Cost-Efficient Strategies for Restraining Rumor Spreading in Mobile Social Networks

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***Abstract*—With the popularity of mobile devices, mobile social networks (MSNs) have become an important platform for information dissemination. However, the spread of rumors in MSNs present a massive social threat. Currently, there are two kinds of methods to address this: blocking rumors at influential users and spreading truth to clarify rumors. However, most existing works either overlook the cost of various methods or only consider different methods individually. This paper proposes a heterogeneous-network-based epidemic model that incorporates the two kinds of methods to describe rumor spreading in MSNs. Moreover, two cost-efficient strategies are designed to restrain rumors. The first strategy is the real-time optimization strategy that minimizes the rumor-restraining cost by optimally combining various rumor-restraining methods such that a rumor can be extinct within an expected time period. The second strategy is the pulse spreading truth and continuous blocking rumor strategy that restrains rumor spreading through spreading truth periodically. The two strategies can restrain rumors in a continuous or periodical manner and guarantee cost efficiency. The experiments toward the Digg2009 data set demonstrate the effectiveness of the proposed model and the efficiency of the two strategies.**

***Index Terms*—Blocking rumors, maximum immunization period, Pontryagin's maximum principle, pulse immunization, spreading truth.**

I. Introduction

With the advance of mobile communication technology, mobile social networks (MSNs) are providing diverse services through interconnecting mobile devices and social networks. Unfortunately, MSNs also pave the way for the spread of rumors, unverified claims, and other kinds of disinformation. It has been shown that rumors spread much faster in MSNs than in other networks and cause more severe consequences [1].

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For example, on April 23, 2013, the rumor "*Two bombs had exploded at the White House and Barack Obama is injured*" spread on Twitter and caused the United States stock market to crash in a few minutes [2].

Currently, there are two kinds of methods for restraining rumor spreading in MSNs: *blocking rumors* at influential users [3]–[6] and *spreading truth* to clarify rumors [7], [8]. Unfortunately, the costs to carry out these two methods are usually overlooked [5], [9]. The first kind of method may violate human rights, and persuading someone to abandon her current opinion is tedious work. Hence, the first kind of methods generally needs various resources such as incentives, supported documents, and so forth. Similarly, *spreading truth* needs cooperation with social media and requires several network resources such as channels. For simplicity, we denote *blocking rumors* at influential users and *spreading truth* to clarify rumors as *immunization* and *cure* in the rest of this paper without confusion, respectively. Moreover, immunization and cure are individually considered in most existing works [3]–[6]. Hence, prior works overestimate the efficiency of the designed countermeasures and are costly. In this paper, we present two strategies to restrain rumor spreading, both of which consider immunization and cure with limited costs. We first propose a real-time optimization (RTO) strategy that minimizes the rumor-restraining cost by optimally combining immunization and cure so that a rumor can be extinct within an expected time period. With the optimization objective, RTO provides the optimized rates for immunization and cure in a real-time manner. Clearly, RTO requires immunization and cure to be continuously conducted in a period. That is, at any time t, $t \in [0, t_f]$ where t_f is an expected time period, both immunization and cure should be carried out with certain rates. Clearly, RTO needs to occupy rumor-restraining resources continuously since it continuously spreads truth to immunize susceptible users (i.e., immunization). However, it is challenging, sometimes impossible, to continuously consume the limited resources in a period among multiparties. Therefore, this challenge motivates us to propose the pulse spreading truth and continuous blocking rumor (PSCB) strategy that carries out immunization in a periodical manner. Therefore, we propose the PSCB strategy that restrains rumor spreading by spreading truth periodically. PSCB intends to find a maximum period so that the cost of spreading truth can be reduced. We denote such a period as the *maximum immunization period*. RTO is applicable when rumor-restraining resources are enough and rumors are intended to be restrained within a certain time period. Comparatively, PSCB is applicable when resources are not enough and when there is no specific time limitation to restrain rumor. The main contributions of this work are summarized as follows.

- To the best of our knowledge, this is the first cost-efficient work that jointly considers various methods for restraining rumor spreading in MSNs.
- We propose a continuous strategy that continuously carries out countermeasures with the smallest cost to restrain rumor spreading within an expected time period.
- We propose a pulse strategy that periodically spreads truth to restrain rumor spreading. This strategy identifies the maximum immunization period.
- We conduct extensive experiments to evaluate the effectiveness of the proposed model and the efficiency of the two strategies toward the real Digg2009 data set.

The remainder of this paper is organized as follows. Section II introduces the network model and the problem definition. Section III presents the rumor spreading model under continuous countermeasures and derives the threshold that determines whether a rumor continuously spreads or becomes extinct. The RTO strategy and the PSCB strategy are illustrated in Sections IV and V, respectively. The validation of the proposed model and strategies are presented in Section VI. The related works are addressed in Section VII. Section VIII concludes this paper.

II. Preliminaries

A. Network Model

We use four states to indicate the different status of users while a rumor spreads in a network. *Susceptible* (S) means a user has not yet been infected by a rumor but is susceptible to it; *Infected* (I) means a user has been infected by a rumor and performs as a rumor spreader; *Recovered* (R) means a user would never be infected by a rumor; *Dead* (D) means a user has no interest in spreading truth or rumors. Since different users have different abilities to spread and absorb information, we further divide users into different groups based on their degrees in a social network. Hence, the network presents degree-based heterogeneity. Although degree-based heterogeneity is not optimal, some recent works have shown that it can effectively characterize information spreading in social networks [4], [6], [10]. The users in a network are then divided into n groups, and the users in one group have the same degree. Let k_i denote the degree of the users in group i $(i = 0, 1, \ldots, n)$. Let $S_{k_i}(t)$, $I_{k_i}(t)$, $R_{k_i}(t)$, and $D_{k_i}(t)$ denote the density of the susceptible, infected, recovered, and dead users in group i at time t, respectively. Hence, at any time t, the active nodes in a network satisfy $S_{k_i}(t) + I_{k_i}(t) + R_{k_i}(t) = 1$. With countermeasures carried out to restrain rumors, a user's state transforms among S, I, and R. For an arbitrary user, the state transition is shown in Fig. 1 with the following rules.

1) A user changes from state S to state I if it believes in the rumor. The rumor acceptance rate of a user in S_{k_i} is $\lambda(k_i)$ $(0 < \lambda(k_i) < 1)$, i.e., an S user is infected with probability $\lambda(k_i)$ if it is connected to an I user. An S user is connected to one or more I users with probability $\Theta(t)$ at time t. Thus, the infected probability for an S user at time t is $\lambda(k_i)\Theta(t)$.
2) In the RTO strategy, at any time t, an S user is immunized and transforms to R with probability $\varepsilon_1(t)$. In the PSCB strategy, an S user is immunized and transforms to R with probability p every other period T. An I user is cured and transforms to R with probability $\varepsilon_2(t)$ at time t.
3) New users concern about the rumor with rate α. Assume new users are susceptible users, i.e., they have not been infected or immunized. As time passes, some users lose interests in spreading either truth or rumors with interest decaying rate μ to become dead users.

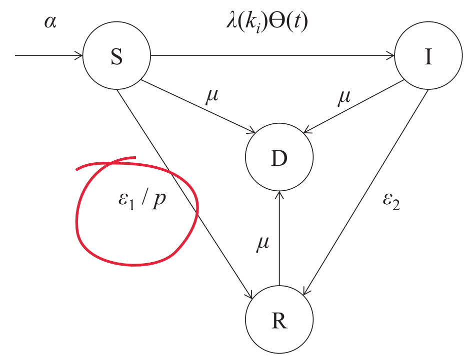

Fig. 1. State transition of a user.

Initially, just a few infected users and most users are S users. As the rumor spreads, S users become I users gradually. Thus, the initial condition of the model is $I_{k_i}(t_0) > 0$, $S_{k_i}(t_0) = 1 - I_{k_i}(t_0)$, and $R_{k_i}(t_0) = D_{k_i}(t_0) = 0$, where $t_0 = 0$.

B. Problem Definition

Based on the aforementioned network model, three questions need to be addressed: 1) What is the threshold that determines whether a rumor continuously spreads or becomes extinct as time passes? 2) If countermeasures can be carried out continuously, what is the real-time strategy to restrain rumor spreading with efficient cost in an expected time period? 3) If countermeasures have to be carried out periodically, what is the cost-efficient pulse strategy? Specifically, the problem studied in this paper is defined as follows.

Case 1—Continuous Strategy:

Input:

1) an MSN with its initial state: $I_{k_i}(t_0)$, $S_{k_i}(t_0)$, $R_{k_i}(t_0)$, and $D_{k_i}(t_0)$, where $t_0 = 0$, $i = 1, 2, \ldots, n$, and they describe the density of the susceptible, infected, recovered, and dead users in group i at time t_0, respectively;
2) the expected time period to restrain a rumor: $[0, t_f]$.

Output:

1) threshold of countermeasures, which determines whether a rumor continuously spreads or becomes extinct as time passes;
2) real-time strategy: $\varepsilon_1^*(t)$ and $\varepsilon_2^*(t)$, $t \in (0, t_f]$, namely, an S user is immunized and transforms to R with probability $\varepsilon_1^*(t)$, and an I user is cured and transforms to R with probability $\varepsilon_2^*(t)$, at time t.

Case 2—Pulse Strategy:

Input:

1) an MSN with its initial state: $I_{k_i}(t_0)$, $S_{k_i}(t_0)$, $R_{k_i}(t_0)$, and $D_{k_i}(t_0)$, where $t_0 = 0, i = 1, 2, \ldots, n$, and they describe the density of the susceptible, infected, recovered, and dead users in group i at time t_0, respectively.

Output:

1) threshold of countermeasures, which determines whether a rumor continuously spreads or becomes extinct as time passes;
2) pulse strategy: maximum immunization period $T_{\max}$.

III. Continuous-Countermeasure-Based Epidemic Model

Here, we first present the rumor spreading model under continuous countermeasures. Then, we analyze the existence and stability of the equilibrium solutions to derive the threshold that determines whether a rumor continuously spreads or becomes extinct.

Ever since the standard model for rumor spreading was introduced by Daley and Kendall in 1965 (i.e., DK model [11]), many variants have been proposed. We also describe the rumor spreading under continuous countermeasures, taking the epidemic model as the basis. Based on the network model introduced in Section II-A, the continuous-countermeasure-based epidemic model can be described as the following system:

$$
\begin{aligned}
\frac{dS_{k_i}(t)}{dt} &= \alpha - \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_1 S_{k_i}(t) - \mu S_{k_i}(t)\\
\frac{dI_{k_i}(t)}{dt} &= \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) - \mu I_{k_i}(t)\\
\frac{dR_{k_i}(t)}{dt} &= \varepsilon_1 S_{k_i}(t) + \varepsilon_2 I_{k_i}(t) - \mu R_{k_i}(t)\\
\frac{dD_{k_i}(t)}{dt} &= \mu S_{k_i}(t) + \mu I_{k_i}(t) + \mu R_{k_i}(t)\\
& i = 1, 2, \ldots, n, \ t > 0.
\end{aligned}
\tag{1}
$$

For convenience, Table I summarizes the major parameters in system (1). The four equations describe the changing rate of $S_{k_i}(t)$, $I_{k_i}(t)$, $R_{k_i}(t)$, and $D_{k_i}(t)$, respectively. As shown by the first equation in system (1), the changing rate of $S_{k_i}(t)$ is determined by four parts. First, new susceptible users join the network with rate α at any time t. Second, $\lambda(k_i)S_{k_i}(t)\Theta(t)$ susceptible users are infected to transform to infected users. Third, $\varepsilon_1 S_{k_i}(t)$ susceptible users are immunized by truth to transform to recovered users. Finally, $\mu S_{k_i}(t)$ susceptible users become dead nodes. The remaining two equations obey the similar rules.

$\Theta(t)$ computes the average degree of all the I users, i.e.,

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$$\Theta(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\omega(k_i)P(k_i)I_{k_i}(t)$$

where $P(k_i)$ is the probability of a user with degree k_i so that $\sum_{i=1}^{n} P(k_i) = 1$, and $\langle k\rangle$ is the average degree of the users in a network; thus, $\langle k\rangle = \sum_{i=1}^{n} k_i P(k_i)$. $\omega(k_i)$ measures the infectivity of a user with degree k_i. Several cases of $\omega(k_i)$ have been considered such as $\omega(k_i) = C$, where C is a constant [12],

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TABLE I
Major Parameters in the Dynamic Model

Parameter	Definition
k_i	Social connectivity of the users in group i (*i.e.*, degree)
α	Rate of new users entering an MSN
μ	User interest decaying rate
$\lambda(k_i)$	Rumor acceptance rate of the susceptible users in group i
$\varepsilon_1(t)$	Proportion of the susceptible users being immunized at time t
$\varepsilon_2(t)$	Proportion of the infected users being blocked at time t
$P(k_i)$	Probability of a node with degree k_i
$\langle k\rangle$	Average degree of an MSN
$\omega(k_i)$	Infectivity of an infected user with degree k_i

$\omega(k_i) = k_i$ [13], or $\omega(k_i)$ is a nonlinear function $k_i^\beta/(1+k_i^\gamma)$ [14]. Intuitively, a user with a larger degree would have larger infectivity. However, the infectivity will saturate when a user's degree reaches some extent. Hence, in this work, we adopt the nonlinear function $k_i^\beta/(1+k_i^\gamma)$ to compute $\omega(k_i)$.

A. Definition of the Equilibrium Solution

If the countermeasures are strong, a rumor no longer spreads. Otherwise, a rumor would continuously spread. These two cases correspond to the zero-equilibrium solution and the positive-equilibrium solution of system (1), respectively. For clarity, the definitions of the equilibrium solutions are presented as follows. More details can be found in [15].

Definition 3.1—Equilibrium Solution: Mathematically, the solution $x^* \in \mathbb{R}$ is an equilibrium solution for the following differential equation:

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$$\frac{dx}{dt} = f(t, x)$$

if $f(t, x^*) = 0$ for all t.

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Definition 3.2—Zero-Equilibrium Solution: E_0 is called a zero-equilibrium solution of system (1) if the solution of system (1) $E = (e_1, e_2, \ldots, e_n)$ converges to $E_0 = (e_1^0, e_2^0, \ldots, e_n^0)$, where $e_i = (S_{k_i}(t), I_{k_i}(t), R_{k_i}(t), D_{k_i}(t))$, $e_i^0 = (S_{k_i}^0, I_{k_i}^0, R_{k_i}^0, D_{k_1}^0)$, and $S_{k_i}^0 > 0$, $R_{k_i}^0 > 0$, $I_{k_i}^0 = 0$, $D_{k_i}^0 > 0$ $(i = 1, 2, \ldots, n)$.

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Definition 3.3—Positive-Equilibrium Solution: E_+ is called a positive-equilibrium solution of system (1) if the solution of system (1) $E = (e_1, e_2, \ldots, e_n)$ converges to $E_+ = (e_1^+, e_2^+, \ldots, e_n^+)$, where $e_i = (S_{k_i}(t), I_{k_i}(t), R_{k_i}(t), D_{k_i}(t))$, $e_i^+ = (S_{k_i}^+, I_{k_i}^+, R_{k_i}^+, D_{k_1}^+)$, and $S_{k_i}^+ > 0$, $R_{k_i}^+ > 0$, $I_{k_i}^+ > 0$, $D_{k_i}^0 > 0$ $(i = 1, 2, \ldots, n)$.

From the given definitions, we find that to compute the threshold that determines whether the rumors continuously spread or not, we should first discuss the existence of the equilibrium solutions of system (1).

B. Existence of the Equilibrium Solution

We simplify $\Theta(t)$ by letting $\varphi(k_i) = \omega(k_i)P(k_i)$. Moreover, since users with state D no longer transforms to other states, for simplicity, we just analyze the active states (i.e., S, I, R) in the following parts. The following theorem shows the existence of the equilibrium solution of system (1).

Theorem 1: For parameter

$$r_0 = \frac{\alpha}{\langle k\rangle}\sum_{i=1}^{n}\frac{\lambda(k_i)\varphi(k_i)}{(\varepsilon_1+\mu)(\varepsilon_2+\mu)}.$$

Case 1: If $r_0 \le 1$, system (1) only has a zero-equilibrium solution denoted by $E_0 = \{(S^0_{k_1}, I^0_{k_1}, R^0_{k_1}), \ldots, (S^0_{k_n}, I^0_{k_n}, R^0_{k_n})\}$, where $S^0_{k_i} = \alpha/(\varepsilon_1 + \mu)$, $I^0_{k_i} = 0$, and $R^0_{k_i} = 1 - \alpha/(\varepsilon_1 + \mu)$ $(i = 1, 2, \ldots, n)$.

Case 2: If $r_0 > 1$, system (1) has both a zero-equilibrium solution and a positive-equilibrium solution denoted by $E_+ = \{(S^+_{k_1}, I^+_{k_1}, R^+_{k_1}), \ldots, (S^+_{k_n}, I^+_{k_n}, R^+_{k_n})\}$, where

$$\begin{aligned} S^+_{k_i} &= \frac{(\varepsilon_2 + \mu) I^+_{k_i}}{\lambda(k_i)\Theta^+} \\ I^+_{k_i} &= \frac{\alpha\lambda(k_i)\Theta^+}{(\varepsilon_2 + \mu)\left(\lambda(k_i)\Theta^+ + \varepsilon_1 + \mu\right)} \\ R^+_{k_i} &= 1 - S^+_{k_i} - I^+_{k_i} \end{aligned}$$

$\Theta^+ = \langle k \rangle^{-1} \sum_{i=1}^n \varphi(k_i) I^+_{k_i} (i = 1, 2, \ldots, n)$.

Proof: Since the first two equations of system (1) do not contain R_{k_i}, we first analyze them and then obtain $R_{k_i}(t)$ based on $S_{k_i}(t) + I_{k_i}(t) + R_{k_i}(t) = 1$ $(i = 1, 2, \ldots, n)$. System (2), shown below, is the simplified system after extracting the first two equations from system (1), i.e.,

$$\begin{aligned} \frac{dS_{k_i}(t)}{dt} &= \alpha - \lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_1 S_{k_i}(t) - \mu S_{k_i}(t) \\ \frac{dI_{k_i}(t)}{dt} &= \lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) - \mu I_{k_i}(t). \end{aligned} \tag{2}$$

When system (2) gets equilibrium solutions $E_* = \{(S^*_{k_1}, I^*_{k_1}, R^*_{k_1}), \ldots, (S^*_{k_n}, I^*_{k_n}, R^*_{k_n})\}$, it indicates that $S_{k_i}(t)$, $I_{k_i}(t)$, and $R_{k_i}(t)$ converge to $S^*_{k_i}$, $I^*_{k_i}$, and $R^*_{k_i}$, respectively. In this case, $dS_{k_i}(t)/dt = 0$, $dI_{k_i}(t)/dt = 0$, and $dR_{k_i}(t)/dt = 0$. Thus, when system (1) gets E_*, system (2) should satisfy

$$\begin{aligned} &\alpha - \lambda(k_i) S^*_{k_i}\Theta^* - \varepsilon_1 S^*_{k_i} - \mu S_{k_i}(t) = 0 \\ &\lambda(k_i) S^*_{k_i}\Theta^* - \varepsilon_2 I^*_{k_i} - \mu I_{k_i}(t) = 0 \end{aligned} \tag{3}$$

where $\Theta^* = \langle k \rangle^{-1} \sum_{i=1}^n \varphi(k_i) I^*_{k_i}$. From system (3), we have

$$I^*_{k_i} = \frac{\alpha\lambda(k_i)\Theta^*}{(\varepsilon_2 + \mu)(\lambda(k_i)\Theta^* + \varepsilon_1 + \mu)}. \tag{4}$$

Clearly, $I^*_{k_i} = 0$ $(i = 1, 2, \ldots, n)$ is always the solution of (4). Substituting $I^*_{k_i} = 0$ in system (3), the corresponding $S^*_{k_i}$ and $R^*_{k_i}$ can be derived as shown in Case 1 of Theorem 1. Substituting (4) in Θ^* and moving the right item to the left, we have

$$\Theta^* \left(1 - \frac{1}{\langle k \rangle} \sum_{i=1}^n \frac{\alpha\lambda(k_i)\varphi(k_i)}{(\varepsilon_2 + \mu)\left(\lambda(k_i)\Theta^* + \varepsilon_1 + \mu\right)} \right) = 0. \tag{5}$$

For (5), let

$$F(\Theta^*) = 1 - \frac{1}{\langle k \rangle} \sum_{i=1}^n \frac{\alpha\lambda(k_i)\varphi(k_i)}{(\varepsilon_2 + \mu)\left(\lambda(k_i)\Theta^* + \varepsilon_1 + \mu\right)}.$$

Since $F'(\Theta^*) > 0$ for all Θ^* and $\lim_{\Theta^* \to \infty} F(\Theta^*) = 1$, there is a nontrivial solution for equation $F(\Theta^*) = 0$ if and only if $\lim_{\Theta^* \to 0^+} F(\Theta^*) < 0$, which is true when $r_0 > 1$. That is, system (1) has a positive-equilibrium solution when $r_0 > 1$. Substituting (4) in system (3), we can obtain the positive-equilibrium solution of system (3) as shown in Case 2 of Theorem 1. □

Theorem 1 shows the restrictive correlation between the existence of an equilibrium solution and the level of countermeasures. However, as pointed out in [16], a differential system only converges to a stable equilibrium solution; hence, Theorem 1 is not enough to determine the spreading dynamics of rumors. The stability of the two equilibrium solutions will be analyzed in the following sections.

C. Stability of the Equilibrium Solution

Two types of stabilities exist: local asymptotic stability and global asymptotic stability. The stability theory [16] shows that the equilibrium solution is locally asymptotically stable if and only if the system variables (i.e., S_{k_i}, I_{k_i}, and $R_{k_i}, i = 1, 2, \ldots, n$) converge to it when the initial values of system variables slightly deviate from the equilibrium solution. Correspondingly, the equilibrium solution is globally asymptotically stable if and only if the system variables converge to it under any value of the initial values. For simplicity, we denote *locally asymptotically stable*, *globally asymptotically stable*, *local asymptotic stability*, and *global asymptotic stability* as *L-stable*, *G-stable*, *L-stability*, and *G-stability*, respectively. Lyapunov's second stability method is generally utilized to determine the G-stability of the equilibrium solution of the differential equation system. For clarity, it is introduced as follows, and more details can be found in [16].

Definition 3.4—Lyapunov's Second Method for Stability: Mathematically, the equilibrium solution $x^* = 0$ is G-stable for the following differential equation:

$$\frac{dx}{dt} = f(t, x)$$

if there is a scalar function $V(x) : \mathbb{R}^{\ltimes} \to \mathbb{R}$, which has continuous first partial derivative and satisfies the following.

1) $V(x) \ge 0$ with equality if and only if $x = 0$.
2) $dV(x)/dt \le 0$ with equality not constrained to only $x = 0$.

We now investigate the L-stability and G-stability of E_0 and E_*, respectively.

1) Stability of E_0: For the stability of system (1) at E_0, we have the following theorems.

Theorem 2: If $r_0 < 1$, E_0 is L-stable. If $r_0 > 1$, E_0 is unstable.

Proof: According to the stability theory [16], if and only if all the eigenvalues of the characteristic equation of $J(E_*)$ are less than zero, the system is L-stable at E_*, where $J(E_*)$ is the Jacobian matrix of the dynamic system at E_*. Thus, to analyze the stability of system (1) at E_0, we need to linearize system (1) first to obtain the eigenvalues of the characteristic equation of $J(E_0)$. The result of linearizing system (2) at E_0 is as follows:

$$\begin{aligned} \frac{dS_{k_i}(t)}{dt} &= J^{1,1}_{k_i}\left(S - S^0_{k_i}\right) + J^{1,2}_{k_i}\left(I - I^0_{k_i}\right) \\ \frac{dI_{k_i}(t)}{dt} &= J^{2,1}_{k_i}\left(S - S^0_{k_i}\right) + J^{2,2}_{k_i}\left(I - I^0_{k_i}\right) \end{aligned}$$

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where $J_{k_i}^{p,q}$ $(p, q \in \{1, 2\}, i = 1, 2, \ldots, n)$ are the elements of the Jacobian matrix of system (2) at E_0 for group i. Then, the Jacobian matrix of system (2) at E_0 can be written as

$$J(E_0) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

For each $A_{i,j}$ where $i = j$, i.e., the diagonal elements of $J(E_0)$, we have

$$A_{i,j} = \begin{pmatrix} -\varepsilon_1 - \mu, & -\frac{\lambda(k_i)\alpha}{\varepsilon_1+\mu}\frac{1}{\langle k\rangle}\varphi(k_i) \\ 0, & \frac{\lambda(k_i)\alpha}{\varepsilon_1+\mu}\frac{1}{\langle k\rangle}\varphi(k_i) - \varepsilon_2 - \mu \end{pmatrix}.$$

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For $A_{i,j}$ where $i \neq j$, we have

$$A_{i,j} = \begin{pmatrix} 0, & -\frac{\lambda(k_i)\alpha}{\varepsilon_1+\mu}\frac{1}{\langle k\rangle}\varphi(k_i) \\ 0, & \frac{\lambda(k_i)\alpha}{\varepsilon_1+\mu}\frac{1}{\langle k\rangle}\varphi(k_i) \end{pmatrix}.$$

The characteristic equation of $J(E_0)$ is

$$(\chi + \varepsilon_1 + \mu)^3(\chi + \varepsilon_2 + \mu)^2\left(\chi - (\Gamma - \varepsilon_2 - \mu)\right) = 0$$

where

$$\Gamma = \frac{\alpha}{\langle k\rangle}\sum_{i=1}^{n}\frac{\lambda(k_i)\varphi(k_i)}{\varepsilon_1 + \mu}.$$

Thus, the eigenvalues of the characteristic equation of $J(E_0)$ are $\chi_1 = -\varepsilon_1 - \mu$, $\chi_2 = -\varepsilon_2 - \mu$, and $\chi_3 = \Gamma - \varepsilon_2 - \mu$. Since $-\varepsilon_1 - \mu < 0$ and $-\varepsilon_2 - \mu < 0$, the local stability of E_0 is completely determined by the sign of $\Gamma - \varepsilon_2 - \mu$. If $r_0 < 1$, we have $\Gamma - \varepsilon_2 - \mu < 0$ so that system (1) is L-stable at E_0. If $r_0 > 1$, we have $\Gamma - \varepsilon_2 - \mu > 0$; thus, system (1) is unstable at E_0. □

To verify the global asymptotic stability of E_0, we first present Lemma 1.

Lemma 1: As system (1) asymptotically converges to E_+ (i.e., $r_0 > 1$), ε_2 should satisfy

$$\lim_{E^* \to E^+} \varepsilon_2 = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\lambda(k_i)\varphi(k_i)S_{k_i}^{+} - \mu. \quad (6)$$

Proof: Based on the definition of $\Theta(t)$ and combining the second equation of system (1), we have

$$\begin{aligned}\Theta'(t) &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)I'_{k_i}(t) \\ &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\left(\lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) - \mu I_{k_i}(t)\right) \\ &= \Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\lambda(k_i)\varphi(k_i)S_{k_i}(t)\right) - \varepsilon_2 - \mu\right). \quad (7)\end{aligned}$$

When system (1) converges to E_+, $\Theta'(t) = 0$. Since $\Theta(t) > 0$, from (7), we can derive (6). □

Based on Lemma 1, the stability of E_0 is stated by Theorem 3.

Theorem 3: If $r_0 < 1$, E_0 is G-stable.

Proof: To investigate the G-stability of system (1) at E_* where E_* is the equilibrium point, we need to construct a Lyapunov function $V(t)$. According to Definition 3.4, we construct the Lyapunov function for E_0 as

$$V(t) = \frac{1}{\varepsilon_2 + \mu}\Theta(t).$$

Then, based on Lemma 1 and combining the equilibrium solution $S_{k_i}^0 = \alpha/(\varepsilon_1 + \mu)$, the time derivative of $V(t)$ computed in the solution space of system (1) for $t > 0$ is

$$\begin{aligned}\frac{dV(t)}{dt} &= \frac{1}{\varepsilon_2 + \mu}\Theta'(t) \\ &= \frac{1}{\varepsilon_2 + \mu}\Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\lambda(k_i)\varphi(k_i)S_{k_i}(t)\right) - \varepsilon_2 - \mu\right) \\ &\leq \frac{1}{\varepsilon_2 + \mu}\Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\lambda(k_i)\varphi(k_i)S_{k_i}^0\right) - \varepsilon_2 - \mu\right) \\ &= \frac{1}{\varepsilon_2 + \mu}\Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{\alpha\lambda(k_i)\varphi(k_i)}{\varepsilon_1 + \mu} - \varepsilon_2 - \mu\right) \\ &= \Theta(t)\left(\frac{\alpha}{\langle k\rangle}\sum_{i=1}^{n}\frac{\lambda(k_i)\varphi(k_i)}{(\varepsilon_1 + \mu)(\varepsilon_2 + \mu)} - 1\right) \\ &= \Theta(t)(r_0 - 1).\end{aligned}$$

When $r_0 < 1$, we have $dV(t)/dt < 0$. Thus, as time approaches infinity, E_0 is G-stable. □

2) Stability of E_+: For the stability of system (1) at E_+, we have the following theorem.

Theorem 4: If $r_0 > 1$, E_+ is G-stable.

Proof: We construct the Lyapunov function $V(t)$ as

$$\begin{aligned}V(t) = \frac{1}{2}\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{S_{k_i}^{+}}\left(\varphi(k_i)\left(S_{k_i}(t) - S_{k_i}^{+}\right)^2\right) \\ + \left(\Theta(t) - \Theta^{+} - \Theta^{+}\ln\left(\frac{\Theta(t)}{\Theta^{+}}\right)\right).\end{aligned}$$

Then, the time derivative of $V(t)$ computed in the solution space of system (1) is

$$\begin{aligned}V'(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)\left(S_{k_i}(t) - S_{k_i}^{+}\right)S'_{k_i}\right) \\ + \frac{\Theta(t) - \Theta^{+}}{\Theta(t)}\Theta'(t). \quad (8)\end{aligned}$$

For clarity, we split (8) into two parts. When system (2) converges to E_+, from the first equation of system (3), we have

$\alpha = \lambda(k_i)S_{k_i}^{+}\Theta^{+} + \varepsilon_1 S_{k_i}^{+}$. Thus, for the first part of (8), we have

$$
\begin{aligned}
&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)\left(S_{k_i}(t)-S_{k_i}^{+}\right)S'_{k_i}\right)\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)\left(S_{k_i}(t)-S_{k_i}^{+}\right)\right.\\
&\qquad\left.\times\left(\alpha-\lambda(k_i)S_{k_i}(t)\Theta(t)-\varepsilon_1 S_{k_i}(t)-\mu S_{k_i}(t)\right)\right)\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)\left(S_{k_i}(t)-S_{k_i}^{+}\right)\right.\\
&\qquad\times\left(\lambda(k_i)S_{k_i}^{+}\Theta^{+}+(\varepsilon_1+\mu)S_{k_i}^{+}-\lambda(k_i)S_{k_i}(t)\Theta(t)\right.\\
&\qquad\left.\left.-(\varepsilon_1+\mu)S_{k_i}(t)\right)\right)\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon_1+\mu)\left(S_{k_i}(t)-S_{k_i}^{+}\right)^2\right.\\
&\qquad\left.-\varphi(k_i)\lambda(k_i)\left(\Theta(t)-\Theta^{+}\right)\left(S_{k_i}(t)-S_{k_i}^{+}\right)\right)
\end{aligned}
\tag{9}
$$

and for the second part of (8), we have

$$
\begin{aligned}
&\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\Theta'(t)\\
&=\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\left[\Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}(t)-\varepsilon_2-\mu\right)\right]\\
&=\left(\Theta(t)-\Theta^{+}\right)\left[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}(t)\right.\\
&\qquad\left.-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}^{+}\right]\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)\left(\Theta(t)-\Theta^{+}\right)\left(S_{k_i}(t)-S_{k_i}^{+}\right).
\end{aligned}
\tag{10}
$$

Combining (9) and (10), we can derive the result of (8) as follows:

$$
\begin{aligned}
V'(t)&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left\{\frac{1}{S_{k_i}^{+}}\varphi(k_i)\left(S_{k_i}(t)-S_{k_i}^{+}\right)S'_{k_i}\right\}+\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\Theta(t)'\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon_1+\mu)\left(S_{k_i}(t)-S_{k_i}^{+}\right)^2\right.\\
&\qquad\left.-\varphi(k_i)\lambda(k_i)\left(\Theta(t)-\Theta^{+}\right)\left(S_{k_i}(t)-S_{k_i}^{+}\right)\right)\\
&\quad+\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)\left(\Theta(t)-\Theta^{+}\right)\left(S_{k_i}(t)-S_{k_i}^{+}\right)\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon_1+\mu)\left(S_{k_i}(t)-S_{k_i}^{+}\right)^2\right)\\
&\le 0.
\end{aligned}
$$

Thus, E_{+} is G-stable. □

Based on the discussion of the existence and stability of equilibrium solutions under continuous countermeasures, we have the following conclusion regarding the threshold.

Theorem 5: If strong continuous countermeasures are carried out making $r_0 < 1$, the rumor would become extinct. Otherwise, if the continuous countermeasures are so weak that $r_0 \ge 1$, the rumor would continuously spread, and the user states would converge to the positive-equilibrium solution.

Theorem 5 indicates that to restrain rumor spread, countermeasures should be carried out to let $r_0 < 1$.

IV. Real-Time Optimization Strategy

Here, we introduce the RTO strategy to restrain rumor spreading at the end of an expected time period with the lowest cost. To specify the cost, we employ c_1 and c_2 to represent the average cost of immunizing an S user and curing an I user, respectively. The expected time period is $(0, t_f]$. Then, the RTO strategy motivates the following objective function:

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$$
\begin{aligned}
J\left(\varepsilon_1(t),\varepsilon_2(t)\right)=\min&\left\{\sum_{i=1}^{n}I_{k_i}(t_f)\right.\\
&\left.+\int_0^{t_f}\sum_{i=1}^{n}\left(c_1F\left(\varepsilon_1(t),S_{k_i}(t)\right)+c_2G\left(\varepsilon_2(t),I_{k_i}(t)\right)\right)dt\right\}
\end{aligned}
\tag{11}
$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are control variables. The feasible region of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ is $U = \{(\varepsilon_1(t), \varepsilon_2(t)) | 0 \le \varepsilon_1(t) \le \varepsilon_1^{\max}, 0 \le \varepsilon_2(t) \le \varepsilon_2^{\max}, t \in (0, t_f]\}$, where $\varepsilon_1^{\max}$ and $\varepsilon_2^{\max}$ are the upper bounds of ε_1 and ε_2, respectively.

The objective function (11) incorporates two optimization objectives. First, $\sum_{i=1}^{n} I_{k_i}(t_f)$ is the number of the I users at t_f so that minimizing it can guarantee that the rumor can be restrained at t_f. Second, $\int_0^{t_f}\sum_{i=1}^{n}(c_1F(\varepsilon_1(t), S_{k_i}(t)) + c_2G(\varepsilon_2(t), I_{k_i}(t)))dt$ is the total cost of the two methods in $(0, t_f]$ within which immunization and cure are carried out. Functions F and G represent the value of immunized S users and I users at each time t, respectively. F and G are quadratic functions that are popular. Thus, the objective function can be rewritten as

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$$
\begin{aligned}
J\left(\varepsilon_1(t),\varepsilon_2(t)\right)=\min&\left\{\sum_{i=1}^{n}I_{k_i}(t_f)\right.\\
&\left.+\int_0^{t_f}\sum_{i=1}^{n}\left(c_1\varepsilon_1^2(t)S_{k_i}^2(t)+c_2\varepsilon_2^2(t)I_{k_i}^2(t)\right)dt\right\}.
\end{aligned}
\tag{12}
$$

The main challenge of computing the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ is that system (1) can be solved on the condition that $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are unknown. Moreover, an exhaustive search in U is impossible since an infinite number of such $\varepsilon_1(t)$'s and $\varepsilon_2(t)$'s are in U. Fortunately, Pontryagin's maximum principle offers an efficient way based on which the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ can be easily derived. Based on Pontryagin's maximum principle [17], our optimized control problem is shown as follows.

Input:

1) Dynamic control system: rumor spreading model, i.e., system (1).
2) Initial conditions: $I_{k_i}(t_0) > 0$, $S_{k_i}(t_0) = 1 - I_{k_i}(t_0)$, and $R_{k_i}(t_0) = 0$, where $t_0 = 0$ and $i = 1, 2, \ldots, n$.
3) Transversality conditions: $\psi_i(t_f) = 0$ and $\phi_i(t_f) = 1$ $(i = 1, 2, \ldots, n)$, where ψ_i and ϕ_i are costate functions of group i.
4) Admissible controls: $\varepsilon_1(t)$ and $\varepsilon_2(t)$, where $t \in (0, t_f]$, $0 \le \varepsilon_1(t) \le \varepsilon_1^{\max}$, $0 \le \varepsilon_2(t) \le \varepsilon_2^{\max}$, and $i = 1, 2, \ldots, n$.

Output: The optimized controls $(\varepsilon_1^*(t), \varepsilon_2^*(t))$ that satisfy the objective function (12).

According to Pontryagin's maximum principle, if there exist continuous and differentiable adjoint functions $\psi_i(t)$ and $\phi_i(t)$ at each $t \in (0, t_f]$, $\varepsilon_1^*(t)$ and $\varepsilon_2^*(t)$ satisfy

$$(\varepsilon_1^*(t), \varepsilon_2^*(t)) \in \arg\max H\{(\psi_i(t), \phi_i(t)), (S_{k_i}(t), I_{k_i}(t)), U\}$$

where the Hamiltonian function H is defined as

$$\begin{aligned} H = &\sum_{i=1}^{n} \left(c_1 \varepsilon_1^2(t) S_{k_i}^2(t) + c_2 \varepsilon_2^2(t) I_{k_i}^2(t)\right) \\ &+ \left(\sum_{i=1}^{n} \psi_i(t)(\alpha - \lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_1(t) S_{k_i}(t) - \mu S_{k_i}(t))\right) \\ &+ \left(\sum_{i=1}^{n} \phi_i(t)(\lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_2(t) I_{k_i}(t) - \mu I_{k_i}(t))\right) \end{aligned}$$

S'

I'

in which adjoint functions $\psi_1(t)$ and $\psi_2(t)$ are defined as follows:

$$\begin{aligned} \frac{d\psi_i(t)}{dt} &= -\frac{\partial H}{\partial S_{k_i}(t)} \\ &= -2c_1 \varepsilon_1^2(t) S_{k_i}(t) - \psi_i(t)\left(-\lambda(k_i)\Theta(t) - \varepsilon_1(t) - \mu\right) \\ &\quad - \phi_i(t)\lambda(k_i)\Theta(t) \\ \frac{d\phi_i(t)}{dt} &= -\frac{\partial H}{\partial I_{k_i}(t)} \\ &= -2c_2 \varepsilon_2^2(t) I_{k_i}(t) + \psi_i(t)\langle k\rangle^{-1}\varphi(k_i)\lambda(k_i) S_{k_i}(t) \\ &\quad - \phi_i(t)\left(\langle k\rangle^{-1}\varphi(k_i)\lambda(k_i) S_{k_i}(t) - \varepsilon_2(t) - \mu\right). \end{aligned}$$

Since the density of I users is minimized at t_f, the transversality conditions are

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$$\psi_i(t_f) = 0, \ \phi_i(t_f) = 1.$$

Since H is a quadratic convex function of $\varepsilon_1(t)$ and $\varepsilon_2(t)$, the maximum value of H is obtained at the stationary point (i.e., $\partial H/\partial \varepsilon_1(t) = 0$ and $\partial H/\partial \varepsilon_2(t) = 0$) or at the start point and end point. We first compute the stationary point of H as

$$\begin{aligned} \frac{\partial H}{\partial \varepsilon_1(t)} &= 2c_1 \varepsilon_1(t) \sum_{i=1}^{n} S_{k_i}^2(t) - \sum_{i=1}^{n} \psi_i(t) S_{k_i}(t) \\ \frac{\partial H}{\partial \varepsilon_2(t)} &= 2c_2 \varepsilon_2(t) \sum_{i=1}^{n} I_{k_i}^2(t) - \sum_{i=1}^{n} \phi_i(t) I_{k_i}(t). \end{aligned} \tag{13}$$

Let $\partial H/\partial \varepsilon_1(t) = 0$ and $\partial H/\partial \varepsilon_2(t) = 0$. From (13), we have

$$\varepsilon_1(t) = \frac{\sum_{i=1}^{n} \psi_i(t) S_{k_i}(t)}{2c_1 \sum_{i=1}^{n} S_{k_i}^2(t)}, \ \varepsilon_2(t) = \frac{\sum_{i=1}^{n} \phi_i(t) I_{k_i}(t)}{2c_2 \sum_{i=1}^{n} I_{k_i}^2(t)}.$$

Finally, the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ under the objective function (12) in $(0, t_f]$ are

$$\begin{aligned} \varepsilon_1^*(t) &= \min\left\{\max\left(0, \varepsilon_1(t)\right), \varepsilon_1^{\max}\right\} \\ \varepsilon_2^*(t) &= \min\left\{\max\left(0, \varepsilon_2(t)\right), \varepsilon_2^{\max}\right\}. \end{aligned} \tag{14}$$

Thus, (14) provide the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ to restrain rumor spreading at the end of the expected time period $(0, t_f]$ with the lowest cost.

V. Pulse Spreading Truth and Continuous Blocking Rumor Strategy

Here, we introduce the PSCB strategy to restrain rumor spreading with efficient cost. PSCB spreads truth in every other period T. Specifically, at any time $t = mT (m = 1, 2, \ldots)$, an S user is immunized to transform to R with a certain probability. With the state transition rule stated in Section II-A, the heterogeneous-network-based epidemic model with pulse immunization is shown as follows:

$$\begin{cases} t \neq mT: \\ \dfrac{dS_{k_i}(t)}{dt} = \alpha - \lambda(k_i) S_{k_i}(t)\Theta(t) - \mu S_{k_i}(t) \\ \dfrac{dI_{k_i}(t)}{dt} = \lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) - \mu I_{k_i}(t) \\ \dfrac{dR_{k_i}(t)}{dt} = \varepsilon_2 I_{k_i}(t) - \mu R_{k_i}(t) \\ \dfrac{dD_{k_i}(t)}{dt} = \mu S_{k_i}(t) + \mu I_{k_i}(t) + \mu R_{k_i}(t) \\ \qquad i = 1, 2, \ldots, n, \ t > 0 \\ t = mT: \\ S_{k_i}(mT^+) = (1 - p) S_{k_i}(mT) \\ I_{k_i}(mT^+) = I_{k_i}(mT) \\ R_{k_i}(mT^+) = R_{k_i}(mT) + p S_{k_i}(mT) \\ D_{k_i}(mT^+) = D_{k_i}(mT). \end{cases} \tag{15}$$

In system (15), pulse immunization is applied to $pS_{k_i}(mT)$ S users so that they can transform to R users at time $mT (m = 1, 2, \ldots)$. It is worth noting that the I users do not have any changes at time $mT (m = 1, 2, \ldots)$ since we carry out continuous cure. When $t = mT$, the state transition rule is similar to system (1).

A. Existence of the Zero-Equilibrium Solution

Similarly, since users of state D no longer transform to other states, for simplicity, we just analyze the active states (i.e., S, I, R) in the following parts. For the existence of the zero-equilibrium solution, we have the following theorem.

Theorem 6: If the immunization period is T, the T-period zero-equilibrium solution of system (15) is $(S_{k_i}^T(t), 0, 1 - S_{k_i}^T(t))$, where

$$S_{k_i}^T(t) = \frac{\alpha}{\mu} + \left(S_{k_i}^* - \frac{\alpha}{\mu}\right) e^{-\mu(t-mT)}$$

$$\text{where } S_{k_i}^* = \frac{\alpha}{\mu}\frac{(1-p)(1-e^{-\mu T})}{1-(1-p)e^{-\mu T}}.$$

Proof: From system (15), we observe that the first and second equations do not include R_{k_i} and D_{k_i} when $t \neq T$. Hence, we first analyze them and then derive $R_{k_i}(t)$ from $S_{k_i}(t) + I_{k_i}(t) + R_{k_i}(t) = 1$. Then, we just analyze the following subsystem:

$$\begin{aligned}
&t \neq mT:\\
&\quad \frac{dS_{k_i}(t)}{dt} = \alpha - \lambda(k_i) S_{k_i}(t)\Theta(t) - \mu S_{k_i}(t)\\
&\quad \frac{dI_{k_i}(t)}{dt} = \lambda(k_i) S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) - \mu I_{k_i}(t)\\
&\qquad\qquad i = 1, 2, \ldots, n,\ t > 0\\
&t = mT:\\
&\quad S_{k_i}(mT^+) = (1-p) S_{k_i}(mT)\\
&\quad I_{k_i}(mT^+) = I_{k_i}(mT).
\end{aligned} \tag{16}$$

According to the definition of the zero-equilibrium solution described in Section III-A, we have $I_{k_i}(t) = 0\ (i = 1, 2, \ldots, n)$ when system (15) gets the zero-equilibrium solution. Substituting $I_{k_i}(t) = 0$ into system (16), we have

$$\begin{aligned}
\frac{dS_{k_i}(t)}{dt} &= \alpha - \mu S_{k_i}(t), \quad t \neq mT\\
S_{k_i}(mT^+) &= (1-p) S_{k_i}(mT), \quad t = mT.
\end{aligned} \tag{17}$$

In time interval $(mT, (m+1)T]$, the solution of system (17) is $S_{k_i}(t) = (\alpha/\mu) + (S_{k_i}(mT^+) - (\alpha/\mu))e^{-\mu(t-mT)}$. Represent $S_{k_i}(mT^+)$ by $S_{k_i}^m$. Meanwhile, in the case of $t = mT$, we have

$$\begin{aligned}
S_{k_i}\left((m+1)T^+\right) &= (1-p) S_{k_i}\left((m+1)T\right)\\
&= (1-p)\left(\frac{\alpha}{\mu} + \left(S_{k_i}(mT^+) - \frac{\alpha}{\mu}\right) e^{-\mu T}\right).
\end{aligned}$$

Hence, we can construct a map $S_{k_i}^{m+1} = f(S_{k_i}^m)$, in which

$$f\left(S_{k_i}\right) = (1-p)\left(\frac{\alpha}{\mu} + \left(S_{k_i} - \frac{\alpha}{\mu}\right) e^{-\mu T}\right). \tag{18}$$

Thus, solving the map function (18), we can obtain the only equilibrium, i.e.,

$$S_{k_i}^* = \frac{\alpha}{\mu}\frac{(1-p)(1-e^{-\mu T})}{1-(1-p)e^{-\mu T}}.$$

Moreover

$$\frac{df}{dS_{k_i}} = (1-p)e^{-\mu T} < 1.$$

Hence, $S_{k_i}^*$ is the T-peroid zero-equilibrium solution of $S_{k_i}(t)$ at a series of time $t = mT$. As such, the T-peroid zero-equilibrium solution of system (17) is

$$S_{k_i}^T(t) = \frac{\alpha}{\mu} + \left(S_{k_i}^* - \frac{\alpha}{\mu}\right) e^{-\mu(t-mT)}.$$

Thus, $(S_{k_i}^T(t), 0, 1 - S_{k_i}^T(t))$ is the $T-$peroid zero-equilibrium solution of system (15). □

B. *Stability of the T—Peroid Zero-Equilibrium Solution*

For the local asymptotical stability of the T—peroid zero-equilibrium solution, we can get the following theorem.

Theorem 7: If $R_0^{k_i} < 1$, the T—peroid zero-equilibrium solution of system (15) is L-stable, where

$$R_0^{k_i} = \frac{\alpha}{\mu}\frac{\lambda(k_i)}{\varepsilon_2 + \mu}\left(1 - \frac{p(1-e^{-\mu T})}{T\mu\left(1-(1-p)e^{-\mu T}\right)}\right).$$

Proof: To verify the L-stability of $(S^T(t), 0)$, we first let

$$\begin{aligned}
S_{k_i}(t) &= S_{k_i}^T(t) + s_{k_i}(t)\\
I_{k_i}(t) &= i_{k_i}(t)
\end{aligned} \tag{19}$$

where $s_{k_i}(t)$ and $i_{k_i}(t)$ are small perturbations on $(S_{k_i}^T(t), 0)$. Substituting (19) into System (16) and expanding by the Taylor Series while neglecting the high-order terms, we can obtain a linear pulse differential system of system (16) as follows:

$$\begin{aligned}
\frac{ds_{k_i}(t)}{dt} &= -\mu s_{k_i}(t) - \lambda(k_i) S_{k_i}^T(t)\theta(t)\\
\frac{di_{k_i}(t)}{dt} &= \lambda(k_i) S_{k_i}^T(t)\theta(t) - (\varepsilon_2 + \mu) i_{k_i}(t)
\end{aligned} \tag{20}$$

where $\theta(t) = (1/\langle k\rangle)\sum_{i=1}^n \omega(k_i)P(k_i) i_{k_i}(t)$. Then, from system (20), we can obtain the fundamental solution matrix of linear system (20)

$$B_{k_i}(t) = \begin{pmatrix} e^{-\mu t} & \gamma_{12}(t)\\ 0 & \gamma_{22}^{k_i}(t)\end{pmatrix}$$

where $\gamma_{22}^{k_i}(t) = e^{\int_0^T \lambda(k_i) S_{k_i}^T(\tau) - \varepsilon_2 - \mu d\tau}$. When $t = nT$, from system (16), we have

$$\begin{pmatrix} s_{k_i}(mT^+)\\ i_{k_i}(mT^+)\end{pmatrix} = \begin{pmatrix} 1-p & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} s_{k_i}(mT)\\ i_{k_i}(mT)\end{pmatrix}.$$

According to the Floquet theorem [18], we have

$$\begin{aligned}
M_{k_i} &= \begin{pmatrix} 1-p & 0\\ 0 & 1\end{pmatrix} B_{k_i}(t)\\
&= \begin{pmatrix} 1-p & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} e^{-\mu T} & \gamma_{12}(t)\\ 0 & e^{\int_0^T \lambda(k_i) S_{k_i}^T(\tau) - \varepsilon_2 - \mu\, d\tau}\end{pmatrix}\\
&= \begin{pmatrix} (1-p)e^{-\mu T} & (1-p)\gamma_{12}(t)\\ 0 & e^{\int_0^T \lambda(k_i) S_{k_i}^T(\tau) - \varepsilon_2 - \mu\, d\tau}\end{pmatrix}.
\end{aligned} \tag{21}$$

Thus, the eigenvalues of (21) are

$$\begin{aligned}
\beta_1^{k_i} &= (1-p)e^{-\mu T}\\
\beta_2^{k_i} &= e^{\int_0^T \lambda(k_i) S_{k_i}^T(\tau) - \varepsilon_2 - \mu\, d\tau}.
\end{aligned}$$

According to the Floquet theorem, we know that the equilibrium state of system (16) is L-stable if $|\beta_j^{k_i}| < 1$, where $j = 1, 2$. Obviously, $0 < |\beta_1^{k_i}| < 1$. Thus, the stability of the T-period zero-equilibrium solution of system (16) just requires $|\beta_2^{k_i}| < 1$, namely, $e^{\int_0^T \lambda(k_i) S_{k_i}^T(t) - \varepsilon_2 - \mu \, dt} < 0$. Solving it, we can obtain $R_0^{k_i} < 1$, and the value of $R_0^{k_i}$ is as described in Theorem 7. □

For the G-stability of the T-peroid zero-equilibrium solution, we have the following theorem.

Theorem 8: If $\widetilde{R}_0^{k_i} < 1$, the T-peroid zero-equilibrium solution of system (15) is G-stable, where

$$\widetilde{R}_0^{k_i} = \frac{\alpha}{\mu} \frac{\lambda(k_i)}{\varepsilon_2 + \mu} \left(1 - \frac{p}{1 - (1-p)e^{-\mu T}}\right).$$

Proof: From the first equation of system (16), we have $dS_{k_i}(t)/dt \le \alpha - \mu S_{k_i}(t)$. Then, we can construct the comparison system as follows:

$$\begin{aligned} x'_{k_i} &= \alpha - \mu x_{k_i}(t), \quad t \ne mT \\ x_{k_i}(mT^+) &= (1-p)x_{k_i}(mT), \quad t = mT. \end{aligned} \tag{22}$$

With the similar procedure, we can get the T-period zero-equilibrium solution of system (22) as

$$x_{k_i}^T(t) = \frac{\alpha}{\mu} + \left(x_{k_i}^* - \frac{\alpha}{\mu}\right) e^{-\mu(t - mT)} \tag{23}$$

where $x_{k_i}^* = (\alpha/\mu)((1-p)(1 - e^{-\mu T}))/(1 - (1-p)e^{-\mu T})$.

According to the differential equation comparison theorem [19] $\forall \kappa > 0, \exists M_1 \in \mathbb{N}_+$, when $m \ge M_1$, we have

$$\begin{aligned} &S_{k_i}(t) \le x_{k_i}^T(t) + \kappa \le x_{k_i}^* + \kappa \\ &t \in (mT, (m+1)T], \; m > M_1. \end{aligned}$$

From the second equation of system (15), we have

$$\frac{dI_{k_i}(t)}{dt} \le \lambda(k_i)\Theta(t)(x^* + \kappa) - (\varepsilon_2 + \mu) I_{k_i}(t). \tag{24}$$

Then, we can construct the comparison equation of (24) as

$$\frac{dy_{k_i}(t)}{dt} \le \lambda(k_i)\Theta_y(t)(x^* + \kappa) - (\varepsilon_2 + \mu) y_{k_i}(t)$$

where $\Theta_y(t) = (1/\langle k \rangle) \sum_{i=1}^n \omega(k_i) P(k_i) y_{k_i}(t)$.

Since $\widetilde{R}_0^{k_i} < 1$, there exists a sufficiently small κ_{k_i} letting $\lambda(k_i)(x_{k_i}^* + \kappa_{k_i}) < (\varepsilon_2 + \mu)$; hence, $\lim_{t\to\infty} y_{k_i}(t) = 0$. According to the differential equation comparison theorem, we also have $\lim_{t\to\infty} I_{k_i}(t) = 0$. That is, for $\forall \kappa > 0, \exists M_2 > M_1$, when $m \ge M_2$, we have $I_{k_i}(t) < \kappa$. Moreover, from the first equation of system (16), we have

$$\frac{S_{k_i}(t)}{dt} \ge \alpha - \lambda(k_i)\kappa S_{k_i}(t) - \mu S_{k_i}(t).$$

Then, we can construct the comparison equation of system (16) as

$$\begin{aligned} \frac{z_{k_i}(t)}{dt} &= \alpha - \lambda(k_i)\kappa z_{k_i}(t) - \mu z_{k_i}(t), \quad t \ne mT \\ z_{k_i}(mT^+) &= (1-p) z_{k_i}(mT), \quad t = mT. \end{aligned} \tag{25}$$

Similar to the T-period zero-equilibrium solution of system (22) [i.e., (23)], we have the T-period zero-equilibrium solution of system (25) in $(nT, (n+1)T]$ as

$$z_{k_i}^T(t) = \frac{\alpha}{\mu + \lambda(k_i)\kappa} + \left(1 - \frac{p e^{-(\mu + \lambda(k_i)\kappa)(t - mT)}}{1 - (1-p)e^{-(\mu + \lambda(k_i)\kappa)T}}\right).$$

According to the comparison theorem [19], for $\forall \kappa > 0, \exists M_3 > M_2$, when $m \ge M_3$, we have $z_{k_i}^T(t) - \kappa \le S_{k_i}(t) \le x_{k_i}^T(t) + \kappa$. Meanwhile, we found that $\lim_{\kappa\to 0} x_{k_i}^T(t) = S_{k_i}^T(t)$ and $\lim_{\kappa\to 0} z_{k_i}^T(t) = S_{k_i}^T(t)$. Hence, we have $\lim_{t\to\infty} S_{k_i}(t) = S_{k_i}^T(t)$. Hence, the T-period zero-equilibrium solution $(S_{k_i}^T(t), 0)$ is G-stable for system (16). That is, $(S_{k_i}^T(t), 0, 1 - S_{k_i}^T(t))$ is G-stable for system (15). □

Based on the given discussion of the existence and stability of the T-period zero-equilibrium solution under pulse countermeasures, we have the following conclusion.

Theorem 9: If strong pulse countermeasures are carried out making $\widetilde{R}_0^{k_i} < 1$, the rumor would become extinct. Otherwise, if the pulse countermeasures are so weak that $\widetilde{R}_0^{k_i} \ge 1$, the rumor would continuously spread.

C. Maximum Immunization Period

Immunizing the S users with the maximum immunization period can minimize the frequency of occupying network resources. Here, we derive the maximum immunization period $T_{\max}(k_i)$ as the threshold that determines whether the T-period zero-equilibrium solution in group i is G-stable, $i = 1, 2, \ldots, n$. Hence, any immunization period $T_{k_i}(T_{k_i} < T_{\max}(k_i))$ would guarantee that the rumors become extinct in group i. On the contrary, any immunization period $T_{k_i}(T_{k_i} > T_{\max}(k_i))$ would guarantee that the rumors continuously spread in group i. Based on Theorem 8, we just need to let $\widetilde{R}_0^{k_i} = 1$ to obtain

$$T_{\max 1}(k_i) = \frac{1}{\mu} \cdot \ln\left((1-p) \cdot \frac{\alpha\lambda(k_i) - \mu(\varepsilon_2 + \mu)}{\alpha\lambda(k_i) - \mu(\varepsilon_2 + \mu) - p\alpha\lambda(k_i)}\right).$$

VI. Model Validation

Here, we validate the proposed rumor spreading model and the RTO and PSCB strategies toward the Digg2009 data set [20]. The Digg2009 data set contains 1 731 658 friendship links of 71 367 users. According to different social connectivity degrees, these 71 367 users are divided into 848 groups. The maximum degree of this data set is 995, and the minimum degree is 1. The average degree of this data set is around 24, i.e., $\langle k \rangle = 24$.

A. Effectiveness of the Thresholds of the RTO Strategy

Theorem 5 shows that the threshold is r_0, which gives the specific relationship between rumor spreading dynamics and the countermeasure levels ε_1 and ε_2. We assume that the rumor acceptance rate grows linearly with the users' degree, namely, $\lambda(k_i) = k_i$. Meanwhile, we take the nonlinear infectivity as illustrated in Section III: $\omega(k_i) = k_i^\beta/(1 + k_i^\gamma)$ with $\beta = 0.5$ and $\gamma = 0.5$. Other parameters in system (1) are set as $\alpha = 0.01$, $\varepsilon_1 = 0.2$, and $\varepsilon_2 = 0.05$. We can compute that $r_0 = 0.7220 < 1$

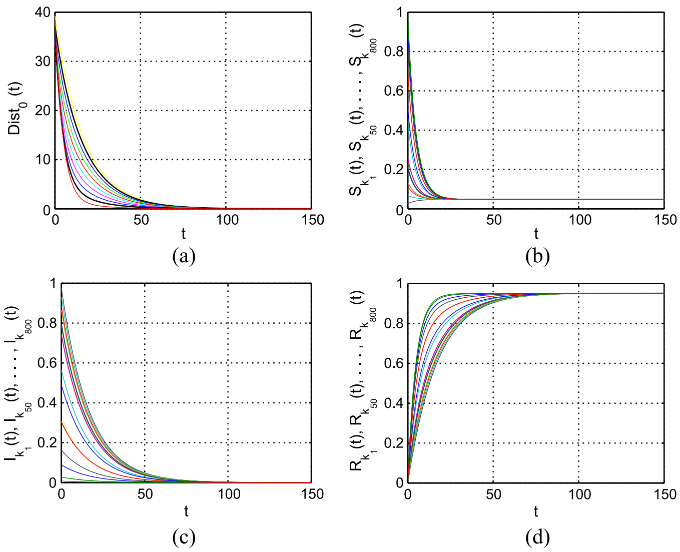

Fig. 2. (a) Evolution of $\mathrm{Dist}_0(t)$ under ten different initial conditions. (b)–(d) Evolutions of $S_{k_i}(t)$, $I_{k_i}(t)$, and $R_{k_i}(t)$, respectively, where $i = 1, 50, \ldots, 800$, and $r_0 = 0.7220 < 1$.

so that E_0 is G-stable (as indicated by Theorem 3). According to Theorem 5, in this case, the rumor will be extinct, and the state of system (1) would converge to E_0.

Assuming that $E(t)$ is an arbitrary solution of system (1), we employ $\mathrm{Dist}_0(t)$ to denote the Euclidean distance between $E(t)$ and E_0, i.e.,

$$\mathrm{Dist}_0(t) = \|E(t) - E_0\|_\infty .$$

Under ten different initial values (i.e., different $S_{k_i}(0)$ and $I_{k_i}(0)$, $R_{k_i}(0) = 0$), the evolution of $\mathrm{Dist}_0(t)$ is shown in Fig. 2(a). We observe that $\mathrm{Dist}_0(t)$ converges to zero under different initial conditions, which means that E_0 is G-stable. Then, Fig. 2(b)–(d) shows the evolutions of S_{k_i}, I_{k_i}, and R_{k_i}, $i = 1, 50, 100, \ldots, 800$ under an arbitrary initial condition, respectively. We observe that the infection is no longer epidemic and the rumor will be extinct with such countermeasures.

Keeping the other parameters unchanged, we set $\alpha = 0.002$, $\varepsilon_1 = 0.002$, and $\varepsilon_2 = 0.0001$ and compute that $r_0 = 2.1661 > 1$. In this case, E_+ is G-stable (as indicated by Theorem 4). According to Theorem 5, the rumor will continuously spread, and system variables converge to E_+. Similarly, to verify it, the Euclidean distance between $E(t)$ and E_+ is measured by

$$\mathrm{Dist}_+(t) = \|E(t) - E_+\|_\infty .$$

The evolution of $\mathrm{Dist}_+(t)$ under ten initial conditions is shown in Fig. 3(a). We observe that $\mathrm{Dist}_+(t)$ converges to zero under different initial conditions, which means that E_+ is G-stable. Then, setting an arbitrary initial condition, the evolutions of S_{k_i}, I_{k_i}, and R_{k_i}, $i = 1, 2, \ldots, 20$ are shown in Fig. 3(b)–(d), respectively. We observe that with such level of countermeasures, a rumor will continuously spread, and the system variables converge to E_+.

B. *Efficiency of the RTO Strategy*

Suppose the expected time period is (0, 100] and the cost of *immunization* is larger than that of *cure*, specifically, $c_1 = 5$ and $c_2 = 10$. Keeping the other parameters unchanged, the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ ($t \in (0, 100]$) are shown in Fig. 4(a). As shown in Fig. 4(a), spreading truth should play a dominant role (i.e., $\varepsilon_1 > \varepsilon_2$) in the initial rumor restraining phase. Then, when approaching the end of the expected time period, blocking rumors should be carried out intensively (i.e., $\varepsilon_1 < \varepsilon_2$).

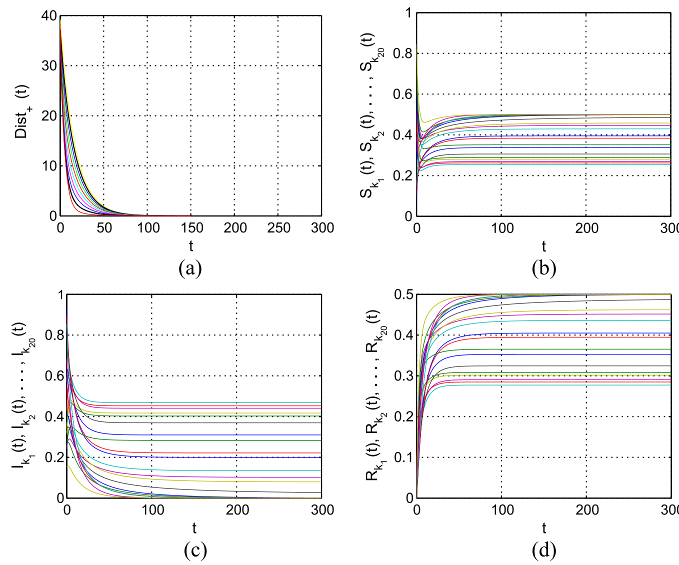

Fig. 3. (a) Evolution of $\mathrm{Dist}_+(t)$ under ten different initial conditions. (b)–(d) Evolutions of $S_{k_i}(t)$, $I_{k_i}(t)$, and $R_{k_i}(t)$, where $i = 1, 2, \ldots, 20$, and $r_0 = 2.1661 > 1$.

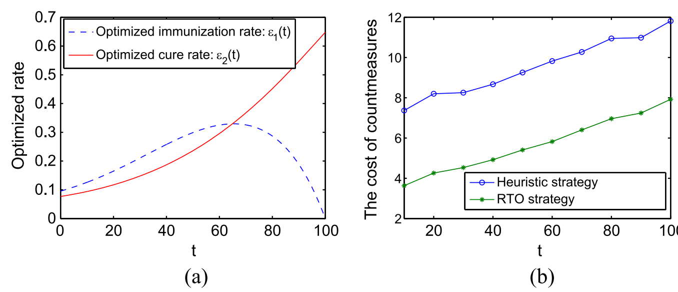

Fig. 4. (a) Evolution of ε_1 and ε_2. (b) Cost comparison of the heuristic and RTO strategies where the horizontal axis is t_f.

To verify the efficiency of the RTO strategy, we compare the cost of the heuristic and RTO strategy when controlling the number of the I users to a same level within a same expected time period. The heuristic strategy restrains rumor spreading just based on the current infection state, i.e., there is no global control. We set a set of t_f such as $t_f = 10, 20, \ldots, 100$ and let the density of the I users at t_f to be less than 0.0001. The cost comparison of these ten different time periods is shown in Fig. 4(b). Compared with the heuristic strategy, the RTO strategy has lower cost while achieving the same effects.

C. *Effectiveness of the PSCB Strategy*

For simplicity, we represent the T-period zero-equilibrium solution of system (15) as E_0^p. According to Theorem 8, the thresholds in group i are determined by threshold $\widetilde{R}_0^{k_i}$. As indicated in Section V-C: If $T_{k_i} < T_{\max}(k_i)$, we expect that the infection in group i is no longer epidemic and that the

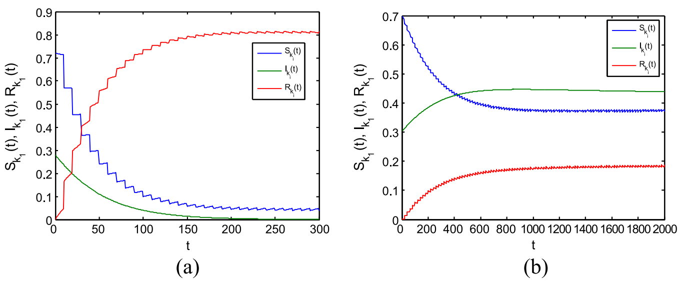

Fig. 5. (a) Evolutions of $S_{k_1}(t), I_{k_1}(t), R_{k_1}(t)$ when $T_{k_1} < T_{\max}(k_1)$. (b) Evolutions of $S_{k_1}(t), I_{k_i}(t), R_{k_1}(t)$ when $T_{k_1} > T_{\max}(k_1)$.

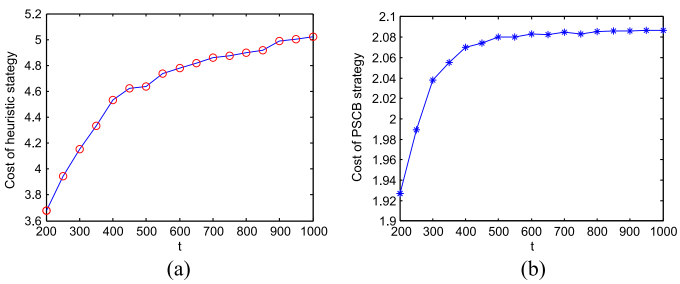

Fig. 6. Cost comparison between (a) the heuristic strategy and (b) the PSCB strategy when both of them have the same level of I users.

rumor will be extinct; otherwise, the rumor will continuously spread if $T_{k_i} > T_{\max}(k_i)$. The reason is that $T_{k_i} < T_{\max}(k_i)$ or $T_{k_i} > T_{\max}(k_i)$ guarantees E_0^p is G-stable (or not). For clarity, we only show the evolutions of S_{k_1}, I_{k_1}, and R_{k_1} under an arbitrary initial condition, respectively. Assuming immunization rate $p = 0.2$ and keeping the other parameters unchanged, we can compute $T_{\max}(k_1) = 12.3770$. Given $T_{k_i} = 5(T_{k_i} < T_{\max}(k_1)$, we expect the rumors in group i to become extinct. Given an arbitrary initial condition, the evolutions of S_{k_1}, I_{k_1}, and R_{k_1} are shown in Fig. 5(a). We observe that with frequent immunization, a rumor will become extinct and the system variables converge to E_0. Keeping the other parameters unchanged, we set $T_{k_1} = 20$, i.e., $T_{k_i} > T_{\max}(k_1)$. In this case, we expect the rumors to continuously spread. Then, setting an arbitrary initial condition, the evolutions of S_{k_1}, I_{k_1}, and R_{k_1} are shown in Fig. 5(b). We observe that with such countermeasures, the rumor continuously spreads.

To verify the cost efficiency of the PSCB strategy, we compare the costs of the heuristic strategy and the PSCB strategy when the densities of the I users reach to a same level within a same expected time period. Similarly, we set a set of t_f such as $t_f = 200, 250, \ldots, 1000$ and let the density of the I users at t_f to be less than 0.0001. The cost of countermeasures of PSCB with these ten different time periods is shown in Fig. 6(b). Compared with the heuristic strategy as shown in Fig. 6(a), the PSCB strategy has lower cost while achieving the same effect.

VII. Related Works

Rumor threats in MSNs have been extensively documented, and a large body of research efforts has been made. For restraining rumor spreading, the popular approach of the existing works is to select several seed users based on their social connectivity (such as degree, core, and betweeness) as truth spreaders or to block rumors at these selected seed users such as blocking the links [3] and removing influential nodes [5] [6]. In [12], it is shown which method has better performance: *blocking rumors at influential users* or *spreading truth to clarify rumors*. Taking users' preference into consideration, an analytical model has been developed in [21] to investigate the temporal dynamics of positive and negative information spreading in MSNs. Moreover, [22] proposes an efficient immunization strategy to restrain rumor spreading in MSNs based on the weighted trust networks.

On the other hand, various variants of epidemic models are proposed to investigate rumor spreading [23]–[25]. To name some, a Susceptible–Infected–Recovered (SIR) model that incorporates the immunization method is proposed [23]. An SIR model is proposed in [24] to investigate the final infected scale of the rumor spreading under different levels of countermeasures. Similarly, the work in [26] utilized the SIR model to describe the rumor spreading in microblog considering different rumor-blocking strategies such as the recovered opinion leader. In [27], the SIR model is extended into a partial differential equation model to describe the temporal–spatial dynamics of malware spreading in sensor networks.

Unfortunately, few existing works consider the implementation cost of the rumor restraining method. In [5], it is shown how to minimize the cost of the *spreading truth to clarify rumors* method, which is an NP-hard problem. In our prior work [28], a cost-efficient countermeasure is proposed. However, in [28], the limitation of rumor-restraining resources is ignored. In this paper, we improve the aforementioned works by combining different methods and investigate the effective and cost-efficient countermeasures.

VIII. Conclusion

In this paper, we have analyzed the rumor spreading dynamics in MSNs and proposed two cost-efficient strategies to restrain rumor spreading. On the basis of our rumor spreading model, the analysis results indicate that whether the rumor continuously spreads or becomes extinct is determined by a threshold, which formulates a restrictive relationship between network properties and countermeasures. In addition, we propose two cost-efficient strategies to restrain rumor spreading with a continuous (RTO) and pulse (PSCB) manner, respectively. For the RTO strategy, we obtain the optimized countermeasures that can efficiently restrain rumors within an expected time period. For the PSCB strategy, we can restrain rumors efficiently by immunizing susceptible users with the maximum immunization period. The experiment results show that both RTO and PSCB outperform the heuristic strategy.

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