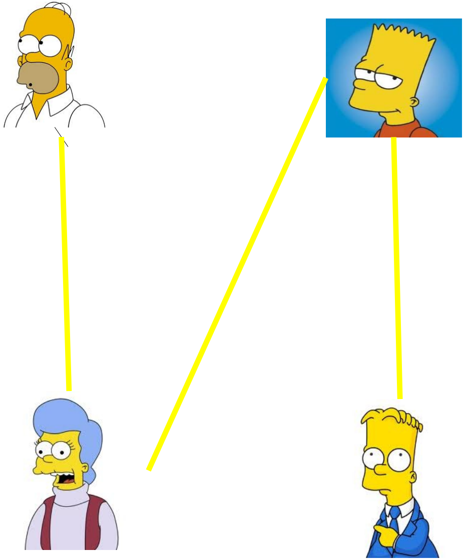
Paths, Trees, and Flowers

Jack Edmonds

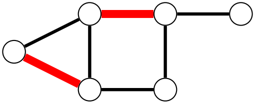
Work scheduling

Schedule workers in pairs. What is the optimal assignment?



Matching

Matching in a graph is a set of edges, no two of which meet at a common vertex.



Maximum matching is a matching of maximum cardinality.

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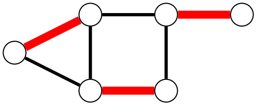
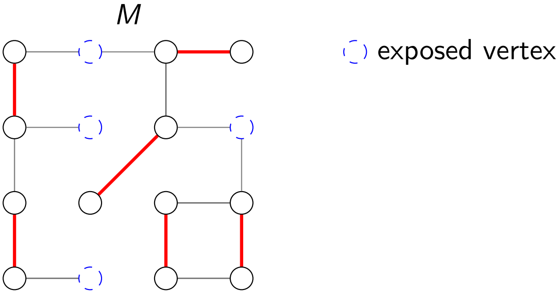


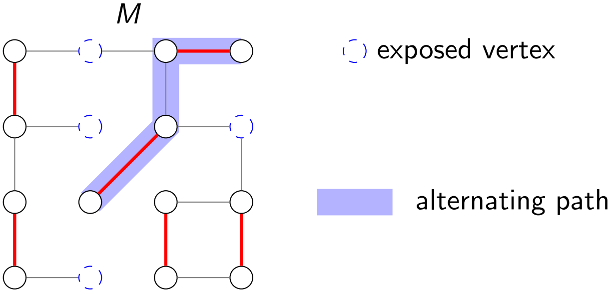
Exposed vertex

Exposed(free) vertex is a vertex that is not incident with any edge in M



Alternating path

Alternating path is a path whose edges are alternately in M and $\overline{M}$



Different kinds of alternating path

An alternating path between two matched vertices

An alternating path between two exposed vertices

An alternating path between a matched vertex and an exposed vertex

An alternating cycle

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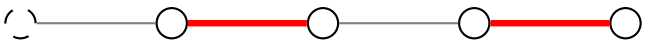

An alternating cycle

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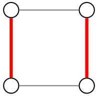
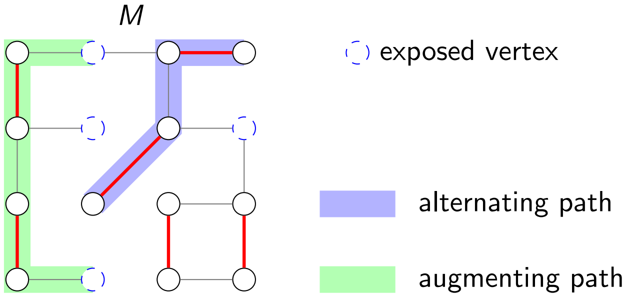


Augmenting path

Augmenting path is a simple alternating path between exposed vertices

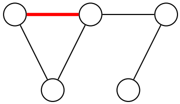


Symmetric difference

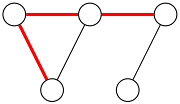
Symmetric difference of two sets D and E is defined as

$$D \oplus E = (D - E) \cup (E - D)$$

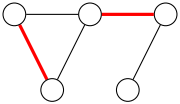
D



E

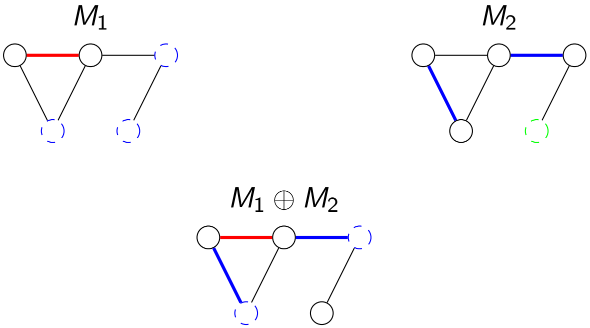


$D \oplus E$



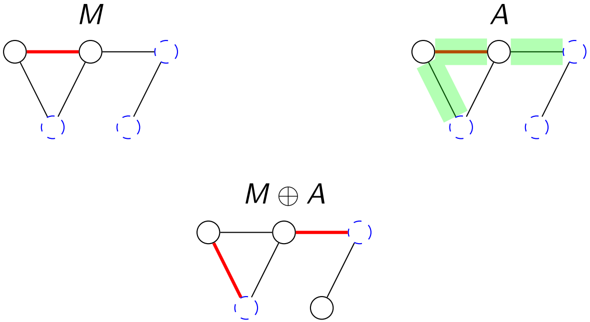
Alternating path

For any two matchings M_1 and M_2 in graph G, the components of the graph formed by $M_1 \oplus M_2$ are paths and circuits which are alternating for M_1 and M_2. Each path end-point is exposed for either M_1 or M_2.



Augmenting path

A is an augmenting path in (G, M) . $M \oplus A$ is a matching of G larger than M by one.



Berge's theorem

Theorem - Berge(1957)

M is not a maximum matching if and only if there exists an augmenting path with respect to M

- Proof: if M_2 is a larger matching than M, some component of graph $M \oplus M_2$ must contain more M_2 edges than M, such a component is an augmenting path for (G, M).

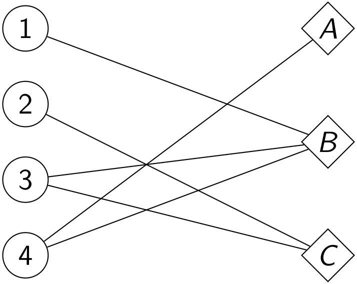
Algorithm

Maximum-Matching(G)

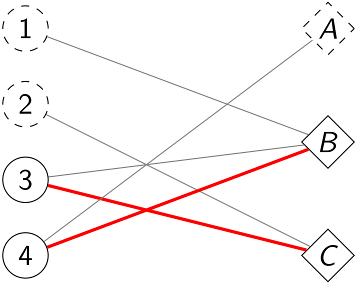
$M = \emptyset$
repeat
    if there is an augmenting path P with respect to M
        $M = M \oplus P$
until there isn't an augmenting path with respect to M
return M

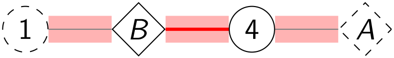
Bipartite Matching

Left circles represent boys, right diamonds represent girls, and edges are their preferences. We want to make maximum number of couples.



Find An Augmenting Path Using BFS





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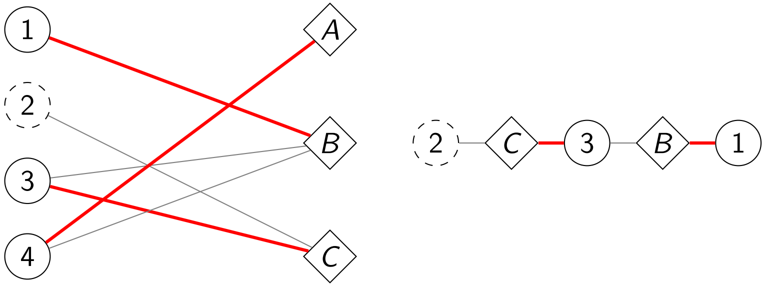
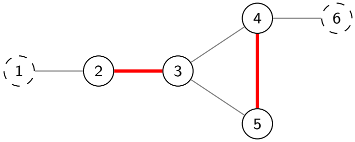
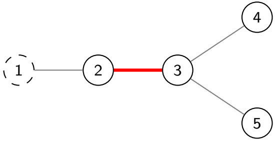


Examples of BFS failure on non-bipartite graphs

- Initial graph



- Using BFS



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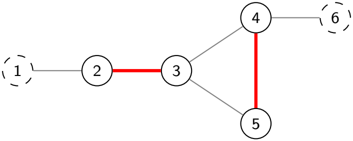
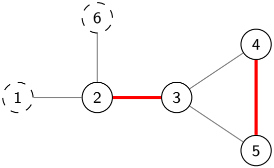


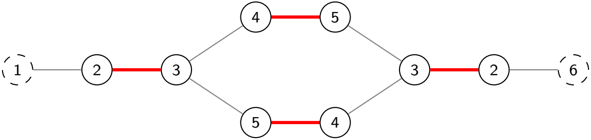
- Allow a vertex to be visited at both odd and even levels, we find an augmenting path

Examples of BFS failure on non-bipartite graphs

- Initial graph



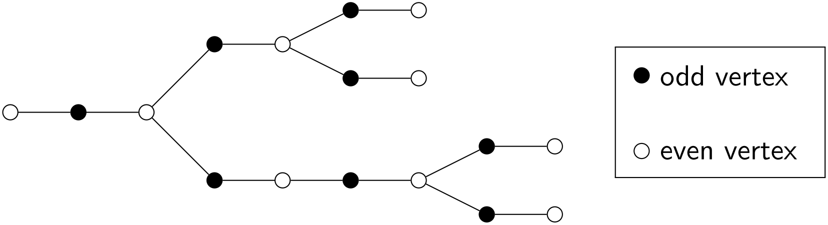
- But the augmenting path we find may not be a simple path



Alternating Tree

Alternating Tree is a tree J, each of whose edges joins an odd vertex to an even vertex so that each odd vertex of J meets exactly two edges of J.

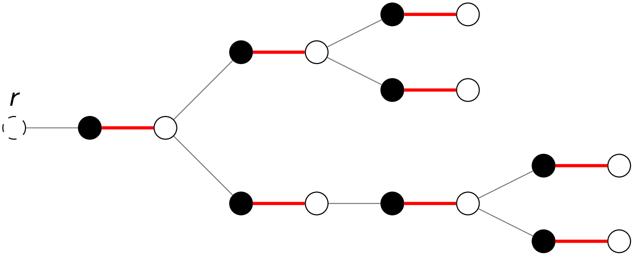
- An alternating tree contains one more even vertex than odd vertices.



Planted Tree

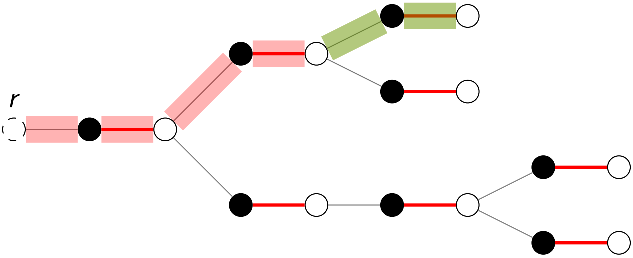
Planted tree is an alternating tree $J(M)$ of G for matching M s.t. $M \cap J$ is a maximum matching of J.

Root is a vertex r in J which is exposed.



Planted Tree

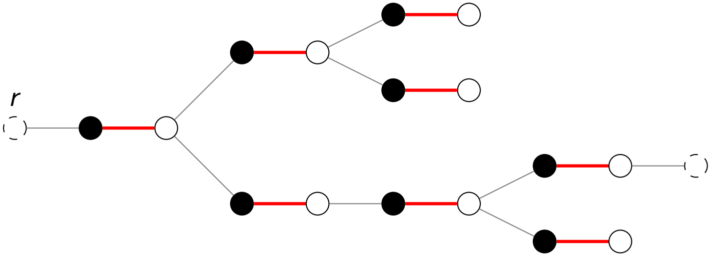
In planted tree $J(M)$ every alternating path $P(M)$, which has even vertex v and the matching edge to v at one of its ends, is a subpath of the alternating path $P_v(M)$ in $J(M)$ which joins v to the root r.



From any vertex in the planted tree we can easily find the path back to the root!

Augmenting Tree

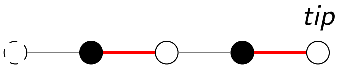
Augmenting tree is a planted tree $J(M)$ plus an edge e of G such that one end-point of e is an even vertex v_1 of J and the other end-point v_2 is exposed not in J.



Stem

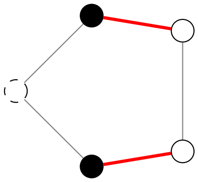
Stem is an alternating path with an exposed vertex at one end and a matching edge at the other end.

Tip is the non-exposed end vertex in the stem.



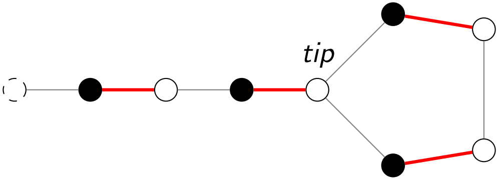
Blossom

Blossom is an odd cycle in G for which $M \cap B$ is a maximum matching in B with one vertex exposed for $M \cap B$.



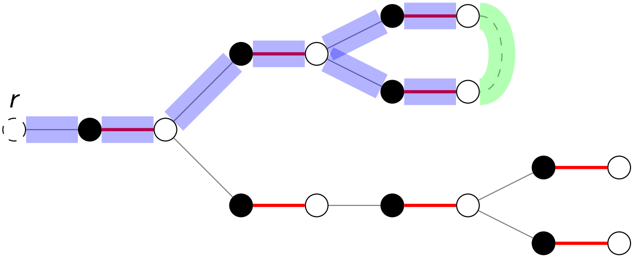
Flower

Flower consists of a blossom and a stem which intersect only at the tip of the stem.



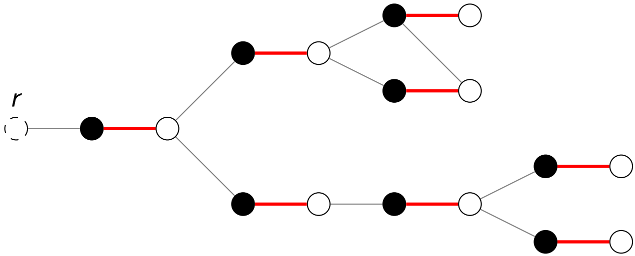
Flowered tree

Flowered tree is a planted tree J plus an edge e of G which joins a pair of even vertices of J. The union of e and the two paths which join its even-vertex end-points to the root of of J is a flower, F.

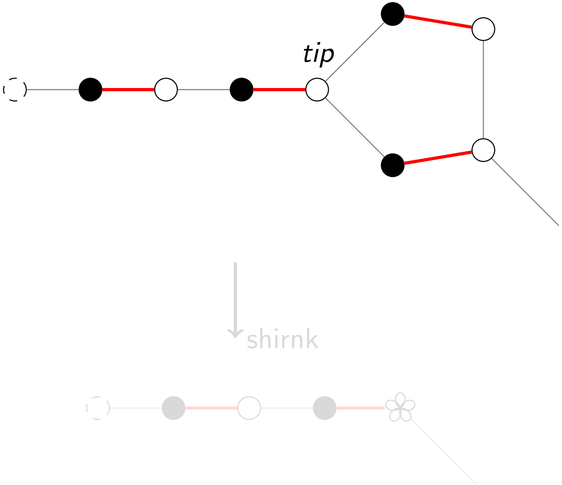


Hungarian tree

Hungarian tree is an alternating tree whose even vertices are joined by edges of G only to its inner vertices.



Shrinking the blossom



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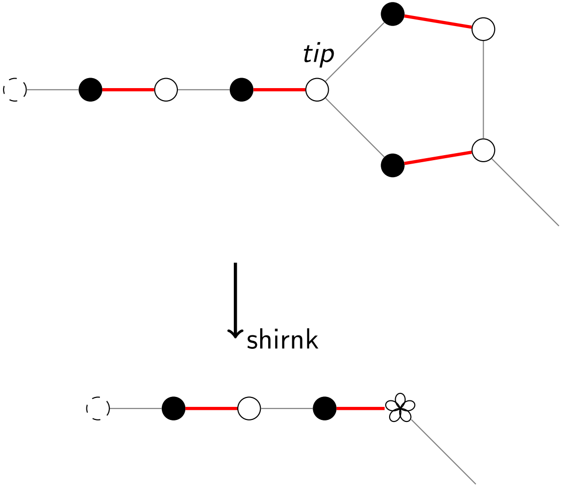
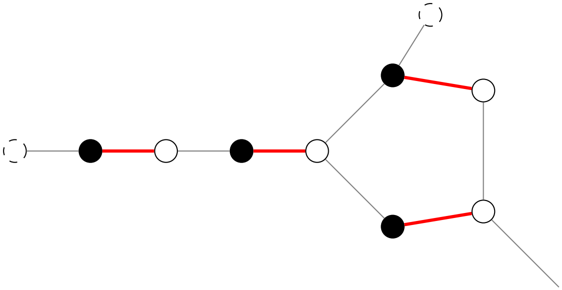


Correctness of the algorithm

Theorem

M is a maximum size matching in G if and only if M/B is maximum size matching in G/B.



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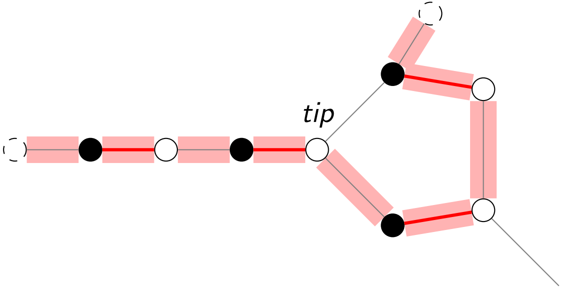
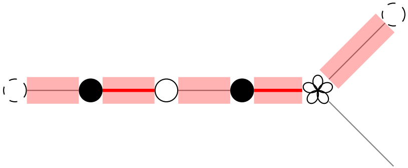
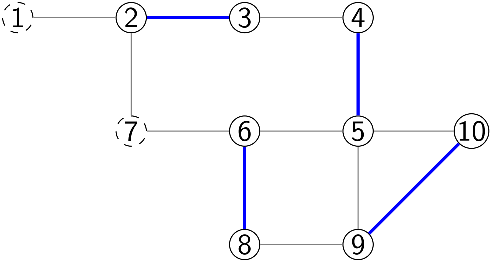




Examples

A graph $G = (V, E)$ and a matching M.



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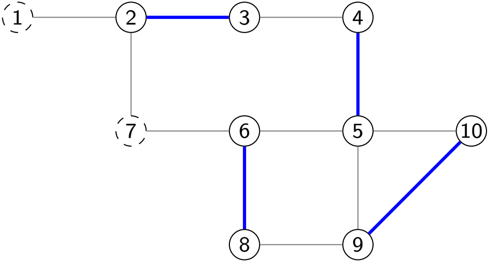

Start a BFS from 1. An exposed vertex is a planted tree.

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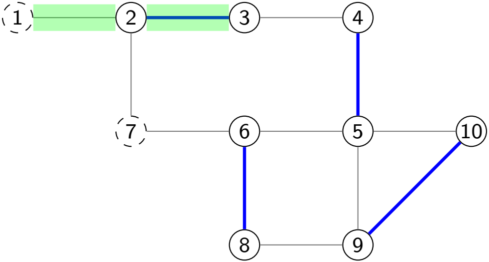
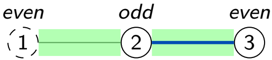


If one vertex connected to the even vertex of the planted tree is in the matching, we can extend to a larger planted tree.



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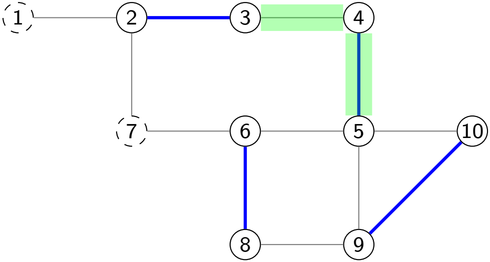
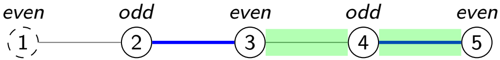


Extend to a larger planted tree..



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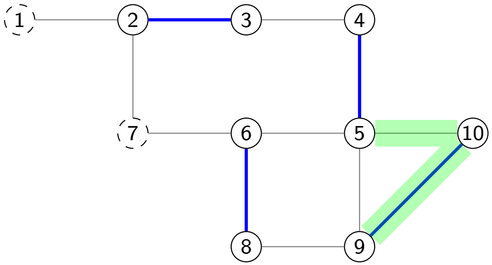

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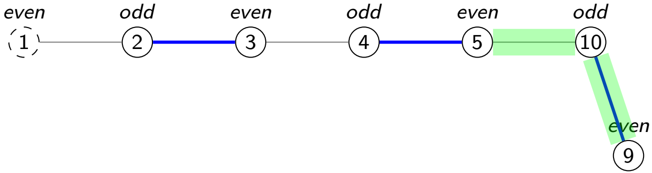
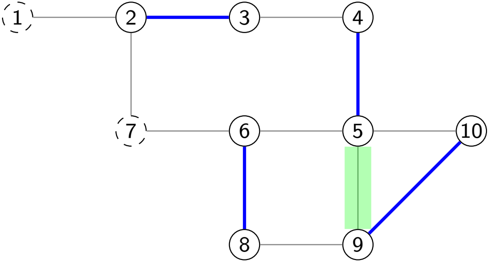
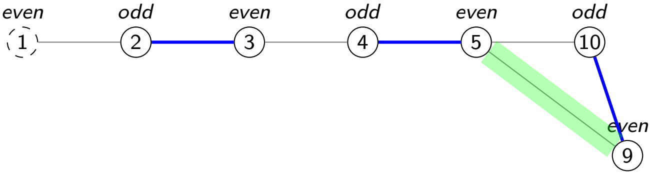


Examples

A graph $G = (V, E)$ and a matching M.



Encounter an edge connecting two even vertices. Find a blossom!



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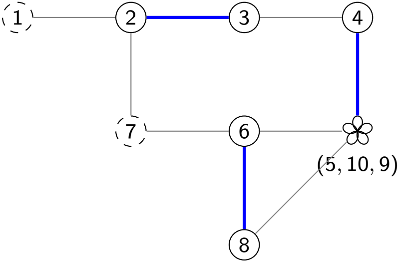
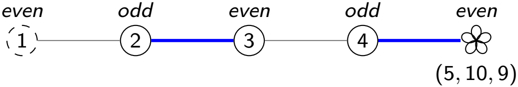


Shrink (5,10,9) into a single macro vertex.



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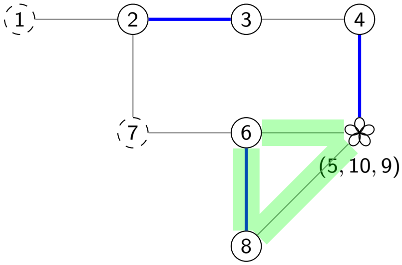
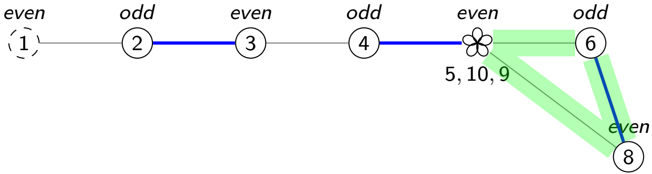


Continue the search, and encounter the cycle (5,10,9)-6-8.



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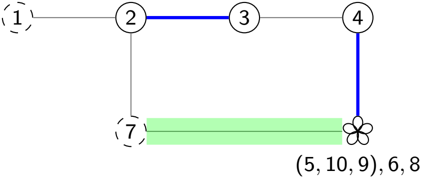
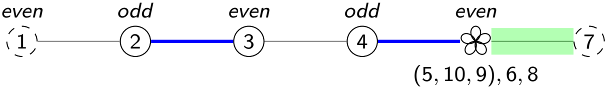


Shrinke the blossom and encounter an exposed vertex 7.



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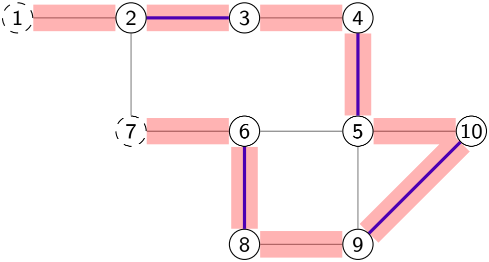
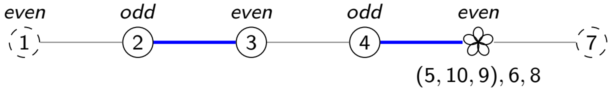
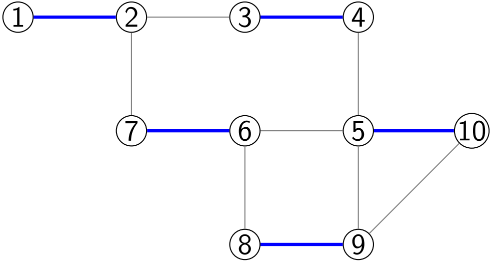


This is an augmenting tree and contains an augmenting path.



Examples

Applying $M = M \oplus P$ yeilds following enlarged matching in the original graph.



Running time

- At most $O(n)$ augmentations.
- Each augmentation will shrink at most $O(n)$ blossoms.
- Constructing the alternating tree takes at most $O(m)$.
- $\therefore O(n^2 m)$

Matching-duality theorem

Linear programming duality theorem

If

$$x \geq 0, Ax \leq c$$
$$y \geq 0, A^T y \geq b$$

for given real vectors b and c and real matrix A, then for real vectors x and y,

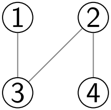
$$max_x(b, x) = min_y(c, y)$$

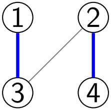
if such extrema exist.

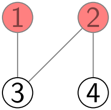
Matching-duality theorem

König theorem

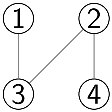
In a bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.







Matching-duality theorem



Maximum matching

maximize

$$x_{13} + x_{23} + x_{24}$$

subject to

$$\begin{array}{ccccccc}
x_{13} & & & & & \leq & 1 \\
 & & x_{23} & + & x_{24} & \leq & 1 \\
x_{13} & + & x_{23} & & & \leq & 1 \\
 & & & & x_{24} & \leq & 1
\end{array}$$

Minimum cover

minimize

$$x_1 + x_2 + x_3 + x_4$$

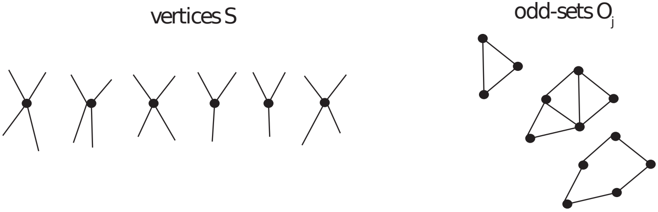
subject to

$$\begin{array}{ccccccc}
x_1 + & & x_3 & & & \geq & 1 \\
 & x_2 + & x_3 & & & \geq & 1 \\
 & x_2 + & & & x_4 & \geq & 1
\end{array}$$

Matching-duality theorem

Odd set cover in a graph consists of a collection of vertices S and odd-cardinality sets of vertices $O_1, O_2, ..., O_t$, disjoint from one another and from S, such that every edge of G is either incident with a vertex in S or lies within an odd set O_j.

The capacity of the odd-set cover is defined as: $|S| + \sum_{j=1}^{t} \frac{|O_j|-1}{2}$



Matching-duality theorem

General König theorem

If M is a maximum matching and C is a minimum odd-set cover then

$$|M| = capacity(C)$$

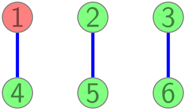
- Proof:

  It is obvious that the capacity-sum of any odd-set cover in G is at least as large as the cardinality of any matching in G, so we have only to prove the existence in G of an odd-set cover and a matching for which the numbers are equal.

Matching-duality theorem

- Proof:

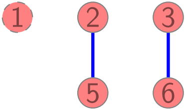
  For a pertect matching M with no exposed vertices



Matching-duality theorem

- Proof:

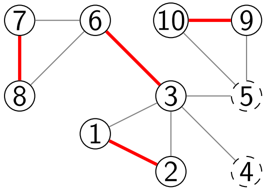
  For a graph which has a matching with one exposed vertices



Matching-duality theorem

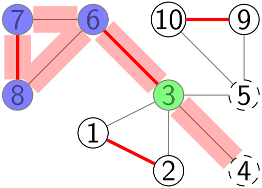
- Proof:

  For a graph with more than one exposed vertex



Matching-duality theorem

- Proof:



Matching-duality theorem

- Proof: