MATH2850J Honors Mathematics III
Functions of Multiple Variables

Summer 2024 Term Project

Date Due: 11:59 PM, Saturday, the 6$^{\text{th}}$ of July 2024

JOINT INSTITUTE
交大密西根学院

General Information

The goal of this term project is to help you apply your knowledge of mathematics in extended tasks that are beyond the scope of ordinary assignments. **It is strongly recommended that you do not leave the entire project to the last minute** but rather commence work on individual parts as soon as you are able to do so.

You will find three project options below: the first project focusses strongly on mechanics and the mathematics of curves, while the second project is based on geometric optics as well as a different aspect of curve geometry. The geometrical aspects predominate. The final project is more purely mathematical / computational. Your group (see below) should choose a project that bets fits your interests.

Group Work

You will be divided into groups of *4–5 students* each.

> Each group member must be familar with and have contributed to each part of the project report. **You may not divide up the work in such a way that only certain members are involved with certain parts.** In the event of an Honor Code violation (plagiarism or other), all members of the group will be held equally responsible for the violation. Exceptions may only be made, at my discretion, in exceptional situations.

It is therefore all group members' duty to ensure that all collaborators' contributions are plausibly their own and to check on all collaborators' work progress and verify their contributions within reason.

Project Report

The term project will be submitted **electronically only** as a typed report. Handwritten submission will not be accepted! It is recommended that you use a professional type-setting program (such as LaTeX) for your report. Unless you are able to ensure a unified font size and style for formulas and text in Microsoft Word, use of Word is *not recommended*.

Your report should have the appearance, style and contents of a professional report. It should be comprehensible without reference to this document and should be comprehensible by any other student in this course. It is strongly suggested that all members of the project team proof-read the report before submission. **The report should not look like the solution to an assignment.** Do not structure the section titles as "Answer to Question i)" or similarly.

The following are a list of projects, one of which should be completed during this term. The goal of the projects is to allow you to perform some long-term in-depth research on an interesting subject, in a much more intensive way than is possible in standard course work. The projects are intentionally open-ended, i.e., after an initial part that will get you started, you determine yourselves in what direction and how far you want to take the project. In particular, you can perform work on a computer, using a computer algebra system to visualize certain aspects, or a programming language (Pascal, C++, java) to write a program that will perform numerical calculations, or even create a web page for your project! This is not specified in the projects, but can be developed as you see fit. If you perform experiments or carry out research, you can also make a video of your activities and include it in your project report. It is up to you how far you develop these projects.

Guidelines

The following guidelines apply:

- You will be randomly assigned into Project Teams of 4-5 students each Each group will then choose a single project, and submit this project as a group. No individual work will be credited, and all members of a group will receive the same grade, except in special cases.
- Each group will create a joint Project Report, typed on a word processor such as Word or LaTeX. The project report must be submitted in a PDF file format and include:
  - formulation of your aims for this project,
  - a detailed report of your research/experiments/other activities,
  - your results and calculations,
  - a conclusion - were the aims of the project achieved? Why or why not?,
  - a bibliography,
  - supplementary material
- Needless to say, your work must be original. Naturally, many groups will choose the same project, but each group should use its own methods and interpretations to deal with the project. If you find that you are running out of ideas to pursue, contact me. Please also read the UM-SJTU JI Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text.

Grading Policy

This term project accounts for 15% of the course grade; it will be scored based on

- **Form (3 points):** Does the report contain essential elements, such as a cover page (with title, date, list of authors), a synopsis (abstract giving the main conclusions of the project), table of contents, clear section headings, introduction, clear division into sections and appendices with informative titles and bibliography (if applicable)? Are the pages numbered? Are the text and formulas composed in a unified font? Are all figures (graphs and images) clearly labeled with identifiable source?
- **Language (3 points):** Is the style of english appropriate for a technical report? Do not treat the project as an assignment and simply number your results like part-exercises. Your text should be a single, coherent whole. The text should be a pleasant read for anyone wanting to find out about the subject matter.

  Errors in grammar and orthography (use a spell-checker!) will be penalized. Make sure that the report is interesting to read. Avoid simply repeating sentences by cut-and-paste.
- **Content (9 points):** Are the mathematical and statistical methods and deductions clearly exhibited and easy to follow? Are the conclusions well-supported by the mathematical analysis? It is important to not just copy calculations from elsewhere, but to fully make them your own, adding details and comments where necessary.

This yields the common project score. **The individual score will be adjusted according to the results of the CATME peer evaluation.** See the next section for details.

However, if a student verifiable does not contribute at all to the group's work, they will receive 0 Marks for the project.

CATME Peer Evaluation

All group members are **required** to participate in the peer evaluation. Any group member who does not participate will receive zero marks for the project.

Based on the evaluation of peers, CATME determines an *Adjustment Factor (without Self)*. This common project score will be multiplied with this factor for each individual group member to determine the final score. The factor generally lies between 0 and 1.05. Based on past evaluations, the overwhelming majority of factors lies between 0.95 and 1.05, but factors of 0.4 or lower are possible in cases where students did not participate in the group work.

The systems detects attempts to manipulate the rating system by giving artificially high or low scores, collusion among two or more team members, and other anomalous scores. Where suspected instances are flagged by CATME, the instructor may override the system's Adjustment Factor.

On Plagiarism

Study JI's Honor Code carefully. **Any** information from third parties (books, web sites, even conversations) that you use in your project must be accounted for in the bibliography, with a reference in the text. Follow the rules regarding the correct attribution of sources that you have learned in your English course (e.g., Vy100, Vy200). All members of a group are jointly responsible for the correct attribution of all sources in all parts of the project essay, i.e., any plagiarism will be considered a violation of the Honor Code by all group members. Every group member has a duty to confirm the origin of any part of the text.

The following list includes some specific examples of plagiarism:

- Use of any passage of three words or longer from another source without proper attribution. Use of any phrase of three words or more must be enclosed in quotation marks ("example, example, example"). This excludes set phrases (e.g., "and so on", "it follows that") and very precise technical terminology (e.g., "without loss of generality", "reject the null hypothesis") that cannot be paraphrased,
- Use of material from an uncredited source, making very minor changes (like word order or verb tense) to avoid the three-word rule.
- Inclusion of facts, data, ideas or theories originally thought of by someone else, without giving that person (organization, etc.) credit.
- Paraphrasing of ideas or theories without crediting the original thinker.
- Use of images, computer code and other tools and media without appropriate credit to their creator and in accordance with relevant copyright laws.

Instances of plagiarism. will be reported to the Honor Council. **Please ensure good communication and close collaboration with your team.** Furthermore, as laid out in the Honor Code,

You may not use any form of generative AI software, such as ChatGPT, Gemini, or any other similar text-, image- or video-generating software at all. You may also not use electronic translation software.

All instances of plagiarism will be reported to the Honor Council. **Please ensure good communication and close collaboration with your team.**

Project 1: A Perfect Pendulum

Pendula[1] have been used for time-keeping and in physical and engineering applications for millennia. Their periodic motion and simple construction made them well-suited for the construction of clockworks. However, they possess a basic drawback when used for these purposes: their periods depend on the initial displacement. The present project investigates this effect in detail.

i) Give a definition of a "mathematical pendulum" of length l and mass m. What is the difference to a "physical pendulum"? Show that the energy of a mathematical pendulum is given by

$$E(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta),$$

where $\theta = \theta(t)$ is the angle of displacement and $\dot{\theta}$ is the time-derivative of θ. Here g is the acceleration due to gravity, assumed constant. Derive the *pendulum equation*

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin(\theta(t)) = 0.$$

ii) Suppose that a pendulum is held at an angle $\theta(0) = \theta_0$ at time $t = 0$ and then let go. Show that

$$|\dot{\theta}| = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

For a single period, the map $t \mapsto \theta$ is bijective and hence invertible. Use the Inverse Function Theorem of Vv186 to derive the equation

$$T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

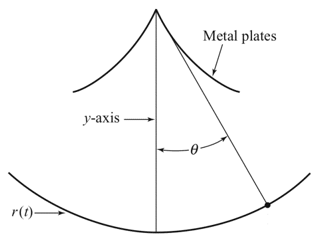
Using an appropriate substitution, show that

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\theta_0/2)\sin^2\phi}}$$

Note that this is a complete elliptic integral of the first kind (see Assignment 12 of Vv186). Give a formula relating the period of the pendulum to the arithmetic-geometric mean. Using an appropriate series expansion, show that the period is approximately given by

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16}\right) \approx 2\pi\sqrt{\frac{l}{g}}.$$

The mass of a simple pendulum moves along a circular path, held there by its string. Suppose the shape of this path could be modified (e.g. by varying the effective length of the string in some way, e.g., using plates). The famous mathematical *tautochrone problem* (sometimes also called the *isochrone problem*) asks whether there exists a path along which the period of such a pendulum would not depend on θ_0. This problem was solved by Christiaan Huygens in 1659 and the main part of this project is concerned with its solution.



Pendulum constrained by plates. Figure taken from [2].

iii) Consider the one-dimensional motion of a point mass on the real axis, travelling along a trajectory $\gamma\colon \mathbb{R} \to \mathbb{R}$ according to Newton's law $F \circ \gamma(t) = m\gamma''(t)$. Suppose that there exists a function $U\colon \mathbb{R} \to \mathbb{R}$ such that $F(x) = U'(x)$ for all $x \in \mathbb{R}$ (called the *potential energy*), $U(0) = 0$ and U is monotonically increasing on $[0, \infty)$.

$$U(\gamma(t)) + \frac{m}{2}\gamma'(t)^2 = \text{const.}$$

Given a potential U, Newton's law together with $\gamma(0) = x_1$ for some $x_1 > 0$, $\gamma'(0) = 0$ can be used to determine a trajectory such that $\gamma(T) = 0$. In general, $T = T(x_1)$. We seek to determine U such that T that is actually independent of x_1. First, show that

$$T(x_1) = \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{1}{\sqrt{U(x_1) - U(x)}}\, dx$$

[1] Either *pendulums* or *pendula* is accepted as the plural of pendulum.

Substitute in the integral as follows:

$$y^2 = \frac{U(x)}{U(x_1)}$$

and then show that U must satisfy the equation $U'(x) = c \cdot \sqrt{U(x)}$ for some $c > 0$. Assuming that $U(0) = 0$ and that U is strictly increasing for $x \geq 0$, this equation has a unique solution on $[0, \infty)$.

iv) Explain why the tautochrone must be a path along which

$$\frac{d^2 s}{dt^2} = -ks, \tag{1}$$

where t is time and s is the path length. Does a simple pendulum (as discussed in parts i) and ii) above) satisfy this relation?

v) Deduce that a curve satisfying (1) is a cycloid and give the parametric equations (parametrization) of this cycloid (cf., for example, [2, Exercise 3.2]). Note that a proof that a cycloid curve satisfies (1) can be commonly found, e.g., in [3]. However, such a proof depends on already knowing the solution curve. You are asked here to deduce the cycloid from the equation alone.

vi) Follow the ideas discussed in Exercises 3.4 and 3.5 of [2] to describe the construction of a tautochronous pendulum.

References

[1] A. Belyaev. Plane and space curves. Curvature. Curvature-based features. www.math.utah.edu/~palais/dnamath/04gm_curves.pdf, 2006. Web. Accessed July 10th, 2016.

[2] R. Knoebel, A. Laubenbacher, R. Lodder, and D. Pengelley. *Mathematical Masterpieces: Further Chronicles by the Explorers*. Undergraduate Texts in Mathematics. Springer, 2007. Several sections can be downloaded here: `http://sofia.nmsu.edu/~history/`.

[3] E. W. Weisstein. Tautochrone problem. From MathWorld–A Wolfram Web Resource. Web. Accessed April 11th, 2012.

Project Option 2: Solar Halos from Ice Crystals

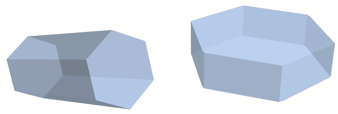

Image ©Zhang Jiayi

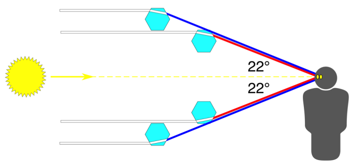

Image ©Mu Kaixuan

A common meteorological phenomenon is that of a "halo", a ring of light appearing around the sun. A beautiful example was photographed by students at SJTU on June 7^{th}, 2021. Two photos, kindly provided by Zhang Jiayi and Mu Kaixuan, are shown at left.

This particular halo is known as a 22° *halo* and is the most commonly observed type. The goal of this project is to investigate this optical effect in detail.

The "short explanation" for this effect is as follows: thin, high-altitude clouds (of *cirrus* or *cirrostratus* type) consist mainly of ice crystals. These crystals are near-regular hexagonal columns or plates, as shown at right.

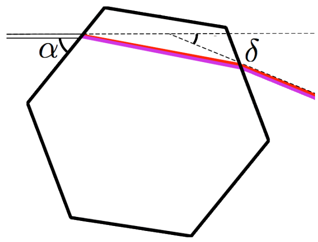

Hexagonal Column and Plate



22° Halo Diagram [3]

Sunlight is refracted through these hexagonal crystals. Even though the ice crystals need not be arranged in any specific configuration (they may float about randomly), we observe the reflected sunlight at 22° (hence the name of this particular halo). Due to the radial symmetry, the halo appears as an annulus. The refraction index of the ice crystals depends slightly on the wavelength, leading to a separation of colors.

i) The angle of refraction δ of a light ray in a hexagonal prism depends on the angle of incidence α of that ray (see sketch at right). Calculate $\delta = \psi(\alpha)$ and plot the solution. Find the minimum deflection angle by solving $\psi'(\alpha) = 0$ and verify that this is indeed a minimum.



Refraction in a hexagonal prism. Derived from [4]

ii) Why is the halo visible at roughly 22°? Why is the interior of the halo darker than the surrounding sky? Research and answer these questions. Start by looking through the references below, such as [1], [5]. A more mathematically-oriented approach is found, for instance in [7, 8].

iii) What is a 46° halo and how is it formed?

iv) Sometimes, a 22° halo is accompanied by two or more bright lights, called *sun dogs* (see [5, Ice halos] for a discussion of the various ice halos. A beautiful example is also shown in [6]). Describe and explain their formation.

v) Ice crystals are also responsible for the appearance of an "inverted rainbow," formally known as a *circumzenithal arc*. For this phenomenon it is important that the ice crystals are not randomly oriented but rather aligned with each other. Feel free to explore any aspects of this phenomenon that interest you or to choose one of the many other halo-related optical phenomena caused by ice crystals.


Circumzenithal Arc. [2]

References

[1] J. A. Adam. *Mathematics in Nature*. Princeton University Press, Princeton and Oxford, 2003.

[2] Wikimedia Commons. File:strange rainbow in sky (7753346820).jpg — wikimedia commons, the free media repository. `https://commons.wikimedia.org/w/index.php?title=File:Strange_Rainbow_in_Sky_(7753346820).jpg&oldid=490836158`, 2020. Web. Acessed June 9^{th}, 2021.

[3] Wikimedia Commons. File:20200712 22 degree halo - diagram.svg — wikimedia commons, the free media repository. `https://commons.wikimedia.org/w/index.php?title=File:20200712_22_degree_halo_-_diagram.svg&oldid=522890463`, 2021. Web. Acessed June 8^{th}, 2021.

[4] Wikimedia Commons. File:path of rays in a hexagonal prism.png — wikimedia commons, the free media repository. `https://commons.wikimedia.org/w/index.php?title=File:Path_of_rays_in_a_hexagonal_prism.png&oldid=526777289`, 2021. Web. Acessed June 8^{th}, 2021.

[5] L. Cowley. Atmospheric optics. `http://www.atoptics.co.uk`, 2015. Web. Acessed June 8$^{\text{th}}$, 2021.

[6] People's Daily. What a spectacular wonder! Five suns are seen shining in the sky in N China's Inner Mongolia. `https://twitter.com/PDChina/status/1228436251528830976`, 5:50 AM on February 15$^{\text{th}}$, 2020. Web. Acessed June 9$^{\text{th}}$, 2021.

[7] W. Tape. Geometry of halo formation. *J. Opt. Soc. Am.*, 69(8):1122–1132, Aug 1979.

[8] W. Tape. Analytic foundations of halo theory. *J. Opt. Soc. Am.*, 70(10):1175–1192, Oct 1980.

Project Option 3: $\pi = 3.14159265\ldots$

The number π has fascinated mathematicians for millenia. From ancient approximations through Archimedes's circle approximations by polygons through techniques based on calculus, advances in mathematics have been accompanied by new methods for finding ever more decimal digits of π. We have already discovered and proven some fascinating identities, such as

- The Leibniz series $\dfrac{\pi}{4} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
- The Euler series $\dfrac{\pi^2}{6} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$
- The Wallis formula $\dfrac{\pi}{2} = \dfrac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9} \cdots$

Unfortunately, these formulas are not very practical for actually calculating π to any significant precision because the infinite series/products converge extremely slowly. Both the Leibniz and the Euler series, after summing 10,000 terms, only give $\pi \approx 3.141$. The goal of this project is to investigate efficient algorithms for decimal digits of π. The following exercises are based entirely on [1].

i) Read Section 11.1 of [1]. The Leibniz formula is based on the arctangent series $\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. *Machin's formula*,

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right),$$

published in 1706 is a significant improvement on the Leibniz series, because on the one hand $1/239$ is very small and the arctangent series converges rapidly and on the other hand $x = 1/5$ is well-suited for decimal calculations in the arctangent series. Prove Machin's formula as follows: Let $\theta = \arctan(1/5)$. Show that

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{5}{12}$$

and $\tan(4\theta) = 1 + \frac{1}{119}$. Finally establish

$$\tan\left(4\theta - \frac{\pi}{4}\right) = \frac{1}{239}.$$

Many similar formulas based on the arctangent were found and used up to the 1970s for calculations of decimal digits of π. Up to a million digits were obtained in this way.

ii) Very efficient algorithms can be derived from the theory of the arithmetic-geometric mean. We will study Algorithm 2.1 of [1]: Let $x_0 = \sqrt{2}$, $\pi_0 = 2 + \sqrt{2}$, $y_1 = \sqrt[4]{2}$. Define

$$x_n = \frac{1}{2}\left(\sqrt{x_{n-1}} + \frac{1}{\sqrt{x_{n-1}}}\right), \qquad y_{n+1} = \frac{y_n\sqrt{x_n} + 1/\sqrt{x_n}}{y_n + 1}, \qquad \pi_n = \frac{x_n + 1}{y_n + 1}\pi_{n-1}$$

for $n \geq 1$. Then $\pi_n \searrow \pi$,

$$\pi_n - \pi < \frac{1}{10}(\pi_{n-1} - \pi)^2 \qquad \text{and} \qquad \pi_n - \pi < 10^{-2^{n+1}}.$$

Thus, in n iterations the algorithm gives at least 2^n decimal digits of π. The theory underlying this algorithm involves various kinds of elliptic integrals. This is the main part of the project: collect the necessary theory from [1] and prove this algorithm. Write an exposition introducing the different types of elliptic integrals and proving the necessary relations. Omit any information that you don't need (for example, theta functions do not play a role here and should not be mentioned). Your derivation should be detailed enough that any other student of MATH285 can follow it without additional references.

iii) Implement the algorithm of ii) in a programming language or a computer algebra system, e.g., Mathematica.

References

[1] J. M. Borwein and P. B. Borwein. *Pi and the AGM.* John Wiley & Sons, New York, 1987. http://as.wiley.com/WileyCDA/WileyTitle/productCd-047131515X.html A copy of this book is available from me upon request.