Selected topics in Mathematical Statistics, Quiz 5

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Problem Description

Quiz 5: Prove (18) under standard normal distribution, where (18) is:

If φ_X is absolutely integrable,

$$(18) \quad f_X(x) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-itx} \varphi_X(t)\, dt$$

Proof of Lemma 1

To prove the statement, first we will need one Lemma:

$$\text{Lemma 1:} \quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2} \, dx = \sqrt{2\pi} \tag{1}$$

We start with

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2} \, dx \tag{2}$$

First, substitute $S = x - it$, we have

$$\int_{-\infty-it}^{\infty-it} e^{-\frac{1}{2}S^2} \, dS \tag{3}$$

We're first taking the integral between α and $-\alpha$ and later take the limits at ∞. Consider the integral on a contour like this:
$\mathcal{C} = \alpha \to -\alpha \to -\alpha - it \to \alpha - it \to \alpha$.
Now since the normal distribution is analytic everywhere, we must have

$$\oint_{\mathcal{C}} f_X(z)\ dz = 0 \tag{4}$$

Writing out all four parts of the contour integral gives

$$\begin{aligned}\oint_{\mathcal{C}} f(S)\ dS = &\int_{\alpha}^{-\alpha} e^{-\frac{S^2}{2}}\ dS \\ &+ \int_{-\alpha}^{-\alpha-it} e^{-\frac{S^2}{2}}\ dS \\ &+ \int_{-\alpha-it}^{\alpha-it} e^{-\frac{S^2}{2}}\ dS \\ &+ \int_{\alpha-it}^{\alpha} e^{-\frac{S^2}{2}}\ dS = 0\end{aligned} \tag{5}$$

As we take the limits for $\alpha \to \infty$, the first term becomes $-\sqrt{2\pi}$ (because we're integrating from right to left), and terms 2 and 4 become zero. The third term is the term we're interested in. As we solve for that term, we get

$$\int_{-\infty-it}^{\infty-it} e^{-\frac{S^2}{2}}\ dS = \sqrt{2\pi} \tag{6}$$

which completes the proof.

Proof of Statement (18)

In order to proof (18), we first compute the characteristic function $\varphi_X(t)$ of the standard normal distribution. From (17) we have

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \tag{7}$$

Looking at the exponent of e, we complete the square in t

$$
\begin{aligned}
\varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2itx-t^2)} e^{-\frac{1}{2}t^2} \; dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \; \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2itx-t^2)} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \; \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2}
\end{aligned} \tag{8}
$$

Now by Lemma 1, the Integral in the last line of (8) is $\sqrt{2\pi}$, and $\varphi_X(t)$ is

$$\varphi_X(t) = e^{-\frac{t^2}{2}} \tag{9}$$

To complete the proof we substitute $\varphi_X(t)$ into (18):

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} \, dt \tag{10}$$

We proceed by completing the squre similarly to (8) and get

$$f_X(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t-ix)^2}\ dt \tag{11}$$

Again by Lemma 1, the integral is $\sqrt{2\pi}$ and we are left with

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{12}$$

which completes the proof.