Selected Topics of Mathematical Statistics: Quiz 6

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Quiz 6

Prove the following property of the characteristic function for the Standard Normal Distribution for $k = 1, 2, 3, 4$:

$$\varphi_X^{(k)}(t)\,\big|_{t=0} = \mathbf{i}^k \cdot \mathrm{E}\left[X^k\right]$$

Characteristic Function of the Standard Normal Distribution

In Quiz 4 it was shown that the characteristic function of a standard normal random variable is of the following form:

$$\varphi_X(t) = \exp\left(-\frac{t^2}{2}\right), \quad t \in \mathbb{R}$$

Proof for k=1

$$
\begin{aligned}
\varphi_X^{(1)}(t)\,|_{t=0} &= \frac{d}{dt}\left\{\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} \\
&= \left\{-t\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} \\
&= 0 \\
&= \mathbf{i}\cdot \mathrm{E}\left[X\right]
\end{aligned}
$$

because $\mathrm{E}\left[X\right] = 0$ for the Standard Normal Distribution.

Proof for k=2

$$
\begin{aligned}
\varphi_X^{(2)}(t)\,|_{t=0} &= \frac{d^2}{dt^2}\left\{\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= \frac{d}{dt}\left\{-t\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= \left\{-\exp\left(-\frac{t^2}{2}\right) + t^2\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= -1 \\
&= \mathbf{i}^2 \cdot \mathrm{Var}(X) = \mathbf{i}^2 \cdot \mathrm{E}\left[X^2\right]
\end{aligned}
$$

because $\mathrm{Var}(X) = 1$ and $\mathrm{E}[X] = 0$ for the Standard Normal Distribution.

Proof for k=3

$$
\begin{aligned}
\varphi_X^{(3)}(t)\,|_{t=0} &= \frac{d^3}{dt^3}\left\{\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= \frac{d}{dt}\left\{-\exp\left(-\frac{t^2}{2}\right) + t^2\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= \left\{t\exp\left(-\frac{t^2}{2}\right) + 2t\exp\left(-\frac{t^2}{2}\right) - t^3\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} \\
&= 0 \\
&= \mathbf{i}^3 \cdot \mathrm{E}\left[X^3\right]
\end{aligned}
$$

because $\mathrm{E}\left[X^3\right] = 0$ for the Standard Normal Distribution.

Proof for k=4

$$
\begin{aligned}
\varphi_X^{(4)}(t)\,|_{t=0} &= \frac{d^4}{dt^4}\left\{\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} \\
&= \frac{d}{dt}\left\{t\exp\left(-\frac{t^2}{2}\right) + 2t\exp\left(-\frac{t^2}{2}\right)\right. \\
&\qquad \left. -t^3\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} \\
&= \frac{d}{dt}\left\{t\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} + \frac{d}{dt}\left\{2t\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} \\
&\qquad - \frac{d}{dt}\left\{t^3\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0}
\end{aligned}
$$

Proof for k=4

It holds:

$$\frac{d}{dt}\left\{t\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} = \left\{\exp\left(-\frac{t^2}{2}\right) - t^2\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} = 1 \tag{1}$$

$$\frac{d}{dt}\left\{2t\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} = 2\left\{\exp\left(-\frac{t^2}{2}\right) - t^2\exp\left(-\frac{t^2}{2}\right)\right\}\Bigg|_{t=0} = 2 \tag{2}$$

Proof for k=4

$$\frac{d}{dt}\left\{t^3\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} = \left\{3t^2\exp\left(-\frac{t^2}{2}\right) - t^4\exp\left(-\frac{t^2}{2}\right)\right\}\Big|_{t=0} = 0 \qquad (3)$$

With (1), (2), (3) it follows:

$$\begin{aligned}\varphi_X^{(4)}(t)\,|_{t=0} &= 1+2-0\\ &= 3\\ &= \mathbf{i}^4\cdot \mathrm{E}\left[X^4\right]\end{aligned}$$

because $\mathbf{i}^4 = \mathbf{i}^2\cdot\mathbf{i}^2 = (-1)\cdot(-1) = 1$ and $\mathrm{E}\left[X^4\right] = 3$ for the Standard Normal Distribution.