## Quiz 12: Proof Theorem 28 and 29.

### Theorem (Theorem 29)

*Let $X, X_1, X_2, ..$ be i.i.d. and $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.*

*Then if $E|X| < \infty$ it holds $\bar{X}_n \stackrel{2nd}{\rightarrow} \mu = E[X]$.*

Proof of Theorem 29 from Björn Bokelmann

For each $i \in \mathbb{N}$ we define $Y_i = X_i - \mu$.
Because we applied a continuous mapping on $X_i$ to get $Y_i$, the random variables $Y_1, Y_2, \ldots$ are i.i.d..
It further holds $E[Y_i] = 0$ for each $i \in \mathbb{N}$ and therefore

$$\sigma^2 = Var[Y_i] = E[Y_i^2] \tag{1}$$

and

$$E[\sum_{i=1}^{n} Y_i] = 0 \tag{2}$$

With Bienaymes Formula we get

$$
\begin{aligned}
E[(\sum_{i=1}^{n} Y_i)^2] &= E[(\sum_{i=1}^{n} Y_i - E[\sum_{i=1}^{n} Y_i])^2] \\
&= Var[\sum_{i=1}^{n} Y_i] \\
&= \sum_{i=1}^{n} Var[Y_i] \\
&= n \cdot \sigma^2
\end{aligned}
$$

Hence it holds

$$
\begin{aligned}
E[|\bar{X}_n - \mu|^2] &= E[(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu))^2] \\
&= \frac{1}{n^2} E[(\sum_{i=1}^{n} Y_i)^2] \\
&= \frac{1}{n^2} \cdot n \cdot \sigma^2 \\
&= \frac{1}{n}\sigma^2 \overset{n\to\infty}{\to} 0
\end{aligned}
$$

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