$$\iint q(\beta,\gamma)\,\log p(y \mid X,\beta,\sigma^2)\; d\beta\, d\gamma$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\,E\left[\left(y - \sum_{k=1}^{K} X\beta_k\right)^T\left(y - \sum_{k=1}^{K} X\beta_k\right)\right]$$

$r = r_1 + \dots + r_K$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{y^Ty}{2\sigma^2} + \frac{y^TXr}{\sigma^2} - \frac{1}{2\sigma^2}\,\underbrace{E\left[\left(\sum_{k=1}^{K} X\beta_k\right)^T\left(\sum_{k=1}^{K} X\beta_k\right)\right]}_{\star}$$

$$\star = E\left[\sum_{k=1}^{K}\sum_{k'=1}^{K} \beta_k^T X^T X \beta_{k'}\right]$$

$$= \sum_{k=1}^{K}\sum_{k'=1}^{K} E\left[\beta_k^T X^T X \beta_{k'}\right] = \sum_{k=1}^{K}\sum_{k'=1}^{K} \mathrm{tr}\left[X^TX\,E[\beta_k \beta_{k'}]\right]$$

$$= \sum_{k=1}^{K}\sum_{k'=1}^{K}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\,E\left[\beta_{jk}\beta_{j'k'}\right]$$

$$= 2\sum_{k \neq k'}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\,\overbrace{\alpha_{jk}\,\alpha_{j'k}\,\mu_{jk}\,\mu_{j'k}}^{=\, r_{jk}\, r_{j'k}}$$

$$+ \sum_{k=1}^{K}\sum_{j=1}^{p} (X^TX)_{jj}\,\underbrace{E\left[\beta_{jk}^2\right]}_{=\,\alpha_{jk}(\mu_{jk}^2 + s_{jk}^2)}$$

$$+ \sum_{k=1}^{K}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\, r_{jk}\, r_{j'k}$$

$$- \sum_{k=1}^{K}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\, r_{jk}\, r_{j'k}$$

$$= \sum_{k=1}^{K}\sum_{k'=1}^{K}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\, r_{jk}\, r_{j'k}$$ ← Combine lines 1 and 3 above.

$$- \sum_{k=1}^{K}\sum_{j=1}^{p}\sum_{j'=1}^{p} (X^TX)_{jj'}\, r_{jk}\, r_{j'k}$$

$$+ \sum_{k=1}^{K}\sum_{j=1}^{p} (X^TX)_{jj}\,\alpha_{jk}\,(\mu_{jk}^2 + s_{jk}^2)$$ ← This is not $\mathrm{Var}[\beta_{jk}]$!

$$= \|Xr\|^2 - \sum_{k=1}^{K}\|Xr_k\|^2 + \sum_{k=1}^{K}\sum_{j=1}^{p} (X^TX)_{jj}\,\alpha_{jk}\,(\mu_{jk}^2 + s_{jk}^2)$$