Selected Topics of Mathematical Statistics: Quiz 4

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Quiz 4

Show several characteristic functions under different CDFs!

Binomial Distribution

Let $X \sim \mathrm{B}(n, p)$. Then:

$$\begin{aligned}
\varphi_X(t) = \mathrm{E}\left[\exp(\mathrm{i}tX)\right] = \sum_{k=0}^{n} \exp(\mathrm{i}tk) \cdot \mathrm{P}(X = k) \\
&= \sum_{k=0}^{n} \exp(\mathrm{i}tk) \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot \{\exp(\mathrm{i}t) \cdot p\}^k \cdot (1-p)^{n-k} \\
&= \{\exp(\mathrm{i}t) \cdot p + (1-p)\}^n
\end{aligned}$$

because $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k \cdot b^{n-k}$ for $n \in \mathbb{N}$.

Poisson Distribution

Let $X \sim \mathrm{Pois}(\lambda)$. Then:

$$
\begin{aligned}
\varphi_X(t) = \mathrm{E}\left[\exp(\mathrm{i}tX)\right] &= \sum_{k=0}^{\infty} \exp(\mathrm{i}tk) \cdot \mathrm{P}(X = k) \\
&= \sum_{k=0}^{\infty} \exp(\mathrm{i}tk) \cdot \exp(-\lambda) \cdot \frac{\lambda^k}{k!} \\
&= \exp(-\lambda) \cdot \sum_{k=0}^{\infty} \frac{\{\lambda \cdot \exp(\mathrm{i}t)\}^k}{k!} \\
&= \exp(-\lambda) \cdot \exp(\lambda \cdot \exp(\mathrm{i}t)) = \exp(\lambda \cdot (\exp(\mathrm{i}t) - 1))
\end{aligned}
$$

because $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$.

Exponential Distribution

Let $X \sim \mathrm{Exp}\,(\lambda)$. Then:

$$
\begin{aligned}
\varphi_X(t) &= \mathrm{E}\left[\exp\left(\mathbf{i}tX\right)\right] \\
&= \int_0^\infty \exp\left(\mathbf{i}tx\right) \cdot f(x)\; dx \\
&= \int_0^\infty \exp\left(\mathbf{i}tx\right) \cdot \lambda \cdot \exp\left(-\lambda x\right)\; dx \\
&= \lambda \cdot \int_0^\infty \exp\left(\mathbf{i}tx - \lambda x\right)\; dx \\
&= \lambda \cdot \int_0^\infty \exp\left(\left(\mathbf{i}t - \lambda\right) \cdot x\right)\; dx
\end{aligned}
$$

Exponential Distribution

$$\begin{aligned}
\varphi_X(t) &= \lambda \cdot \int_0^{\infty} \exp\left(-\left(\lambda - \mathbf{i}t\right) \cdot x\right)\ dx \\
&= \lambda \cdot \left\{ -\frac{1}{\lambda - \mathbf{i}t} \cdot \exp\left(-\left(\lambda - \mathbf{i}t\right) \cdot x\right) \right\} \Bigg|_0^{\infty} \\
&= -\frac{\lambda}{\lambda - \mathbf{i}t} \cdot \left\{ \exp\left(-\left(\lambda - \mathbf{i}t\right) \cdot \infty\right) - \exp\left(-\left(\lambda - \mathbf{i}t\right) \cdot 0\right) \right\} \\
&= -\frac{\lambda}{\lambda - \mathbf{i}t} \cdot \left\{ \lim_{a \to \infty} \exp\left(-\left(\lambda - \mathbf{i}t\right) \cdot a\right) - 1 \right\} \\
&\overset{(1)}{=} -\frac{\lambda}{\lambda - \mathbf{i}t} \cdot (0 - 1) = \frac{\lambda}{\lambda - \mathbf{i}t}
\end{aligned}$$

Exponential Distribution

To show (1), we prove that

$$\lim_{a\to\infty} \exp\left(-\left(\lambda - \mathbf{i}t\right)\cdot a\right) = 0 \qquad (*)$$

It holds:

$$\begin{aligned}\lim_{a\to\infty} \exp\left(-\left(\lambda - \mathbf{i}t\right)\cdot a\right) &= \lim_{a\to\infty} \exp\left(-\lambda a + \mathbf{i}ta\right)\\ &= \lim_{a\to\infty} \exp\left(-\lambda a\right)\cdot \exp\left(\mathbf{i}ta\right)\\ &= 0\end{aligned}$$

because $\lim_{a\to\infty} \exp\left(-\lambda a\right) = 0$ and $\left|\exp\left(\mathbf{i}ta\right)\right| = 1$. Hence, $(*)$ was proved and we get $\varphi_X\left(t\right) = \frac{\lambda}{\lambda - \mathbf{i}t}$.

Standard Normal Distribution

Let $X \sim \mathrm{N}(0,1)$. Then:

$$
\begin{aligned}
\varphi_X(t) = \mathrm{E}\left[\exp(\mathbf{i}tX)\right] &= \int_{-\infty}^{\infty} \exp(\mathbf{i}tx) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_{-\infty}^{0} \exp(\mathbf{i}tx) \cdot \exp\left(-\frac{x^2}{2}\right)\ dx \right. \\
&\qquad \left. + \int_{0}^{\infty} \exp(\mathbf{i}tx) \cdot \exp\left(-\frac{x^2}{2}\right)\ dx \right\} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_{0}^{\infty} \exp(-\mathbf{i}tx) \cdot \exp\left(-\frac{x^2}{2}\right)\ dx \right. \\
&\qquad \left. + \int_{0}^{\infty} \exp(\mathbf{i}tx) \cdot \exp\left(-\frac{x^2}{2}\right)\ dx \right\}
\end{aligned}
$$

Standard Normal Distribution

$$
\begin{aligned}
\varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \cdot \left\{ \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \cdot \exp(\mathbf{i}tx) \right. \\
&\qquad \left. + \ \exp\left(-\frac{x^2}{2}\right) \cdot \exp(-\mathbf{i}tx) \ dx \right\} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \cdot \{\exp(\mathbf{i}tx) + \exp(-\mathbf{i}tx)\} \ dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \cdot 2 \cdot \cos(tx) \ dx \\
&= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \cdot \cos(tx) \ dx \overset{(1)}{=} \exp\left(-\frac{t^2}{2}\right)
\end{aligned}
$$

Standard Normal Distribution

To prove (1) let

$$F(t) \stackrel{\text{def}}{=} \varphi_X(t) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \cdot \cos(tx)\ dx$$

We show that

$$F'(t) = -t \cdot F(t) \tag{$*$}$$

This ordinary differential equation has the solution $F(t) = c \cdot \exp\left(-\frac{t^2}{2}\right)$, where c is a constant. Since we have

$$F(0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)\ dx = 1$$

we get $c = 1$. Hence, $\varphi_X(t) = F(t) = \exp\left(-\frac{t^2}{2}\right)$.

Standard Normal Distribution

Proof of $(*)$:

$$F'(t) = \frac{d}{dt}\left\{\frac{2}{\sqrt{2\pi}}\cdot\int_0^\infty \exp\left(-\frac{x^2}{2}\right)\cdot\cos(tx)\ dx\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\cdot\int_0^\infty \exp\left(-\frac{x^2}{2}\right)\cdot\frac{d}{dt}\cos(tx)\ dx$$

$$= \frac{2}{\sqrt{2\pi}}\cdot\int_0^\infty -\exp\left(-\frac{x^2}{2}\right)\cdot x\cdot\sin(tx)\ dx$$

Integration by parts:

$u(x) = \exp\left(-\frac{x^2}{2}\right)$ $\qquad u'(x) = -x\cdot\exp\left(-\frac{x^2}{2}\right)$

$v(x) = \sin(tx)$ $\qquad v'(x) = t\cdot\cos(tx)$

Standard Normal Distribution

$$
\begin{aligned}
F'(t) &= \frac{2}{\sqrt{2\pi}} \cdot \left[\left\{ \exp\left(-\frac{x^2}{2}\right) \cdot \sin(tx) \right\} \Big|_0^\infty \right. \\
&\qquad \left. - \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot t \cdot \cos(tx) \; dx \right] \\
&= -\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left(-\frac{x^2}{2}\right) \cdot t \cdot \cos(tx) \; dx \\
&= -t \cdot \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \cdot 2 \cdot \cos(tx) \; dx \\
&= -t \cdot F(t)
\end{aligned}
$$

Hence, $(*)$ was proved and we get $\varphi_X(t) = \exp\left(-\frac{t^2}{2}\right)$.