

# Paths, Trees, and Flowers



*Jack Edmonds*

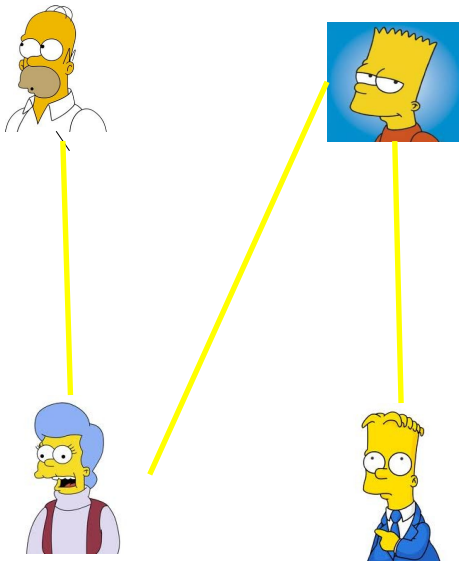
Introduction and background

Edmonds maximum matching algorithm

Matching-duality theorem

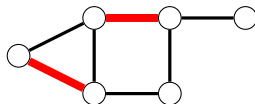
# Work scheduling

Schedule workers in pairs. What is the optimal assignment?

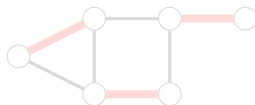


# Matching

Matching in a graph is a set of edges, no two of which meet at a common vertex.

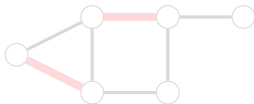


Maximum matching is a matching of maximum cardinality.

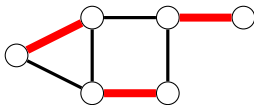


# Matching

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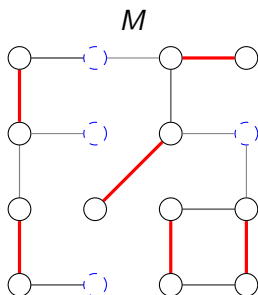


Maximum matching is a matching of maximum cardinality.



# Exposed vertex

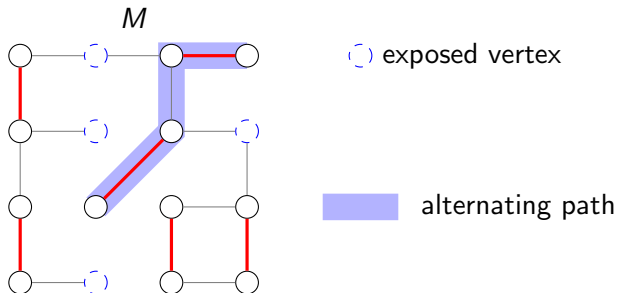
Exposed(free) vertex is a vertex that is not incident with any edge in  $M$



 exposed vertex

## Alternating path

**Alternating path** is a path whose edges are alternately in  $M$  and  $\overline{M}$



# Different kinds of alternating path

An alternating path between two matched vertices



An alternating path between two exposed vertices



An alternating path between a matched vertex and an exposed vertex



An alternating cycle





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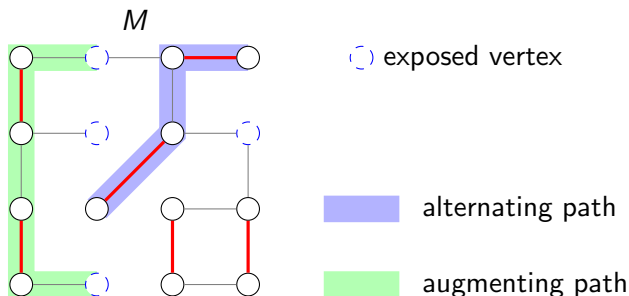


An alternating cycle



# Augmenting path

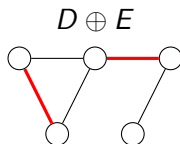
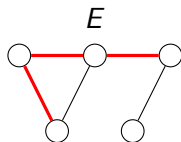
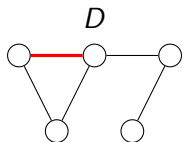
Augmenting path is a simple alternating path between exposed vertices



# Symmetric difference

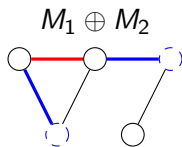
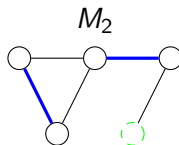
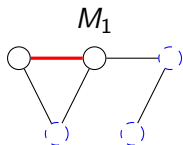
Symmetric difference of two sets  $D$  and  $E$  is defined as

$$D \oplus E = (D - E) \cup (E - D)$$



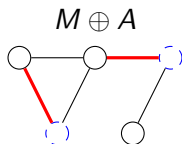
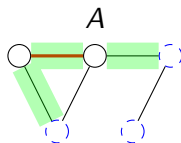
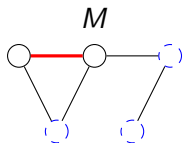
## Alternating path

For any two matchings  $M_1$  and  $M_2$  in graph  $G$ , the components of the graph formed by  $M_1 \oplus M_2$  are paths and circuits which are alternating for  $M_1$  and  $M_2$ . Each path end-point is exposed for either  $M_1$  or  $M_2$ .



# Augmenting path

$A$  is an augmenting path in  $(G, M)$ .  $M \oplus A$  is a matching of  $G$  larger than  $M$  by one.



# Berge's theorem

## Theorem - Berge(1957)

$M$  is not a maximum matching if and only if there exists an augmenting path with respect to  $M$

- Proof: if  $M_2$  is a larger matching than  $M$ , some component of graph  $M \oplus M_2$  must contain more  $M_2$  edges than  $M$ , such a component is an augmenting path for  $(G, M)$ .





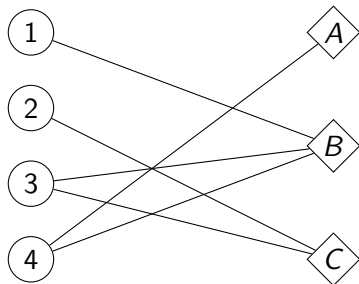
# Algorithm

MAXIMUM-MATCHING( $G$ )

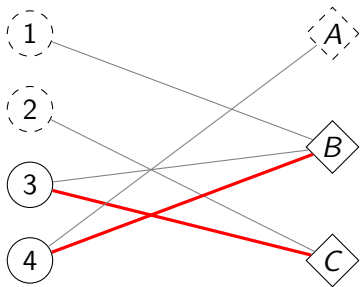
```
1   $M = \emptyset$ 
2  repeat
3      if there is an augmenting path  $P$  with respect to  $M$ 
4           $M = M \oplus P$ 
5  until there isn't an augmenting path with respect to  $M$ 
6  return  $M$ 
```

# Bipartite Matching

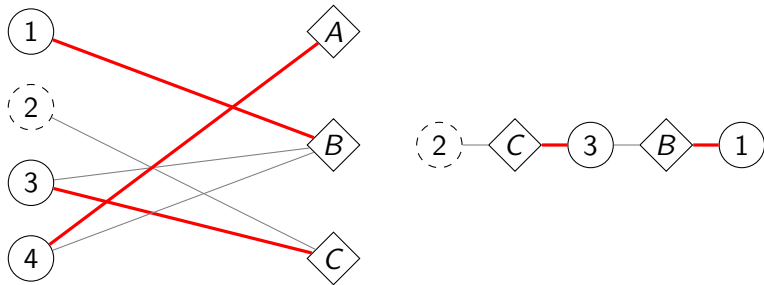
Left circles represent boys, right diamonds represent girls, and edges are their preferences. We want to make maximum number of couples.



## Find An Augmenting Path Using BFS

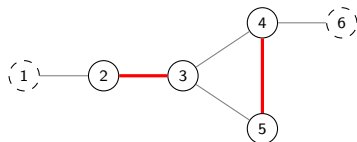


## Find An Augmenting Path Using BFS

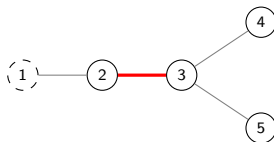


# Examples of BFS failure on non-bipartite graphs

- Initial graph

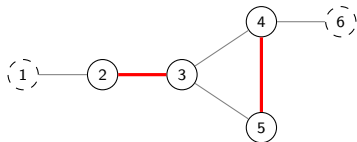


- Using BFS



# Examples of BFS failure on non-bipartite graphs

- Initial graph

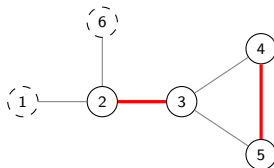


- Allow a vertex to be visited at both odd and even levels, we find an augmenting path

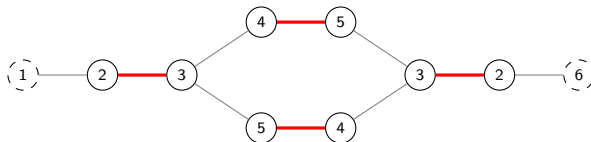


# Examples of BFS failure on non-bipartite graphs

- Initial graph



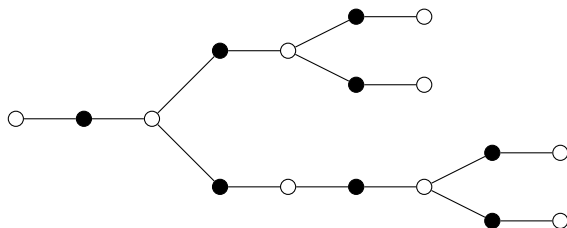
- But the augmenting path we find may not be a simple path



# Alternating Tree

**Alternating Tree** is a tree  $J$ , each of whose edges joins an odd vertex to an even vertex so that each odd vertex of  $J$  meets exactly two edges of  $J$ .

- An alternating tree contains one more even vertex than odd vertices.



● odd vertex

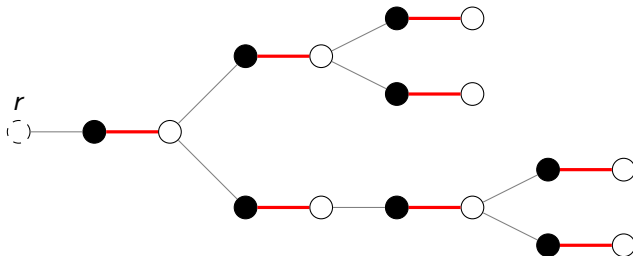
○ even vertex



# Planted Tree

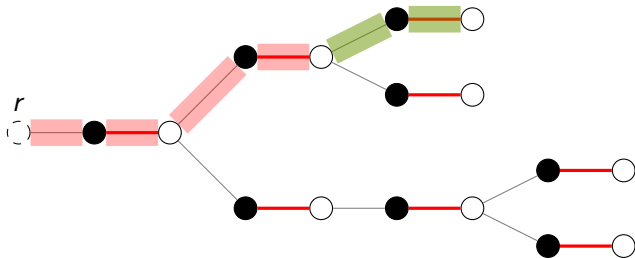
**Planted tree** is an alternating tree  $J(M)$  of  $G$  for matching  $M$  s.t.  
 $M \cap J$  is a maximum matching of  $J$ .

**Root** is a vertex  $r$  in  $J$  which is exposed.



# Planted Tree

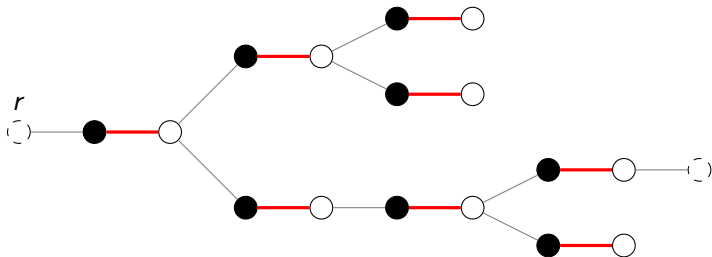
In planted tree  $J(M)$  every alternating path  $P(M)$ , which has even vertex  $v$  and the matching edge to  $v$  at one of its ends, is a subpath of the alternating path  $P_v(M)$  in  $J(M)$  which joins  $v$  to the root  $r$ .



From any vertex in the planted tree we can easily find the path back to the root!

## Augmenting Tree

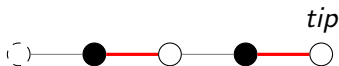
**Augmenting tree** is a planted tree  $J(M)$  plus an edge  $e$  of  $G$  such that one end-point of  $e$  is an even vertex  $v_1$  of  $J$  and the other end-point  $v_2$  is exposed not in  $J$ .



# Stem

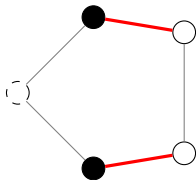
**Stem** is an alternating path with an exposed vertex at one end and a matching edge at the other end.

**Tip** is the non-exposed end vertex in the stem.



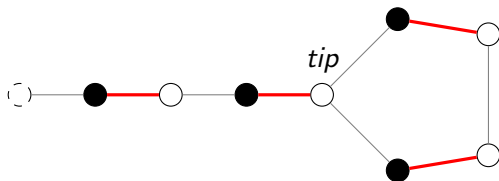
# Blossom

**Blossom** is an odd cycle in  $G$  for which  $M \cap B$  is a maximum matching in  $B$  with one vertex exposed for  $M \cap B$ .



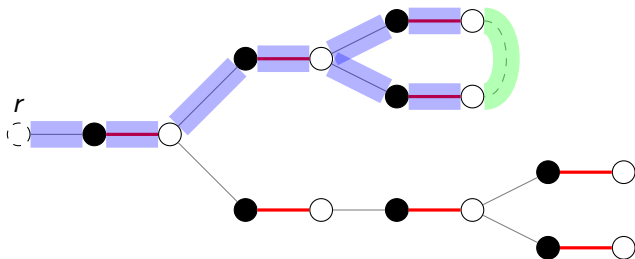
# Flower

Flower consists of a blossom and a stem which intersect only at the tip of the stem.



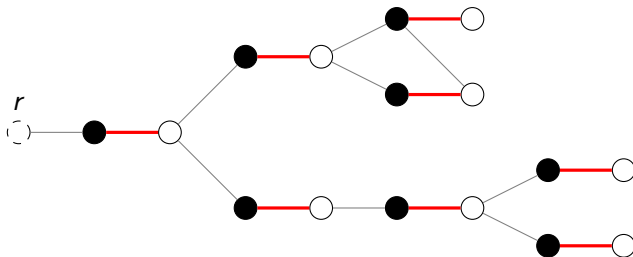
# Flowered tree

**Flowered tree** is a planted tree  $J$  plus an edge  $e$  of  $G$  which joins a pair of even vertices of  $J$ . The union of  $e$  and the two paths which join its even-vertex end-points to the root of  $J$  is a flower,  $F$ .



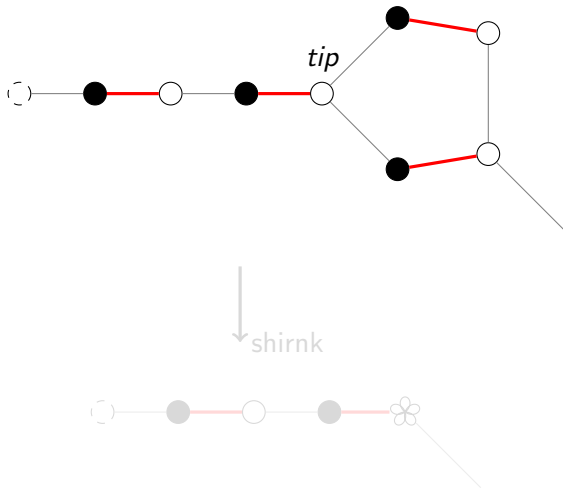
# Hungarian tree

**Hungarian tree** is an alternating tree whose even vertices are joined by edges of  $G$  only to its inner vertices.

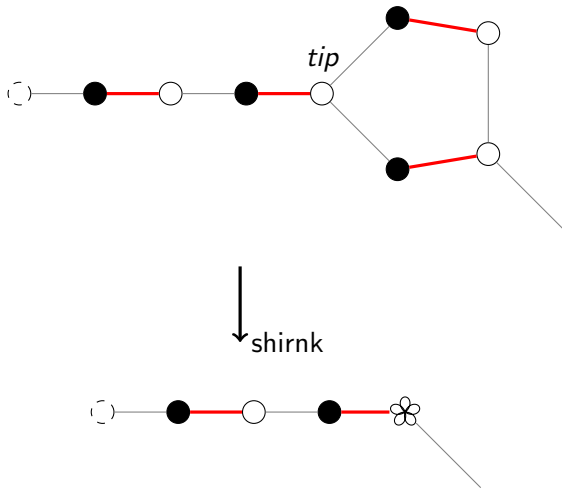




## Shrinking the blossom



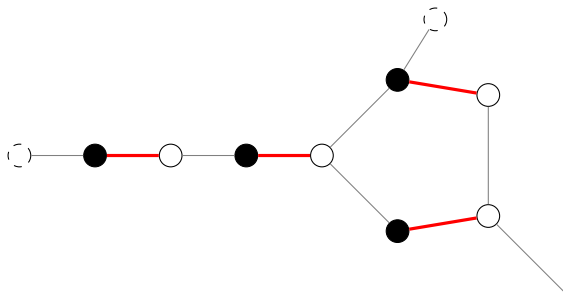
## Shrinking the blossom



# Correctness of the algorithm

## Theorem

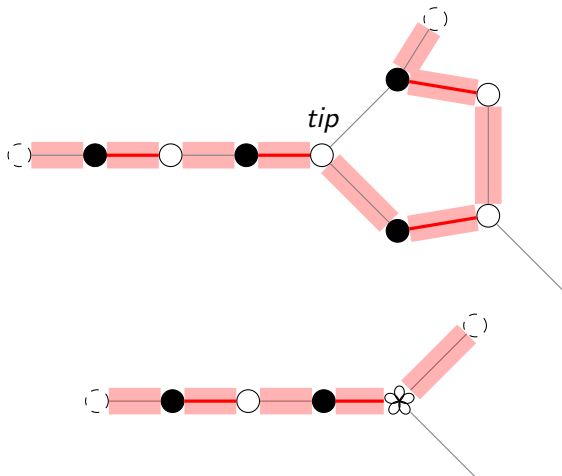
$M$  is a maximum size matching in  $G$  if and only if  $M/B$  is maximum size matching in  $G/B$ .



# Correctness of the algorithm

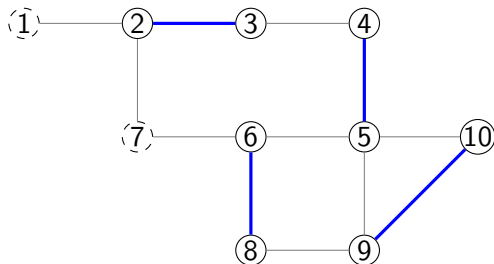
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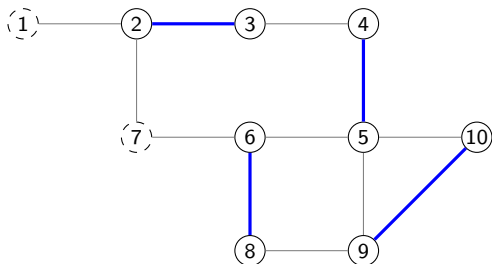
# Examples

A graph  $G = (V, E)$  and a matching  $M$ .



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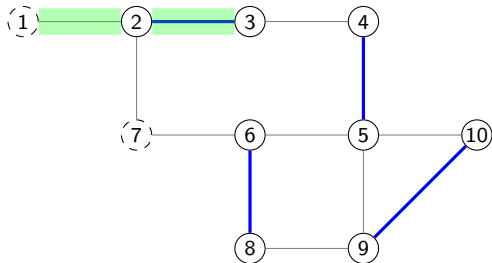
Start a BFS from 1. An exposed vertex is a planted tree.

*even*

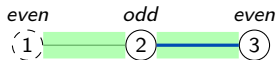


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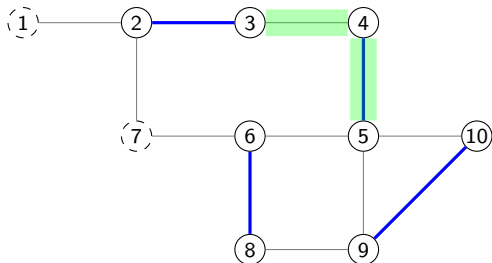


If one vertex connected to the even vertex of the planted tree is in the matching, we can extend to a larger planted tree.

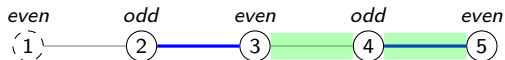


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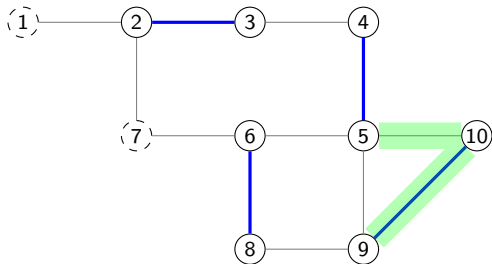
Extend to a larger planted tree..



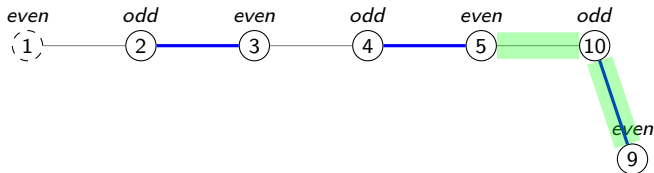


# Examples

A graph  $G = (V, E)$  and a matching  $M$ .

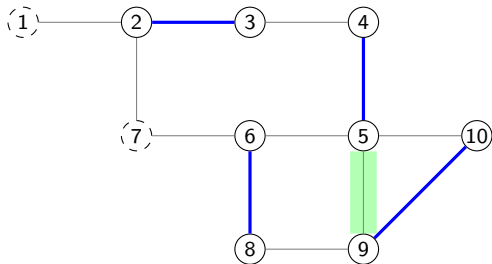


Extend to a larger planted tree....

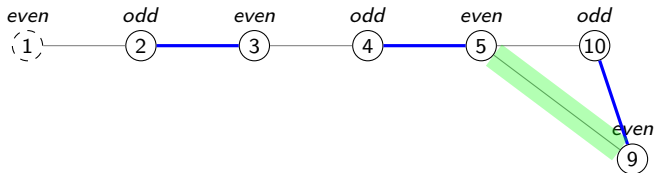


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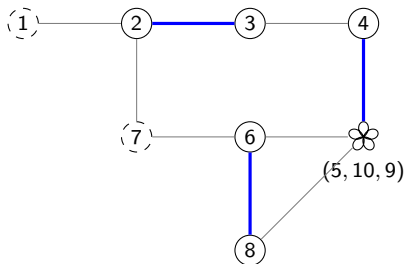


Encounter an edge connecting two even vertices. Find a blossom!

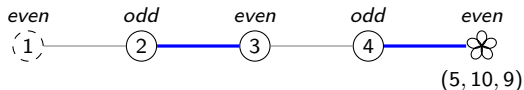


# Examples

A graph  $G = (V, E)$  and a matching  $M$ .

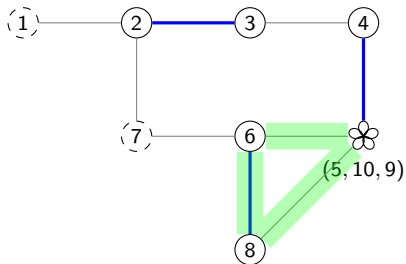


Shrink (5,10,9) into a single macro vertex.

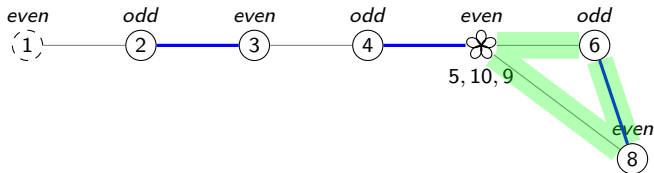


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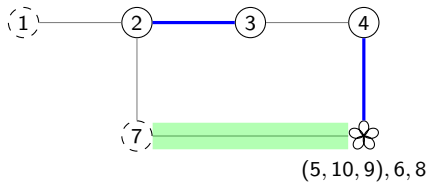


Continue the search, and encounter the cycle  $(5,10,9)-6-8$ .

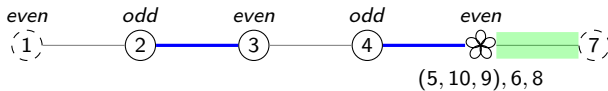


# Examples

A graph  $G = (V, E)$  and a matching  $M$ .

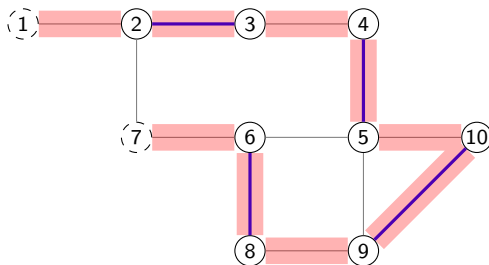


Shrink the blossom and encounter an exposed vertex 7.

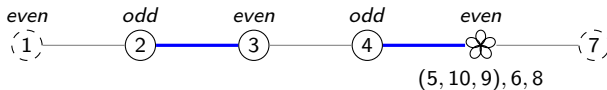


# Examples

A graph  $G = (V, E)$  and a matching  $M$ .

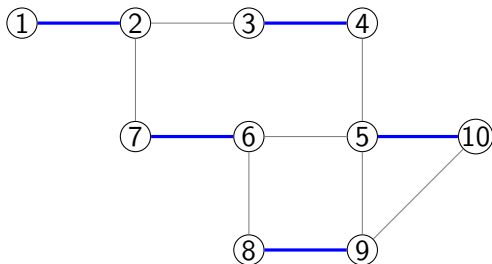


This is an augmenting tree and contains an augmenting path.



## Examples

Applying  $M = M \oplus P$  yeilds following enlarged matching in the original graph.



# Matching-duality theorem

## Linear programming duality theorem

If

$$\begin{aligned}x &\geq 0, Ax \leq c \\ y &\geq 0, A^T y \geq b\end{aligned}$$

for given real vectors  $b$  and  $c$  and real matrix  $A$ , then for real vectors  $x$  and  $y$ ,

$$\max_x (b, x) = \min_y (c, y)$$

if such extrema exist.

## König theorem

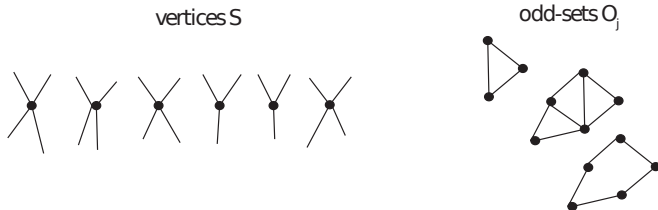
In a bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.



# Matching-duality theorem

**Odd set cover** in a graph consists of a collection of vertices  $S$  and odd-cardinality sets of vertices  $O_1, O_2, \dots, O_t$ , disjoint from one another and from  $S$ , such that every edge of  $G$  is either incident with a vertex in  $S$  or lies within an odd set  $O_j$ .

**The capacity** of the odd-set cover is defined as:  $|S| + \sum_{j=1}^t \frac{|O_j|-1}{2}$



# Matching-duality theorem

## General König theorem

If  $M$  is a maximum matching and  $C$  is a minimum odd-set cover then

$$|M| = \text{capacity}(C)$$

► Proof:

It is obvious that the capacity-sum of any odd-set cover in  $G$  is at least as large as the cardinality of any matching in  $G$ , so we have only to prove the existence in  $G$  of an odd-set cover and a matching for which the numbers are equal.

# Matching-duality theorem

► Proof:

For a perfect matching  $M$  with no exposed vertices, the odd-set cover consists of two sets, one set containing one of the vertices, the other containing all the other vertices.

For a graph which has a matching with one exposed vertex, the odd-set cover may be taken as one set containing all the vertices of the graph.

The general case can be proven by induction.