11 - Designing Interpolating Curves

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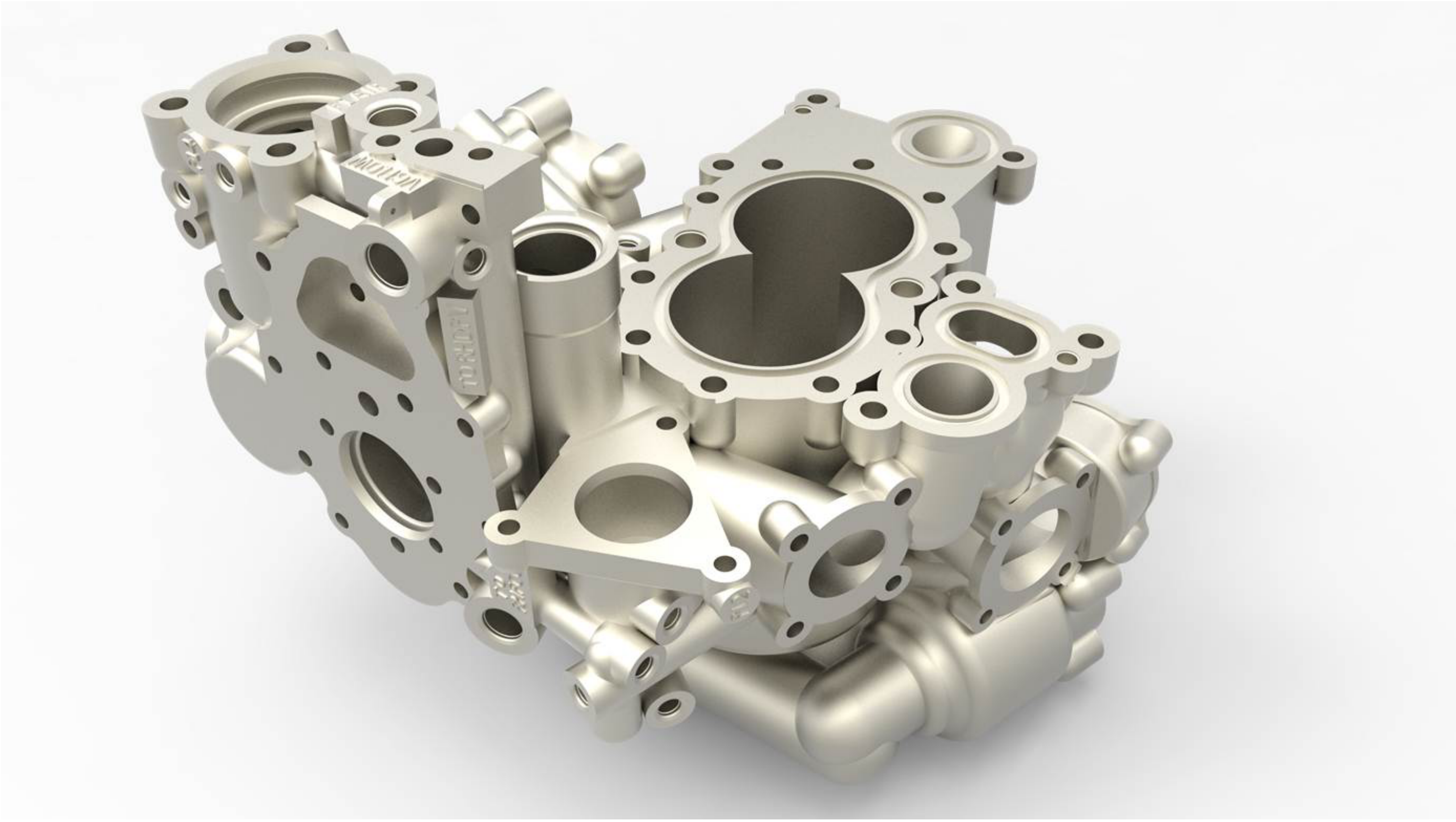
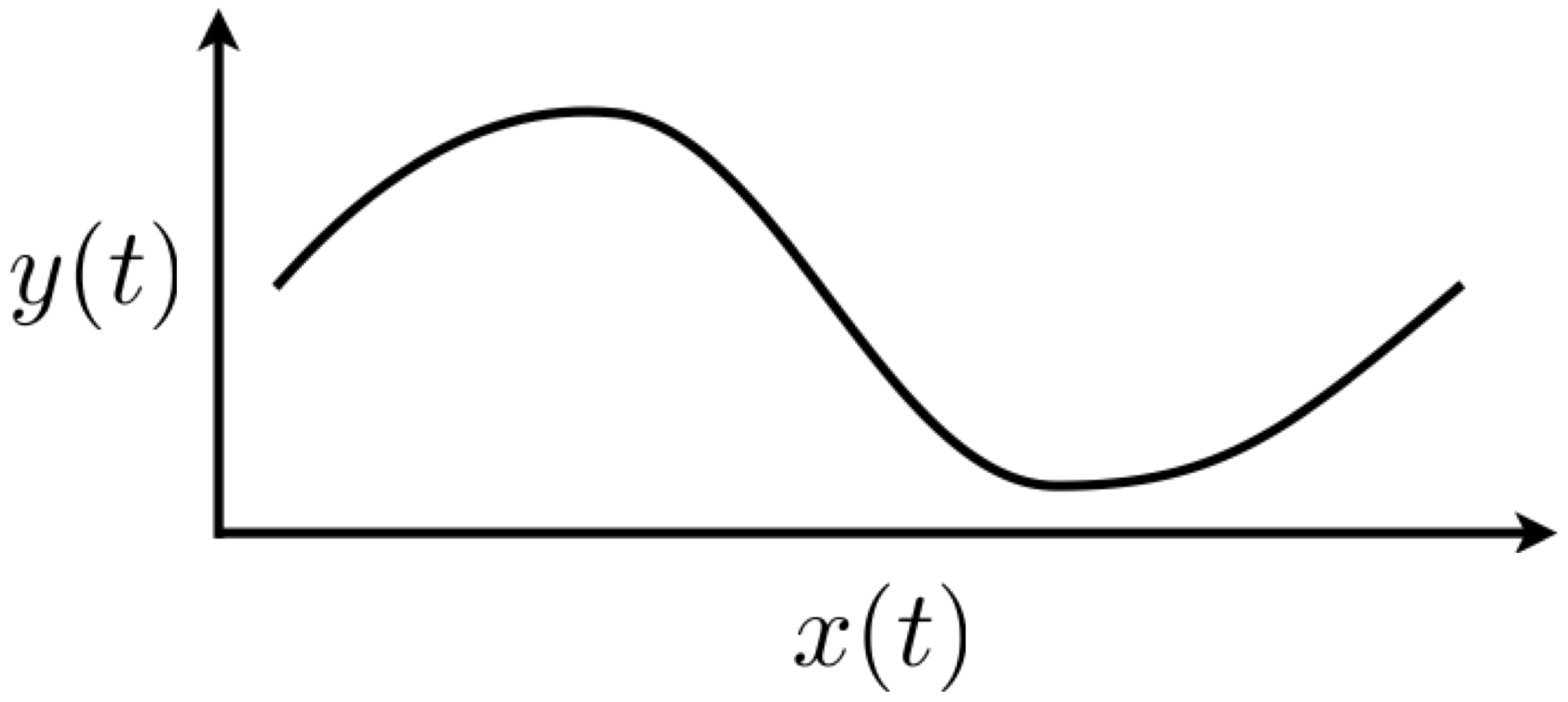
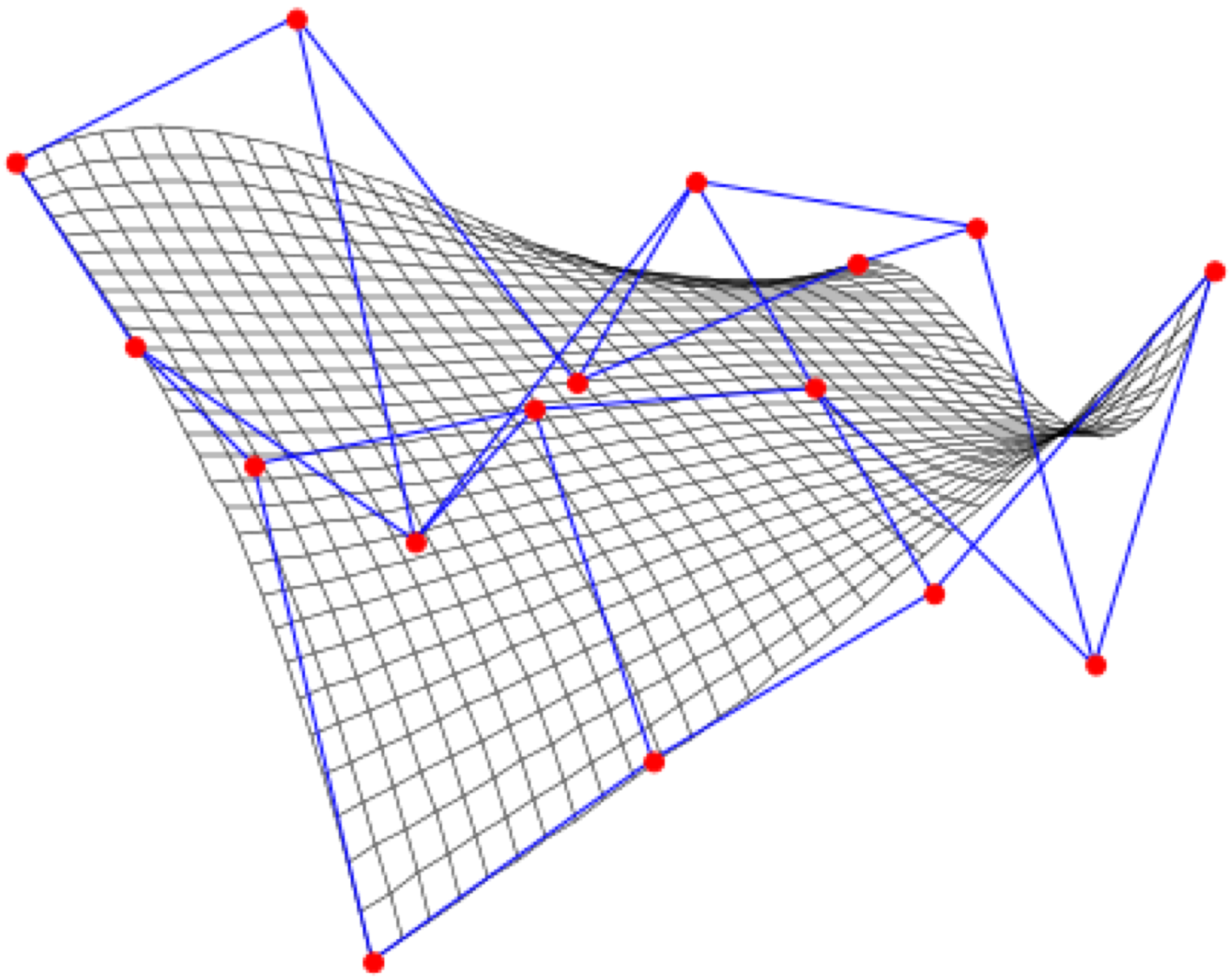
How do we model shapes?



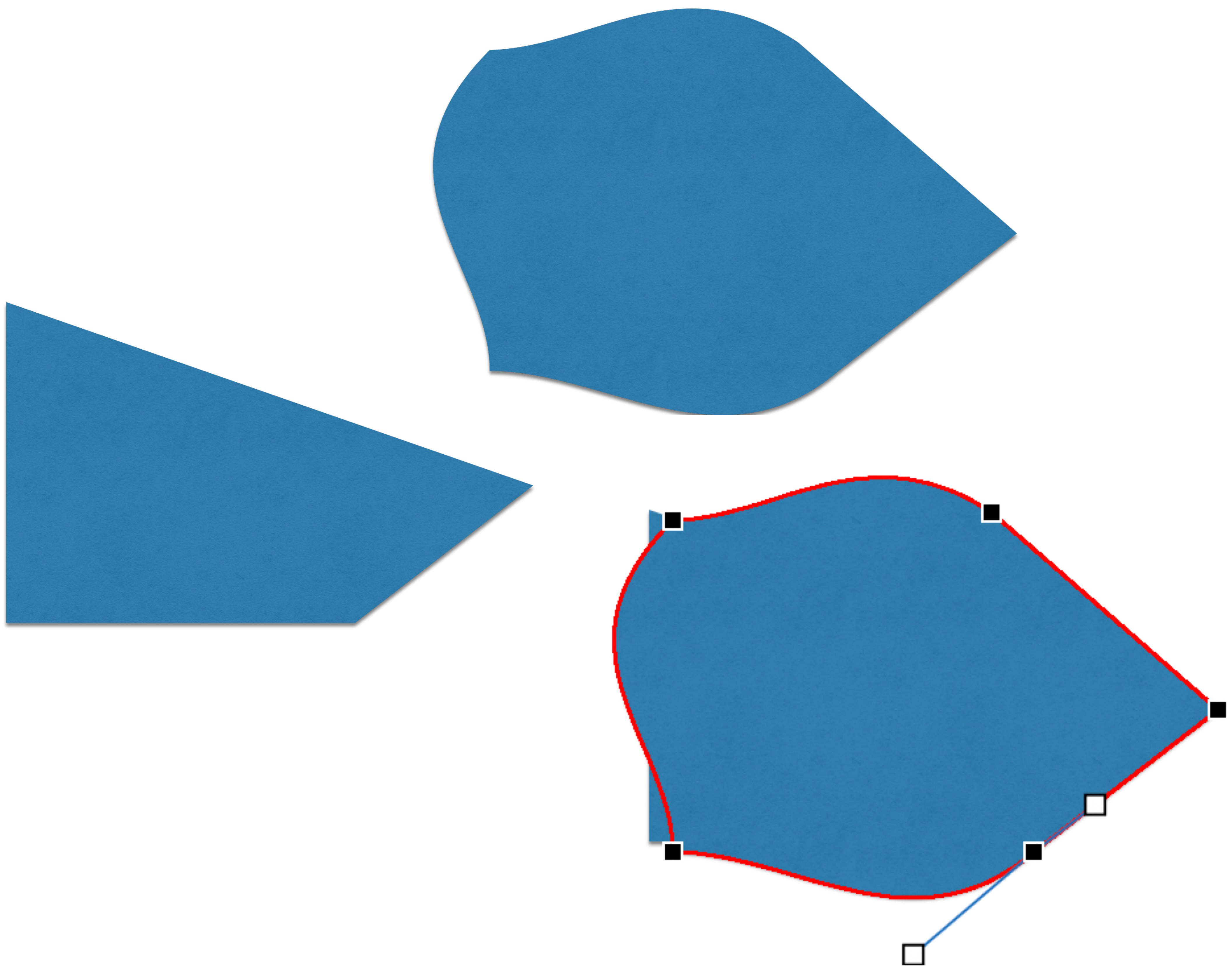
Building blocks: curves and surfaces





A modeling session

- Demo with PowerPoint



Modeling curves

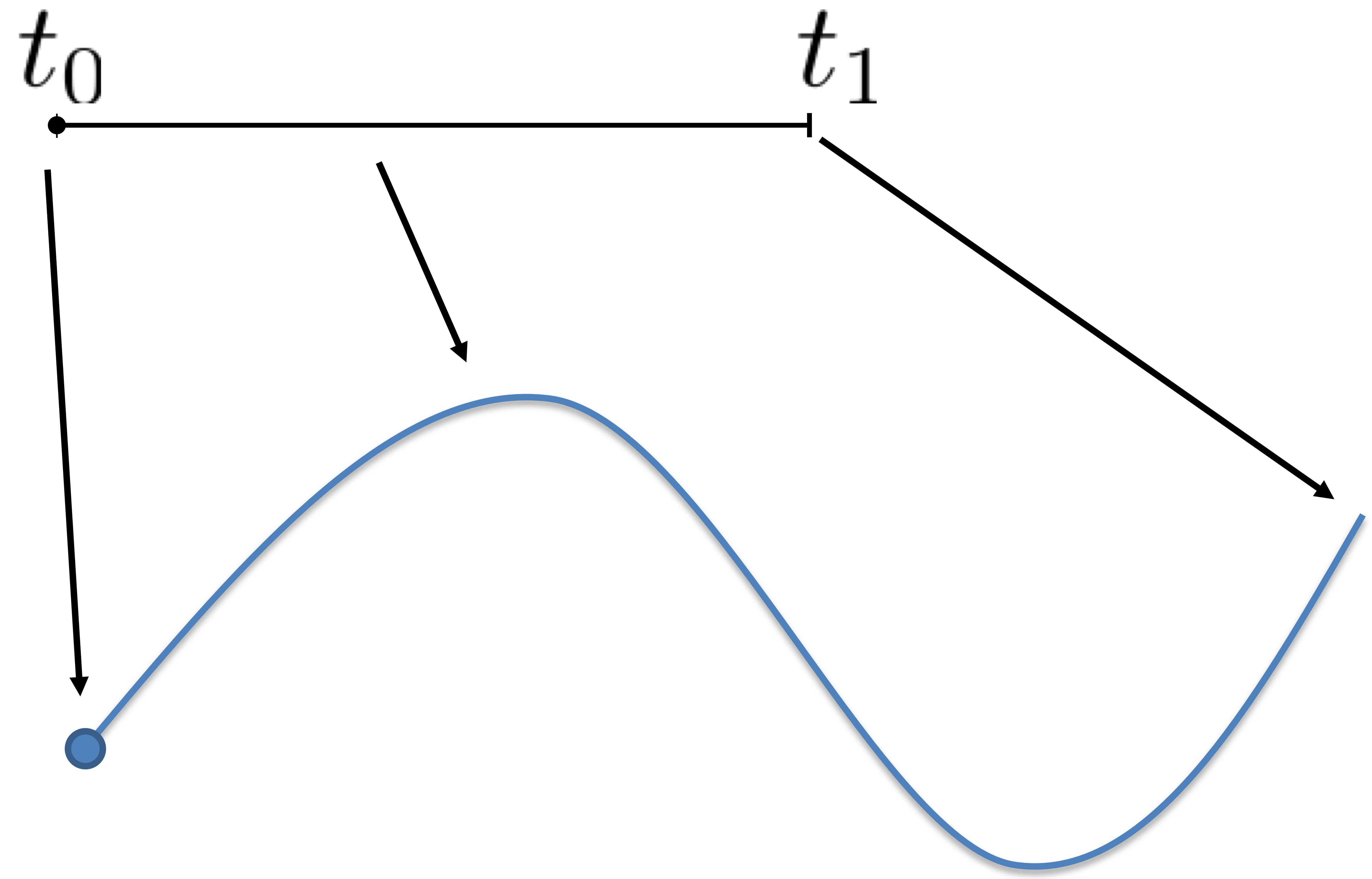
- We need **mathematical concepts** to characterize the desired curve properties

- Notions from **curve geometry** help with designing user interfaces for curve creation and editing

2D parametric curve

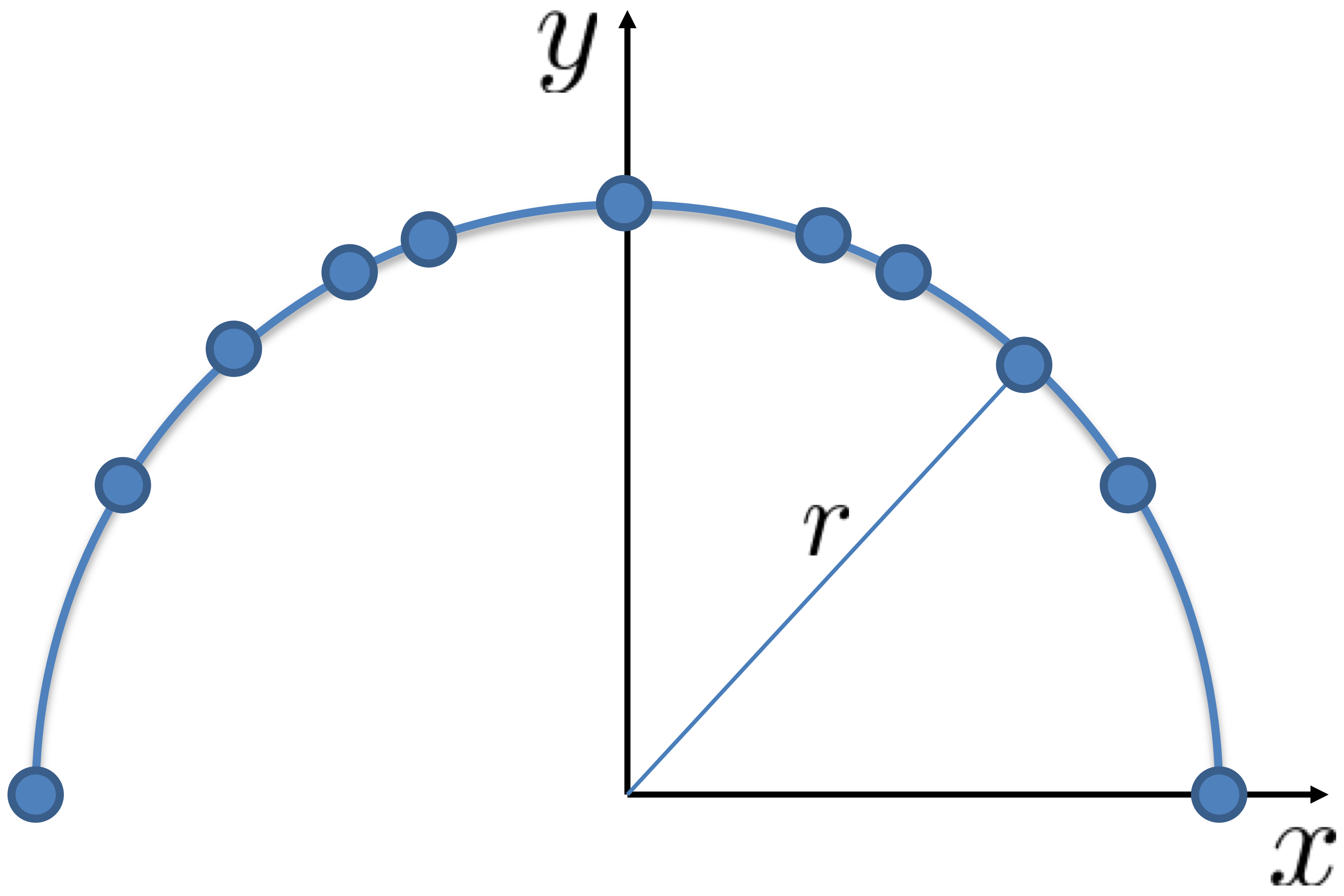
$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \; t \in [t_0, t_1]$$

- $\mathbf{p}(t)$ must be continuous

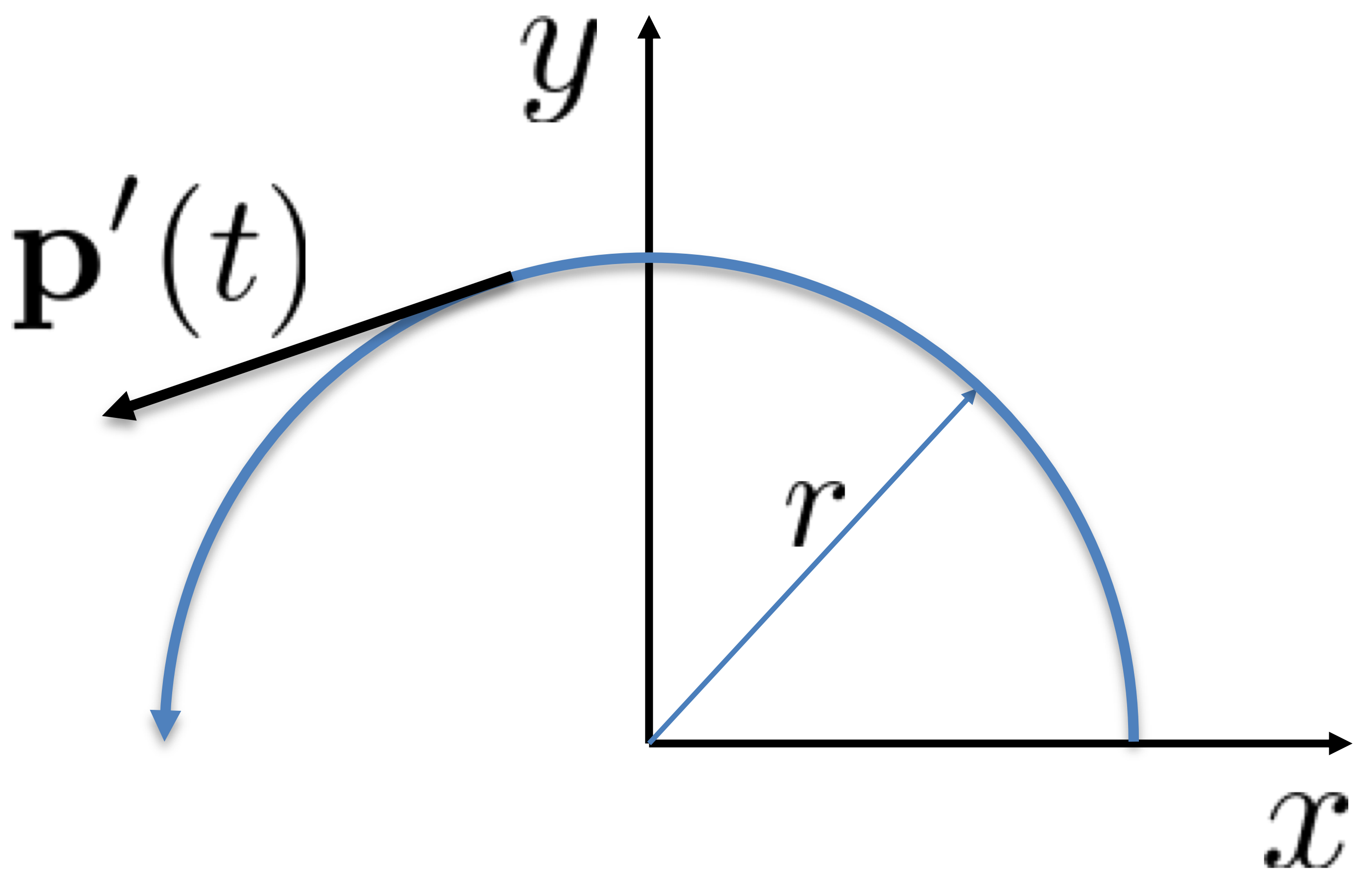


A curve can be parameterized in many different ways

$$\begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}, t \in [0, \pi] \qquad \begin{pmatrix} -rt \\ r\sqrt{1-t^2} \end{pmatrix}, t \in [-1, 1]$$



Tangent vector



$$\mathbf{p}(t) = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

$$\mathbf{p}'(t) = \begin{pmatrix} -r\sin t \\ r\cos t \end{pmatrix}$$

$$\|\mathbf{p}'(t)\| = \text{speed}$$

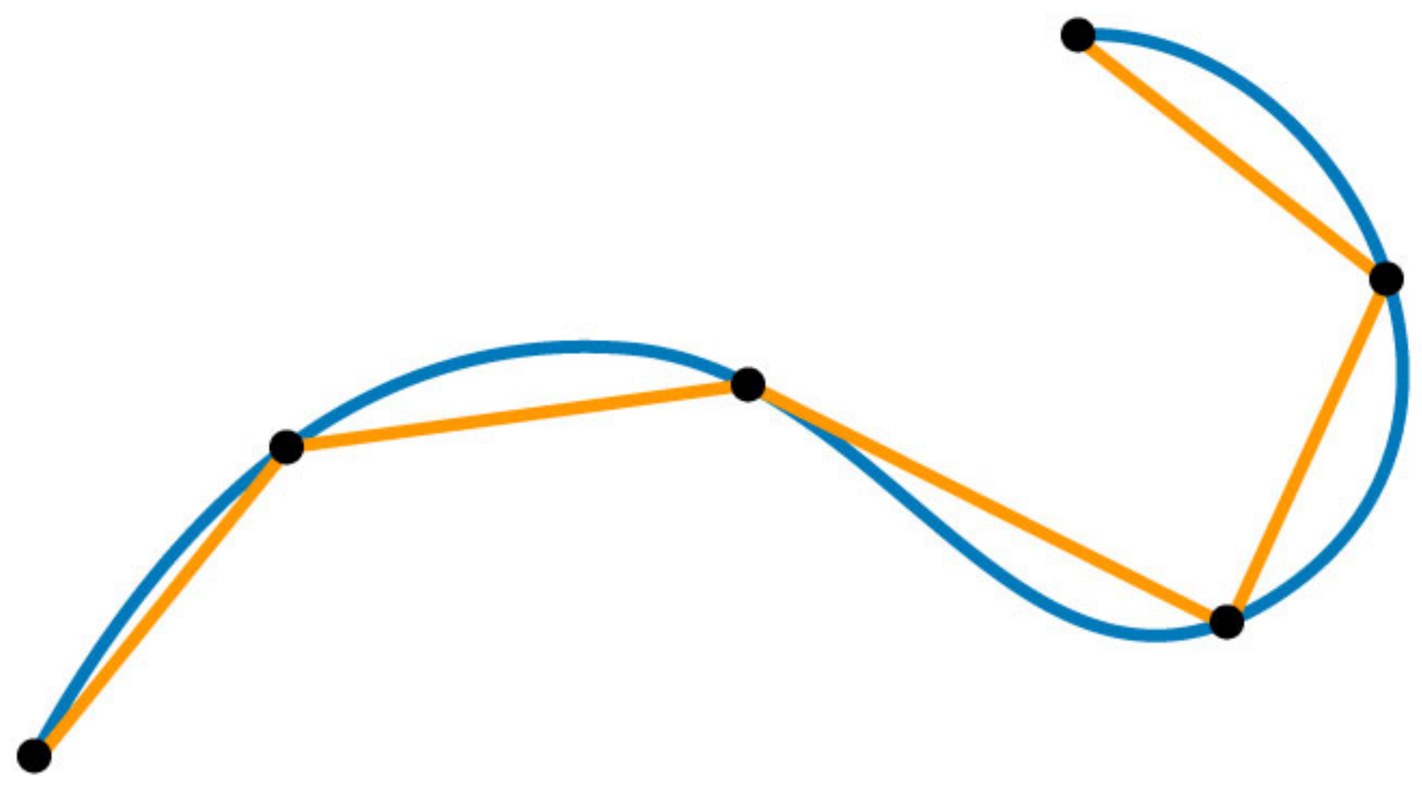
Parametrization-independent!

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t) = \text{unit tangent}$$

Arc length

- How long is the curve between t_0 and t? How far does the particle travel?
- Speed is $\|\mathbf{p}'(t)\|$, so:

$$s(t) = \int_{t_0}^{t} \|\mathbf{p}'(t)\| dt$$

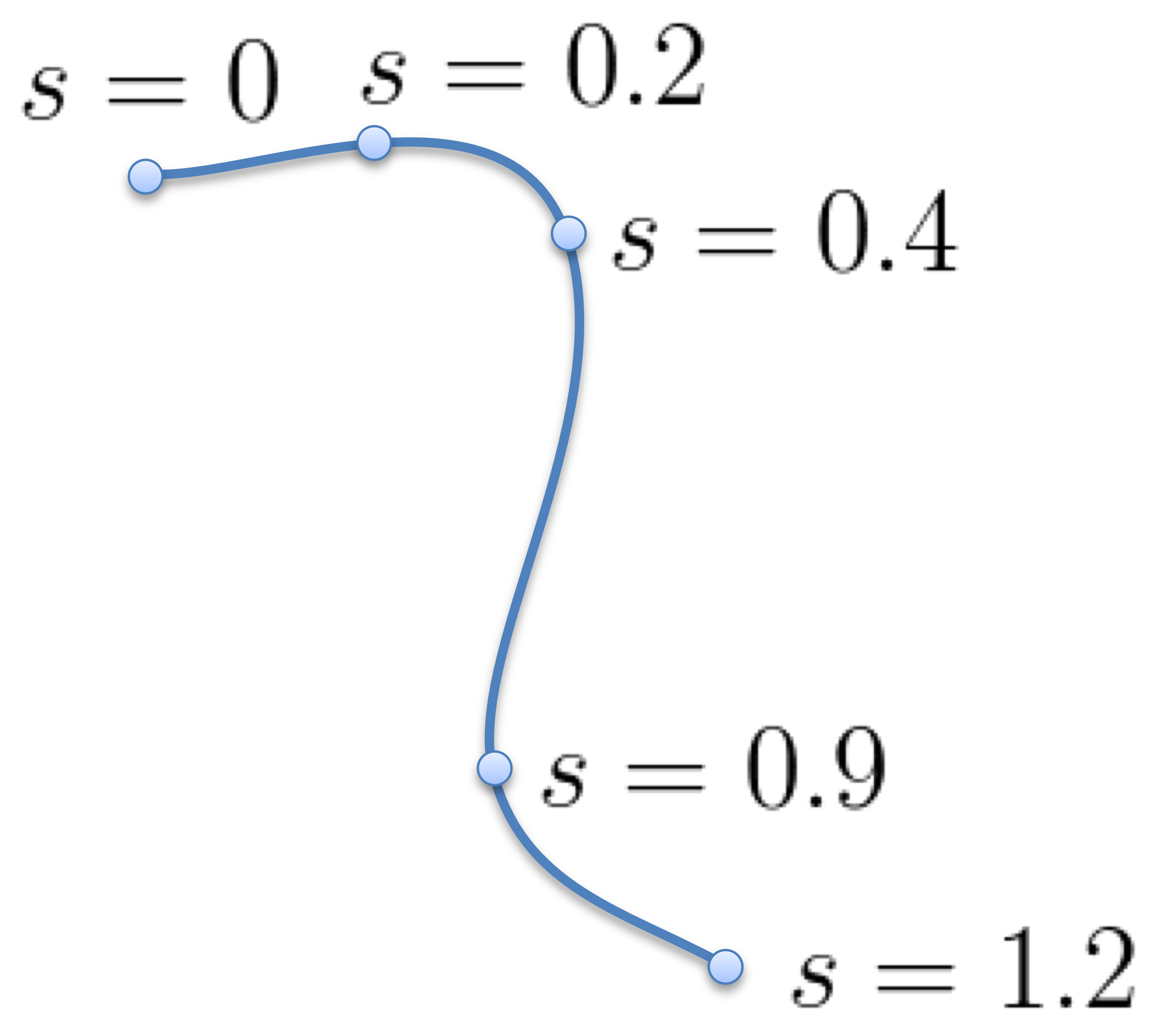


- Speed is nonnegative, so $s(t)$ is non-decreasing

Arc length parameterization

- Every curve has a natural parameterization:

$$\mathbf{p}(s), \text{ such that } \|\mathbf{p}'(s)\| = 1$$



Arc length parameterization

- Every curve has a natural parameterization:

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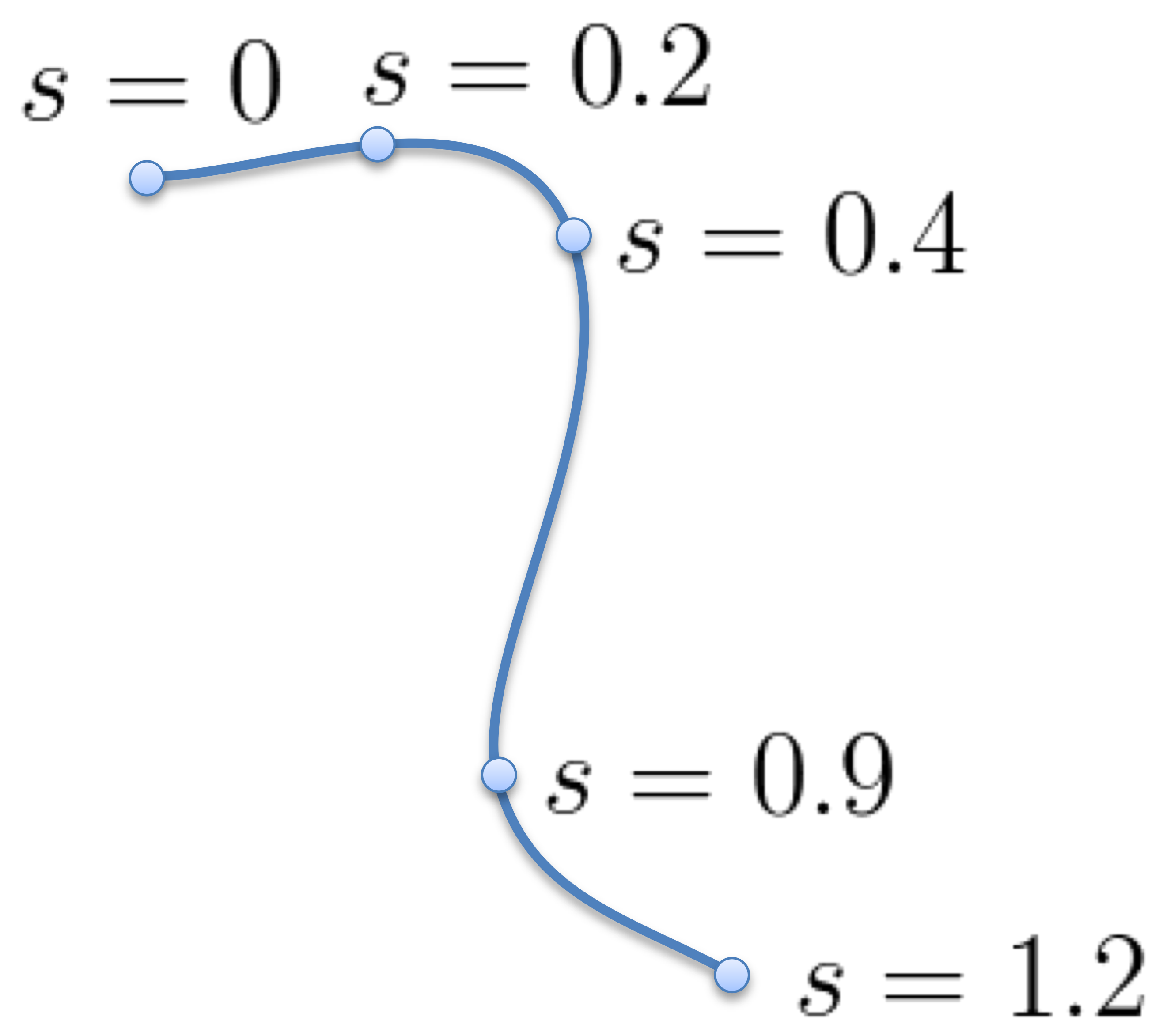
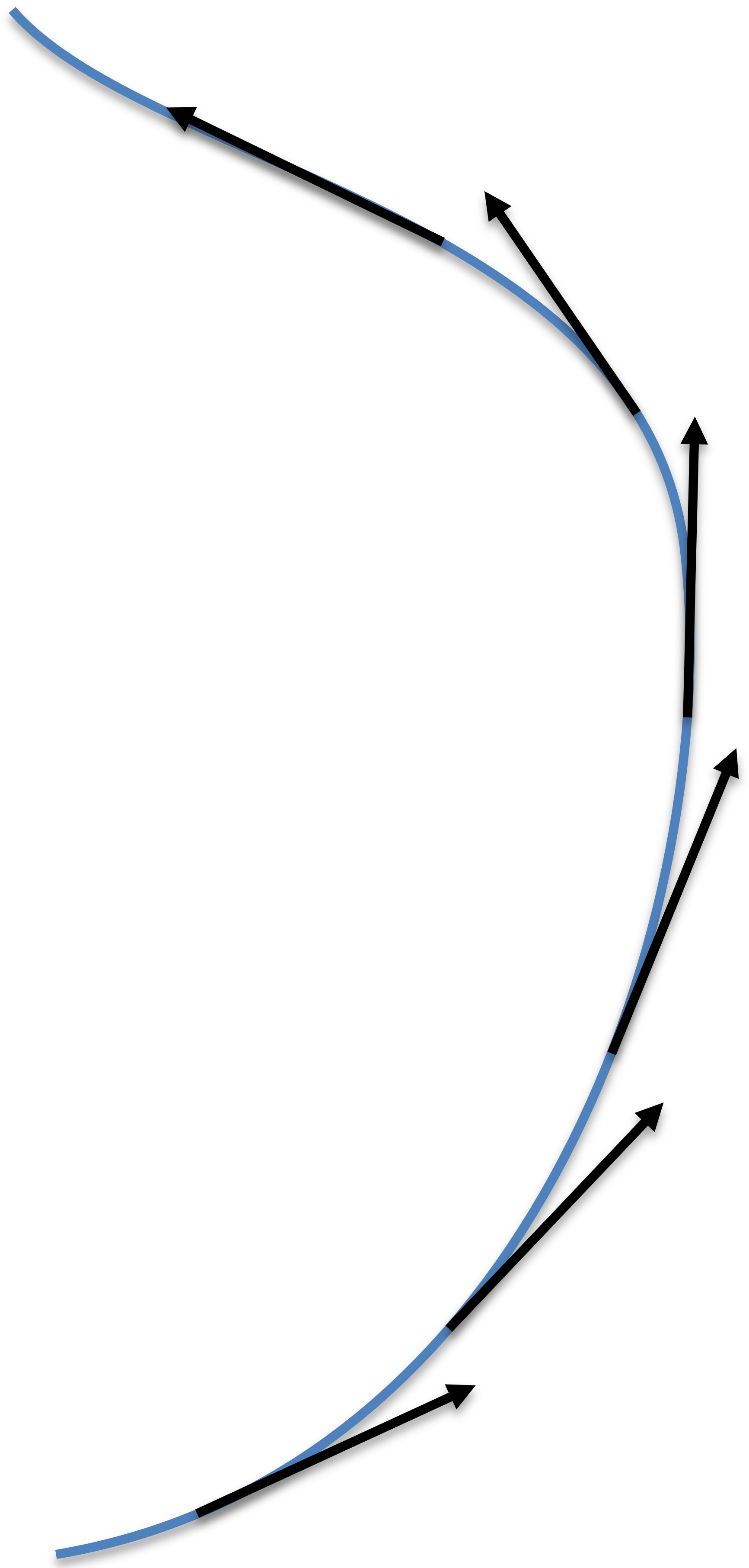


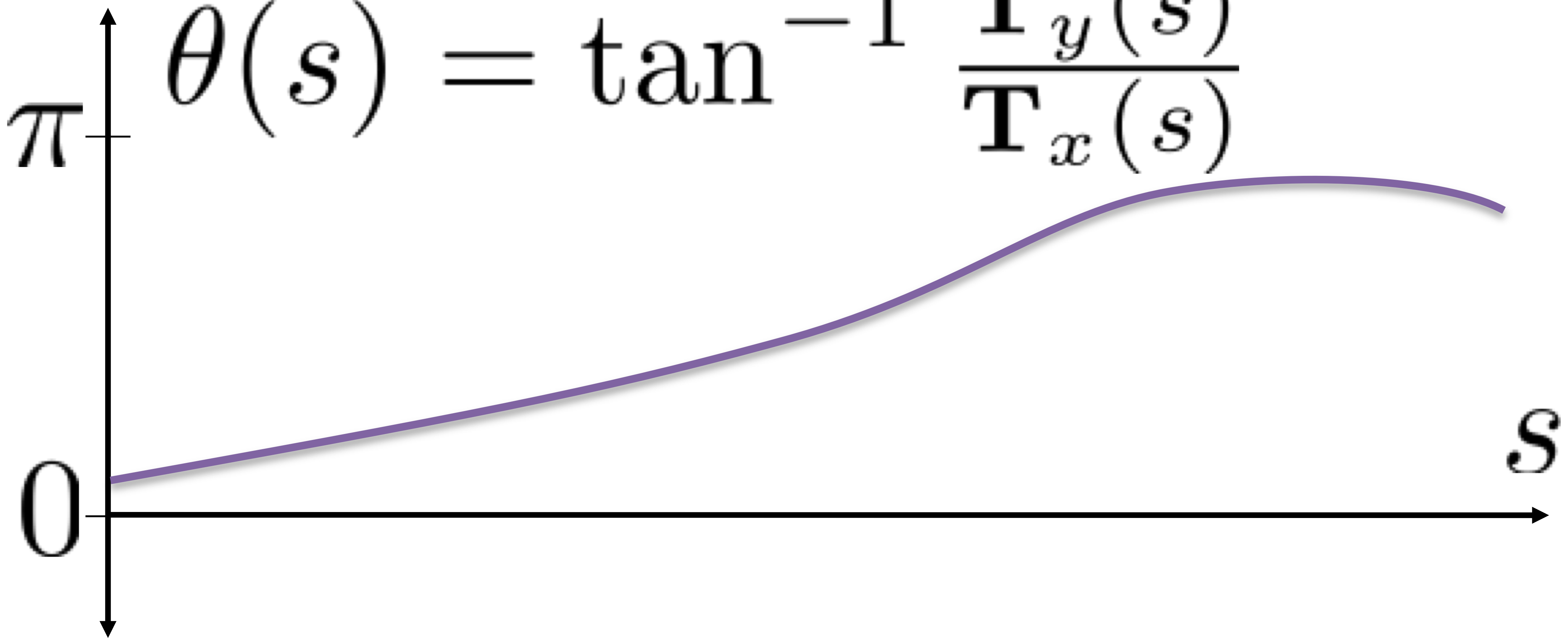
- Isometry between parameter domain and curve
- Tangent vector is unit-length: $\mathbf{p}'(s) = \mathbf{T}(s)$

Curvature

- How much does the curve turn per unit s ?

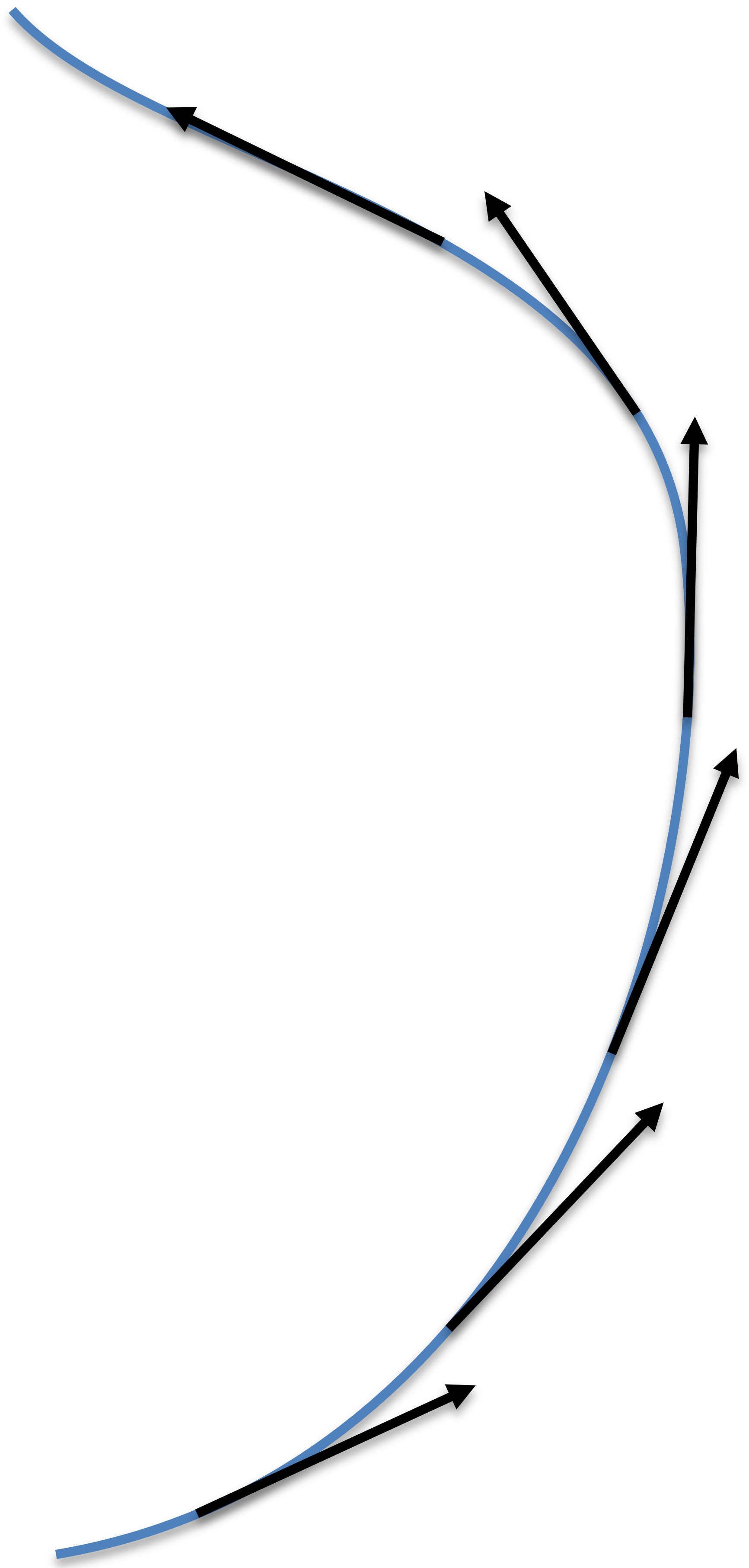


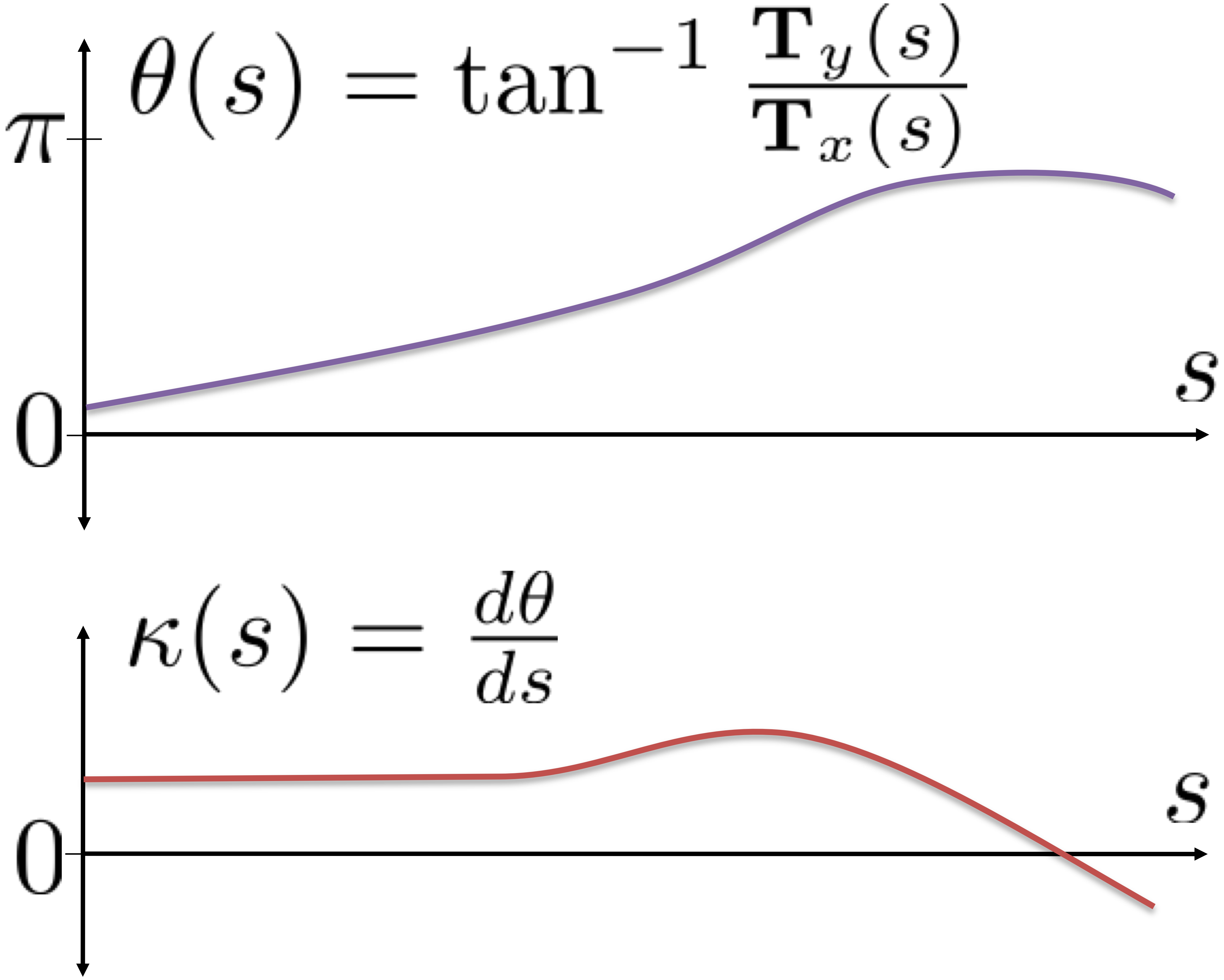
$$\theta(s) = \tan^{-1} \frac{\mathbf{T}_y(s)}{\mathbf{T}_x(s)}$$



Curvature

- How much does the curve turn per unit s?





Curvature profile

- Given $\kappa(s)$, we can get $\theta(s)$ up to a constant by integration.

- Integrating

$$\mathbf{p}(s) = \mathbf{p}_0 + \int_{s_0}^{s_1} \begin{pmatrix} \cos\theta(s) \\ \sin\theta(s) \end{pmatrix} ds$$

reconstructs the curve up to rigid motion.

Curvature of a circle

- Curvature of a circle:

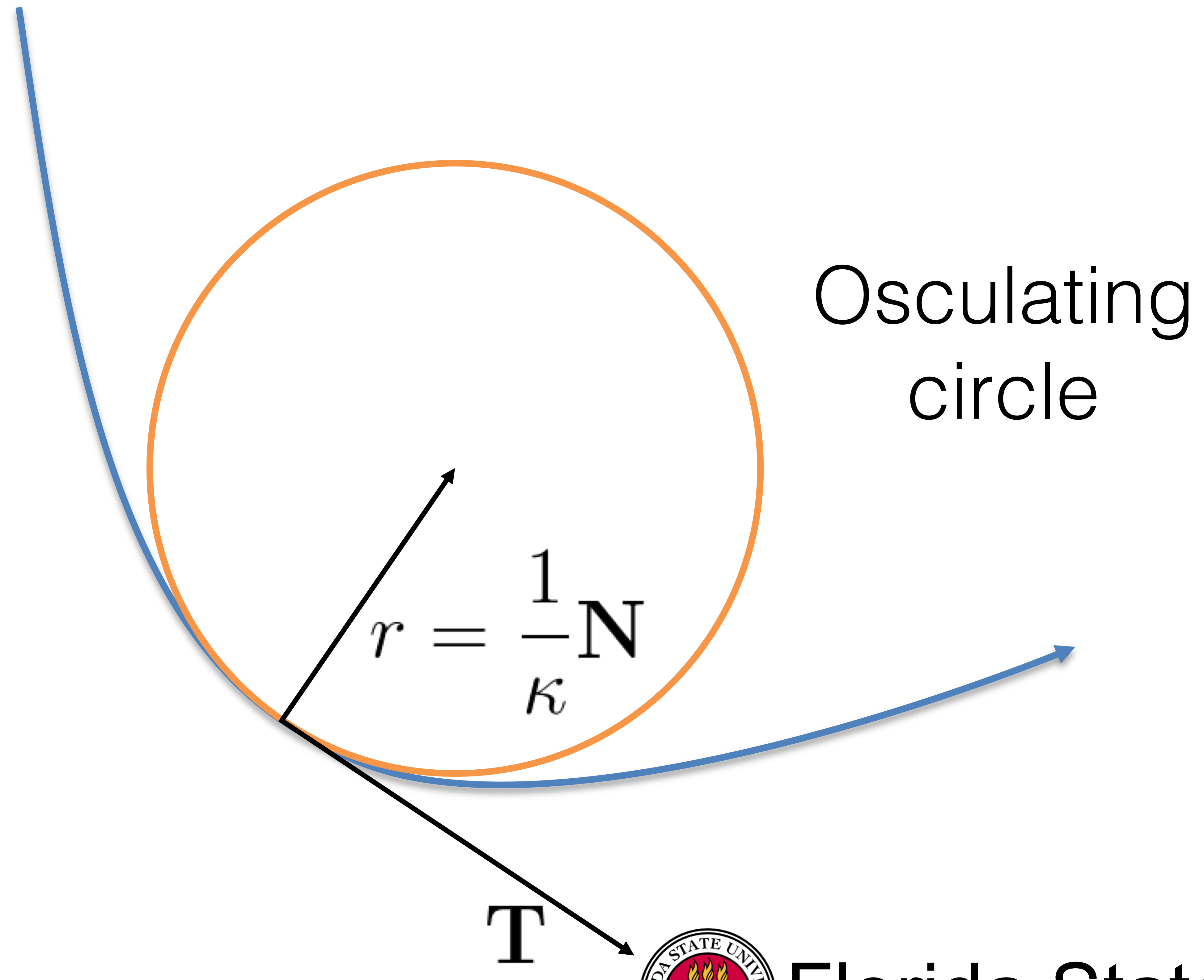
$$\mathbf{p}(s) = \begin{pmatrix} r\cos(s/r) \\ r\sin(s/r) \end{pmatrix}$$

$$\mathbf{p}'(s) = \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \end{pmatrix}$$

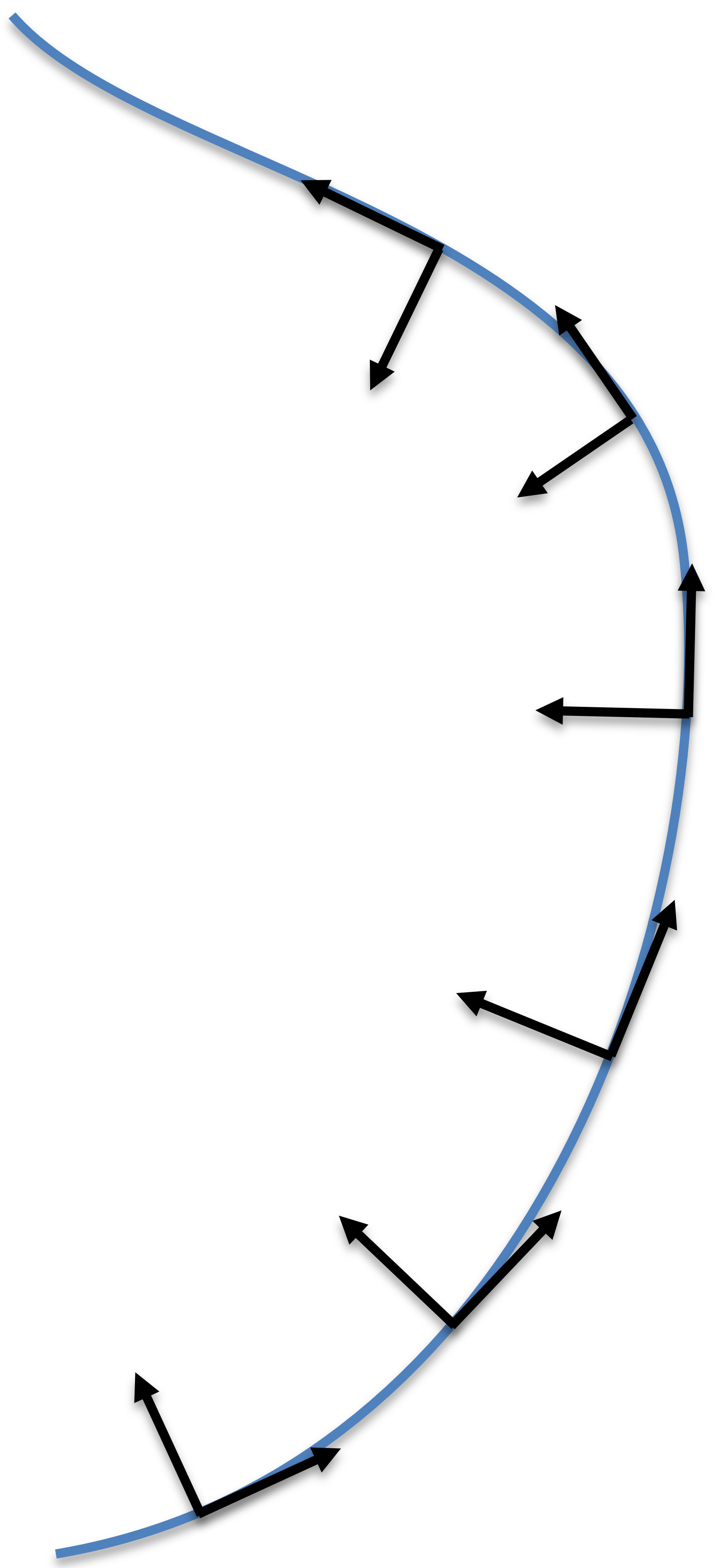
$$\theta(s) = \tan^{-1}\frac{\cos(s/r)}{-\sin(s/r)} \\ = s/r - \pi/2$$

$$\kappa(s) = 1/r$$

$$\mathbf{N}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T}(t)$$ = Unit Normal



Frenet Frame

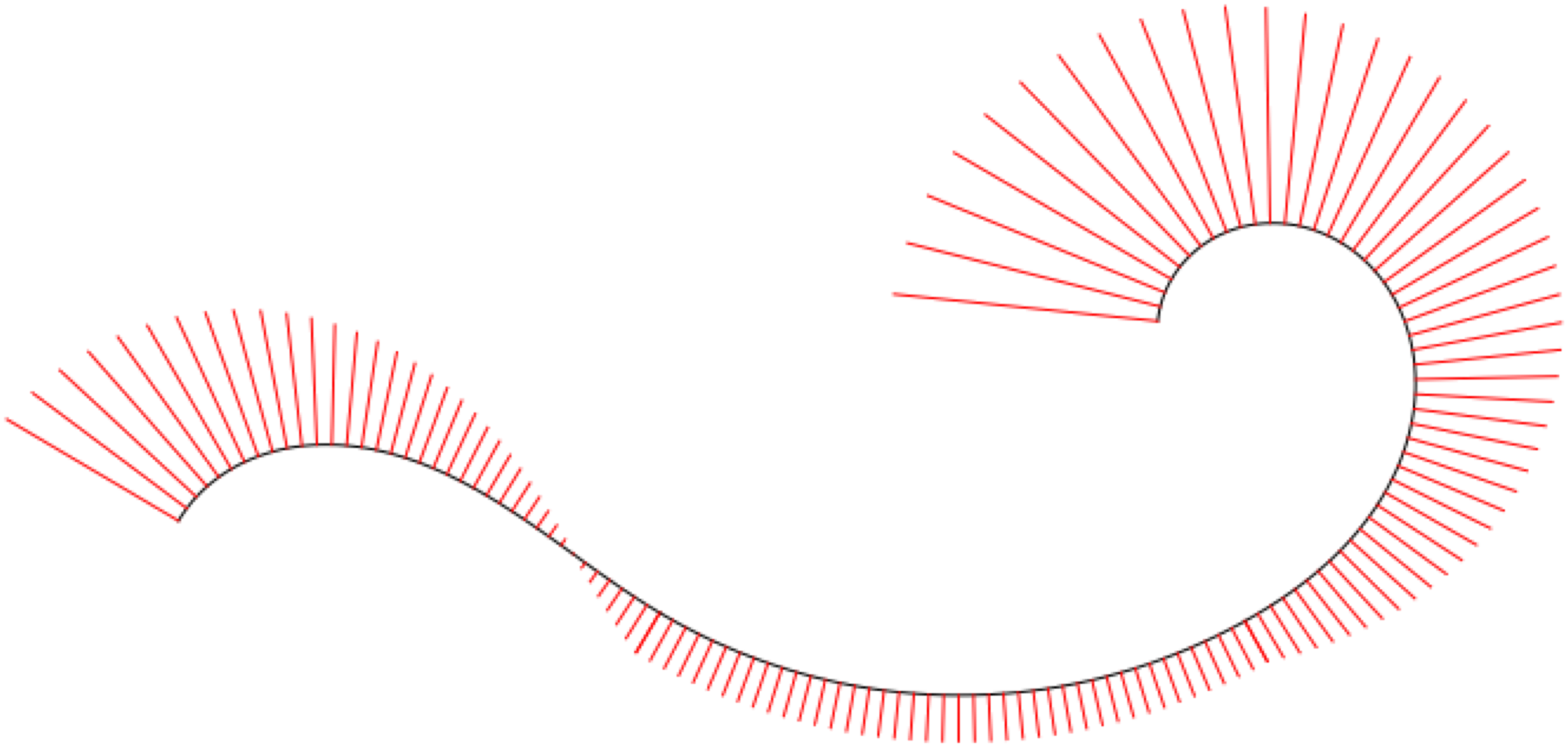


$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$
$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s)$$

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}$$

Curvature normal

- Points inward $$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$
- $-\kappa(s)\mathbf{N}(s)$ useful for evaluating curve quality



Smoothness

Two kinds, parametric and geometric:

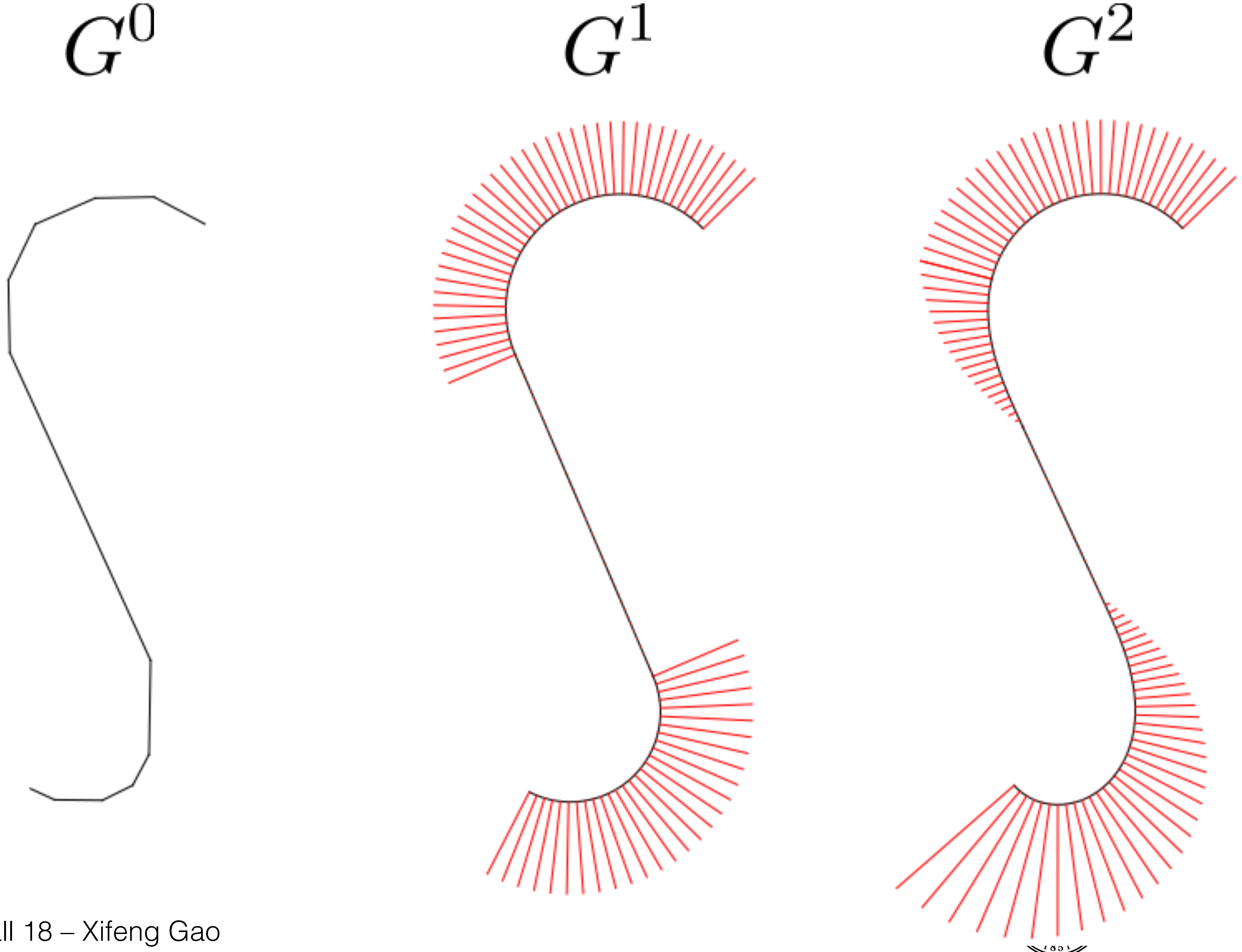
$$C^1 : \mathbf{p}(t) \text{ is continuously differentiable}$$
$$G^1 : \mathbf{p}(s) \text{ is continuously differentiable}$$

Parametrization-Independent

$$C^1 \qquad \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

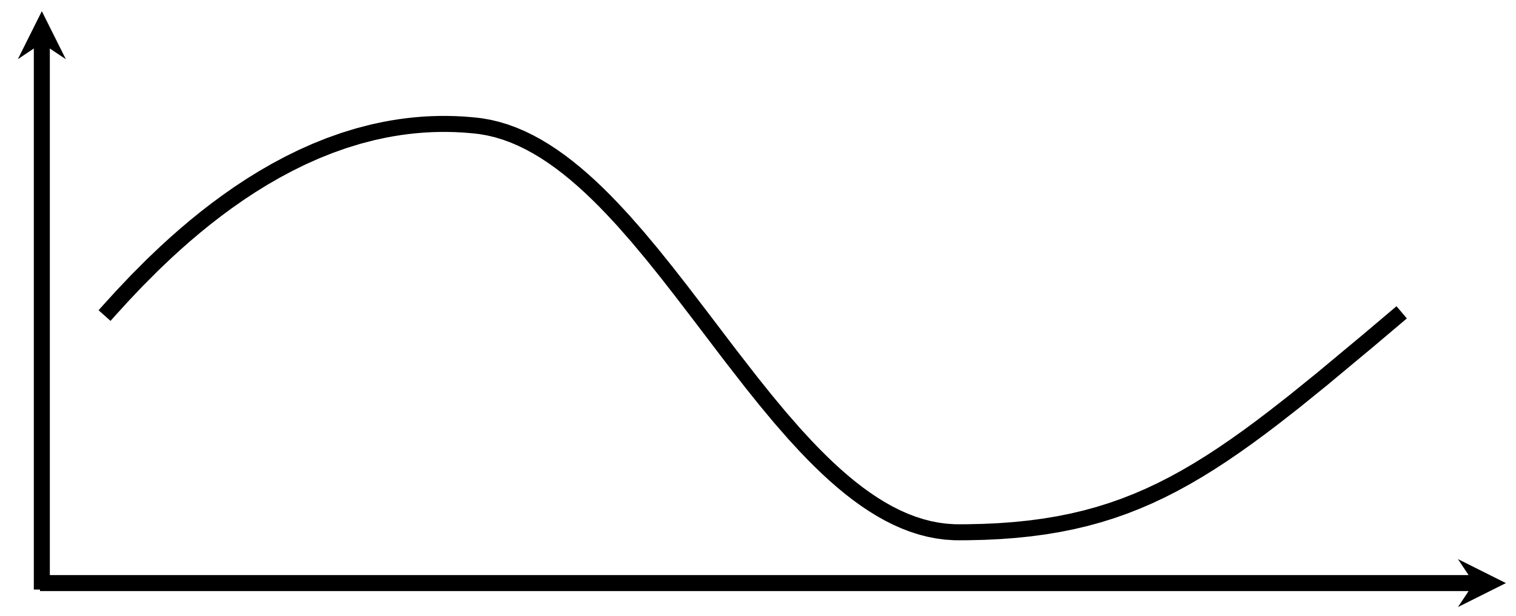
$$G^1 \qquad \begin{pmatrix} \cos \hat{t} \\ \sin \hat{t} \end{pmatrix}, \hat{t} = \begin{cases} t+1 & \text{if } t < 1 \\ 2t & \text{if } t \geq 1 \end{cases}$$

Smoothness example



Recap on parametric curves

$$\mathbf{p}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \; t \in [t_0, t_1]$$



$$\|\mathbf{p}'(t)\| = \text{speed}$$

$$s(t) = \int_{t_0}^{t} \|\mathbf{p}'(t)\| dt$$

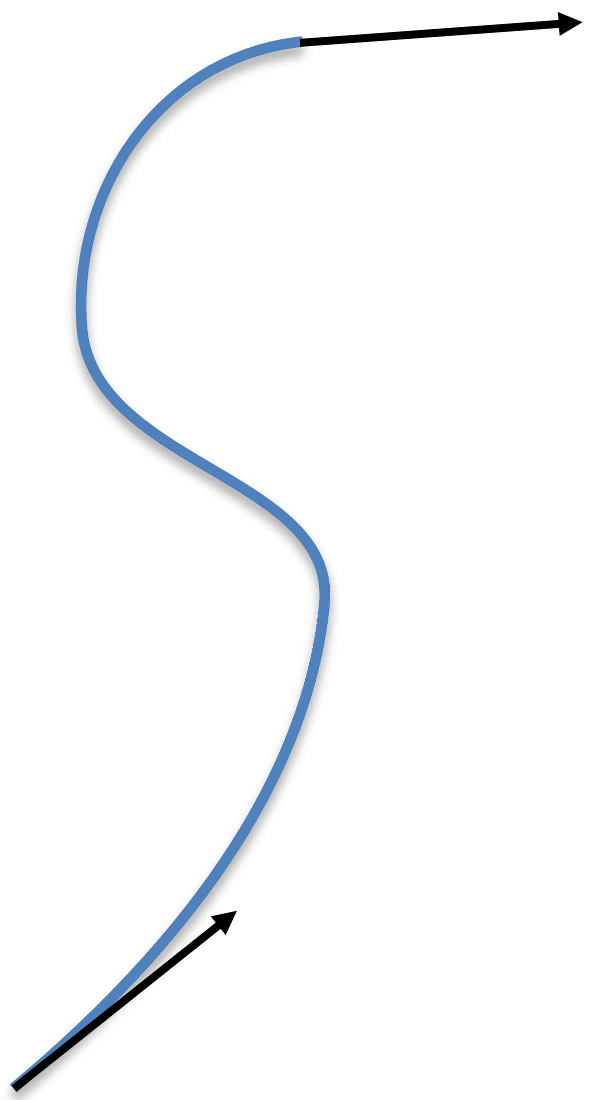
$$\kappa(s) = \frac{d\theta}{ds} = \mathbf{T}'(s) \cdot \mathbf{N}(s)$$

$$\frac{\mathbf{p}'(t)}{\|\mathbf{p}'(t)\|} = \mathbf{T}(t)$$

Turning

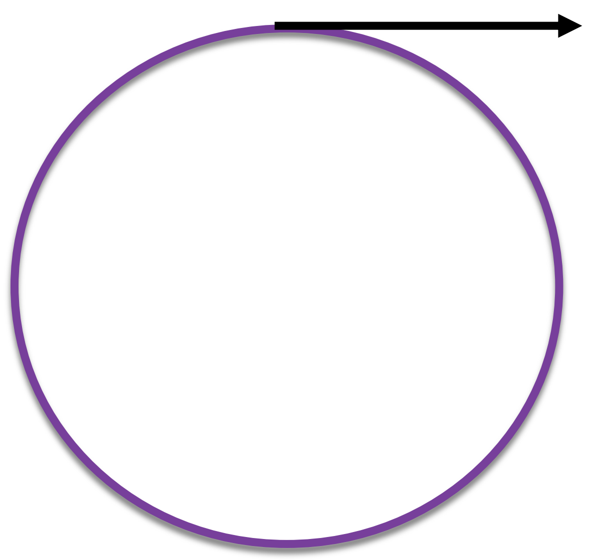
- Angle from start tangent to end tangent:

$$\int_{s_0}^{s_1} \kappa(s)ds = \int_{t_0}^{t_1} \kappa(t)\|\mathbf{p}'(t)\|dt$$



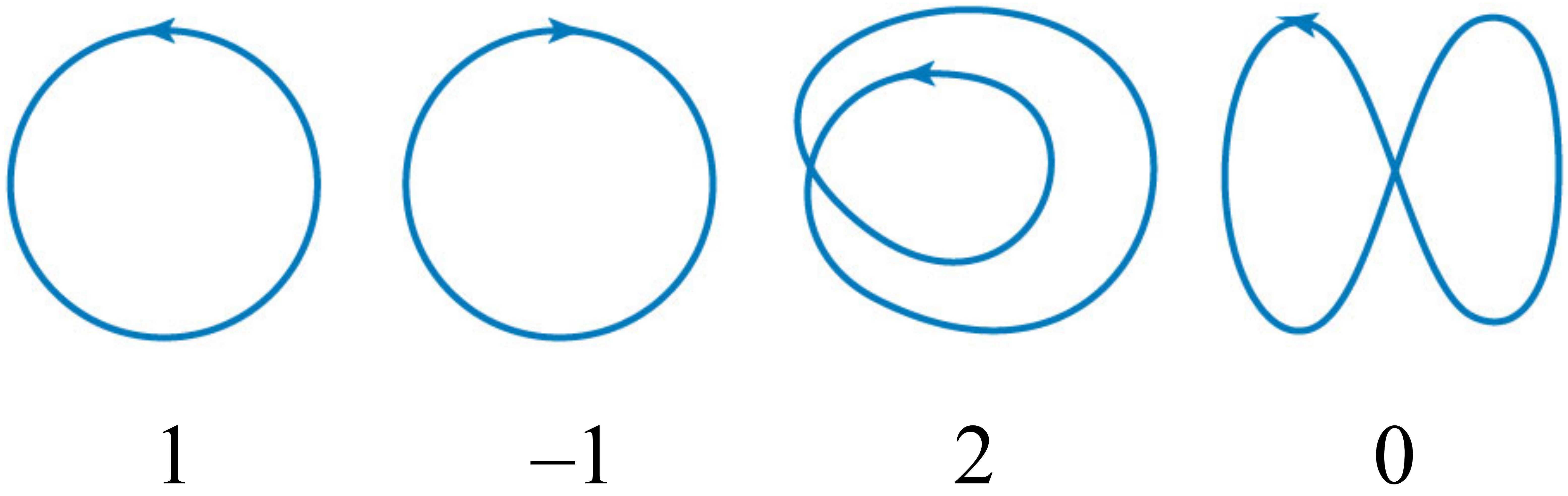
- If curve is closed, the tangent at the beginning is the same as the tangent at the end

$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$



Turning numbers

$$\oint_{\mathbf{p}} \kappa(s)ds = 2\pi n$$



- n measures how many full turns the tangent makes.

Space curves (3D)

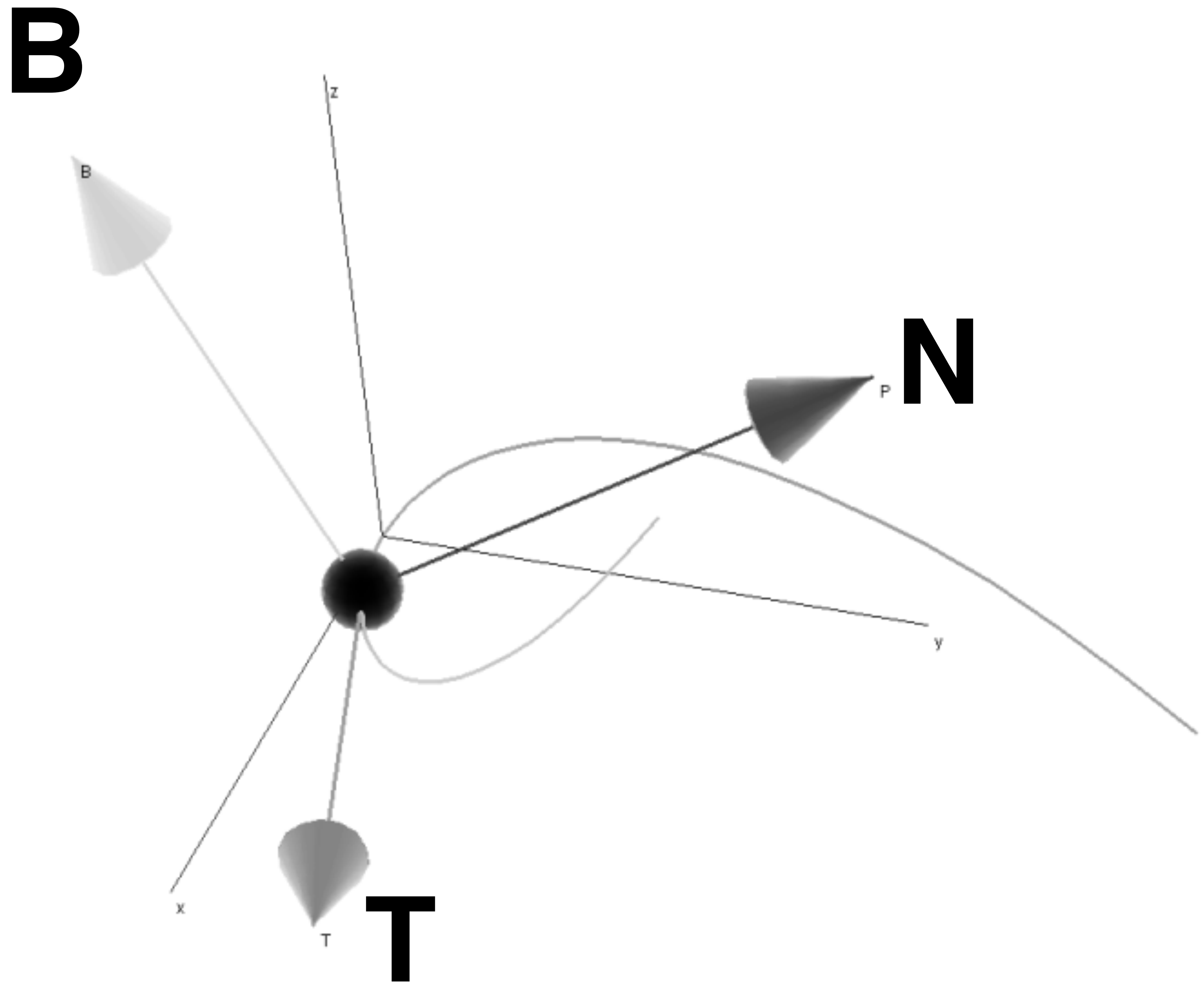
- In 3D, many vectors are orthogonal to **T**

$$\mathbf{N}(s) := \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$$

$$\mathbf{B}(s) := \mathbf{T}(s) \times \mathbf{N}(s)$$

- $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the “Frenet frame”
- τ is torsion: non-planarity

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$



References

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4th Edition **by Steve Marschner, Peter Shirley**

Chapter 15