Selected Topics of Mathematical Statistics: Quiz 6

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Quiz 6

Prove the following property of the characteristic function for the Standard Normal Distribution for $k = 1, 2, 3, 4$:

$$\varphi_X^{(k)}(t)\big|_{t=0} = i^k \cdot \mathbb{E}[X^k]$$

Characteristic Function of Standard Normal Distribution

In Quiz 4 it was shown that the characteristic function for a random variable $X \sim N(0,1)$ is of the following form:

$$\varphi_X(t) = e^{-\frac{1}{2}t^2}, \quad t \in \mathbb{R}$$

k=1

$$\begin{aligned}
\varphi_X^{(1)}(t) \mid_{t=0} &= \frac{d}{dt} \, [e^{-\frac{t^2}{2}}] \mid_{t=0} \\
&= [-t \cdot e^{-\frac{t^2}{2}}] \mid_{t=0} \\
&= 0 \\
&= i^1 \cdot \mathbb{E}[X]
\end{aligned}$$

because $\mathbb{E}[X] = 0$ for the Standard Normal Distribution.

k=2

$$\begin{aligned}
\varphi_X^{(2)}(t) \mid_{t=0} &= \frac{d^2}{dt^2} \, [e^{-\frac{t^2}{2}}] \mid_{t=0} \\
&= \frac{d}{dt} \, [-t \cdot e^{-\frac{t^2}{2}}] \mid_{t=0} \\
&= [-e^{-\frac{t^2}{2}} + t^2 \cdot e^{-\frac{t^2}{2}}] \mid_{t=0} \\
&= -1 \\
&= i^2 \cdot Var(X) \\
&= i^2 \cdot \mathbb{E}[X^2]
\end{aligned}$$

because $Var(X) = 1$ and $\mathbb{E}[X] = 0$ for the Standard Normal Distribution.

k=3

$$
\begin{aligned}
\varphi_X^{(3)}(t)\ |_{t=0} &= \frac{d^3}{dt^3}\ [e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= \frac{d}{dt}\ [-e^{-\frac{t^2}{2}} + t^2 \cdot e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= \frac{d}{dt}\ [-e^{-\frac{t^2}{2}}]\ |_{t=0} + \frac{d}{dt}\ [t^2 \cdot e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= [t \cdot e^{-\frac{t^2}{2}}]\ |_{t=0} + [2t \cdot e^{-\frac{t^2}{2}} - t^3 \cdot e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= 0 \\
&= i^3 \cdot \mathbb{E}[X^3]
\end{aligned}
$$

because $\mathbb{E}[X^3] = 0$ for the Standard Normal Distribution.

k=4

$$
\begin{aligned}
\varphi_X^{(4)}(t)\ |_{t=0} &= \frac{d^4}{dt^4}\ [e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= \frac{d}{dt}\ [t\cdot e^{-\frac{t^2}{2}} + 2t\cdot e^{-\frac{t^2}{2}} - t^3\cdot e^{-\frac{t^2}{2}}]\ |_{t=0} \\
&= \frac{d}{dt}\ [te^{-\frac{t^2}{2}}]\ |_{t=0} + \frac{d}{dt}\ [2te^{-\frac{t^2}{2}}]\ |_{t=0} - \frac{d}{dt}\ [t^3e^{-\frac{t^2}{2}}]\ |_{t=0}
\end{aligned}
$$

k=4

It holds:

$$\frac{d}{dt}\left[t \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = \left[e^{-\frac{t^2}{2}} - t^2 \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = 1 \qquad (1)$$

$$\frac{d}{dt}\left[2t \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = 2 \cdot \left[e^{-\frac{t^2}{2}} - t^2 \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = 2 \qquad (2)$$

$$\frac{d}{dt}\left[t^3 \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = \left[3 \cdot t^2 \cdot e^{-\frac{t^2}{2}} - t^4 \cdot e^{-\frac{t^2}{2}}\right]\Big|_{t=0} = 0 \qquad (3)$$

k=4

With (1), (2), (3) it follows:

$$\begin{aligned} \varphi_X^{(4)}(t) \mid_{t=0} &= 1 + 2 - 0 \\ &= 3 \\ &= i^4 \cdot \mathbb{E}[X^4] \end{aligned}$$

because $i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1$ and $\mathbb{E}[X^4] = 3$ for the Standard Normal Distribution.