HUMBOLDT UNIVERSITY BERLIN

Quiz 3

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1 Problem Description

Quiz 3: Show that the Kullback-Leibler-Divergence $K(\theta,\theta')$ satisfies for any θ, θ':

$$\begin{aligned}
&1.\ K(\theta,\theta')|_{\theta,\theta'} = 0 \\
&2.\ \frac{d}{d\theta'} K(\theta,\theta')|_{\theta,\theta'} = 0 \\
&3.\ \frac{d^2}{d\theta'^2} K(\theta,\theta')|_{\theta,\theta'} = \int_{-\infty}^{\infty} \frac{p'(x,\theta)^2}{p(x,\theta)}\, dx
\end{aligned} \tag{1.1}$$

2 Solution

2.1 Proof of statement 1

Notice that the quantity in the logarithm is always one, therefore the logarithm evaluates to zero:

$$K(\theta,\theta')|_{\theta,\theta'} = \int_{-\infty}^{\infty} \log 1\ p(x,\theta)dx = \int_{-\infty}^{\infty} 0 * p(x,\theta)dx = 0 \tag{2.1}$$

2.2 Proof of statement 2

For this proof we'll assume that $p(x,\theta) = p(x,\theta')$ and its derivative with respect to θ is continuous over $\mathbb{R}$, in which case we can evaluate the derivative within the integral according to

Leibniz' rule:

$$
\begin{aligned}
\frac{d}{d\theta'}\ K(\theta,\theta')|_{\theta,\theta'} &= \frac{d}{d\theta'}\ \int_{-\infty}^{\infty} \log\frac{p(x,\theta)}{p(x,\theta')}\ p(x,\theta)dx \\
&= \int_{-\infty}^{\infty} p(x,\theta)\ \frac{d}{d\theta'}\left(log\frac{p(x,\theta)}{p(x,\theta')}\right)dx \\
&= \int_{-\infty}^{\infty} p(x,\theta)\ \frac{d}{d\theta'} - log(p(x,\theta'))\ dx \\
&= \int_{-\infty}^{\infty} -p(x,\theta)\ \frac{1}{p(x,\theta'}\ \frac{d}{d\theta'}p(x,\theta')\ dx \\
&= \int_{-\infty}^{\infty} -\frac{d}{d\theta'}p(x,\theta')\ dx \\
&= \frac{d}{d\theta'}\int_{-\infty}^{\infty} -p(x,\theta')\ dx \\
&= \frac{d}{d\theta'}(-1) \\
&= 0
\end{aligned}
\tag{2.2}
$$

This completes the proof.

2.3 PROOF OF STATEMENT 2

Similar to part 2.2, we assume now that also the second derivative of $p(x,\theta)$ with respect to θ is continuous, again evaluating the derivative in the integral:

$$
\begin{aligned}
\frac{d^2}{d\theta'^2} K(\theta,\theta')|_{\theta,\theta'} &= \frac{d^2}{d\theta'^2} \int_{-\infty}^{\infty} \log \frac{p(x,\theta)}{p(x,\theta'} \; p(x,\theta) dx \\
&= \int_{-\infty}^{\infty} p(x,\theta) \; \frac{d^2}{d\theta'^2} \left(log \frac{p(x,\theta)}{p(x,\theta')} \right) dx \\
&= \int_{-\infty}^{\infty} -p(x,\theta) \; \frac{d^2}{d\theta'^2} \; log p(x,\theta') \; dx \\
&= \int_{-\infty}^{\infty} -p(x,\theta) \; \frac{d}{d\theta'} \left[\frac{1}{p(x,\theta')} \; \frac{d}{d\theta'} p(x,\theta') \right] dx \\
&= \int_{-\infty}^{\infty} -p(x,\theta) \left[-\frac{1}{p(x,\theta')^2} \left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2 + \frac{1}{p(x,\theta')} \; \frac{d^2}{d\theta'^2} \; p(x,\theta') \right] dx \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')} + \frac{d^2}{d\theta'^2} \; p(x,\theta') \; dx \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')} \; dx + \frac{d^2}{d\theta'^2} \; \int_{-\infty}^{\infty} p(x,\theta') \; dx \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')} \; dx + 0 \\
&= \int_{-\infty}^{\infty} \frac{\left\{ \frac{d}{d\theta'} p(x,\theta') \right\}^2}{p(x,\theta')} \; dx \\
&= \int_{-\infty}^{\infty} \frac{p'(x,\theta)^2}{p(x,\theta)} \; dx
\end{aligned}
\tag{2.3}
$$

And the proof is complete.