Selected Topics of Mathematical Statistics: Quiz 5

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Quiz 5

Prove the following equation for the Standard Normal Distribution:

$$f(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-\mathbf{i}tx) \cdot \varphi_X(t)\, dt \qquad (*)$$

Proof

Let $X \sim \mathrm{N}(0,1)$. Define

$$G(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-\mathbf{i}tx) \cdot \varphi_X(t)\ dt$$

In Quiz 4, it was shown that

$$\varphi_X(t) \stackrel{\text{def}}{=} \mathrm{E}[\exp(\mathbf{i}tX)] = \exp\left(-\frac{t^2}{2}\right) \tag{1}$$

with $t \in \mathbb{R}$.

Proof

Hence, we get

$$
\begin{aligned}
G(x) &\overset{(1)}{=} \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp\left(-\mathbf{i}tx\right) \cdot \exp\left(-\frac{t^2}{2}\right) \, dt \\
&= \left\{ \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp\left(-\mathbf{i}tx\right) \cdot \exp\left(-\frac{t^2}{2}\right) \, dt \right\} \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \\
&= \frac{\sqrt{2\pi}}{2\pi} \cdot \int_{-\infty}^{\infty} \exp\left(-\mathbf{i}tx\right) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) \, dt
\end{aligned}
$$

Proof

Let f_X be the probability density function of the Standard Normal Distribution. Then:

$$
\begin{aligned}
G(x) &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \exp(-\mathbf{i}tx) \cdot f_X(t)\ dt \\
&= \frac{1}{\sqrt{2\pi}} \cdot \mathrm{E}\left[\exp(-\mathbf{i}xX)\right] \\
&\overset{(1)}{=} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)
\end{aligned}
$$

Hence, $(*)$ was proved.