Homework Assignment #1 (due Tuesday, Oct. 6th)

In all programming exercises for the homework problems of this course, please represent real numbers in the double precision format unless otherwise explicitly stated.

1. Write a C or C++ program to check when your computer fails to tell the difference between two numbers. Initialize a real number with two, say $c = 2$. Keep reducing the number by $c := (1 + c)/2$. Check after how many times the computer fails to recognize the number is actually greater than one (or the computer starts to treat the number as equal to one). Please repeat the experiment with single precision format for the real numbers.

2. Write C or C++ programs to approximately compute $\ln 2$ with a series.

   (a) Apply the series
   $$\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$
   to compute $\ln 2$. Find the minimum number of terms in the series for the computed value to have an absolute error less than 10^{-6}.

   (b) Apply the series
   $$\ln \frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n - 1} + \cdots\right)$$
   with $x = 1/3$ to compute $\ln 2$. Find the minimum number of terms in the series for the computed value to have an absolute error less than 10^{-6}.

3. Write C or C++ programs to numerically compute the integral $I_n = \displaystyle\int_0^1 \frac{x^n}{x + 10}\, dx$ for integer $n > 0$.

   (a) Note that for $n = 0$, $I_0 = \ln \frac{11}{10}$. Use the recursion
   $$I_k = \frac{1}{k} - 10\, I_{k-1} \qquad \text{for } k = 1, 2, 3, \cdots, n$$
   to compute the integrals, I_{10} and I_{20}.

   (b) Note that
   $$\frac{1}{11\,(n + 1)} < I_n < \frac{1}{10\,(n + 1)}.$$
   The integral I_n can be approximated by the average of the lower and upper bounds. For positive integers $m > n$, we have the recursion
   $$I_{k-1} = \frac{1}{10}\left(\frac{1}{k} - I_k\right) \qquad \text{for } k = m, m - 1, \cdots, n + 1.$$
   Let $m = 40$ and approximate the integral at $m = 40$ by $I_m \approx \dfrac{21}{220(m + 1)}$. Use the recursive relation above to compute the integrals, I_{10} and I_{20}.

(c) Do the previous two problems again. But this time please represent real numbers in the single precision format in your programs. Compare the results with different precision formats.

4. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix whose leading principal sub-matrices are all invertible. Please give detailed proof for the following two statements.

   (a) There is a unique LU decomposition of the matrix $\mathbf{A}$ so that $\mathbf{A} = \mathbf{LU}$ for a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$ with the diagonal entries of $\mathbf{L}$ all equal to one.

   (b) If the matrix $\mathbf{A}$ is symmetric and positive definite, it has a unique decomposition in the form of $\mathbf{A} = \mathbf{LDL}^{\mathrm{T}}$. Here, $\mathbf{D}$ is a diagonal matrix with all diagonal entries positive and $\mathbf{L}$ is a lower triangular matrix with all diagonal entries equal to one.

5. Please make LU decomposition by hand for the following matrices.

$$(a).\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix} \qquad (b).\ \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

6. Let $n > 0$ be an integer, and $\mathbf{A}$ be the dense matrix (every entry is nonzero) given by

$$\mathbf{A} = \frac{1}{n+1} \begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2 \\ n-2 & 2(n-2) & 3(n-2) & \ddots & 6 & 3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-1) & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}_{n\times n}.$$

Write a C/C++ computer program to find the inverse of the matrix $\mathbf{A}$. Print out the inverse matrices $\mathbf{A}^{-1}$ for $n = 4$ and $n = 8$. Report the computer times used by your program to find the inverse for $n = 100, 200, 400, 800$. Find the relation of the computer time used by your program with the dimension of the matrix.