

Selected Topics of Mathematical Statistics: Quiz 4

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Quiz 4

**Show several characteristic functions under
different CDFs!**



Binomial Distribution

Let $X \sim B(n, p)$. Then:

$$\begin{aligned}\varphi_X(t) &= E(\exp\{itX\}) = \sum_{k=0}^n \exp\{itk\} \cdot P(X = k) \\ &= \sum_{k=0}^n \exp\{itk\} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} \cdot (\exp\{it\} \cdot p)^k \cdot (1-p)^{n-k} \\ &= [\exp\{it\} \cdot p + \{1-p\}]^n\end{aligned}$$

because $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$ for $n \in \mathbb{N}$.



Poisson Distribution

Let $X \sim \text{Pois}(\lambda)$. Then:

$$\begin{aligned}\varphi_X(t) &= E(\exp\{itX\}) = \sum_{k=0}^{\infty} \exp\{itk\} \cdot P(X = k) \\&= \sum_{k=0}^{\infty} \exp\{itk\} \cdot \exp\{-\lambda\} \cdot \frac{\lambda^k}{k!} \\&= \exp\{-\lambda\} \cdot \sum_{k=0}^{\infty} \frac{(\lambda \cdot \exp\{it\})^k}{k!} \\&= \exp\{-\lambda\} \cdot \exp\{\lambda \cdot \exp\{it\}\} = \exp\{\lambda \cdot (\exp\{it\} - 1)\}\end{aligned}$$

because $\exp\{x\} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$.



Standard Normal Distribution

Let $X \sim N(0, 1)$. Then:

$$\begin{aligned}\varphi_X(t) &= E(\exp\{itX\}) = \int_{-\infty}^{\infty} \exp\{itx\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 \exp\{itx\} \cdot \exp\left\{-\frac{x^2}{2}\right\} dx \right. \\&\quad \left. + \int_0^{\infty} \exp\{itx\} \cdot \exp\left\{-\frac{x^2}{2}\right\} dx \right] \\&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} \exp\{-itx\} \cdot \exp\left\{-\frac{x^2}{2}\right\} dx \right. \\&\quad \left. + \int_0^{\infty} \exp\{itx\} \cdot \exp\left\{-\frac{x^2}{2}\right\} dx \right]\end{aligned}$$



Standard Normal Distribution

$$\begin{aligned}\varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \cdot \left[\int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \exp\{itx\} \right. \\ &\quad \left. + \exp\left\{-\frac{x^2}{2}\right\} \cdot \exp\{-itx\} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot [\exp\{itx\} + \exp\{-itx\}] dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot 2 \cdot \cos(tx) dx \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \cos(tx) dx \stackrel{(1)}{=} \exp\left\{-\frac{t^2}{2}\right\}\end{aligned}$$



Standard Normal Distribution

To prove (1) let

$$F(t) \stackrel{\text{def}}{=} \varphi_X(t) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left\{-\frac{x^2}{2}\right\} \cdot \cos(tx) \, dx$$

We show that

$$F'(t) = -t \cdot F(t) \quad (*)$$

This ordinary differential equation has the solution

$F(t) = c \cdot \exp\left\{-\frac{t^2}{2}\right\}$, where c is a constant. Since we have

$$F(0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{x^2}{2}\right\} \, dx = 1$$

we get $c = 1$. Hence, $\varphi_X(t) = F(t) = \exp\left\{-\frac{t^2}{2}\right\}$.



Standard Normal Distribution

$$\begin{aligned} F'(t) &= \frac{d}{dt} \left[\frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp \left\{ -\frac{x^2}{2} \right\} \cdot \cos(tx) \, dx \right] \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp \left\{ -\frac{x^2}{2} \right\} \cdot \frac{d}{dt} [\cos(tx)] \, dx \\ &= \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} -\exp \left\{ -\frac{x^2}{2} \right\} \cdot x \cdot \sin(tx) \, dx \end{aligned}$$

Integration by parts:

$$\begin{aligned} u(x) &= \exp \left\{ -\frac{x^2}{2} \right\} & u'(x) &= -x \cdot \exp \left\{ -\frac{x^2}{2} \right\} \\ v(x) &= \sin(tx) & v'(x) &= t \cdot \cos(tx) \end{aligned}$$



Standard Normal Distribution

$$\begin{aligned} F'(t) &= \frac{2}{\sqrt{2\pi}} \cdot \left[\left(\exp \left\{ -\frac{x^2}{2} \right\} \cdot \sin(tx) \right) \Big|_0^\infty \right. \\ &\quad \left. - \int_0^\infty \exp \left\{ -\frac{x^2}{2} \right\} \cdot t \cdot \cos(tx) \, dx \right] \\ &= -\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp \left\{ -\frac{x^2}{2} \right\} \cdot t \cdot \cos(tx) \, dx \\ &= -t \cdot \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} \cdot 2 \cdot \cos(tx) \, dx \\ &= -t \cdot F(t) \end{aligned}$$

Hence, (*) was proved and we get $\varphi_X(t) = \exp \left\{ -\frac{t^2}{2} \right\}$.

