HUMBOLDT UNIVERSITY BERLIN

Quiz 1

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1 PROBLEM DESCRIPTION

Quiz 1: Relate Interquartile Range (IQR) to Standard Deviation (SD) under some distributions.

2 SOLUTION

2.1 UNIFORM DISTRIBUTION

Probability density function:

$$f_X(x) = \frac{1}{b-a} \tag{2.1}$$

Cumulative density function:

$$F_X(x) = \frac{x-a}{b-a} \tag{2.2}$$

Quantile function:

$$F_X^{-1}(p) = a + p(b-a) \tag{2.3}$$

The quantiles:

$$\begin{aligned} \xi_{\frac{1}{4}} &= F^{-1}(\tfrac{1}{4}) = a + \frac{1}{4}(b-a) \\ \xi_{\frac{3}{4}} &= F^{-1}(\tfrac{3}{4}) = a + \frac{3}{4}(b-a) \end{aligned} \tag{2.4}$$

So, the Interquartile Range is

$$IQR = \frac{1}{2}(b-a) \tag{2.5}$$

And the Standard Deviation of the uniform distribution is known to be $\frac{(b-a)}{\sqrt{12}}$, so the ratio of IQR to SD is

$$\frac{IQR}{SD} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{12}}} = \sqrt{3} \tag{2.6}$$

2.2 EXPONENTIAL DISTRIBUTION

Probability density function:

$$f_X(x) = \lambda e^{-\lambda x} \tag{2.7}$$

Cumulative density function:

$$F_X(x) = 1 - e^{-\lambda x} \tag{2.8}$$

Deriving the quantile function:

$$\begin{aligned} F_X^{-1}(p) &= x_p \\ p &= 1 - e^{-\lambda x_p} \\ e^{-\lambda x_p} &= 1 - p \\ x_p &= -\frac{\log(1-p)}{\lambda} \end{aligned} \tag{2.9}$$

So, $F_X^{-1}(p)$ is:

$$F_X^{-1}(p) = -\frac{\log(1-p)}{\lambda} \tag{2.10}$$

$$\begin{aligned} \xi_{\frac{1}{4}} &= F^{-1}(\tfrac{1}{4}) = -\frac{\log(1-\frac{1}{4})}{\lambda} = \frac{\log\frac{4}{3}}{\lambda} \\ \xi_{\frac{3}{4}} &= F^{-1}(\tfrac{3}{4}) = -\frac{\log(1-\frac{3}{4})}{\lambda} = \frac{\log 4}{\lambda} \end{aligned} \tag{2.11}$$

So, the Interquartile Range is

$$IQR = \frac{\log 4 - \log\frac{4}{3}}{\lambda} = \frac{\log 3}{\lambda} \tag{2.12}$$

And the Standard Deviation of the exponential distribution is known to be $\frac{1}{\lambda^2}$, so the ratio of IQR to SD is

$$\frac{IQR}{SD} = \lambda \log 3 \approx \lambda * 1.098 \tag{2.13}$$

2.3 Standard Normal Distribution

For the Normal Distribution, the cumulative and quantile distributions can't be expressed in elementary functions, therefore one has to use numerical approximations.
The first and third quantiles are

$$\begin{aligned} \xi_{\frac{1}{4}} &= \Phi^{-1}(\tfrac{1}{4}) = -0.68 \\ \xi_{\frac{3}{4}} &= \Phi^{-1}(\tfrac{3}{4}) = 0.68 \end{aligned} \tag{2.14}$$

So the interquartile range is $IQR = 0.68 - (-0.68) = 1.36$. Since the standard deviation of the standard normal distribution is 1, the ratio is

$$\frac{IQR}{SD} = \frac{1.36}{1} = 1.36 \tag{2.15}$$