

Selected Topics of Mathematical Statistics: Quiz 17

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Quiz 17: Question

Proof of Theorem 37

For each $-\infty < x < \infty$,

$$\sqrt{n}\{F_n(x) - F(x)\} \xrightarrow{\mathcal{L}} N[0, F(x)\{1 - F(x)\}]$$

where $F_n(x)$ is the empirical distribution function of an *iid* sequence $\{X_i\}$ with cdf F .



Quiz 17: Solution

- Set $Y_i := I(X_i \leq x)$, $i \in \mathbb{N}$. $\{Y_i\}$ is therefore also an *iid* sequence, since $\{Y_i\}$ is *iid* and indicate function is measurable, and

$$E[Y_i] = E[I(X_i \leq x)] = P(X_i \leq x) = F(x)$$

$$\begin{aligned} \text{Var}(Y_i) &= E[\underbrace{(I(X_i \leq x))^2}_{I(X_i \leq x)}] - (E[I(X_i \leq x)])^2 \\ &= F(x) - (F(x))^2 = F(x)(1 - F(x)) \end{aligned}$$



Quiz 17: Solution

- (i) If $F(x) \notin \{0, 1\}$ for all $x \in \mathbb{R}$, we can get from Theorem 31 that

$$\sqrt{n}(\bar{Y}_n - F(x)) = \sqrt{n}(F_n(x) - F(x)) \xrightarrow{\mathcal{L}} N(0, F(x)(1 - F(x)))$$

where

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(X_i \leq x) = F_n(x)$$



Quiz 17: Solution

- (ii) If $F(x) = P(X_i \leq x) = 0$ for all $x \in \mathbb{R}$, so that $Y_i = I(X_i \leq x) = 0$ a.s. That means $Y_i \rightarrow 0$ a.s., and that

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x) = 0 \quad a.s.$$

So that

$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{a.s.} 0$$

and thus

$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{\mathcal{L}} 0$$



Quiz 17: Solution

- (iii) If $F(x) = P(X_i \leq x) = 1$ for all $x \in \mathbb{R}$, so that $Y_i = I(X_i \leq x) = 1$ a.s. That means $Y_i \rightarrow 0$ a.s., and that

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x) = \frac{n}{n} = 1 \quad a.s.$$

So that

$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{a.s.} 0$$

and thus

$$\sqrt{n}(F_n(x) - F(x)) \xrightarrow{\mathcal{L}} 0$$

