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# Receding horizon $H_\infty$ control for discrete-time Markovian jump linear systems

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**Abstract:** Receding horizon $H_\infty$ control scheme which can deal with both the $H_\infty$ disturbance attenuation and mean square stability is proposed for a class of discrete-time Markovian jump linear systems when minimizing a given quadratic performance criteria. First, a control law is established for jump systems based on pontryagin's minimum principle and it can be constructed through numerical solution of iterative equations. The aim of this control strategy is to obtain an optimal control which can minimize the cost function under the worst disturbance at every sampling time. Due to the difficulty of the assurance of stability, then the above mentioned approach is improved by determining terminal weighting matrix which satisfies cost monotonicity condition. The control move which is calculated by using this type of terminal weighting matrix as boundary condition naturally guarantees the mean square stability of the closed-loop system. A sufficient condition for the existence of the terminal weighting matrix is presented in linear matrix inequality (LMI) form which can be solved efficiently by available software toolbox. Finally, a numerical example is given to illustrate the feasibility and effectiveness of the proposed method.

**Keywords:** Markov jump linear systems, receding horizon $H_\infty$ control, mean square stability, terminal weighting matrix, pontryagin's minimum principle, current time jump mode.

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## 1. Introduction

Markov jump linear systems (MJLS) are class of systems which have received much attention because of its theoretical and applicable value. Many real systems, such as solar thermal receivers, economic systems and networked control systems (NCS) may experience random abrupt changes in their inputs, internal variables and other system parameters and they have successfully been represented by MJLS [1]. The stabilization and other control problems of MJLS attract many researchers from the control and mathematical communities so that a lot of interesting results are reported [1−5].

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On the other hand, receding horizon control (RHC) (also known as model predictive control MPC) [6−10] which is a form of control algorithm has received much attention as a powerful tool for the control of process systems including MJLS. Compared with other control strategies, RHC is convenient to deal with both the constraint and performance optimality of the system.

RHC has been used for discrete-time MJLS without model uncertainty [11]. The robust RHC which considered uncertainty in both model parameters and transition probabilities for discrete-time MJLS is proposed in [12]. This RHC approach gave a linear state feedback gain for each mode at every sampling time and showed several advantages over conventional minimization of an upper bound on the worst-case cost function. The robust one-step RHC [13] is studied while the minimum value of the worst-case cost and a nonlinear control sequence are obtained by using an one-step receding horizon cost function. In [13], the guaranteed cost control, robust RHC, and one-step RHC are compared within a numerical example which showed that the last one can provide the best performance. Based on the results of [12], the infinite time RHC scheme is extended to uncertain MJLS subject to actuator saturation [14] and constrained RHC for MJLS was also discussed in [15,16] by introducing invariant sets which depend on the jump mode.

However, there are few results on RHC for discrete-time MJLS while $H_\infty$ disturbance attenuation is considered. Different from the optimization issues in [12−14], where the uncertainty is contained in the system matrix, the extra disturbance appears as another term in the performance criterion and it can not be sampled as available information. So we can not treat the RH $H_\infty$ control problem in a similar way as in the above mentioned references. First, a control strategy which can minimize the cost function under the worst disturbance is obtained based on the pontryagin's minimum principle [8]. However, the terminal weighting matrix must be obtained through cut and try method and the stability of the system can not be guaranteed under the control law. Then we develop an linear matrix inequality (LMI) based method to determine the terminal weighting matrix which depends on the cost monotonicity condition that leads to the mean square stability of the system. The control strategy can be implemented under the assumption that information of the current time jump mode is available. To the best of our knowledge, there is no similar one proposed for MJLS. Since the RHC scheme is based on the optimal control [7], a good $H_\infty$ control performance could be expected.

## 2. Problem formulation

Consider discrete-time MJLS described as follows

$$\begin{cases} x_{k+1} = A(r_k)x_k + B(r_k)u_k + B_w(r_k)w_k \\ z_k = C(r_k)x_k \end{cases} \tag{1}$$

$$\Pr\{r_{k+1} = j \mid r_k = i\} = p(i,j) \tag{2}$$

where $x_k \in R^n$ denotes system states, $u_k \in R^m$ control inputs, $w_k \in R^p$ disturbance, $z_k \in R^q$ controlled outputs. $A(r_k)$, $B(r_k)$, $C(r_k)$ and $B_w(r_k)$ are matrices with appropriate dimensions which depend on the jump mode $r_k$. Suppose the initial state and initial jump mode are $x(0) = x_0$ and $r(0) = r_0$, respectively. The mode process $\{r_k : k = 0, 1, \ldots\}$ is a finite-state Markov chain taking values in $S = \{1, 2, \ldots, s\}$ with transition probabilities $p(i,j)$. We always assume the complete access to the current time jump mode $r_k$. For $r_k = i \in S$, $A(r_k)$, $B(r_k)$, $C(r_k)$ and $B_w(r_k)$ can be noted as $A_i$, $B_i$, $C_i$ and $B_{wi}$ for simplicity.

To derive the cost function of RH $H_\infty$ control for MJLS (1), we first construct the system as

$$\begin{cases} x_{k+1} = A(r_k)x_k + B(r_k)u_k + B_w(r_k)w_k \\ \tilde{z}_k = \begin{bmatrix} Q^{\frac{1}{2}}x_k \\ R^{\frac{1}{2}}u_k \end{bmatrix} \end{cases} \tag{3}$$

where $\tilde{z}_k \in R^{q+n}$ is the controlled variable which needs to be regulated, $Q = Q^{\mathrm{T}} > 0$ and $R = R^{\mathrm{T}} > 0$ are weighting matrices. System (3) is introduced only for the purpose that we can combined the traditional quadratic cost function and $H_\infty$ criterion into a unified cost function which is to be optimized [8]. The $H_\infty$ norm of $T_{\tilde{z}w}(\mathrm{e}^{\mathrm{j}w})$ can be represented as

$$\|T_{\tilde{z}w}(\mathrm{e}^{\mathrm{j}w})\|_\infty = \mathrm{E}\{\sup_{w_{k+f}} \sigma(T_{\tilde{z}w}(\mathrm{e}^{\mathrm{j}w}))|x_0, r_0\} =$$

$$\mathrm{E}\left\{ \sup_{w_{k+f}} \frac{\sum_{f=0}^{\infty}[x_{k+f}^{\mathrm{T}} Q x_{k+f} + u_{k+f}^{\mathrm{T}} R u_{k+f}]}{\sum_{f=0}^{\infty}[w_{k+f}^{\mathrm{T}} w_{k+f}]} |x_0, r_0 \right\} \tag{4}$$

where $T_{\tilde{z}w}(\mathrm{e}^{\mathrm{j}w})$ can be viewed as a transfer function from variable $w_k$ to $\tilde{z}_k$ and symbol $\sigma$ represents the maximum singular value. Indexes $k$ and $f$ represent current time index and predictive step, respectively. However, it is hard to achieve an optimal $H_\infty$ control, so we introduce a sub-optimal $H_\infty$ performance criterion such that $\|T_{\tilde{z}w}(\mathrm{e}^{\mathrm{j}w})\|_\infty < \gamma^2$ for the given positive scalar $\gamma^2$. From simple algebraic calculation and also adding weighting matrix $R_w = R_w^{\mathrm{T}} > 0$ to $w_{k+f}$, we have

$$\sup_{w_{k+f}} \mathrm{E}\{\sum_{f=0}^{\infty}[\|x_{k+f}\|_Q^2 + \|u_{k+f}\|_R^2 - \gamma^2\|w_{k+f}\|_{R_w}^2]|x_0, r_0\} < 0 \tag{5}$$

where $\|x\|_Q^2$ is defined as $x^{\mathrm{T}}Qx$. Our purpose is to obtain an optimal control action to minimize the cost function

$$\sup_{w_{k+f}} \mathrm{E}\{\sum_{f=0}^{\infty}[\|x_{k+f}\|_Q^2 + \|u_{k+f}\|_R^2 - \gamma^2\|w_{k+f}\|_{R_w}^2]|x_0, r_0\}$$

(5) represents the infinite horizon case. When dealing with the finite horizon case, which is more common in application, a free terminal cost is usually included [7], such as

$$\sup_{w_{k+f}} \mathrm{E}\{\sum_{f=0}^{N-1}[\|x_{k+f}\|_Q^2 + \|u_{k+f}\|_R^2 - \gamma^2\|w_{k+f}\|_{R_w}^2] + \|x_{k+N}\|_{P(r_k)}^2|x_0, r_0\} < 0 \tag{6}$$

where $P(r_k)$ is the terminal weighting matrix which corresponding to the current time jump mode $r_k$ at every sampling time $k$. When jump mode $r_k = i$, we denote $P(r_k) = P_i = P_i^{\mathrm{T}} > 0$. We try to find the best control $u_{k+f}^*$ under the worst disturbance $w_{k+f}^*$. A feasible solution $u_{k+f}$ in (6) can be found from the following min-max problem

$$\min_{u_{k+f}} \max_{w_{k+f}} J(u_{k+f}, w_{k+f}, x_{k+f}) \tag{7}$$

where

$$J(u_{k+f}, w_{k+f}, x_{k+f}) = \mathrm{E}\{\sum_{f=0}^{N-1}[\|x_{k+f}\|_Q^2 + \|u_{k+f}\|_R^2 - \gamma^2\|w_{k+f}\|_{R_w}^2] + \|x_{k+N}\|_{P(r_k)}^2|x_0, r_0\} \tag{8}$$

The RH $H_\infty$ control move can be obtained by solving the min-max problem (7). $J(u_{k+f}, w_{k+f}, x_{k+f})$ is also called quadratic cost function of system (1). $\tilde{z}_k$ in (3) is chosen to make the quadratic cost function and $z_k$ in (1) is expected to be regulated into a mean square stable sense.

To obtain the control law which can guarantee the closed-loop stability, we also define the mean square stability which is selected as the stability concept in this paper.

**Definition 1** [13] The system (1) is mean square stable, if for any initial state $x_0$ and initial mode $r_0$

$$\mathrm{E}\{x_k^{\mathrm{T}} x_k | x_0, r_0\} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \tag{9}$$

## 3. Main results

Now we direct our attention to derive an RH $H_\infty$ control scheme for discrete-time MJLS.

**Theorem 1** An RH $H_\infty$ controller for discrete-time jump system (1) can be given in the form

$$u_k^* = -R^{-1} B_i^{\mathrm{T}} \varOmega_{k+1,k+N}^{-1} M_{k+1,k+N} A_i x_k \tag{10}$$

$$w_k^* = \gamma^{-2} R_w^{-1} B_{wi}^{\mathrm{T}} \varOmega_{k+1,k+N}^{-1} M_{k+1,k+N} A_i x_k \tag{11}$$

$$\text{s.t.} -\gamma^2 R_w + B_{wi}^{\mathrm{T}} M_{k+1,k+N} B_{wi} < 0 \tag{12}$$

where $\varOmega_{k+1,k+N}^{-1}$ and $M_{k,k+N}$ can be obtained from

$$\varOmega_{k+1,k+N} = I + M_{k+1,k+N} \cdot$$
$$[B_i R^{-1} B_i^{\mathrm{T}} - \gamma^{-2} B_{wi} R_w^{-1} B_{wi}^{\mathrm{T}}] \tag{13}$$

$$M_{k,k+N} = A_i^{\mathrm{T}} \varOmega_{k+1,k+N}^{-1} M_{k+1,k+N} A_i + Q \tag{14}$$

Boundary condition

$$M_{k+N,k+N} = P_i \tag{15}$$

$w_k^*$ is the worst disturbance and $\gamma^2$ is a given scalar. The computation of (14) is made in a backward way when the boundary condition is described as terminal weighting matrix $P_i$ depends on the jump mode. Obviously, $P_i$ can be set at every sampling time or just given as a constant matrix for simplicity.

**Proof** From pontryagin's minimum principle [8], we first form the following Hamilton function

$$\hbar_{k,f} = [\|x_{k+f}\|_Q^2 + \|u_{k+f}\|_R^2 - \gamma^2 \|w_{k+f}\|_{R_w}^2] +$$
$$q_{k+f+1}^{\mathrm{T}} [A(r_{k+f}) x_{k+f} + B(r_{k+f}) u_{k+f} +$$
$$B_w(r_{k+f}) w_{k+f}], \quad f = 0, 1, \ldots, N-1$$

The necessary conditions for $u_{k+f}$ and $w_{k+f}$ to be the saddle points are

$$x_{k+f+1} = \frac{\partial \hbar_{k,f}}{\partial q_{k+f+1}} = A(r_{k+f}) x_{k+f} + B(r_{k+f}) u_{k+f} +$$
$$B_w(r_{k+f}) w_{k+f} \tag{16}$$

$$q_{k+f} = \frac{\partial \hbar_{k,f}}{\partial x_{k+f}} = 2Q x_{k+f} + A^{\mathrm{T}}(r_{k+f}) q_{k+f+1} \tag{17}$$

$$0 = \frac{\partial \hbar_{k,f}}{\partial u_{k+f}} = 2R u_{k+f} + B^{\mathrm{T}}(r_{k+f}) q_{k+f+1} \tag{18}$$

$$0 = \frac{\partial \hbar_{k,f}}{\partial w_{k+f}} = -2\gamma^2 R_w w_{k+f} + B_w^{\mathrm{T}}(r_{k+f}) q_{k+f+1} \tag{19}$$

$$q_{k+N} = \frac{\partial h(x_{k+N})}{\partial x_{k+N}} = 2 P_i x_{k+N} \tag{20}$$

where

$$h(x_{k+N}) = x_{k+N}^{\mathrm{T}} P_i x_{k+N} \tag{21}$$

Assume

$$q_{k+f} = 2 M_{k+f,k+N} x_{k+f} + 2 g_{k+f,k+N} \tag{22}$$

From (16), (18) and (22), we have

$$\frac{\partial \hbar_{k,f}}{\partial u_{k+f}} = 2R u_{k+f} + B^{\mathrm{T}}(r_{k+f}) q_{k+f+1} =$$
$$2R u_{k+f} + 2B^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} x_{k+f+1} +$$
$$2B^{\mathrm{T}}(r_{k+f}) g_{k+f+1,k+N} =$$
$$2R u_{k+f} + 2B^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} [A(r_{k+f}) x_{k+f} +$$
$$B(r_{k+f}) u_{k+f} + B_w(r_{k+f}) w_{k+f}] +$$
$$2B^{\mathrm{T}}(r_{k+f}) g_{k+f+1,k+N}$$

Therefore

$$\frac{\partial^2 \hbar_{k,f}}{\partial u_{k+f}^2} = 2R + 2B^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} B(r_{k+f})$$

Obviously, $\dfrac{\partial^2 \hbar_{k,f}}{\partial u_{k+f}^2}$ must satisfy $\dfrac{\partial^2 \hbar_{k,f}}{\partial u_{k+f}^2} > 0$. Similarly, we have

$$\frac{\partial \hbar_{k,f}}{\partial w_{k+f}} = -2\gamma^2 R_w w_{k+f} + B_w^{\mathrm{T}}(r_{k+f}) q_{k+f+1} =$$
$$-2\gamma^2 R_w w_{k+f} + 2B_w^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} x_{k+f+1} +$$
$$2B_w^{\mathrm{T}}(r_{k+f}) g_{k+f+1,k+N} = -2\gamma^2 R_w w_{k+f} +$$
$$2B_w^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} [A(r_{k+f}) x_{k+f} + B(r_{k+f}) u_{k+f} +$$
$$B_w(r_{k+f}) w_{k+f}] + 2B_w^{\mathrm{T}}(r_{k+f}) g_{k+f+1,k+N}$$

Therefore

$$\frac{\partial^2 \hbar_{k,f}}{\partial w_{k+f}^2} = -2\gamma^2 R_w + 2B_w^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} B_w(r_{k+f})$$

Obviously, $\dfrac{\partial^2 \hbar_{k,f}}{\partial w_{k+f}^2}$ must satisfy $\dfrac{\partial^2 \hbar_{k,f}}{\partial w_{k+f}^2} < 0$.

From the above mentioned analysis, the min-max problem (7) has a unique solution if and only if

$$2R + 2B^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} B(r_{k+f}) > 0 \tag{23}$$

$$-2\gamma^2 R_w + 2B_w^{\mathrm{T}}(r_{k+f}) M_{k+f+1,k+N} B_w(r_{k+f}) < 0 \tag{24}$$

Now, we proceed to obtain the optimal control move $u^*_{k+f}$ and worst disturbance $w^*_{k+f}$. According to (16), (18), (19) and with simple arrangement, yields

$$x_{k+f+1} = A(r_{k+f})x_{k+f} + \frac{1}{2}[-B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) + \gamma^{-2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})]q_{k+f+1} \tag{25}$$

From (22) and (25), we obtain

$$q_{k+f+1} = 2M_{k+f+1,k+N}x_{k+f+1} + 2g_{k+f+1,k+N} = 2M_{k+f+1,k+N}A(r_{k+f})x_{k+f} + M_{k+f+1,k+N}[-B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) + \gamma^{-2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})]q_{k+f+1} + 2g_{k+f+1,k+N}$$

By simple arrangement, we have

$$q_{k+f+1} = 2\{I + M_{k+f+1,k+N}[B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) - \gamma^{2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})]\}^{-1}\cdot [M_{k+f+1,k+N}A(r_{k+f})x_{k+f} + g_{k+f+1,k+N}]$$

Note

$$\Omega_{k+f+1,k+N} = I + M_{k+f+1,k+N}[B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) - \gamma^{-2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})] \tag{26}$$

Then $q_{k+f+1}$ can be written as

$$q_{k+f+1} = 2\Omega^{-1}_{k+f+1,k+N}\cdot [M_{k+f+1,k+N}A(r_{k+f})x_{k+f} + g_{k+f+1,k+N}] \tag{27}$$

If we substitute (27) into (17), then we obtain

$$q_{k+f} = 2Qx_{k+f} + 2A^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}\cdot [M_{k+f+1,k+N}A(r_{k+f})x_{k+f} + g_{k+f+1,k+N}] = 2[A^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}M_{k+f+1,k+N}A(r_{k+f}) + Q]x_{k+f} + 2A^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}g_{k+f+1,k+N} \tag{28}$$

According to (22) and (28), yields

$$M_{k+f,k+N} = A^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}\cdot M_{k+f+1,k+N}A(r_{k+f}) + Q \tag{29}$$

$$g_{k+f,k+N} = A^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}g_{k+f+1,k+N} \tag{30}$$

Combined with (20) and (22), boundary conditions are given by

$$M_{k+N,k+N} = P_i \tag{31}$$

$$g_{k+N,k+N} = 0 \tag{32}$$

From (31) and (32), we have $g_{k+f,k+N} = 0$ and the boundary condition is only depended on the terminal weighting matrix $P_i$.

Pre-and post-multiplying on both sides of (24) by $R_w^{-1/2}$, we have

$$I - \gamma^{-1}R_w^{-1/2}B_{wi}^{\mathrm{T}}M^{1/2}_{k+f+1,k+N}\cdot \gamma^{-1}M^{1/2}_{k+f+1,k+N}B_{wi}R_w^{-1/2} > 0 \tag{33}$$

$$I - \gamma^{-2}M^{1/2}_{k+f+1,k+N}B_{wi}R_w^{-1}B_{wi}^{\mathrm{T}}M^{1/2}_{k+f+1,k+N} > 0 \tag{34}$$

where (34) comes from the fact that $I - MM^{\mathrm{T}} > 0$ implies $I - M^{\mathrm{T}}M > 0$.

$\Omega^{-1}_{k+f+1,k+N}M_{k+f+1,k+N}$ on the right side of (29) can be written as

$$\Omega^{-1}_{k+f+1,k+N}M_{k+f+1,k+N} = \{I + M_{k+f+1,k+N}[B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) - \gamma^{-2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})]\}^{-1}M_{k+f+1,k+N} = M^{1/2}_{k+f+1,k+N}\{I + M^{1/2}_{k+f+1,k+N}[B(r_{k+f})R^{-1}B^{\mathrm{T}}(r_{k+f}) - \gamma^{-2}B_w(r_{k+f})R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})]M^{1/2}_{k+f+1,k+N}\}^{-1}\cdot M^{1/2}_{k+f+1,k+N} > 0$$

where the last inequality holds because (34) holds. And then it is obviously to see that $M_{k+f+1,k+N}$ generated by (29) is always non-negative definite and the optimal condition (23) is naturally guaranteed by condition (24).

Considering (18), (19), (27) and $g_{k+f,k+N} = 0$, the receding horizon $H_\infty$ control can be given as

$$u^*_{k+f} = -\frac{1}{2}R^{-1}B^{\mathrm{T}}(r_{k+f})q_{k+f+1} = -R^{-1}B^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}\cdot M_{k+f+1,k+N}A(r_{k+f})x_{k+f} \tag{35}$$

The worst disturbance

$$w^*_{k+f} = \frac{1}{2}\gamma^{-2}R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})q_{k+f+1} = \gamma^{-2}R_w^{-1}B_w^{\mathrm{T}}(r_{k+f})\Omega^{-1}_{k+f+1,k+N}M_{k+f+1,k+N}A(r_{k+f})x_{k+f} \tag{36}$$

However, jump mode in the future time can not be available because of random property of Markovian process. Since the RHC implemented to the real object is always given by the first control action, we note predictive step $f = 0$ and the control (35) can be written as (10). Similarly we have (11)−(14) from (36), (24), (26) and (29). Boundary condition (15) is the same as (31). □

**Remark 1** It can be seen that in Theorem 1, the control law is constructed through numerical solution of an iterative equation. It seems that the mean square stability of the system can not be guaranteed under the control move. Moreover, the choice of terminal matrix $P_i$ which constitutes the boundary condition in Theorem 1 also could be a problem. So, based on the results in theorem1, we try to develop improved method which can deal with both the mean square stabilizability and $H_\infty$ disturbance attenuation while minimizing the quadratic cost function (8).

**Remark 2** It should be pointed out that the transition probability (2) which determines the system evolution has not been considered in Theorem 1 because the control move described by (10) is an optimal control at the current sampling time $k$ and at every sampling time, jump system (1) just behaves as a linear time invariant system under each operation mode. The transition probability will be a significant factor when mean square stabilizability is considered in below.

It is noted that $u_k^*$ can be represented as

$$u_k^* = F_i x_k \tag{37}$$

where

$$F_i = -R^{-1}B_i^{\mathrm{T}}\Omega_{k+1,k+N}^{-1}M_{k+1,k+N}A_i \tag{38}$$

$F_i$ can be viewed as a feedback gain matrix depends on the current time jump mode $r_k$.

We are now able to discuss the mean square stabilizabity of system (1) under the control (37) where control gain $F_i$ is assumed not to be given or calculated.

**Theorem 2** For jump linear system (1) and given scalar $\gamma^2$, if there exists matrices $X_i$, $Y_i$ satisfy the following coupled LMI

$$\begin{bmatrix} \gamma^2 R_w & 0 & U_i^{\mathrm{T}} & 0 & 0 \\ 0 & X_i & V_{1i}^{\mathrm{T}} & X_i Q^{\frac{1}{2}} & Y_i^{\mathrm{T}} R^{\frac{1}{2}} \\ U_i & V_{1i} & \bar{Z} & 0 & 0 \\ 0 & Q^{\frac{1}{2}} X_i & 0 & I & 0 \\ 0 & R^{\frac{1}{2}} Y_i & 0 & 0 & I \end{bmatrix} \geqslant 0 \tag{39}$$

then the closed-loop system driven by some control move $F_i$ is mean square stable with $\gamma$ disturbance attenuation, where,

$$U_i^{\mathrm{T}} = \left[\sqrt{p_{i,1}}B_{wi}^{\mathrm{T}} \cdots \sqrt{p_{i,s}}B_{wi}^{\mathrm{T}}\right], \quad X_i = P_i^{-1}$$

$$V_{i1}^{\mathrm{T}} = \left[\sqrt{p_{i,1}}(A_iX_i + B_iY_i)^{\mathrm{T}} \cdots \sqrt{p_{i,s}}(A_iX_i + B_iY_i)^{\mathrm{T}}\right]$$

$$\bar{Z} = \mathrm{diag}(X_1, X_2, \ldots, X_s), \ \ Y_i = F_iX_i, \ \ i = 1, 2, \ldots, s$$

**Proof** We assume that $u_k^*$, $x_k^*$ are optimal solutions for cost $J(x_\tau, \tau, \xi+1)$ and $u_k^\circ$, $x_k^\circ$ are optimal for $J(x_\tau, \tau, \xi)$, where $\tau$ and $\xi$ represent time index and terminal time index, respectively. The optimal cost function can be written as

$$J^*(x_\tau, \tau, \xi+1) = \mathrm{E}\{\sum_{k=\tau}^{\xi}[x_k^{*\mathrm{T}}Qx_k^* + u_k^{*\mathrm{T}}Ru_k^* - \gamma^2 w_k^{*\mathrm{T}}R_w w_k^*] + x_{\xi+1}^{*\mathrm{T}}P(r_{k+1})x_{\xi+1}^* \,|x_0, r_0\} \tag{40}$$

$$J^*(x_\tau, \tau, \xi) = \mathrm{E}\{\sum_{k=\tau}^{\xi-1}[x_k^{\circ\mathrm{T}}Qx_k^\circ + u_k^{\circ\mathrm{T}}Ru_k^\circ - \gamma^2 w_k^{\circ\mathrm{T}}R_w w_k^\circ] + x_\xi^{\circ\mathrm{T}}P(r_k)x_\xi^\circ \,|x_0, r_0\} \tag{41}$$

$$\delta J^*(x_\tau, \xi) = J^*(x_\tau, \tau, \xi+1) - J^*(x_\tau, \tau, \xi) \tag{42}$$

The pair $(u_k^*, w_k^*)$ constitutes a saddle-point solution if

$$J(u_k^*, w_k) \leqslant J(u_k^*, w_k^*) \leqslant J(u, w_k^*) \tag{43}$$

In (40) and (41), the pair $(u_k^*, w_k^*)$ is a saddle-point solution for $J(x_\tau, \tau, \xi+1)$ and the pair $(u_k^\circ, w_k^\circ)$ is the one for $J(x_\tau, \tau, \xi)$. Note that state $\tilde{x}_k$ is the trajectory associated with $u_k^\circ$ and $w_k^*$, state $x_k^*$ with $u_k^*$ and $w_k^*$, state $x_k^\circ$ with $u_k^*$ and $w_k^\circ$. $u_\xi^\circ = F_i\tilde{x}_\xi$, $w_\xi^* = \Gamma_i\tilde{x}_\xi$. To make the proof clear, we note

$$A_{cl1} = A_i + B_iF_i + B_{wi}\Gamma_i, \quad \Gamma_i = \gamma^{-2}R_w^{-1}B_{wi}^{\mathrm{T}}\Omega_iP_iA_i$$

$$\Omega_i = I + P_i[B_iR^{-1}B_i^{\mathrm{T}} - \gamma^{-2}B_{wi}R_w^{-1}B_{wi}^{\mathrm{T}}] \tag{44}$$

Based on the results of Theorem 1, we have

$$\tilde{x}_{\xi+1} = A_{cl1}\tilde{x}_\xi$$

Due to $J(u_k^*, w_k^*) \leqslant J(u_k^\circ, w_k^*)$, we replace $u_k^*$ by $u_k^\circ$ up to $\xi - 1$ and the following inequality are obtained

$$\begin{aligned} &\mathrm{E}\{\sum_{k=\tau}^{\xi}[x_k^{*\mathrm{T}}Qx_k^* + u_k^{*\mathrm{T}}Ru_k^* - \gamma^2 w_k^{*\mathrm{T}}R_w w_k^*] + x_{\xi+1}^{*\mathrm{T}}P(r_{k+1})x_{\xi+1}^* \,|x_0, r_0\} \leqslant \\ &\mathrm{E}\{\sum_{k=\tau}^{\xi-1}[\tilde{x}_k^{\mathrm{T}}Q\tilde{x}_k + u_k^{\circ\mathrm{T}}Ru_k^\circ - \gamma^2 w_k^{*\mathrm{T}}R_w w_k^*] + \tilde{x}_\xi^{\mathrm{T}}Q\tilde{x}_\xi + u_\xi^{\circ\mathrm{T}}Ru_\xi^\circ - \gamma^2 w_\xi^{*\mathrm{T}}R_w w_\xi^* + \tilde{x}_{\xi+1}^{\mathrm{T}}P(r_{k+1})\cdot \\ &\tilde{x}_{\xi+1}|x_0, r_0\} = \mathrm{E}\{\sum_{k=\tau}^{\xi-1}[\tilde{x}_k^{\mathrm{T}}Q\tilde{x}_k + u_k^{\circ\mathrm{T}}Ru_k^\circ - \gamma^2 w_k^{*\mathrm{T}}R_w w_k^*] + \\ &\tilde{x}_\xi^{\mathrm{T}}Q\tilde{x}_\xi + \tilde{x}_\xi^{\mathrm{T}}F_i^{\mathrm{T}}RF_i\tilde{x}_\xi - \tilde{x}_\xi^{\mathrm{T}}\gamma^2\Gamma_i^{\mathrm{T}}R_w\Gamma_i\tilde{x}_\xi + \sum_{j=1}^{s}p_{i,j}\tilde{x}_\xi^{\mathrm{T}}A_{cl1}^{\mathrm{T}}P_jA_{cl1}\tilde{x}_\xi \,|x_0, r_0\} \end{aligned} \tag{45}$$

Due to $J(u_k^*, w_k^\circ) \leqslant J(u_k^*, w_k^*)$, we replace $w_k^\circ$ by $w_k^*$ up to $\xi - 1$ and this yields

$$\begin{aligned} &\mathrm{E}\{\sum_{k=\tau}^{\xi-1}[x_k^{\circ\mathrm{T}}Qx_k^\circ + u_k^{\circ\mathrm{T}}Ru_k^\circ - \gamma^2 w_k^{\circ\mathrm{T}}R_w w_k^\circ] + x_\xi^{\circ\mathrm{T}}P_ix_\xi^\circ \,|x_0, r_0\} \geqslant \mathrm{E}\{\sum_{k=\tau}^{\xi-1}[\tilde{x}_k^{\mathrm{T}}Q\tilde{x}_k + u_k^{\circ\mathrm{T}}Ru_k^\circ - \\ &\gamma^2 w_k^{*\mathrm{T}}R_w w_k^*] + \tilde{x}_\xi^{\mathrm{T}}P_i\tilde{x}_\xi \,|x_0, r_0\} \end{aligned} \tag{46}$$

According to (45) and (46), we know that system (1) under the control move could be mean square stable if

$$\delta J^*(x_\tau,\xi) \leqslant \mathrm{E}\{\sum_{\mathrm{k}=\tau}^{\xi-1}\tilde{x}_\xi^{\mathrm{T}}[Q+F_i^{\mathrm{T}}RF_i-$$

$$\Gamma_i^{\mathrm{T}}\gamma^2R_w\Gamma_i+A_{cl1}^{\mathrm{T}}\sum_{j=1}^{s}p_{i,j}P_jA_{cl1}-P_i]\tilde{x}_\xi\,|x_0\,,r_0\}\leqslant 0$$

That is

$$Q+F_i^{\mathrm{T}}RF_i-\varGamma_i^{\mathrm{T}}\gamma^2R_w\varGamma_i+A_{cl1}^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_jA_{cl1}-P_i\leqslant 0 \tag{47}$$

(47) is also known as cost monotonicity condition and it can be written as

$$\varGamma_i^{\mathrm{T}}(\gamma^2R_w-B_{wi}^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_jB_{wi})\varGamma_i-\varGamma_i^{\mathrm{T}}B_{wi}^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}$$

$$P_j(A_i+B_iF_i)-(A_i+B_iF_i)^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_jB_{wi}\varGamma_i+\phi_1\geqslant 0$$

where

$$\phi_1=P_i-Q-F_i^{\mathrm{T}}RF_i-$$
$$(A_i+B_iF_i)^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_j(A_i+B_iF_i)$$

By some simple arrangements, we have

$$\begin{bmatrix}\varGamma_i\\ I\end{bmatrix}^{\mathrm{T}}\varTheta_i\begin{bmatrix}\varGamma_i\\ I\end{bmatrix}\geqslant 0$$

where

$$\varTheta_i=\begin{bmatrix}\gamma^2R_w-B_{wi}^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_jB_{wi} & B_{wi}^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_j(A_i+B_iF_i)\\ (A_i+B_iF_i)^{\mathrm{T}}\sum_{i=1}^{s}p_{i,j}P_jB_{wi} & \phi_1\end{bmatrix}\geqslant 0 \tag{48}$$

From (48), it can be seen that we only have to find $P_i$ ($i=1,2,\ldots,s$) such that

$$\begin{bmatrix}\gamma^2R_w & 0\\ 0 & P_i-Q-F_i^{\mathrm{T}}RF_i^{\mathrm{T}}\end{bmatrix}-$$

$$\begin{bmatrix}B_{wi}^{\mathrm{T}}\\ (A_i+B_iF_i)^{\mathrm{T}}\end{bmatrix}\sum_{i=1}^{s}p_{i,j}P_j\begin{bmatrix}B_{wi}^{\mathrm{T}}\\ (A_i+B_iF_i)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\geqslant 0 \tag{49}$$

which is equivalent to

$$\begin{bmatrix}\gamma^2R_w & 0 & B_{wi}^{\mathrm{T}}\\ 0 & X_i^{-1}-Q-F_i^{\mathrm{T}}RF_i & (A_i+B_iF_i)^{\mathrm{T}}\\ B_{wi} & A_i+B_iF_i & \sum_{i=1}^{s}p_{i,j}^{-1}X_j\end{bmatrix}\geqslant 0 \tag{50}$$

Pre-and post-multiplying both sides of (50) by the matrix $\mathrm{diag}\{I,X_i,I\}$, yields

$$\begin{bmatrix}\gamma^2R_w & 0 & U_i^{\mathrm{T}}\\ 0 & X_i-X_iQX_i-(F_iX_i)^{\mathrm{T}}RF_iX_i & V_{1i}^{\mathrm{T}}\\ U_i & V_{1i} & \bar{Z}\end{bmatrix}\geqslant 0 \tag{51}$$

By some simple matrix manipulations and using schur complement, we obtain sufficient condition (39) in Theorem 2. Since condition (39) is equivalent to (47), it also can be viewed as cost monotonicity condition.

The (47) implies that

$$\mathrm{E}\{x_k^{\mathrm{T}}P(r_k)x_k\,|x_0\,,r_0\}\geqslant\mathrm{E}\{x_k^{\mathrm{T}}P(r_{k+1})x_k\,|x_0\,,r_0\}+$$
$$\mathrm{E}\{Q+F_i^{\mathrm{T}}RF_i-\varGamma_i^{\mathrm{T}}\gamma^2R_w\varGamma_i\,|x_0\,,r_0\} \tag{52}$$

It follows from (52) that

$$\mathrm{E}\{x_k^{\mathrm{T}}P(r_k)x_k\,|x_0\,,r_0\}=\mathrm{E}\{x_k^{\mathrm{T}}P(r_{k+1})x_k\,|x_0\,,r_0\}$$

$$\mathrm{E}\{Q+F_i^{\mathrm{T}}RF_i-\gamma^2\varGamma_i^{\mathrm{T}}R_w\varGamma_i\,|x_0\,,r_0\}\to 0,\text{as}\quad k\to\infty$$

It is easy to obtain that

$$\mathrm{E}\{x_k^{\mathrm{T}}\vartheta_1(r_k)x_k\,|x_0\,,r_0\}\to 0\quad\text{as}\quad k\to\infty \tag{53}$$

where

$$\vartheta_1(r_k)=Q+F^{\mathrm{T}}(r_k)RF(r_k)-\varGamma^{\mathrm{T}}(r_k)\gamma^2R_w\varGamma(r_k)$$

$\varGamma(r_k)$ can be obtained by replacing $i$ by $r_k$ in (44). From [13], we have

$$\mathrm{E}\{x_k^{\mathrm{T}}\vartheta_1(r_k)x_k\,|x_0,r_0\}\geqslant\vartheta_1\mathrm{E}\{x_k^{\mathrm{T}}x_k\,|x_0,r_0\} \tag{54}$$

where $\vartheta_1=\min_{r_k=1,2,\ldots,s}\lambda_{\min}(\vartheta_1(r_k))$, and $\lambda$ is a singular value. Since the natural positive property of cost function, it is easy to conclude that $\vartheta_1(r_k)$ is positive which implies that $\vartheta_1$ is positive. Then from (53) and (54), we obtain

$$\mathrm{E}\{x_k^{\mathrm{T}}x_k\,|x_0,r_0\}\to 0\quad\text{as}\quad k\to\infty$$

which means the closed-loop system is mean square stable with $\gamma$ disturbance attenuation under the cost monotonicity condition. $\square$

**Remark 3** Obviously the cost monotonicity condition leads to the mean square stability and the terminal weighting matrix plays an important role in the mean square stabilizability.

**Remark 4** It can be seen from Theorem 2 that the control move calculated by $F_i = Y_i X_i^{-1}$ can stabilize the closed-loop system but can not be the optimal control which can minimize the cost function (8) under the worst disturbance. However, terminal weighting matrix $P_i = X_i^{-1}$ which satisfies the cost monotonicity condition (39) can be used to construct the mean square stabilizing $H_\infty$ controller which is discussed in Theorem 1.

**Remark 5** Obviously, (39) is a sufficient condition to obtain the terminal weighting matrix, that is to say if LMI (39) has a feasible solution then the controller (10) can be obtained by the resulting terminal weighting matrix; conversely, if LMI (39) can not find a feasible solution, it doesn't mean that the stabilized terminal weighting matrix does not exist and it also can be found via heuristic method.

Combined with Theorem 1 and Theorem 2, we have the following theorem.

**Theorem 3** An RH $H_\infty$ controller which minimize the cost (8) under the worst disturbance and stabilize the closed-loop system in the mean square sense with $\gamma$ disturbance attenuation can be obtained in the form of (10)−(15), where the terminal weighting matrix $P_i$ is determined by (39).

## 4. Numerical example

To illustrate efficiency of the proposed RH $H_\infty$ control scheme for discrete-time MJLS, a numerical example is presented in the following.

System (1) is considered, where $r_k = i = 1, 2$. The system matrices

$$A_1 = \begin{bmatrix} 1 & 2.5 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5 & 0 \\ 1.5 & 0 \\ 0 & 1.5 \\ 0 & 0.5 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_{w1} = B_{w2} = \begin{bmatrix} 0.16 & 0.02 \\ 0.10 & 0.09 \\ 0.08 & 0 \\ 0 & 0.05 \end{bmatrix}$$

The weighting matrices

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R_w = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

Initial state $x_0 = [-1 \quad 0 \quad 1 \quad 2]^{\mathrm{T}}$, initial mode $r_0 = 1$, the transition probability matrix for system (1) is chosen as $P_T = \begin{bmatrix} 0.67 & 0.33 \\ 0.30 & 0.70 \end{bmatrix}$. The receding horizon length and sampling time interval are taken as $N = 5$ and $t_s = 0.1$, respectively. Disturbance attenuation is set as $\gamma^2 = 1.5$ and simulation step is chosen as $\mathrm{length} = 30$.

By solving (39), we obtain terminal weighting matrix

$$P_1 = X_1^{-1} = \begin{bmatrix} 5.756\,3 & 10.178\,7 & 0.256\,7 & -1.470\,9 \\ 10.178\,7 & 23.170\,1 & 0.593\,3 & -3.481\,3 \\ 0.256\,7 & 0.593\,3 & 5.231\,6 & -31.937\,0 \\ -1.470\,9 & -3.481\,3 & -31.937\,0 & 275.376\,4 \end{bmatrix}$$

$$P_2 = X_2^{-1} = \begin{bmatrix} 12.197\,4 & -0.190\,7 & 0.000\,1 & 2.574\,2 \\ -0.190\,7 & 0.607\,4 & 0.025\,3 & -1.171\,1 \\ 0.000\,1 & 0.025\,3 & 1.133\,4 & -0.065\,9 \\ 2.574\,2 & -1.171\,1 & -0.065\,9 & 4.146\,2 \end{bmatrix}$$

Then we calculate control move (10) through (11)−(15). The simulation results are shown in Figs. 1−3.

Obviously, the system without control input is not mean square stable under the Markov jump law. Fig. 2 and Fig. 3 show that the RH $H_\infty$ control strategy guarantees the mean square stability of closed-loop system under a given disturbance attenuation level $\gamma^2$.

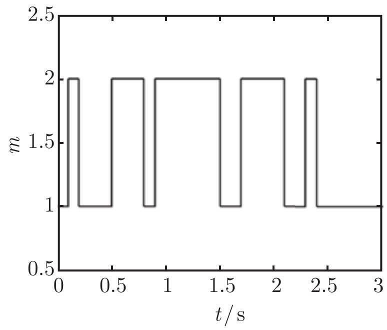


**Fig. 1 Jump mode**

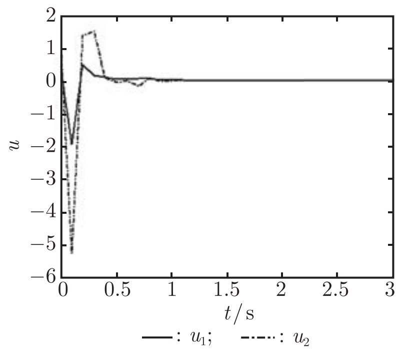


**Fig. 2 Control inputs**

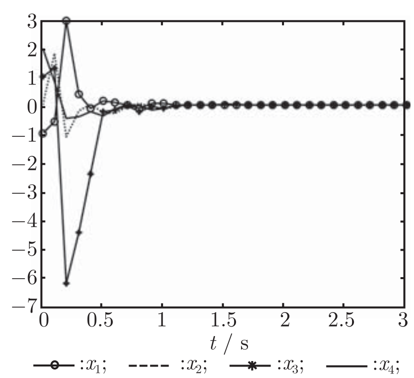


**Fig. 3 State responses under RH $H_\infty$ control**

## 5. Conclusions

RH $H_\infty$ control strategy for discrete-time MJLS is presented. The control law can be obtained by solving some iterative equations while terminal weighting matrix which noted as boundary condition is determined by a set of coupled LMIs. All of the results established depend on the assumption of complete access to the jump mode at the current time. Numerical example is used to demonstrate the design procedures.

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