

Selected Topics of Mathematical Statistics: Quiz 4

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Quiz 4

**Show several characteristic functions under
different CDFs!**



Binomial Distribution

Let $X \sim B(n, p)$. Then:

$$\begin{aligned}\varphi_X(t) &= E[\exp(itX)] = \sum_{k=0}^n \exp(itk) \cdot P(X = k) \\&= \sum_{k=0}^n \exp(itk) \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\&= \sum_{k=0}^n \binom{n}{k} \cdot \{\exp(it) \cdot p\}^k \cdot (1-p)^{n-k} \\&= \{\exp(it) \cdot p + (1-p)\}^n\end{aligned}$$

because $(a + b)^n = \sum_{k=0}^n \binom{n}{k} \cdot a^k \cdot b^{n-k}$ for $n \in \mathbb{N}$.



Poisson Distribution

Let $X \sim \text{Pois}(\lambda)$. Then:

$$\begin{aligned}\varphi_X(t) &= E[\exp(itX)] = \sum_{k=0}^{\infty} \exp(itk) \cdot P(X = k) \\&= \sum_{k=0}^{\infty} \exp(itk) \cdot \exp(-\lambda) \cdot \frac{\lambda^k}{k!} \\&= \exp(-\lambda) \cdot \sum_{k=0}^{\infty} \frac{\{\lambda \cdot \exp(it)\}^k}{k!} \\&= \exp(-\lambda) \cdot \exp(\lambda \cdot \exp(it)) = \exp(\lambda \cdot [\exp(it) - 1])\end{aligned}$$

because $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$.



Exponential Distribution

Let $X \sim \text{Exp}(\lambda)$. Then:

$$\begin{aligned}\varphi_X(t) &= E[\exp(itX)] \\&= \int_0^{\infty} \exp(itx) \cdot f(x) \, dx \\&= \int_0^{\infty} \exp(itx) \cdot \lambda \cdot \exp(-\lambda x) \, dx \\&= \lambda \cdot \int_0^{\infty} \exp(itx - \lambda x) \, dx \\&= \lambda \cdot \int_0^{\infty} \exp([it - \lambda] \cdot x) \, dx\end{aligned}$$



Exponential Distribution

$$\begin{aligned}\varphi_X(t) &= \lambda \cdot \int_0^{\infty} \exp(-[\lambda - it] \cdot x) \, dx \\&= \lambda \cdot \left\{ -\frac{1}{\lambda - it} \cdot \exp(-[\lambda - it] \cdot x) \right\} \Big|_0^{\infty} \\&= -\frac{\lambda}{\lambda - it} \cdot \{ \exp(-[\lambda - it] \cdot \infty) - \exp(-[\lambda - it] \cdot 0) \} \\&= -\frac{\lambda}{\lambda - it} \cdot \left\{ \lim_{a \rightarrow \infty} \exp(-[\lambda - it] \cdot a) - 1 \right\} \\&\stackrel{(1)}{=} -\frac{\lambda}{\lambda - it} \cdot (0 - 1) = \frac{\lambda}{\lambda - it}\end{aligned}$$



Exponential Distribution

To show (1), we prove that

$$\lim_{a \rightarrow \infty} \exp(-[\lambda - it] \cdot a) = 0 \quad (*)$$

It holds:

$$\begin{aligned} \lim_{a \rightarrow \infty} \exp(-[\lambda - it] \cdot a) &= \lim_{a \rightarrow \infty} \exp(-\lambda a + ita) \\ &= \lim_{a \rightarrow \infty} \exp(-\lambda a) \cdot \exp(ita) \\ &= 0 \end{aligned}$$

because $\lim_{a \rightarrow \infty} \exp(-\lambda a) = 0$ and $|\exp(ita)| = 1$. Hence, (*) was proved and we get $\varphi_X(t) = \frac{\lambda}{\lambda - it}$.



Standard Normal Distribution

Let $X \sim N(0, 1)$. Then:

$$\begin{aligned}\varphi_X(t) &= E[\exp(itX)] = \int_{-\infty}^{\infty} \exp(itx) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) dx \\&= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_{-\infty}^0 \exp(itx) \cdot \exp\left(-\frac{x^2}{2}\right) dx \right. \\&\quad \left. + \int_0^{\infty} \exp(itx) \cdot \exp\left(-\frac{x^2}{2}\right) dx \right\} \\&= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_0^{\infty} \exp(-itx) \cdot \exp\left(-\frac{x^2}{2}\right) dx \right. \\&\quad \left. + \int_0^{\infty} \exp(itx) \cdot \exp\left(-\frac{x^2}{2}\right) dx \right\}\end{aligned}$$



Standard Normal Distribution

$$\begin{aligned}\varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \exp(itx) \right. \\ &\quad \left. + \exp\left(-\frac{x^2}{2}\right) \cdot \exp(-itx) \, dx \right\} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \{\exp(itx) + \exp(-itx)\} \, dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot 2 \cdot \cos(tx) \, dx \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \cos(tx) \, dx \stackrel{(1)}{=} \exp\left(-\frac{t^2}{2}\right)\end{aligned}$$



Standard Normal Distribution

To prove (1) let

$$F(t) \stackrel{\text{def}}{=} \varphi_X(t) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \cos(tx) \, dx$$

We show that

$$F'(t) = -t \cdot F(t) \quad (*)$$

This ordinary differential equation has the solution

$F(t) = c \cdot \exp\left(-\frac{t^2}{2}\right)$, where c is a constant. Since we have

$$F(0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \, dx = 1$$

we get $c = 1$. Hence, $\varphi_X(t) = F(t) = \exp\left(-\frac{t^2}{2}\right)$.



Standard Normal Distribution

Proof of (*):

$$\begin{aligned} F'(t) &= \frac{d}{dt} \left\{ \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \cos(tx) \, dx \right\} \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot \frac{d}{dt} \cos(tx) \, dx \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty -\exp\left(-\frac{x^2}{2}\right) \cdot x \cdot \sin(tx) \, dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} u(x) &= \exp\left(-\frac{x^2}{2}\right) & u'(x) &= -x \cdot \exp\left(-\frac{x^2}{2}\right) \\ v(x) &= \sin(tx) & v'(x) &= t \cdot \cos(tx) \end{aligned}$$



Standard Normal Distribution

$$\begin{aligned} F'(t) &= \frac{2}{\sqrt{2\pi}} \cdot \left[\left\{ \exp\left(-\frac{x^2}{2}\right) \cdot \sin(tx) \right\} \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot t \cdot \cos(tx) \, dx \right] \\ &= -\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot t \cdot \cos(tx) \, dx \\ &= -t \cdot \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot 2 \cdot \cos(tx) \, dx \\ &= -t \cdot F(t) \end{aligned}$$

Hence, (*) was proved and we get $\varphi_X(t) = \exp\left(-\frac{t^2}{2}\right)$.

