Selected Topics of Mathematical Statistics: Quiz 4

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Quiz 4

Show several characteristic functions under different CDFs!

Binomial Distribution

Let $X \sim Bin(n, p)$. Then:

$$\begin{aligned}
\varphi_X(t) = \mathbb{E}[e^{itX}] &= \sum_{k=0}^{n} e^{itk} \cdot \mathbb{P}(X = k) \\
&= \sum_{k=0}^{n} e^{itk} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot (e^{it} \cdot p)^k \cdot (1-p)^{n-k} \\
&= [e^{it} \cdot p + (1-p)]^n
\end{aligned}$$

because $(a+b)^n = \sum_0^n \binom{n}{k} a^k b^{n-k}$ for $n \in \mathbb{N}$.

Poisson Distribution

Let $X \sim Pois(\lambda)$. Then:

$$\begin{aligned}
\varphi_X(t) = \mathbb{E}[e^{itX}] &= \sum_{k=0}^{\infty} e^{itk} \cdot \mathbb{P}(X = k) \\
&= \sum_{k=0}^{\infty} e^{itk} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{(\lambda \cdot e^{it})^k}{k!} \\
&= e^{-\lambda} \cdot e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}
\end{aligned}$$

because $e^x = \sum_0^\infty \frac{x^k}{k!}$ for $x \in \mathbb{R}$.

Standard Normal Distribution

Let $X \sim N(0,1)$. Then:

$$
\begin{aligned}
\varphi_X(t) = \mathbb{E}[e^{itX}] &= \int_{-\infty}^{\infty} e^{itx} \cdot f(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{itx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} [\int_{-\infty}^{0} e^{itx} \cdot e^{-\frac{x^2}{2}}\ \mathrm{d}x + \int_{0}^{\infty} e^{itx} \cdot e^{-\frac{x^2}{2}}\ \mathrm{d}x] \\
&= \frac{1}{\sqrt{2\pi}} [\int_{0}^{\infty} e^{-itx} \cdot e^{-\frac{x^2}{2}}\ \mathrm{d}x + \int_{0}^{\infty} e^{itx} \cdot e^{-\frac{x^2}{2}}\ \mathrm{d}x]
\end{aligned}
$$

Standard Normal Distribution

$$
\begin{aligned}
\varphi x(t) &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot e^{itx} + e^{-\frac{x^2}{2}} \cdot e^{-itx} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot [e^{itx} + e^{-itx}] \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot 2 \cdot cos(tx) \, \mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot cos(tx) \, \mathrm{d}x \\
&\overset{(1)}{=} e^{-\frac{t^2}{2}}
\end{aligned}
$$

Standard Normal Distribution

To prove (1) let

$$F(t) := \varphi_X(t) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot cos(tx)\, \mathrm{d}x$$

We show that

$$F'(t) = -t \cdot F(t) \qquad (*)$$

This ordinary differential equation has the solution $F(t) = c \cdot e^{-\frac{t^2}{2}}$, where c is a constant. Since we have

$$F(0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, \mathrm{d}x = 1$$

we get $c = 1$. Hence, $\varphi_X(t) = F(t) = e^{-\frac{t^2}{2}}$.

Standard Normal Distribution

$$
\begin{aligned}
F'(t) &= \frac{d}{dt}\,\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot cos(tx)\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot \frac{d}{dt} cos(tx)\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty -e^{-\frac{x^2}{2}} \cdot x \cdot sin(tx)\mathrm{d}x
\end{aligned}
$$

Integration by parts:

$u(x) = e^{-\frac{x^2}{2}}$, $u'(x) = -x \cdot e^{-\frac{x^2}{2}}$ and

$v(x) = sin(tx)$, $v'(x) = tcos(tx)$

Standard Normal Distribution

$$
\begin{aligned}
F'(t) &= \frac{2}{\sqrt{2\pi}} \cdot \left[\left(e^{-\frac{x^2}{2}} \cdot sin(tx) \right) \Big|_0^\infty - \int_0^\infty e^{-\frac{x^2}{2}} \cdot t \cdot cos(tx) \,\mathrm{d}x \right] \\
&= -\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-\frac{x^2}{2}} \cdot t \cdot cos(tx) \,\mathrm{d}x \\
&= -t \cdot \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot 2 \cdot cos(tx) \,\mathrm{d}x \\
&= -t \cdot F(t)
\end{aligned}
$$

Hence, $(*)$ was proved and we get $\varphi_X(t) = e^{-\frac{t^2}{2}}$.