HUMBOLDT UNIVERSITY BERLIN

Quiz 5

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1 Problem Description

Quiz 5: Prove (18) under standard normal distribution, where (18) is:
If φ_X is absolutely integrable,

$$(18)\ f_X(x) = \frac{1}{(2\pi)^p} \int_{-\infty}^{\infty} e^{-it^\top x} \varphi_X(t)\, dt \tag{1.1}$$

2 Solution

To prove the statement, first we will need one Lemma:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2}\, dx = \sqrt{2\pi} \tag{2.1}$$

2.1 Proof of Lemma 1

We start with

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2}\, dx \tag{2.2}$$

First, substitute $S = x - it$, we have

$$\int_{-\infty-it}^{\infty-it} e^{-\frac{1}{2}S^2}\, dS \tag{2.3}$$

Now, remember that, on a closed contour if the function within the contour is analytic,

$$\oint f(z)\, dz = 0 \tag{2.4}$$

We're first taking the integral between α and $-\alpha$ and later take the limits at ∞. Consider the integral on a contour like this: $\mathscr{C} = \alpha \to -\alpha \to -\alpha - it \to \alpha - it \to \alpha$.
Now since the normal distribution is analytic everywhere, we must have

$$\oint_{\mathscr{C}} f_X(z)\, dz = 0 \tag{2.5}$$

Writing out all four parts of the contour integral gives

$$\oint_{\mathscr{C}} f(S)\, dS = \int_{\alpha}^{-\alpha} e^{-\frac{S^2}{2}}\, dS + \int_{-\alpha}^{-\alpha-it} e^{-\frac{S^2}{2}}\, dS + \int_{-\alpha-it}^{\alpha-it} e^{-\frac{S^2}{2}}\, dS + \int_{\alpha-it}^{\alpha} e^{-\frac{S^2}{2}}\, dS = 0 \tag{2.6}$$

As we take the limits for $\alpha \to \infty$, the first term becomes $-\sqrt{2\pi}$ (because we're integrating from right to left), and terms 2 and 4 become zero. The third term is the term we're interested in. As we solve for that term, we get

$$\int_{-\infty-it}^{\infty-it} e^{-\frac{S^2}{2}}\, dS = \sqrt{2\pi} \tag{2.7}$$

which completes the proof.

2.2 Proof of (18)

In order to proof (18), we first compute the characteristic function $\varphi_X(t)$ of the standard normal distribution. From (17) we have

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx \tag{2.8}$$

Looking at the exponent of e, we complete the square in t

$$\begin{aligned} \varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2itx-t^2)} e^{-\frac{1}{2}t^2}\ dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2itx-t^2)} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2} \end{aligned} \tag{2.9}$$

Now by Lemma 1, the Integral in the last line of (2.9) is $\sqrt{2\pi}$, and $\varphi_X(t)$ is

$$\varphi_X(t) = e^{-\frac{t^2}{2}} \tag{2.10}$$

To complete the proof we substitute $\varphi_X(t)$ into (18):

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}}\ dt \tag{2.11}$$

We proceed by completing the squre similarly to (2.9) and get

$$f_X(x) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t-ix)^2}\ dt \tag{2.12}$$

Again by Lemma 1, the integral is $\sqrt{2\pi}$ and we are left with

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.13}$$

which completes the proof.