Paths, Trees, and Flowers

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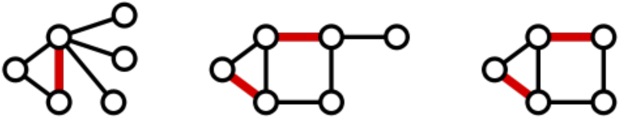
Matching

Matching in a graph is a set of edges, no two of which meet at a common vertex.

Maximum matching is a matching of maximum cardinality.



Alternating Path

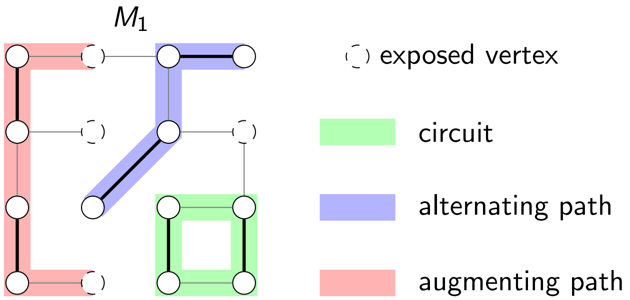
Exposed(free) vertex is a vertex that is not incident with any edge in M

Alternating path is a path whose edges are alternately in M and $\overline{M}$

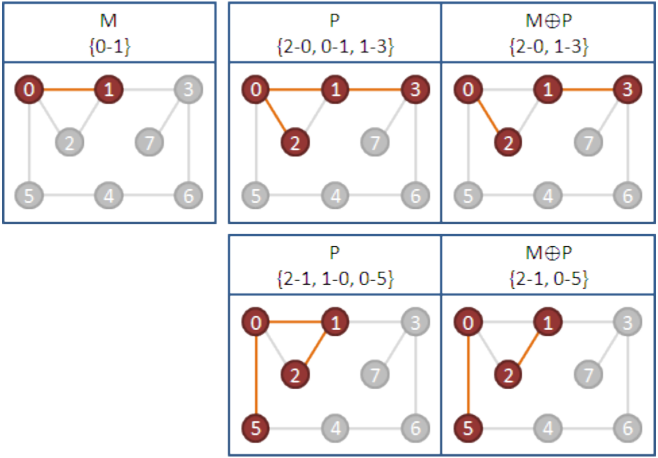
Augmenting path is a simple alternating path between exposed vertices

Symmetric difference of two sets D and E is defined as $D \oplus E = (D - E) \cup (E - D)$

Alternating Path



Alternating Path



Alternating Path

For any two matchings M_1 and M_2 in graph G, the components of the graph formed by $M_1 \oplus M_2$ are paths and circuits which are alternating for M_1 and M_2. Each path end-point is exposed for either M_1 or M_2.

A is an augmenting path in (G, M) . $M \oplus A$ is a matching of G larger than M by one.

Alternating Path

Theorem - Berge(1957)

M is not a maximum matching if and only if there exists an augmenting path with respect to M

- Proof: if M_2 is a larger matching than M, some component of graph $M \oplus M_2$ must contain more M_2 edges than M, such a component is an augmenting path for (G, M).

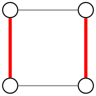
Alternating Path

An alternating path between two matched vertices

An alternating path between two exposed vertices

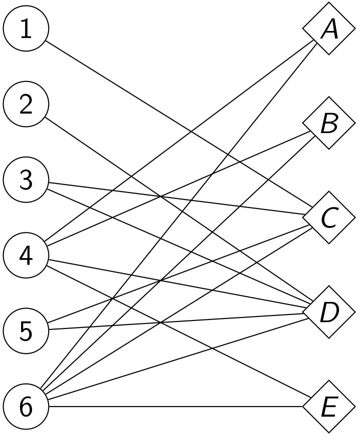
An alternating path between a matched vertex and an exposed vertex

An alternating cycle

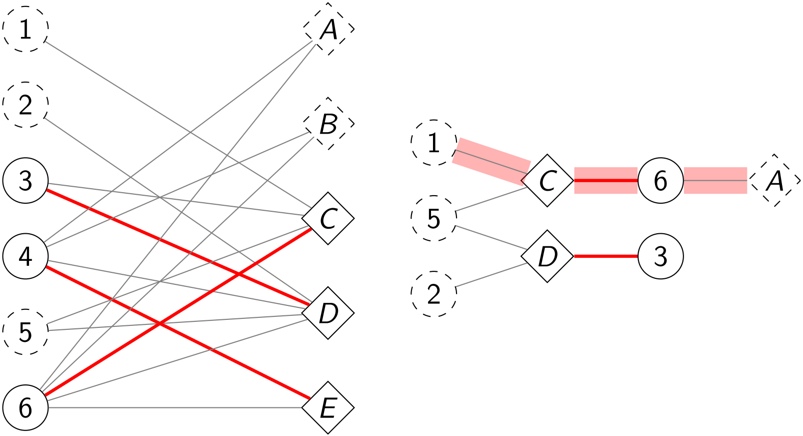


Bipartite Matching

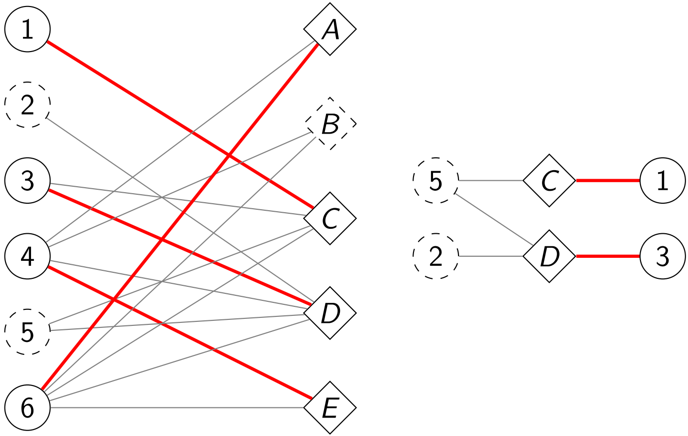
Left circles represent boys, right diamonds represent girls, and edges are their preferences. We want to make maximum number of couples.



Find An Augmenting Path Using BFS

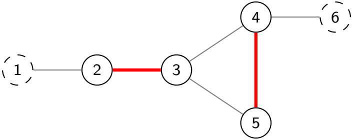


Find An Augmenting Path Using BFS

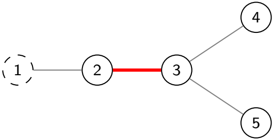


Examples of BFS Failure on non-bipartite graphs

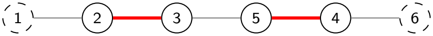
- Initial graph



- Using BFS



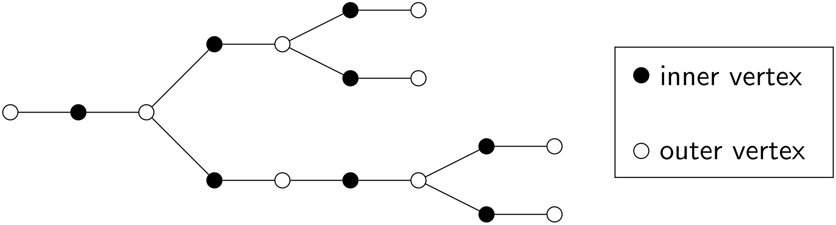
- Actual augmenting path



Alternating Tree

Alternating Tree is a tree J, each of whose edges joins an inner vertex to an outer vertex so that each inner vertex of J meets exactly two edges of J.

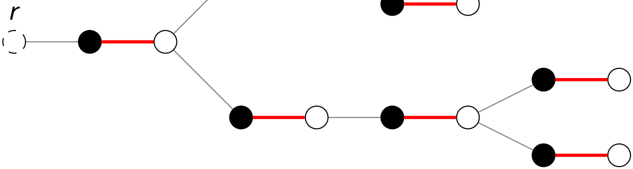
- An alternating tree contains one more outer vertex than inner vertices.



Planted Tree

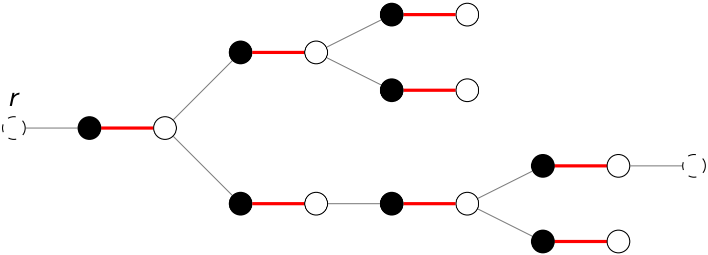
Planted tree is an alternating tree $J(M)$ of G for matching M s.t. $M \cap J$ is a maximum matching of J.

Root is a vertex r in J which is exposed.



Augmenting Tree

Augmenting tree is a planted tree $J(M)$ plus an edge e of G such that one end-point of e is an outer vertex v_1 of J and the other end-point v_2 is exposed not in J.

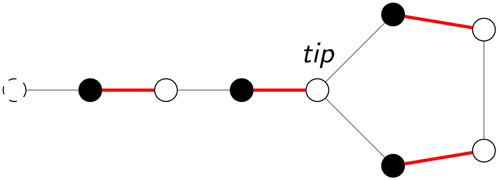


Flower

Stem is an alternating path with an exposed vertex at one end and a matching edge at the other end.

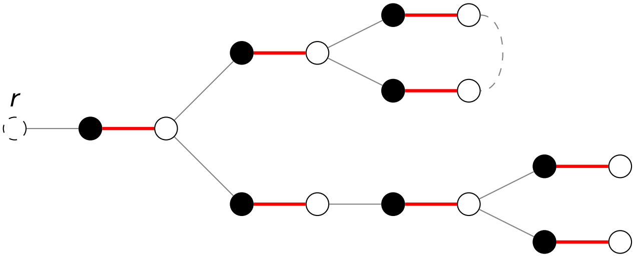
Blossom is an odd cycle in G for which $M \cap B$ is a maximum matching in B with *tip* exposed for $M \cap B$.

Flower consists of a blossom and a stem which intersect only at the tip of the stem.

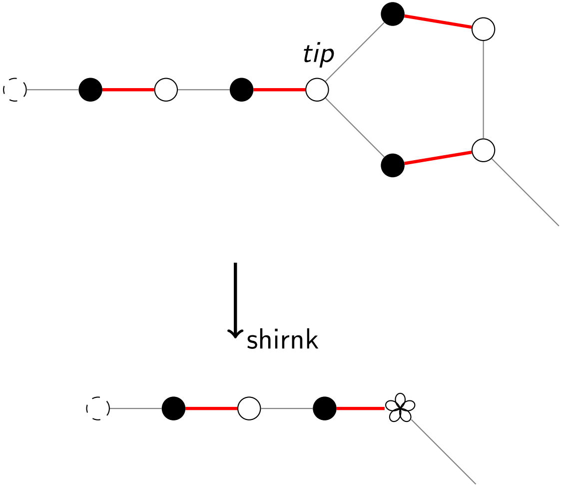


Flowered Tree

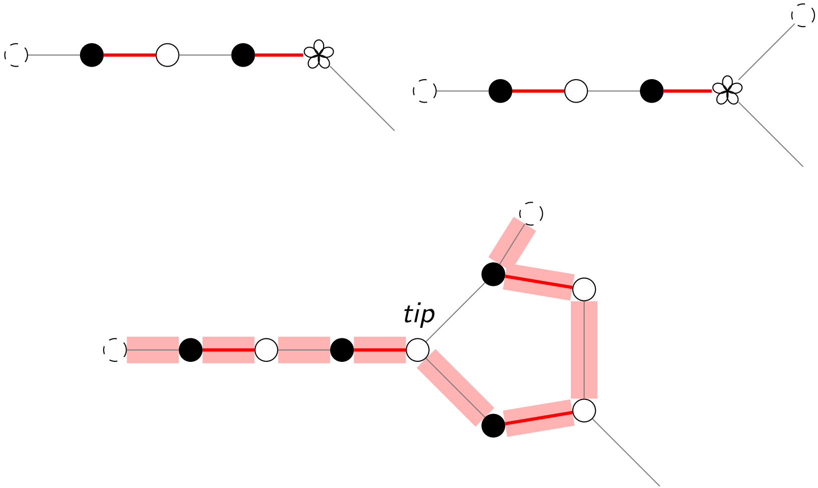
Flowered tree is a planted tree J plus an edge e of G which joins a pair of outer vertices of J. The union of e and the two paths which join its outer-vertex end-points to the root of of J is a flower, F.



Shrinking the Blossom



Finding Augmenting Path



Matching

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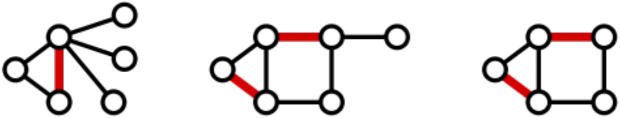
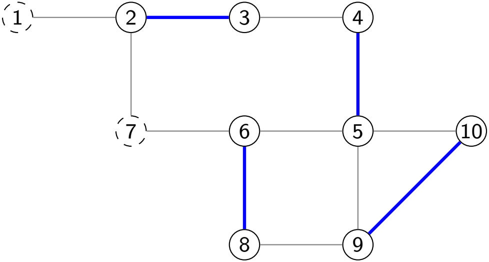
Maximum matching is a matching of maximum cardinality.

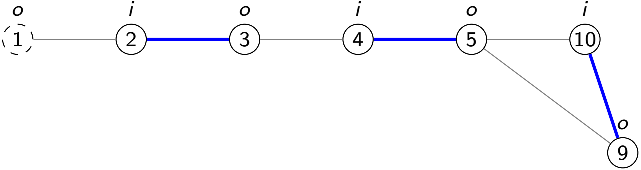


Examples

A graph $G = (V, E)$ and a matching M.

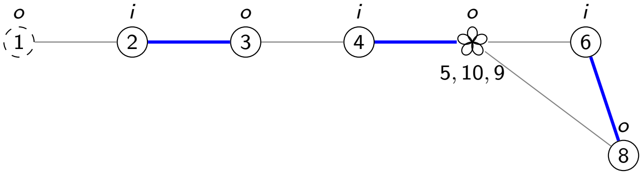


Start a BFS from 1, encounter the odd cycle 5-10-9.

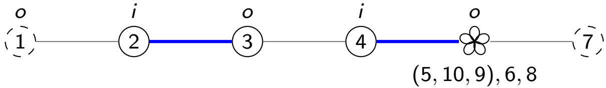


Examples

Shrink 5,10,9 into a single macrovertex. Continue the search, and encounter the cycle (5,10,9)-6-8.

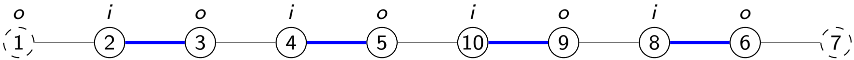


Shrink (5,10,9),6,8 into a single macro vertex. Continue the serch, and encounter a exposed vertex 7.

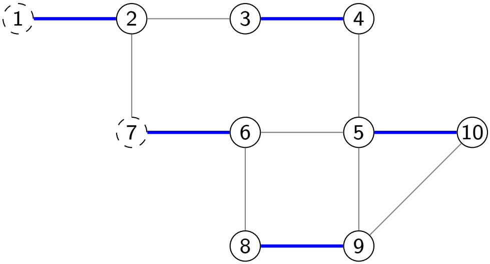


Examples

An augmenting path in the shrunken graph is found. By unshrinking, we get the following augmenting path P in the original graph.



Applying $M = M \oplus P$ yeilds following enlarged matching in the original graph.



Matching-duality theorem

Linear programming duality theorem

If

$$x \geq 0, Ax \leq c$$
$$y \geq 0, A^T y \geq b$$

for given real vectors b and c and real matrix A, then for real vectors x and y,

$$max_x(b, x) = min_y(c, y)$$

when such extrema exist.

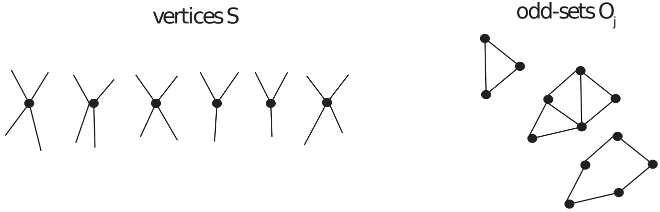
König theorem

In a bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.

Matching-duality theorem

Odd set cover in a graph consists of a collection of vertices S and odd-cardinality sets of vertices $O_1, O_2, ..., O_t$, disjoint from one another and from S, such that every edge of G is either incident with a vertex in S or lies within an odd set O_j.

The capacity of the odd-set cover is defined as: $|S| + \sum_{j=1}^{t} \frac{|O_j|-1}{2}$



Matching-duality theorem

General König theorem

If M is a maximum matching and C is a minimum odd-set cover then

$$|M| = capacity(C)$$

- Proof:

  It is obvious that the capacity-sum of any odd-set cover in G is at least as large as the cardinality of any matching in G, so we have only to prove the existence in G of an odd-set cover and a matching for which the numbers are equal.

Matching-duality theorem

- Proof:

  For a pertect matching M with no exposed vertices, the odd-set cover consists of two sets, one set containing one of the vertices, the other containing all the other vertices.

  For a graph which has a matching with one exposed vertices, the odd-set cover may be taken as one set containing all the vertices of the graph.

  The general case can be proven by induction.