Selected topics in Mathematical Statistics, Quiz 3

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Problem Description

Quiz 3: Show that the Kullback-Leibler-Divergence $K(\theta, \theta')$ satisfies for any θ, θ':

$$
\begin{aligned}
&1.\ K(\theta, \theta')|_{\theta=\theta'} = 0 \\
&2.\ \frac{d}{d\theta'}\, K(\theta, \theta')|_{\theta=\theta'} = 0 \\
&3.\ \frac{d^2}{d\theta'^2}\, K(\theta, \theta')|_{\theta=\theta'} = \int_{-\infty}^{\infty} \frac{p'(x, \theta)^2}{p(x, \theta)}\, dx
\end{aligned}
\tag{1}
$$

Proof of statement 1

Statement 1 was:

$$K(\theta, \theta')|_{\theta=\theta'} = 0 \tag{2}$$

$$\tag{3}$$

Notice that the quantity in the logarithm is always one, therefore the logarithm evaluates to zero:

$$K(\theta, \theta')|_{\theta=\theta'} = \int_{-\infty}^{\infty} \ln 1 \; p(x, \theta) dx = \int_{-\infty}^{\infty} 0 * p(x, \theta) dx = 0 \tag{4}$$

Proof of statement 2

Statement 2 was:

$$\frac{d}{d\theta'}\; K(\theta, \theta')|_{\theta=\theta'} = 0 \tag{5}$$

$$\tag{6}$$

For this proof we'll assume that $p(x, \theta) = p(x, \theta')$ and its derivative with respect to θ is continuous over $\mathbb{R}$, in which case we can evaluate the derivative within the integral according to Leibniz' rule.

Use Leibniz' rule:

$$\begin{aligned}\frac{d}{d\theta'}\ K(\theta,\theta')|_{\theta,\theta'} &= \frac{d}{d\theta'}\ \int_{-\infty}^{\infty} \ln \frac{p(x,\theta)}{p(x,\theta')}\ p(x,\theta)dx \\ &= \int_{-\infty}^{\infty} p(x,\theta)\ \frac{d}{d\theta'}\left(\ln \frac{p(x,\theta)}{p(x,\theta')}\right) dx\end{aligned} \tag{7}$$

$p(x,\theta)$ is independent of θ', so by rewriting the logarithm we get

$$= \int_{-\infty}^{\infty} p(x,\theta)\ \frac{d}{d\theta'}\Big\{ -\ln\left(p(x,\theta')\right)\Big\}\ dx \tag{8}$$

Evaluating the derivate further:

$$= \int_{-\infty}^{\infty} -p(x,\theta)\ \frac{1}{p(x,\theta')}\ \frac{d}{d\theta'} p(x,\theta')\ dx \tag{9}$$

Since $\theta = \theta'$, the first two factors cancel

$$\begin{aligned} &= \int_{-\infty}^{\infty} -\frac{d}{d\theta'} p(x,\theta')\ dx \\ &= \frac{d}{d\theta'} \int_{-\infty}^{\infty} -p(x,\theta')\ dx \end{aligned} \tag{10}$$

and the integral is over a probability distribution, so

$$\begin{aligned} &= \frac{d}{d\theta'} \int_{-\infty}^{\infty} -p(x, \theta') \; dx \\ &= \frac{d}{d\theta'}(-1) \\ &= 0 \end{aligned} \tag{11}$$

Proof of statement 3

Similar to part 2.2, we assume now that also the second derivative of $p(x, \theta)$ with respect to θ is continuous, and use Leibniz rule:

$$\begin{aligned} \frac{d^2}{d\theta'^2} K(\theta, \theta')|_{\theta,\theta'} &= \frac{d^2}{d\theta'^2} \int_{-\infty}^{\infty} \ln \frac{p(x,\theta)}{p(x,\theta')} \, p(x,\theta) dx \\ &= \int_{-\infty}^{\infty} p(x,\theta) \, \frac{d^2}{d\theta'^2} \left\{ \ln \frac{p(x,\theta)}{p(x,\theta')} \right\} dx \end{aligned} \tag{12}$$

again making use of the fact that $p(x, \theta)$ is independent of θ', and evaluating the derivative:

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} p(x,\theta)\ \frac{d^2}{d\theta'^2}\ \left\{ \ln \frac{p(x,\theta)}{p(x,\theta')} \right\}\ dx \\
&= \int_{-\infty}^{\infty} -p(x,\theta)\ \frac{d^2}{d\theta'^2}\ \ln p(x,\theta')\ dx \qquad (13) \\
&= \int_{-\infty}^{\infty} -p(x,\theta)\ \frac{d}{d\theta'}\ \left[\frac{1}{p(x,\theta')}\ \frac{d}{d\theta'} p(x,\theta') \right]\ dx
\end{aligned}
$$

and evaluating the derivative again

$$= \int_{-\infty}^{\infty} -p(x,\theta)\ \frac{d}{d\theta'}\ \left[\frac{1}{p(x,\theta')}\ \frac{d}{d\theta'}p(x,\theta')\right]\ dx$$

$$= \int_{-\infty}^{\infty} -p(x,\theta)\ \left[-\frac{1}{p(x,\theta')^2}\ \left\{\frac{d}{d\theta'}p(x,\theta')\right\}^2\right. \tag{14}$$

$$+ \qquad \left.\frac{1}{p(x,\theta')}\ \frac{d^2}{d\theta'^2}\ p(x,\theta')\right]\ dx$$

multiplying by $-p(x,\theta)$ in front of the brackets:

$$= \int_{-\infty}^{\infty} \frac{\left\{\frac{d}{d\theta'}p(x,\theta')\right\}^2}{p(x,\theta')} - \frac{d^2}{d\theta'^2}\ p(x,\theta')\ dx \tag{15}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')}\ dx - \frac{d^2}{d\theta'^2} \int_{-\infty}^{\infty} p(x,\theta')\ dx \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')}\ dx - 0 \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')}\ dx
\end{aligned}
\tag{16}
$$

In a different notation, this is:

$$= \int_{-\infty}^{\infty} \frac{p'(x,\theta)^2}{p(x,\theta)} \, dx \tag{17}$$

which proves statement three.