

Selected Topics of Mathematical Statistics: Quiz 5

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Quiz 5

Prove the following equation for the Standard Normal Distribution:

$$f(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-itx) \cdot \varphi_X(t) dt \quad (*)$$



Proof

Let $X \sim N(0, 1)$. Define

$$G(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-itx) \cdot \varphi_X(t) \, dt$$

In Quiz 4, it was shown that

$$\varphi_X(t) \stackrel{\text{def}}{=} E[\exp(itX)] = \exp\left(-\frac{t^2}{2}\right) \quad (1)$$

with $t \in \mathbb{R}$.



Proof

Hence, we get

$$\begin{aligned} G(x) &\stackrel{(1)}{=} \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-itx) \cdot \exp\left(-\frac{t^2}{2}\right) dt \\ &= \left\{ \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \exp(-itx) \cdot \exp\left(-\frac{t^2}{2}\right) dt \right\} \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \\ &= \frac{\sqrt{2\pi}}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{t^2}{2}\right) dt \end{aligned}$$



Proof

Let f_X be the probability density function of the Standard Normal Distribution. Then:

$$\begin{aligned} G(x) &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \exp(-itx) \cdot f_X(t) \, dt \\ &= \frac{1}{\sqrt{2\pi}} \cdot E[\exp(ixX)] \\ &\stackrel{(1)}{=} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \end{aligned}$$

Hence, (*) was proved.

