Theory exam

Mars Gao

April 2021

1 Sufficient Statistics

Theorem 1 (Factorization Theorem). *$T \equiv T(X)$ is sufficient for $\mathcal{P}$ if and only if $f_\theta(x)$ factors as follows:*

$$f_\theta(x) = g_\theta(T(x)) \cdot h(x),$$

where $g_\theta(T(x))$ depends on x only through $T(x)$ and $h(x)$ does not depend on θ.

Theorem 2 (Lehmann-Scheffe). *Suppose that X has pdf $f_\theta(x), \theta \in \Omega$ and that $T^*(X)$ satisfies the following:*

$$\frac{f_\theta(y)}{f_\theta(x)} \; is \; \theta \; free \iff T^*(x) = T^*(y).$$

2 Theory of UMVUE

Theorem 3 (Rao-Blackwell-Lehmann-Scheffe (RBLS)). *Let $T = T(X)$ be complete and sufficient for θ. If there exists at least one unbiased estimator $\tilde{\tau} \equiv \tilde{\tau}(X)$ for $\tau(\theta)$, then there exists a unique UMVUE $\hat{\tau} \equiv \hat{\tau}(T)$ for $\tau(\theta)$, namely,*

$$\hat{\tau}(T) \equiv \mathbb{E}\left[\tilde{\tau}(X) | T\right].$$

Corollary 1 (The UMVUE Supermarket). *Let $T \equiv T(X)$ be complete and sufficient for θ. Then any function $\phi(T)$ is the UMVUE of its expectation $\mathbb{E}_\theta\left[\phi(T)\right] \equiv \tau(\theta)$.*

3 The Information Inequality

Theorem 4 (The Cramer-Rao-Frechet Lower Bound in 1d). *Let $T \equiv T(X)$ be complete and suffic Assume that $I_X(\theta) > 0$. Then,*

$$Var_\theta[T(X)] \geq \frac{\left(\frac{d}{d\theta}\mathbb{E}_\theta\left[T(X)\right]\right)^2}{I_X(\theta)},$$

where

$$I_X(\theta) = \mathbb{E}_\theta\left[\left[\frac{d\log f_\theta(X)}{d\theta}\right]^2\right] = -\mathbb{E}_\theta\left[\frac{d^2\log f_\theta(X)}{d\theta^2}\right].$$

Equality holds iff $\{f_\theta(x)\}$ is a 1-parameter exponential family of the form

$$f_\theta(x) = e^{A(\theta)}e^{B(\theta)T(x)}e^{c(x)}.$$

.

Corollary 2 (Unbiased estimator CR-bound)**.** *Suppose that $T(X)$ is an unbiased estimator of $\tau(\theta)$, a smooth function of θ. Then,*

$$Var_\theta(T(X)) \geq \frac{[\tau'(\theta)]^2}{I_X(\theta)}.$$

4 Hypothesis Testing

Definition 1 (NP criterion)**.** *Fix $0 < \alpha < 1$. A test ϕ is level α (size α) for H_0 if*

$$\pi_\phi(\theta_0) \equiv \mathbb{E}_{\theta_0}[\phi(X)] \leq \alpha.$$

A level α test for $H_0 : \theta \in \Omega_0$ is most powerful level α for H_1 if

$$\pi_\phi(\theta_1) \equiv \mathbb{E}_{\theta_1}[\phi(X)] = \sup_{\phi'} \pi_{\phi'}(\theta_1).$$

Theorem 5 (NP theorem)**.** *Let ϕ be a LR test of the form in previous definition with $c < \infty$ and set $\alpha = \mathbb{E}_{\theta_0}[\phi(X)]$. Then ϕ is a MP level α test for testing H_0 vs H_1. That is, if ϕ' is any other test such that*

$$\mathbb{E}_{\theta_0}[\phi'(X)] \leq \alpha,$$

then

$$\mathbb{E}_{\theta_1}[\phi'(X)] \leq \mathbb{E}_{\theta_1}[\phi(X)].$$

Theorem 6 (Wilks theorem)**.** *Let $X_1, ..., X_n$ be i.i.d. samples from a regular family that satisfies Fisher-Cramer and Wald conditions. The MLE $\hat{\theta}$ is CANE for θ. Let λ_n be the Likelihood ratio test statistics for testing. Then, under H_0,*

$$-2\log\lambda_n \xrightarrow{d} \chi^2_{d-d_0}.$$

5 Checklist of examples

- 12.14 UMVUE of Uniform.
- 14.3 MLE of Uniform.
- 14.4 MLE of Gaussian μ, σ.
- 12.12 UMVUE of Gaussian μ, σ.
- 14.5 MLE of Cauchy.
- 14.25 Multinomial Multivariate MLE.
- 14.34 General location-scale family MLE.
- 13.7 UMVUE of Binomial.
- 12.13 UMVUE of Poisson.
- 11.11 Mimimal sufficient statistics of exponential/truncation family.
- Proposition 12.1 and 12.2 for exponential family completeness.
- Theorem 18.6 NP hypothesis testing.
- 2019.1 Confidence interval asymptotic.
- 2019.3 Hypothesis testing LRT, p-value, posterior testing.
- 2019.1 MLE of Poisson.
- 2019.2 Unbiased estimation of Poisson.
- 2019.6 Uniform $[-\theta, \theta^2]$.
- 2020.2 Definition of consistent and unbiased.
- 2020.2 Bernoulli unbiased estimator.
- 2020.3 Multinomial. MLE, MSS.
- 2020.4 Sample median and MLE of Uniform (0, 1).
- 2020.5 2d unit disk variable transformation.
- 2020.6 Bayesian for what??
- 2018.1 Mixture of normal.
- 2018.2 Proof of Chebyshev.
- 2018.3 Poisson UMVUE.
- 2018.4 Logit model.

- 2018.5 Bernoulli MLE. Bayes estimator with MSE.
- 2018.6 MLE of Beta. Information lower bound.
- 2017.1 MGF of Normal and Normal mean.
- 2017.2 All hypothesis testing for exponential.
- 2017.3 Convergence in distribution and probability.
- 2017.4 Variable transformation.
- 2017.5 Exponential (change) MLE, information lower bound. Existence of MLE.
- 2017.6 Mixture of two distribution.
- 2016.1 Uniform k-th moment UMVUE.
- 2016.2 Binomial mean. Binomial cannot find unbiased (polynomial proof).
- 2016.3 Distribution Chi-square and testing.
- 2016.4 Linear model and lowest MSE.
- 2016.5 Multi-nomial, complete statistics, conditional distribution and UMVUE.
- 2016.6 Poisson with 0 events. Unbiased estimator.