Selected Topics of Mathematical Statistics: Quiz 17

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Quiz 17: Question

Proof of Theorem 37

For each $-\infty < x < \infty$,

$$\sqrt{n}\{F_n(x) - F(x)\} \xrightarrow{\mathcal{L}} N[0, F(x)\{1 - F(x)\}]$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(X_i \leqslant x)$ is the empirical distribution function of an *iid* sequence $\{X_i\}$ with cdf F.

Quiz 17: Solution

- Set $Yi := \mathbf{I}(X_i \leqslant x)$, $i \in \mathbb{N}$. $\{Y_i\}$ is therefore also an *iid* sequence, since $\{Y_i\}$ is *iid* and indicate function is measurable, and

$$E[Y_i] = E[\mathbf{I}(X_i \leqslant x)] = P(X_i \leqslant x) = F(x)$$

$$\begin{aligned} Var(Y_i) &= E[\underbrace{(\mathbf{I}(X_i \leqslant x))^2}_{\mathbf{I}(X_i \leqslant x)}] - (E[\mathbf{I}(X_i \leqslant x)])^2 \\ &= F(x) - (F(x))^2 = F(x)(1 - F(x)) \end{aligned}$$

Quiz 17: Solution

(i) If $F(x) \in (0,1)$, we can get from Theorem 31 that

$$\sqrt{n}(\bar{Y}_n - F(x)) = \sqrt{n}(F_n(x) - F(x)) \xrightarrow{\mathcal{L}} N(0, F(x)(1 - F(x))$$

where

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}(X_i \leqslant x) = F_n(x)$$

Quiz 17: Solution

(ii) If $F(x) = P(X_i \leqslant x) = 0$, so that $Y_i = \mathbf{I}(X_i \leqslant x) = 0$ a.s. That means

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(X_i \leqslant x) = 0 \quad a.s.$$

So that

$$\sqrt{n}(F_n(x) - F(x)) \stackrel{a.s.}{\to} 0$$

and thus

$$\sqrt{n}(F_n(x) - F(x)) \stackrel{\mathcal{L}}{\to} 0$$

Quiz 17: Solution

(iii) If $F(x) = P(X_i \leqslant x) = 1$, so that $Y_i = \mathrm{I}(X_i \leqslant x) = 1$ a.s. That means

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}(X_i \leqslant x) = \frac{n}{n} = 1 \quad a.s.$$

So that

$$\sqrt{n}(F_n(x) - F(x)) \stackrel{a.s.}{\to} 0$$

and thus

$$\sqrt{n}(F_n(x) - F(x)) \stackrel{\mathcal{L}}{\to} 0$$