Selected Topics of Mathematical Statistics: Quiz 4

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Quiz 4

Show several characteristic functions under different CDFs!

Binomial Distribution

Let $X \sim B(n,p)$. Then:

$$\begin{aligned}
\varphi_X(t) = \mathrm{E}\left(\exp\{\mathrm{i}tX\}\right) &= \sum_{k=0}^{n} \exp\{\mathrm{i}tk\} \cdot \mathrm{P}(X = k) \\
&= \sum_{k=0}^{n} \exp\{\mathrm{i}tk\} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot (\exp\{\mathrm{i}t\} \cdot p)^k \cdot (1-p)^{n-k} \\
&= [\exp\{\mathrm{i}t\} \cdot p + \{1-p\}]^n
\end{aligned}$$

because $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ for $n \in \mathbb{N}$.

Poisson Distribution

Let $X \sim \text{Pois}(\lambda)$. Then:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}\left(\exp\{\mathrm{i}tX\}\right) = \sum_{k=0}^{\infty} \exp\{\mathrm{i}tk\} \cdot \mathrm{P}(X = k) \\
&= \sum_{k=0}^{\infty} \exp\{\mathrm{i}tk\} \cdot \exp\{-\lambda\} \cdot \frac{\lambda^k}{k!} \\
&= \exp\{-\lambda\} \cdot \sum_{k=0}^{\infty} \frac{(\lambda \cdot \exp\{\mathrm{i}t\})^k}{k!} \\
&= \exp\{-\lambda\} \cdot \exp\{\lambda \cdot \exp\{\mathrm{i}t\}\} = \exp\{\lambda \cdot (\exp\{\mathrm{i}t\} - 1)\}
\end{aligned}$$

because $\exp\{x\} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$.

Standard Normal Distribution

Let $X \sim N(0,1)$. Then:

$$
\begin{aligned}
\varphi_X(t) &= \mathrm{E}\left(\exp\{\mathrm{i}tX\}\right) = \int_{-\infty}^{\infty} \exp\{\mathrm{i}tx\} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} \exp\{\mathrm{i}tx\} \cdot \exp\left\{-\frac{x^2}{2}\right\}\, dx \right. \\
&\qquad \left. + \int_{0}^{\infty} \exp\{\mathrm{i}tx\} \cdot \exp\left\{-\frac{x^2}{2}\right\}\, dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \exp\{-\mathrm{i}tx\} \cdot \exp\left\{-\frac{x^2}{2}\right\}\, dx \right. \\
&\qquad \left. + \int_{0}^{\infty} \exp\{\mathrm{i}tx\} \cdot \exp\left\{-\frac{x^2}{2}\right\}\, dx \right]
\end{aligned}
$$

Standard Normal Distribution

$$
\begin{aligned}
\varphi_X(t) &= \frac{1}{\sqrt{2\pi}} \cdot \left[\int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \exp\{\mathbf{i}tx\} \right. \\
&\qquad \left. + \ \exp\left\{-\frac{x^2}{2}\right\} \cdot \exp\{-\mathbf{i}tx\} \ dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \left[\exp\{\mathbf{i}tx\} + \exp\{-\mathbf{i}tx\}\right] \ dx \\
&= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot 2 \cdot \cos(tx) \ dx \\
&= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \cos(tx) \ dx \overset{(1)}{=} \exp\left\{-\frac{t^2}{2}\right\}
\end{aligned}
$$

Standard Normal Distribution

To prove (1) let

$$F(t) \stackrel{\text{def}}{=} \varphi_X(t) = \frac{2}{\sqrt{2\pi}} \cdot \int_0^{\infty} \exp\left\{-\frac{x^2}{2}\right\} \cdot \cos(tx)\ dx$$

We show that

$$F'(t) = -t \cdot F(t) \tag{$*$}$$

This ordinary differential equation has the solution $F(t) = c \cdot \exp\left\{-\frac{t^2}{2}\right\}$, where c is a constant. Since we have

$$F(0) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{x^2}{2}\right\}\ dx = 1$$

we get $c = 1$. Hence, $\varphi_X(t) = F(t) = \exp\left\{-\frac{t^2}{2}\right\}$.

Standard Normal Distribution

$$F'(t) = \frac{d}{dt}\left[\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \cos(tx)\ dx\right]$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{-\frac{x^2}{2}\right\} \cdot \frac{d}{dt}\ [\cos(tx)]\ dx$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty -\exp\left\{-\frac{x^2}{2}\right\} \cdot x \cdot \sin(tx)\ dx$$

Integration by parts:

$u(x) = \exp\left\{-\frac{x^2}{2}\right\}$ $\qquad$ $u'(x) = -x \cdot \exp\left\{-\frac{x^2}{2}\right\}$

$v(x) = \sin(tx)$ $\qquad$ $v'(x) = t \cdot \cos(tx)$

Standard Normal Distribution

$$
\begin{aligned}
F'(t) &= \frac{2}{\sqrt{2\pi}} \cdot \left[\left(\exp\left\{ -\frac{x^2}{2} \right\} \cdot \sin(tx) \right) \Big|_0^\infty \right. \\
&\qquad \left. - \int_0^\infty \exp\left\{ -\frac{x^2}{2} \right\} \cdot t \cdot \cos(tx) \ dx \right] \\
&= -\frac{2}{\sqrt{2\pi}} \cdot \int_0^\infty \exp\left\{ -\frac{x^2}{2} \right\} \cdot t \cdot \cos(tx) \ dx \\
&= -t \cdot \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2} \right\} \cdot 2 \cdot \cos(tx) \ dx \\
&= -t \cdot F(t)
\end{aligned}
$$

Hence, $(*)$ was proved and we get $\varphi_X(t) = \exp\left\{ -\frac{t^2}{2} \right\}$.