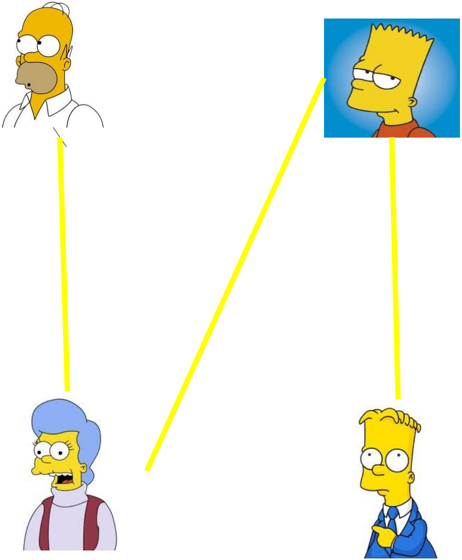
Paths, Trees, and Flowers

Jack Edmonds

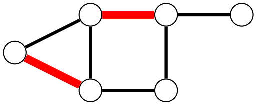
Work scheduling

Schedule workers in pairs. What is the optimal assignment?



Matching

Matching in a graph is a set of edges, no two of which meet at a common vertex.



Maximum matching is a matching of maximum cardinality.

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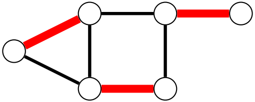
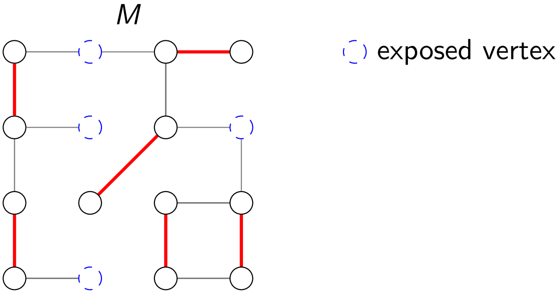


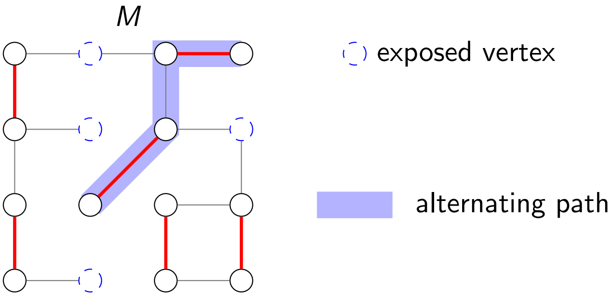
Exposed vertex

Exposed(free) vertex is a vertex that is not incident with any edge in M



Alternating path

Alternating path is a path whose edges are alternately in M and $\overline{M}$



Different kinds of alternating path

An alternating path between two matched vertices

An alternating path between two exposed vertices

An alternating path between a matched vertex and an exposed vertex

An alternating cycle

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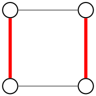
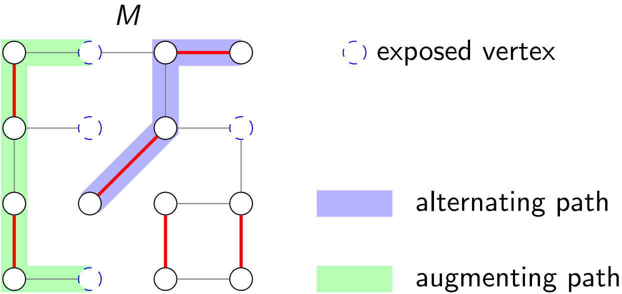


Augmenting path

Augmenting path is a simple alternating path between exposed vertices

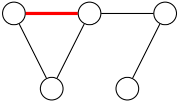


Symmetric difference

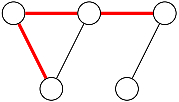
Symmetric difference of two sets D and E is defined as

$$D \oplus E = (D - E) \cup (E - D)$$

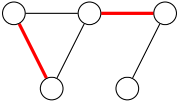
D



E

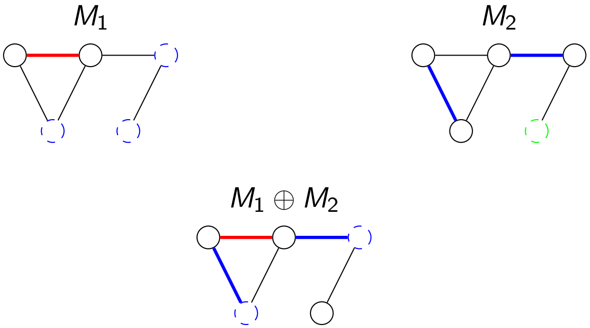


$D \oplus E$



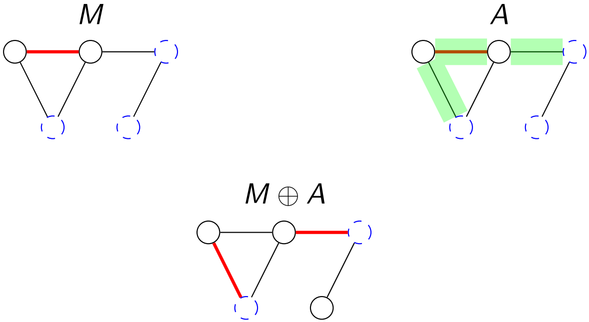
Alternating path

For any two matchings M_1 and M_2 in graph G, the components of the graph formed by $M_1 \oplus M_2$ are paths and circuits which are alternating for M_1 and M_2. Each path end-point is exposed for either M_1 or M_2.



Augmenting path

A is an augmenting path in (G, M) . $M \oplus A$ is a matching of G larger than M by one.



Berge's theorem

Theorem - Berge(1957)

M is not a maximum matching if and only if there exists an augmenting path with respect to M

- Proof: if M_2 is a larger matching than M, some component of graph $M \oplus M_2$ must contain more M_2 edges than M, such a component is an augmenting path for (G, M).

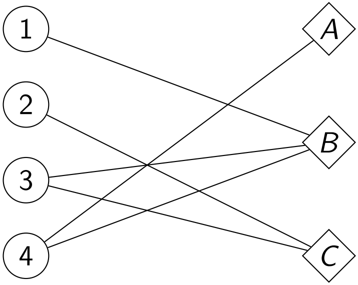
Algorithm

MAXIMUM-MATCHING(G)

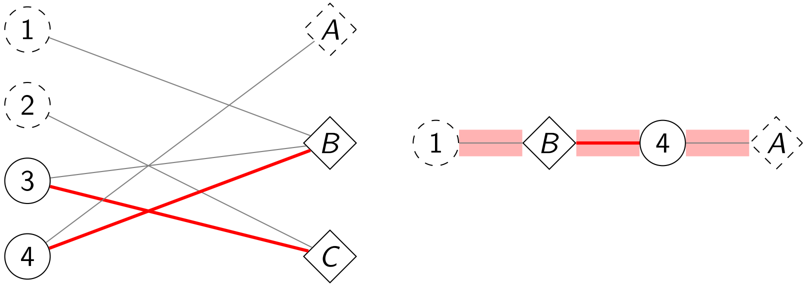
$M = \emptyset$
repeat
    if there is an augmenting path P with respect to M
        $M = M \oplus P$
until there isn't an augmenting path with respect to M
return M

Bipartite Matching

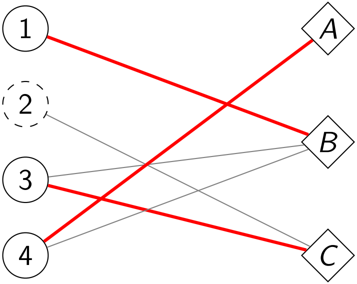
Left circles represent boys, right diamonds represent girls, and edges are their preferences. We want to make maximum number of couples.

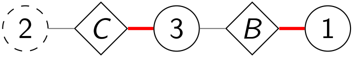


Find An Augmenting Path Using BFS



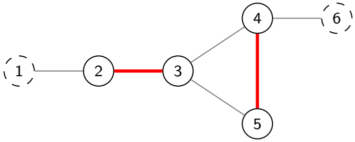
Find An Augmenting Path Using BFS



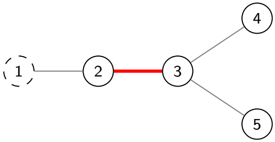


Examples of BFS failure on non-bipartite graphs

- Initial graph



- Using BFS



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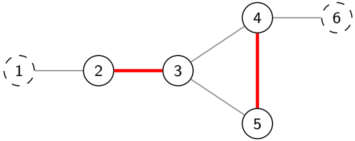
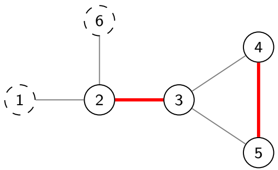
- Initial graph



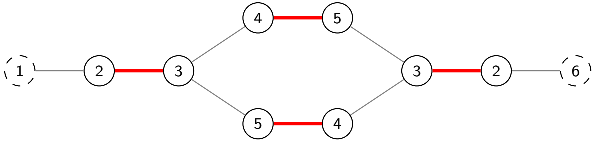
- Allow a vertex to be visited at both odd and even levels, we find an augmenting path

Examples of BFS failure on non-bipartite graphs

- Initial graph



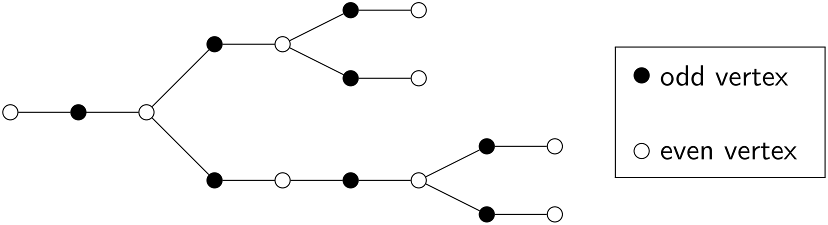
- But the augmenting path we find may not be a simple path



Alternating Tree

Alternating Tree is a tree J, each of whose edges joins an odd vertex to an even vertex so that each odd vertex of J meets exactly two edges of J.

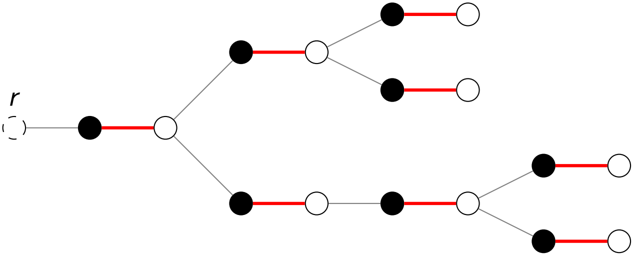
- An alternating tree contains one more even vertex than odd vertices.



Planted Tree

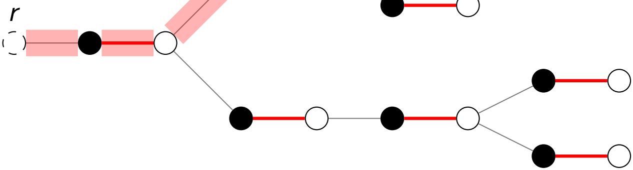
Planted tree is an alternating tree $J(M)$ of G for matching M s.t. $M \cap J$ is a maximum matching of J.

Root is a vertex r in J which is exposed.



Planted Tree

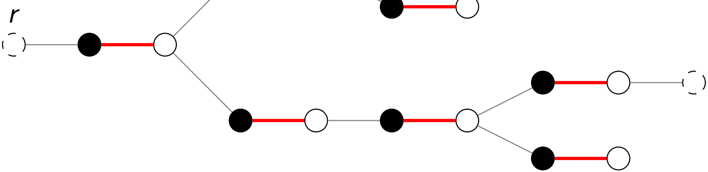
In planted tree $J(M)$ every alternating path $P(M)$, which has even vertex v and the matching edge to v at one of its ends, is a subpath of the alternating path $P_v(M)$ in $J(M)$ which joins v to the root r.



From any vertex in the planted tree we can easily find the path back to the root!

Augmenting Tree

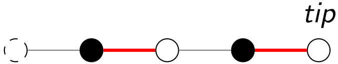
Augmenting tree is a planted tree $J(M)$ plus an edge e of G such that one end-point of e is an even vertex v_1 of J and the other end-point v_2 is exposed not in J.



Stem

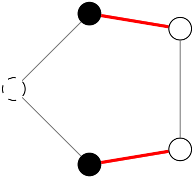
Stem is an alternating path with an exposed vertex at one end and a matching edge at the other end.

Tip is the non-exposed end vertex in the stem.



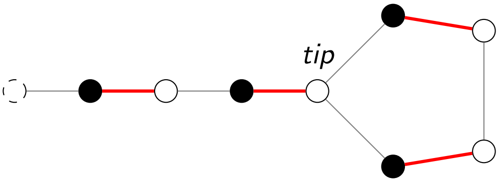
Blossom

Blossom is an odd cycle in G for which $M \cap B$ is a maximum matching in B with one vertex exposed for $M \cap B$.



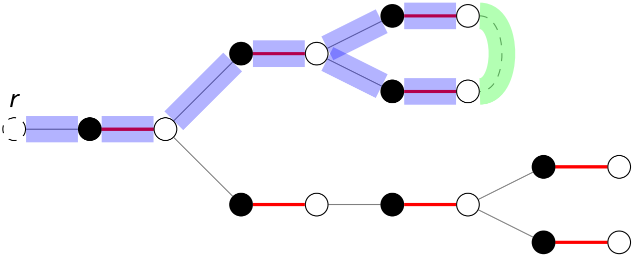
Flower

Flower consists of a blossom and a stem which intersect only at the tip of the stem.



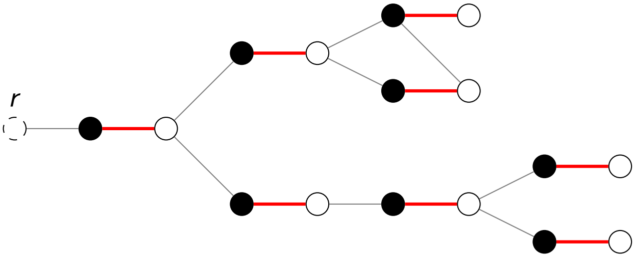
Flowered tree

Flowered tree is a planted tree J plus an edge e of G which joins a pair of even vertices of J. The union of e and the two paths which join its even-vertex end-points to the root of of J is a flower, F.

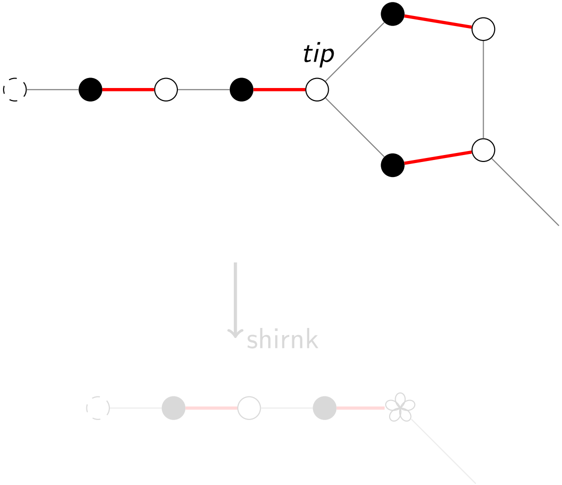


Hungarian tree

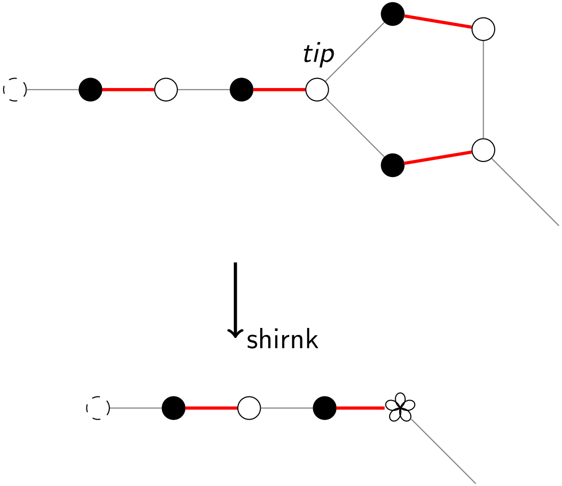
Hungarian tree is an alternating tree whose even vertices are joined by edges of G only to its inner vertices.



Shrinking the blossom



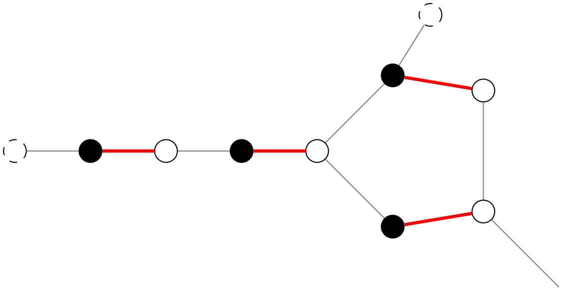
Shrinking the blossom



Correctness of the algorithm

Theorem

M is a maximum size matching in G if and only if M/B is maximum size matching in G/B.



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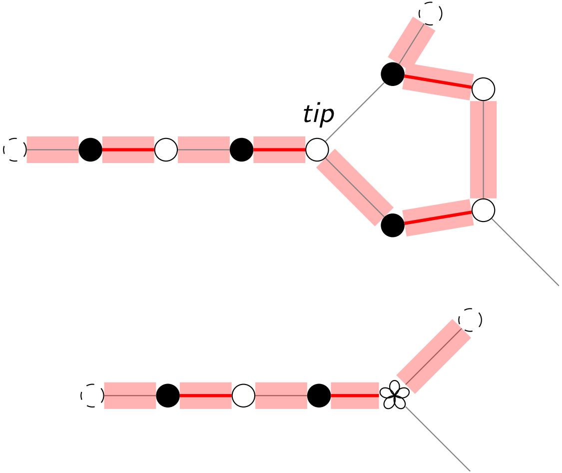
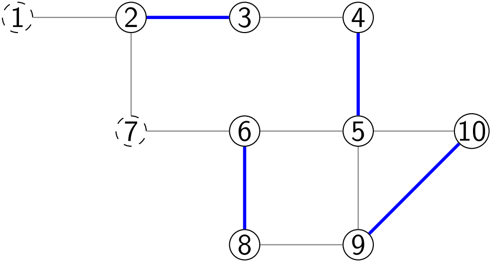


Examples

A graph $G = (V, E)$ and a matching M.



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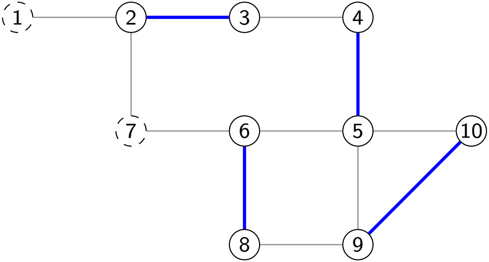

Start a BFS from 1. An exposed vertex is a planted tree.

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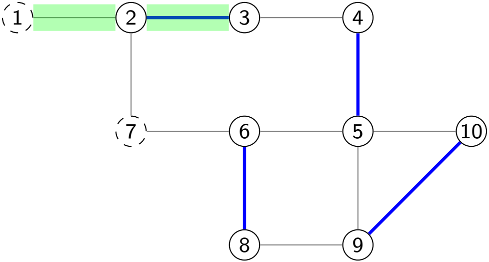
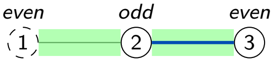
A graph $G = (V, E)$ and a matching M.



If one vertex connected to the even vertex of the planted tree is in the matching, we can extend to a larger planted tree.



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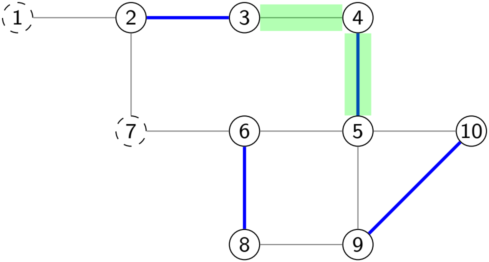
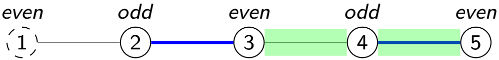


Extend to a larger planted tree..



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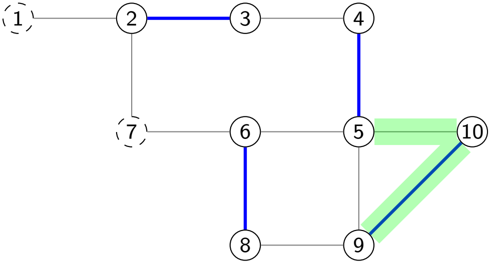

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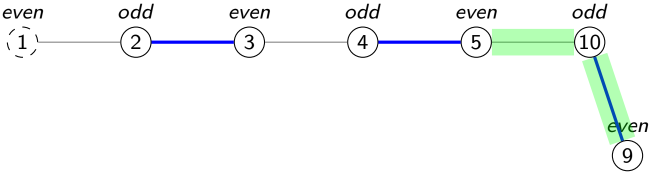

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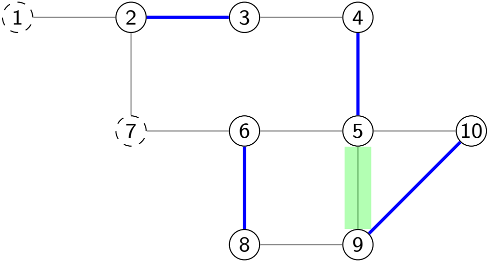
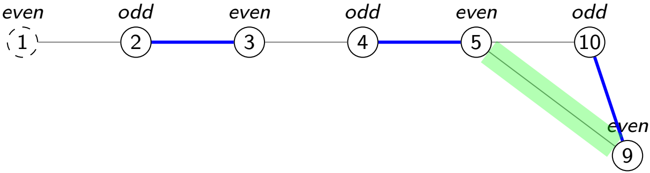
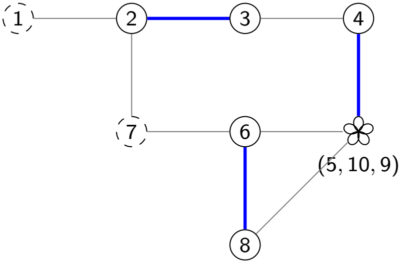


Encounter an edge connecting two even vertices. Find a blossom!

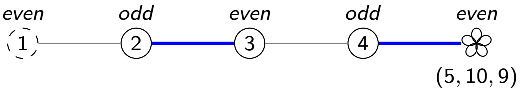


Examples

A graph $G = (V, E)$ and a matching M.

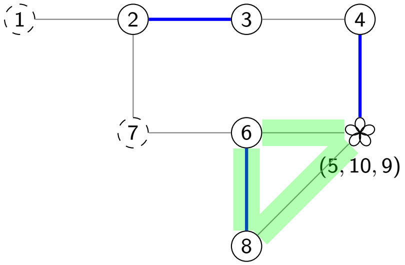


Shrink (5,10,9) into a single macro vertex.

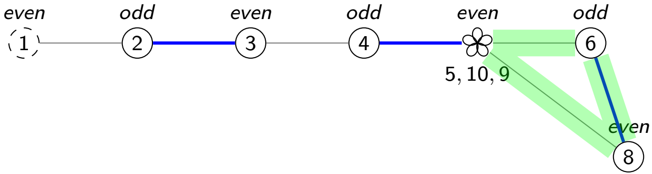


Examples

A graph $G = (V, E)$ and a matching M.



Continue the search, and encounter the cycle (5,10,9)-6-8.



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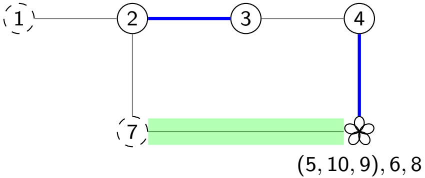
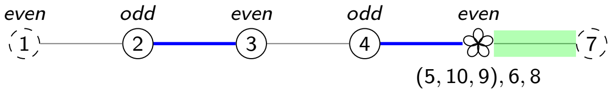
A graph $G = (V, E)$ and a matching M.



Shrinke the blossom and encounter an exposed vertex 7.



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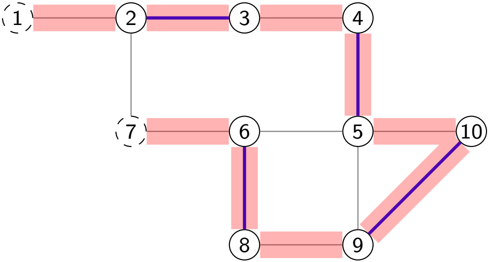
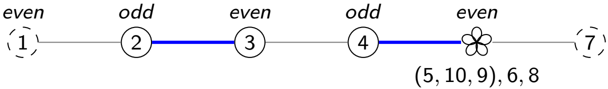
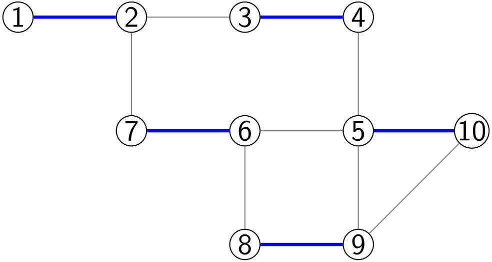


This is an augmenting tree and contains an augmenting path.



Examples

Applying $M = M \oplus P$ yeilds following enlarged matching in the original graph.



Matching-duality theorem

Linear programming duality theorem

If

$$x \geq 0, Ax \leq c$$
$$y \geq 0, A^T y \geq b$$

for given real vectors b and c and real matrix A, then for real vectors x and y,

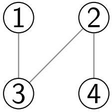
$$max_x(b, x) = min_y(c, y)$$

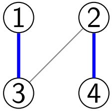
if such extrema exist.

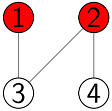
Matching-duality theorem

König theorem

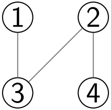
In a bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.







Matching-duality theorem



Maximum matching

maximize

$$x_{13} + x_{23} + x_{24}$$

subject to

$$\begin{array}{ccccccc}
x_{13} & & & & & \leq & 1 \\
 & & x_{23} & + & x_{24} & \leq & 1 \\
x_{13} & + & x_{23} & & & \leq & 1 \\
 & & & & x_{24} & \leq & 1
\end{array}$$

Minimum cover

minimize

$$x_1 + x_2 + x_3 + x_4$$

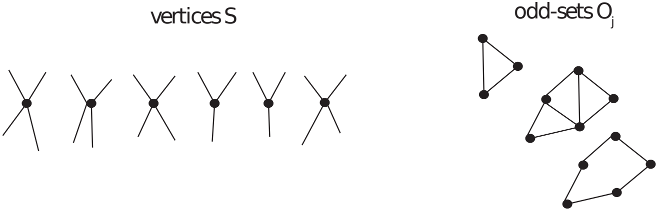
subject to

$$\begin{array}{cccccc}
x_1+ & & x_3 & & \geq & 1 \\
 & x_2+ & x_3 & & \geq & 1 \\
 & x_2+ & & x_4 & \geq & 1
\end{array}$$

Matching-duality theorem

Odd set cover in a graph consists of a collection of vertices S and odd-cardinality sets of vertices $O_1, O_2, ..., O_t$, disjoint from one another and from S, such that every edge of G is either incident with a vertex in S or lies within an odd set O_j.

The capacity of the odd-set cover is defined as: $|S| + \sum_{j=1}^{t} \frac{|O_j|-1}{2}$



Matching-duality theorem

General König theorem

If M is a maximum matching and C is a minimum odd-set cover then

$$|M| = capacity(C)$$

- Proof:

  It is obvious that the capacity-sum of any odd-set cover in G is at least as large as the cardinality of any matching in G, so we have only to prove the existence in G of an odd-set cover and a matching for which the numbers are equal.

Matching-duality theorem

- Proof:

  For a pertect matching M with no exposed vertices, the odd-set cover consists of two sets, one set containing one of the vertices, the other containing all the other vertices.

  For a graph which has a matching with one exposed vertices, the odd-set cover may be taken as one set containing all the vertices of the graph.

  The general case can be proven by induction.