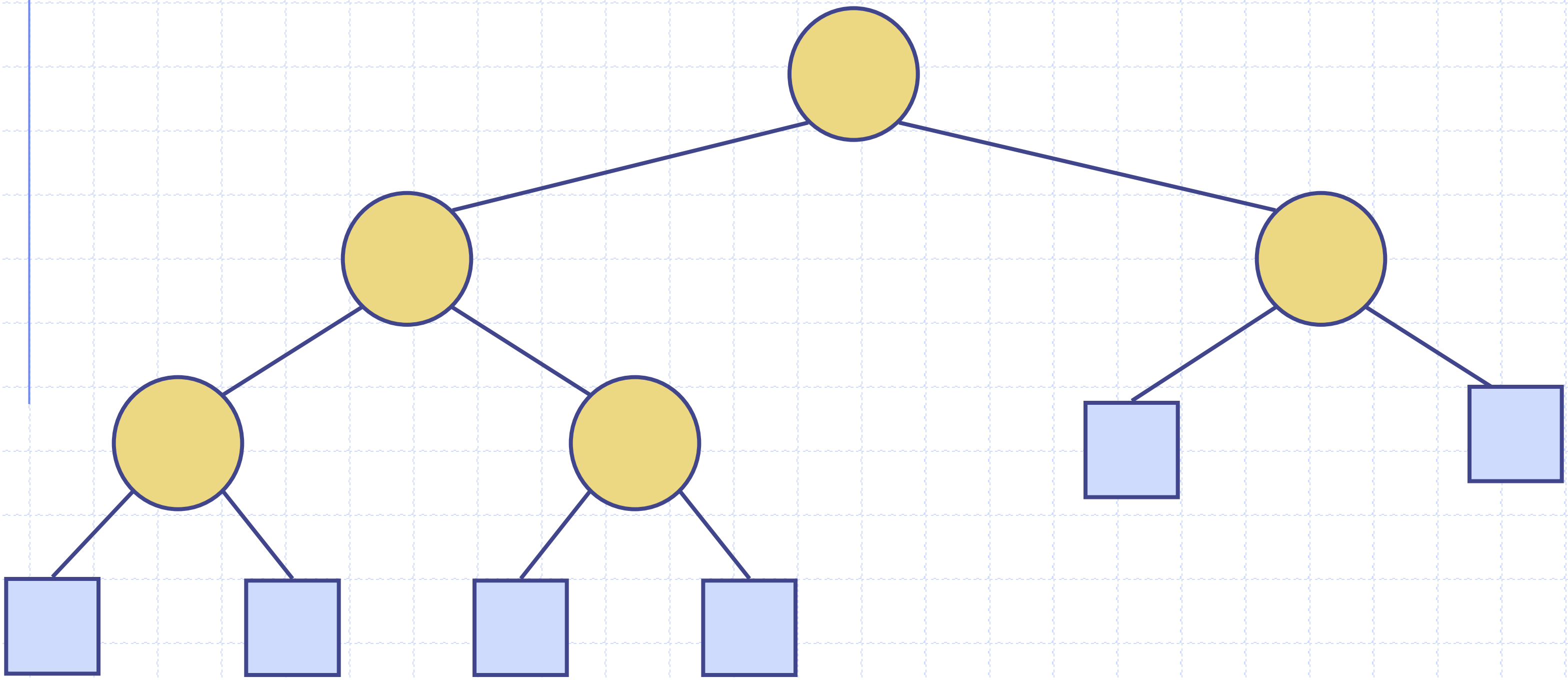
Example



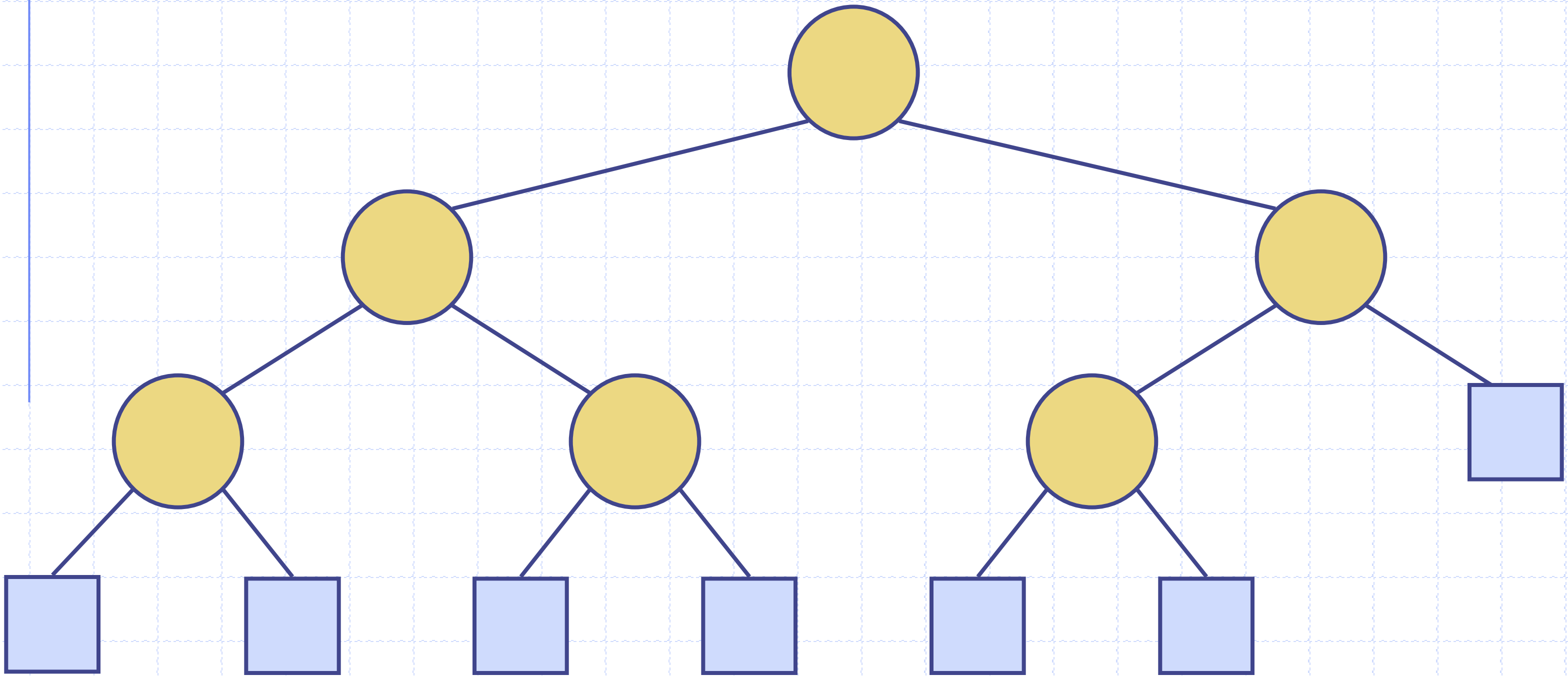
Example



Example



Example



Accessing the Queue

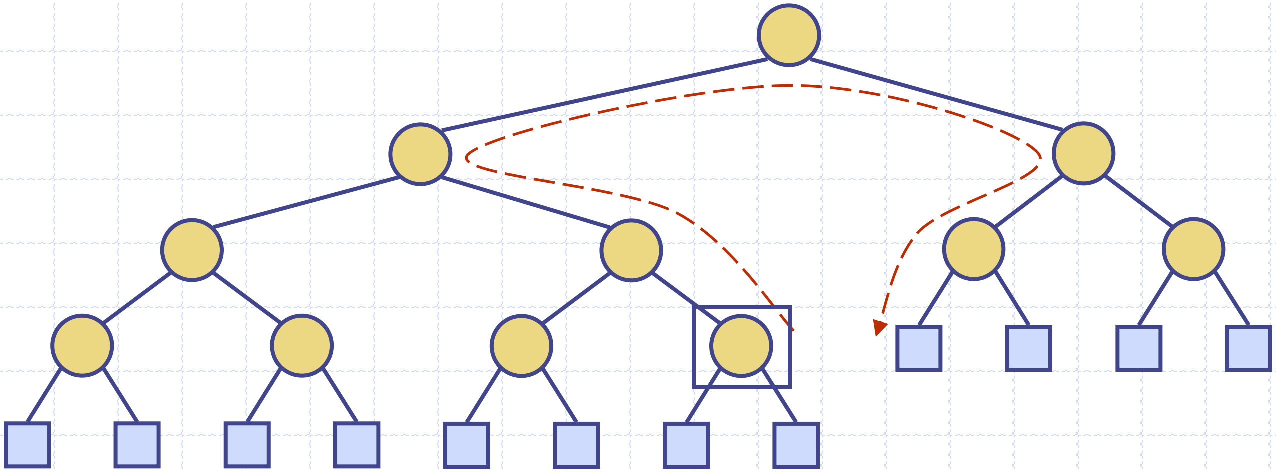
- In a regular queue, you can explicitly keep
  - the head-index and tail-index, or
  - the head-index and the size
- In a priority queue, you can explicitly keep
  - the head-pointer (root) and the tail-pointer (last node), or
  - the head-pointer and the size

Question:

- [TEAMS] How do you update the last node (“tail”) pointer or get it from the queue size?
- Two answers…

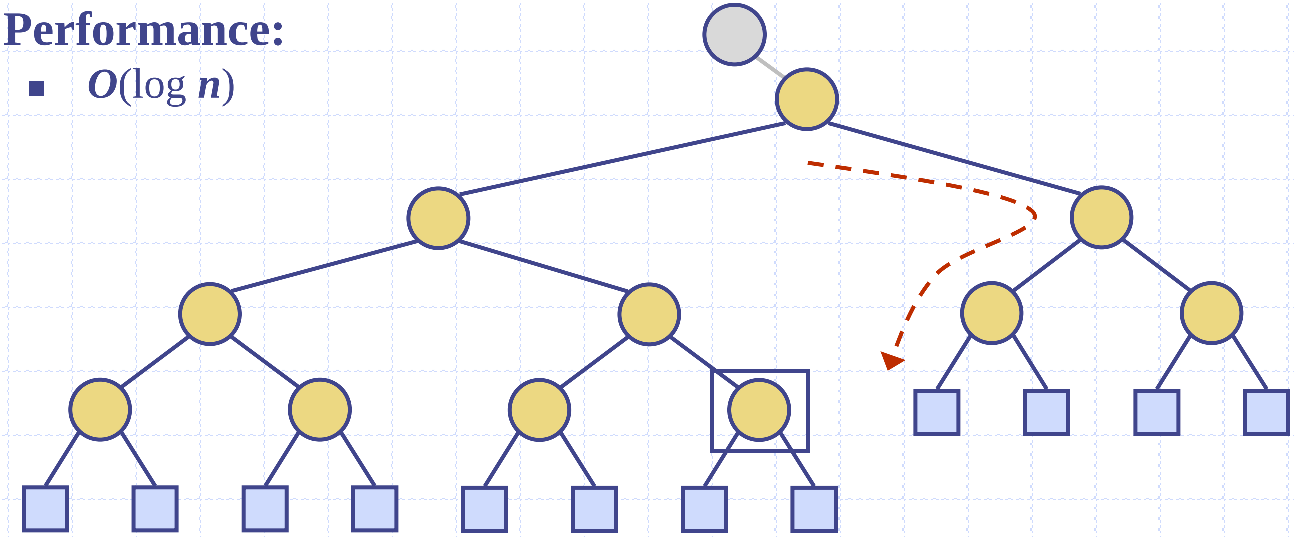
1=Updating Last Node Pointer

- The insertion node can be found by traversing a path:
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal
- **Performance:**
  - $O(\log n)$



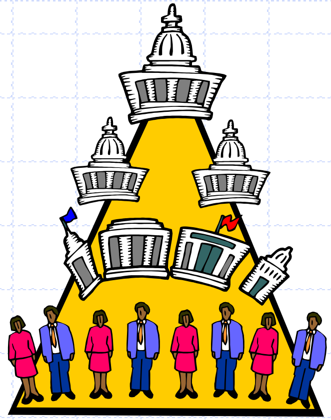
2=Finding Last Node Pointer

- The insertion node can be found by traversing a path without needing an explicit tail pointer:
  - Start at the root and using the binary number equivalent of the new number of nodes
    - Assume the root to be the right-child of an imaginary parent
    - Starting with MSB, traverse using 0=left and 1=right
  - Prevents the need to keep a last node pointer around
  - Asymptotically same performance, but half the cost
- Similar algorithm for updating the last node after a removal
- **Performance:**
  - $O(\log n)$



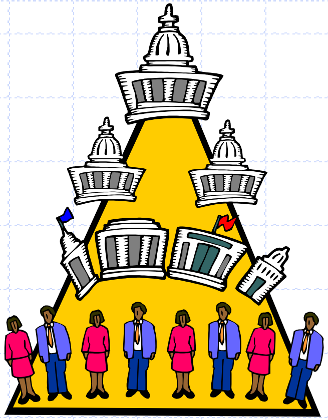
Examples

Heap-Sort



- Consider a priority queue with n items implemented by means of a heap
  - the space used is
    - $O(n)$
  - methods insertItem and removeMin take time
    - $O(\log n)$
  - methods size, isEmpty, minKey, and minElement take time
    - $O(1)$
- Using a heap-based priority queue, we can sort a sequence of n elements in time
  - $O(n \log n)$
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

Heap-Sort



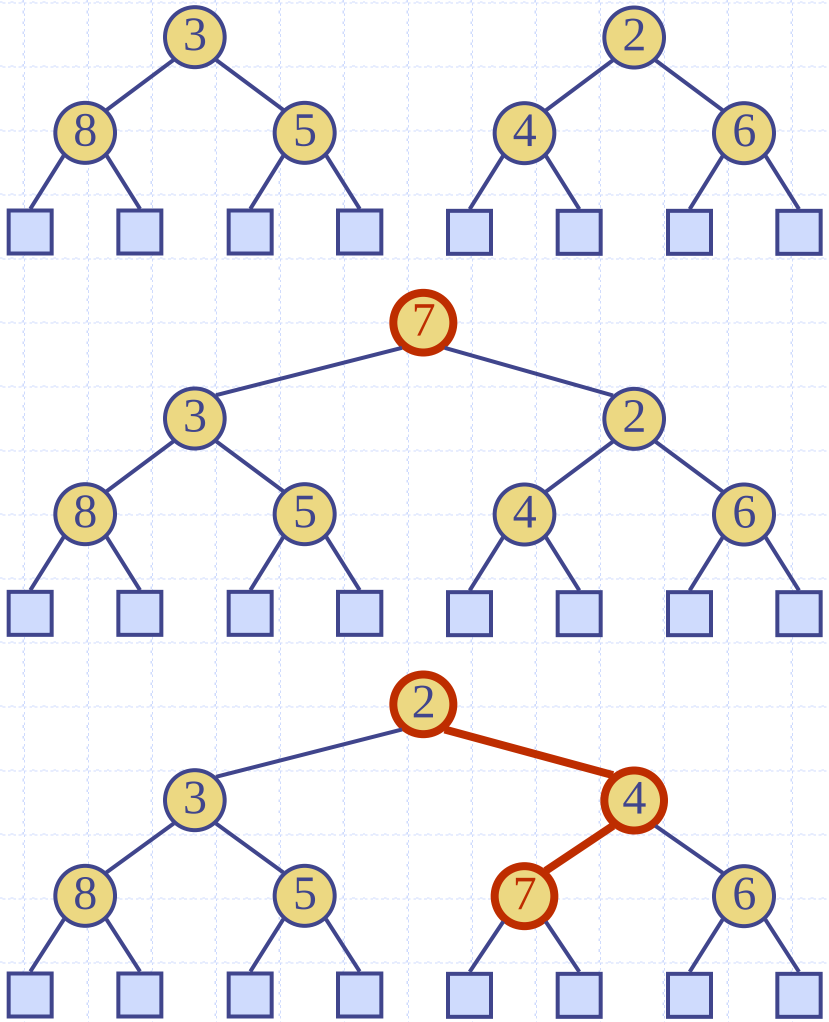
- More explicitly, how much time does it take to construct a heap?
  - $\mathbf{n}$ items, each requiring up to $log\mathbf{n}$ swaps during "up-heap" operations
    - $O(\mathbf{n}log\mathbf{n})$
- How much time does it take to "destruct" a heap (or remove items in sorted order)
  - $\mathbf{n}$ items, each requiring up to $log\mathbf{n}$ swaps during "down-heap" operations
    - $O(\mathbf{n}log\mathbf{n})$
- Thus Heap-Sort is
  - $\mathbf{n}log\mathbf{n}+\mathbf{n}log\mathbf{n} = O(\mathbf{n}log\mathbf{n})$

Heap Construction

- Can you do better than $O(n \log n)$?
- How?
- Why do we care?
  - We only want to find the few smallest keys among many items
  - We want to quickly start "using the items" in sorted order but the sorting can continue while I start using the first items, e.g.: real-time OS, games, simulations, etc.

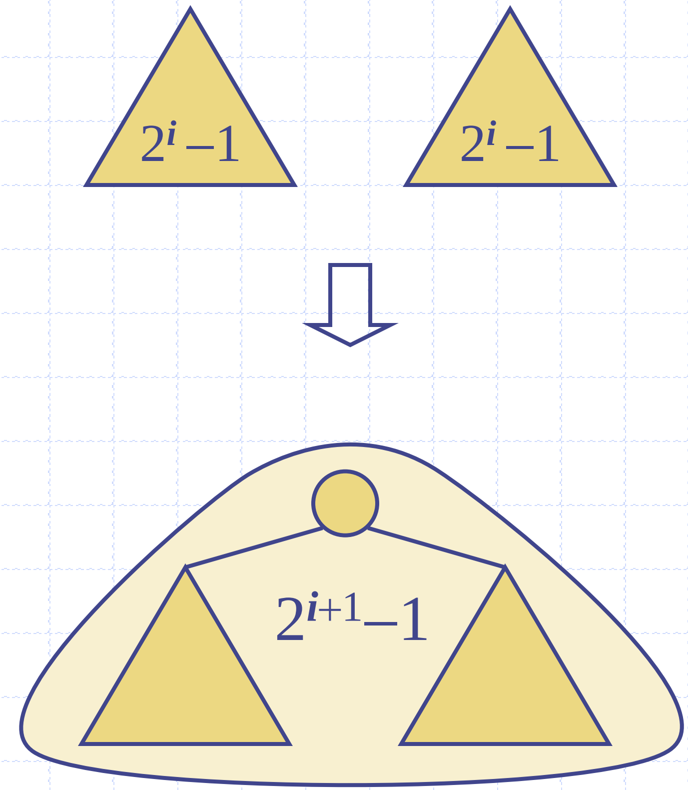
First: Merging Two Heaps

- We are given two two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



Then: Bottom-up Heap Construction

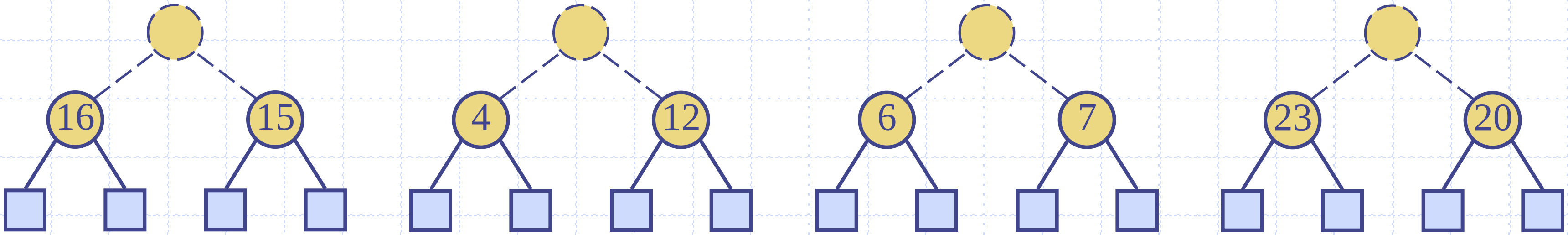
- We can construct a heap storing n given keys using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ keys are merged into heaps with $2^{i+1} - 1$ keys



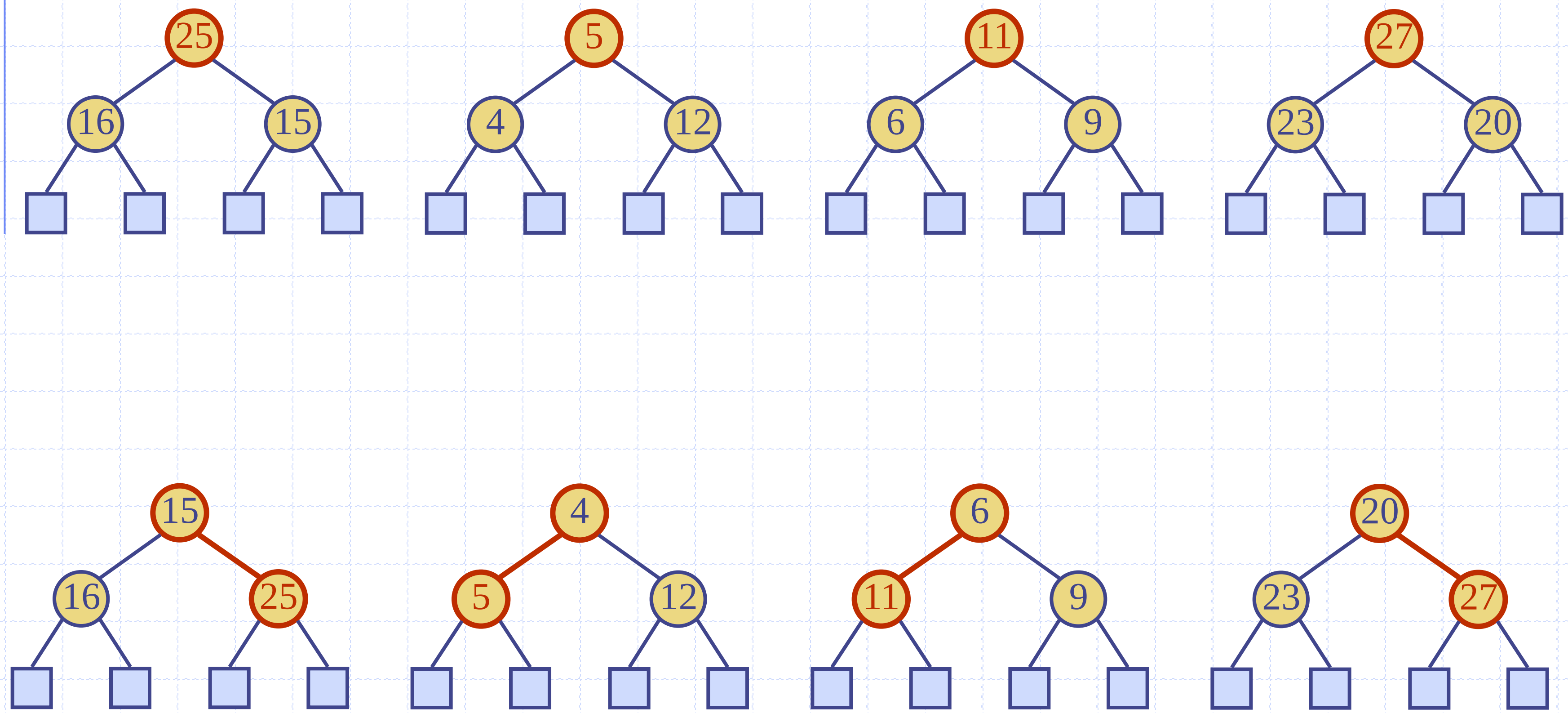
Example

- Goal: to create a heap of N elements
- Assume $N = 2^H - 1$ for some integer H and thus the heap (tree) is "full"
- In a first step, we construct (N+1)/2 basic heap structures
  - One key and two empty children pointers

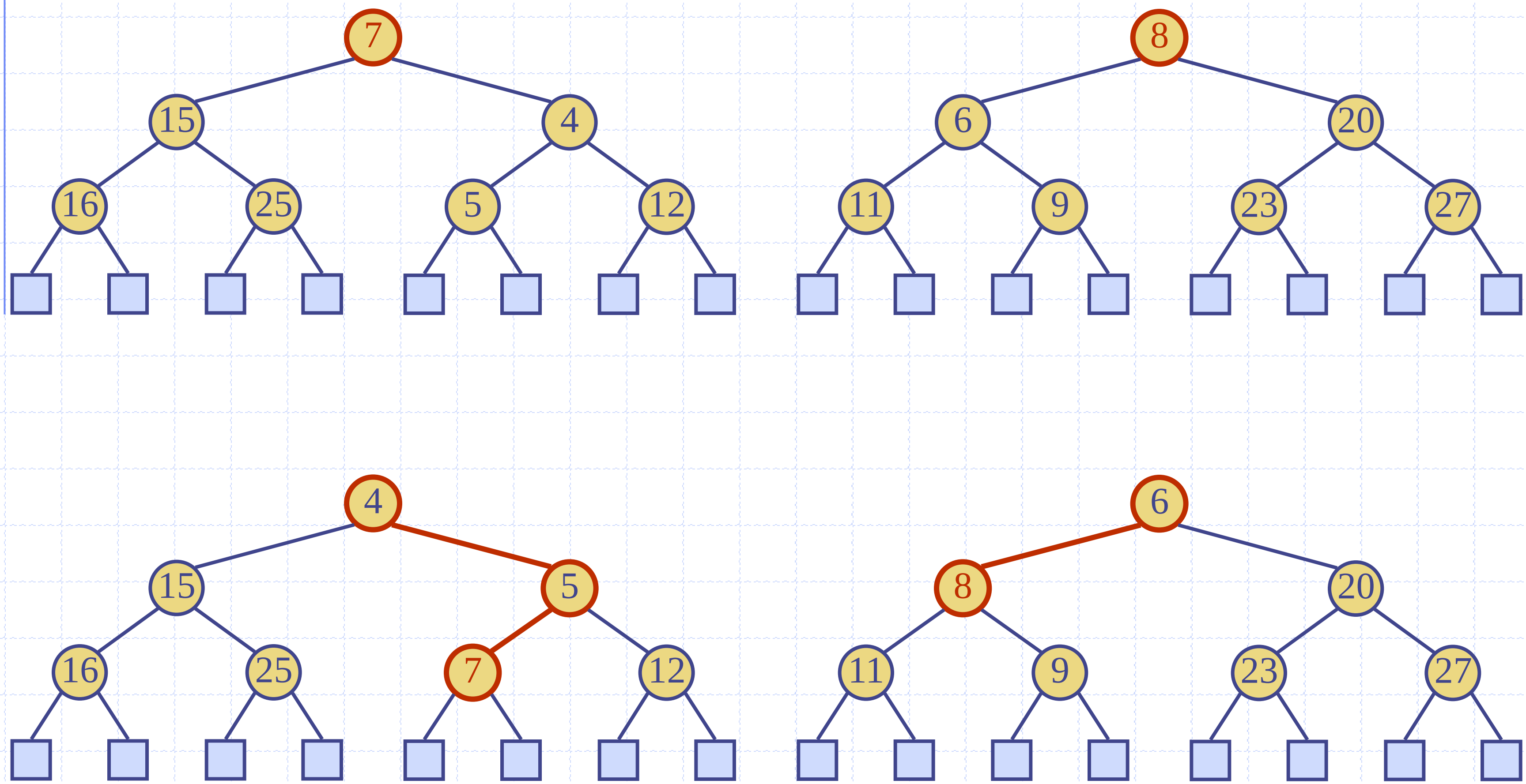
Example



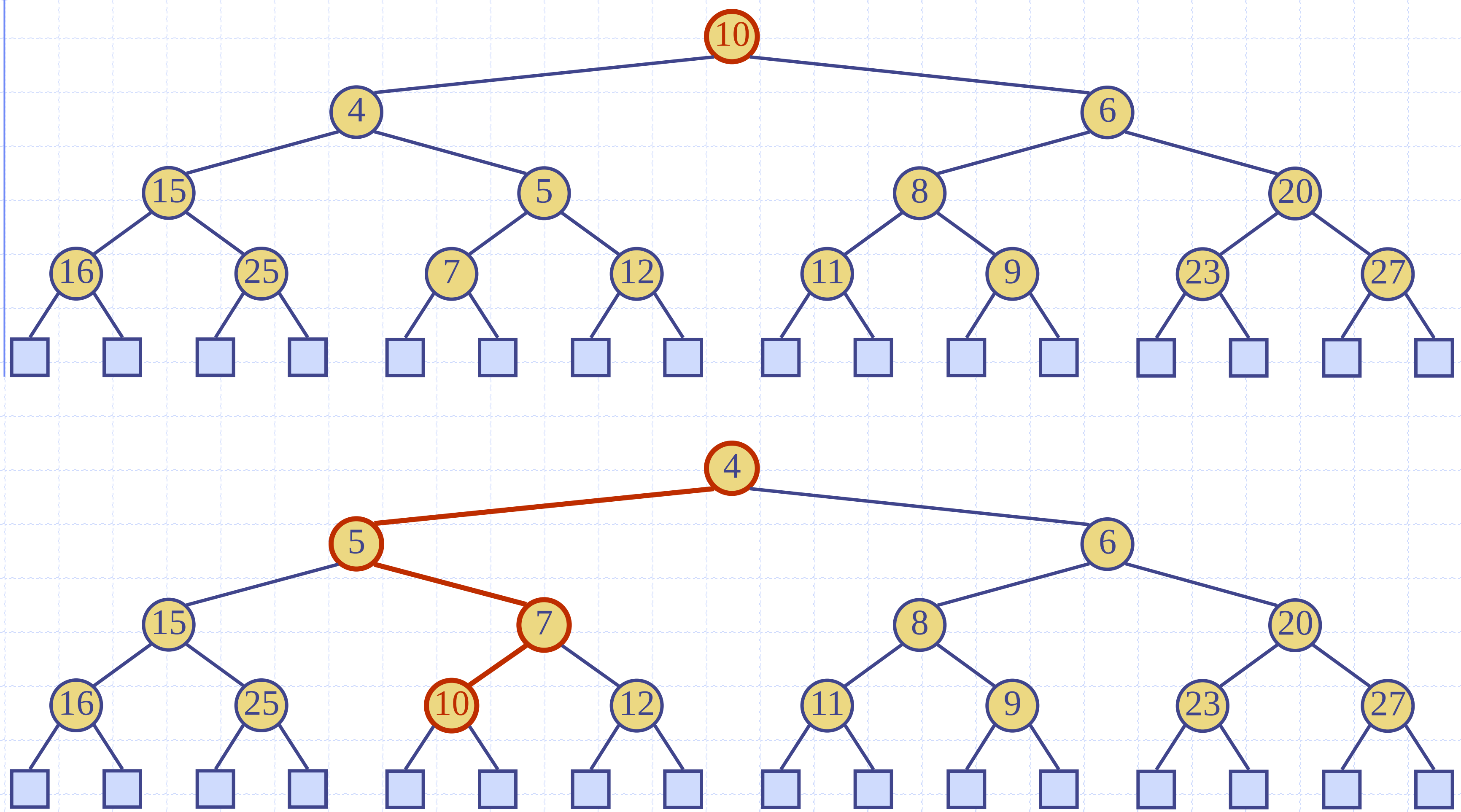
Example (contd.)



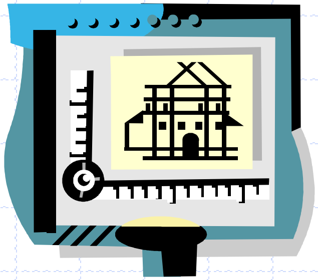
Example (contd.)



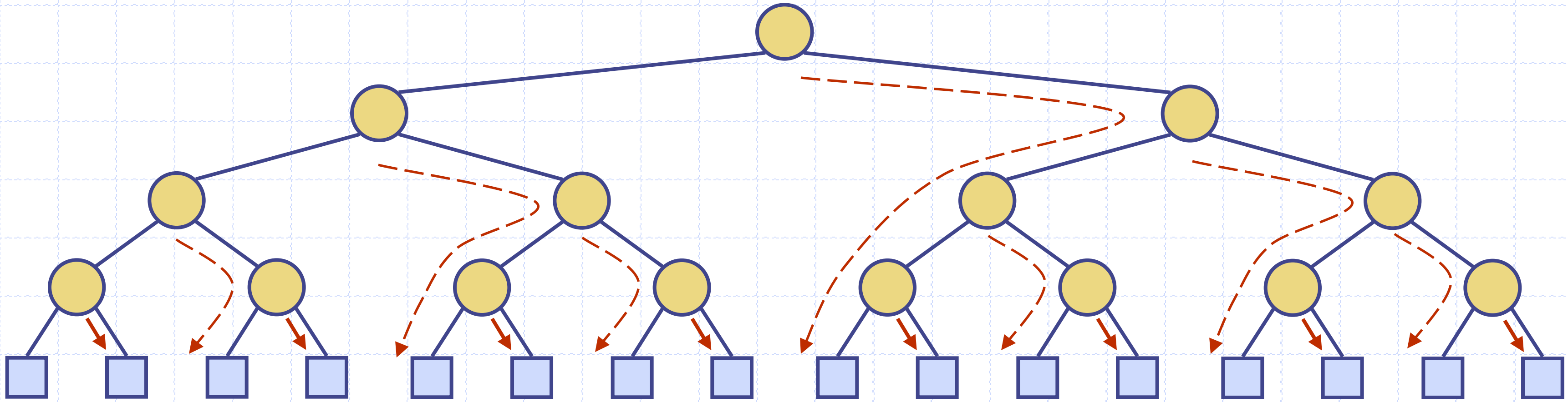
Example (end)



Analysis: What is the performance?

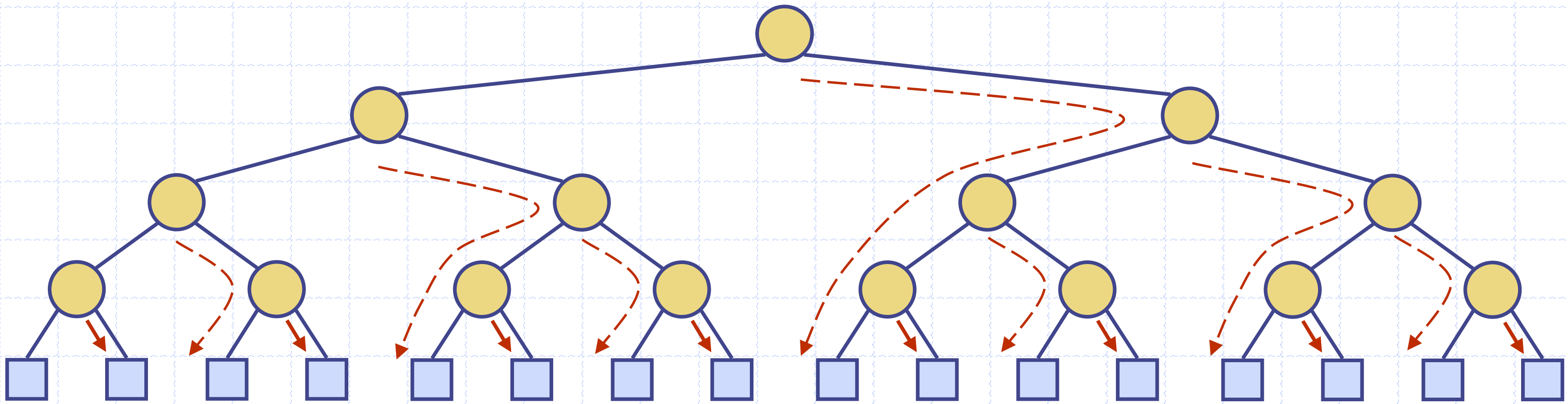


- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
  - Or, similarly, each edge of the tree is visited once and since the total number of edges is (2n-1), then $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
  - Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort

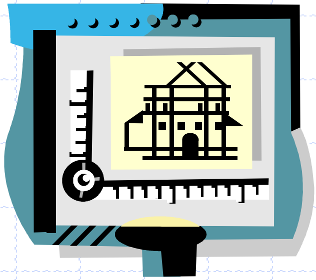


Analysis: What is the performance?

- Thus, we can start using the first results of sorting after $O(n)$ time and using $O(n)$ space
  - Groovy!



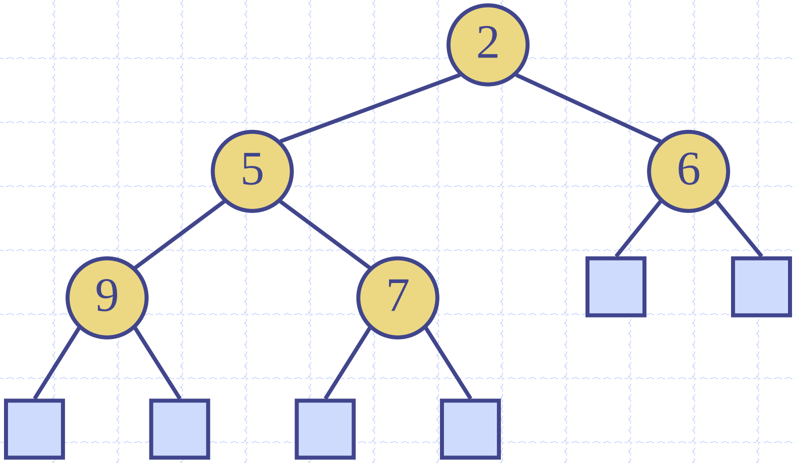
Analysis: Why is this important again?



- Consider the Internet
  - You have $N=10^9$ pages you want to sort and know the top results as soon as possible
  - Waiting $O(N^2) = 10^{18}$ before knowing the top results
    - is too long…
    - Even is you choose a very small time unit, e.g.:
      - you assume a 1-GigaHz computer to do 1-Giga operations per second, you will take 10^9 seconds, or 31 years!
  - Waiting $O(N\log N) = 30 \times 10^9$
    - is doable, maybe it means 30 seconds
  - Waiting $O(N) = 10^9$
    - **is more doable**, maybe meaning 1 second!!!

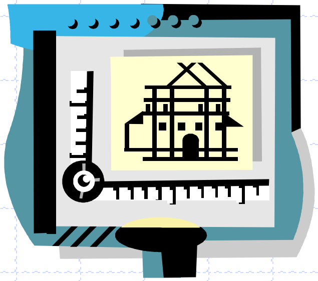
Vector-based Heap Implementation

- We can represent a heap with n keys by means of a vector of length $n+1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i+1$
- Links between nodes are not explicitly stored
- The leaves are not represented
- Operation insertItem corresponds to inserting at rank $n+1$
- Operation removeMin corresponds to removing at rank n
- **Yields in-place heap-sort!**



	2	5	6	9	7
0	1	2	3	4	5

Lets look at this again



- Consider the Internet
  - We looked at sorting the pages
    - $O(N^2)$, $O(N\log N)$, $O(N)$ (for first keys of the sort and $O(N\log N)$ to complete it)
  - How fast can we find any particular page we want in an initially unsorted set?
    - $O(N^2)$?
    - $O(N\log N)$?
    - $O(N)$?
    - $O(1)$? ← It is possible! (kinda)

Coming next!