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Modeling Propagation Dynamics and Developing Optimized Countermeasures for Rumor Spreading in Online Social Networks

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***Abstract*—The spread of rumors in Online Social Networks (OSNs) poses great challenges to the social peace and public order. It is imperative to model propagation dynamics of rumors and develop corresponding countermeasures. Most of the existing works either overlook the heterogeneity of social networks or do not consider the cost of countermeasures. Motivated by these issues, this paper proposes a heterogeneous network based epidemic model that incorporates both the network heterogeneity and various countermeasures. Through analyzing the existence and stability of equilibrium solutions of the proposed ODE (Ordinary Differential Equation) system, the critical conditions that determine whether a rumor continuously propagates or becomes extinct are derived. Moreover, we concern about the cost of the main two types of countermeasures, *i.e.*, *blocking rumors* at influential users and *spreading truth* to clarify rumors. Employing the Pontryagin's maximum principle, we obtain the optimized countermeasures that ensures a rumor can become extinct at the end of an expected time period with lowest cost. Both the critical conditions and the optimized countermeasures provide a real-time decision reference to restrain the rumor spreading. Experiments based on Digg2009 dataset are conducted to evaluate the effectiveness of the proposed dynamic model and the efficiency of the optimized countermeasures.**

I. Introduction

Online Social Networks (OSNs), such as Facebook, Twitter and Google+, are driving new forms for people to swap ideas, disseminate breaking news, keep contact with friends, *etc*. Unfortunately, OSNs also pave a road for the propagation of rumors, gossips and other kinds of misinformation. Rumors, viewed as unverified information or deliberately falsified news, generally spread much faster than ordinary news as they cater to public curiosity [1]. They have been responsible for serious damage in a variety of human affairs and continue to present a great challenge to the social peace and public order. For example, the rumor of "*Two bombs had exploded at the White House and Barack Obama is injured* (April 23, 2013)" resulted in 10 billion USD losses [2]. As another example, the rumors about the flight MH370 tragedy, involving accident causes, passenger information, searching process and so forth, caused great panic and depression to families.

Restraining the spread of rumors in OSNs has attracted increasing attentions. Generally, there are mainly two types of methods including *blocking rumors* at influential users or *spreading truth* to clarify the rumors. The former method first identifies the influential users based on their *Degree*, *Betweenness* or *Core*, then trains these users to help them distinguish between the rumor and truth. Thus, a rumor can be blocked at those influential users, *i.e.*, "Rumor ends with Sage" [3] [4] [5] [6]. The latter method is to spread an "anti-rumor" to public by identifying a set of influential users or media and then persuade them to spread "anti-rumor" [7] [8]. Thus, a rumor would be overwhelmed by the spread of "anti-rumor". Note that, the implementation cost of these two methods are expensive and different. Blocking rumors at influential users may incur backfires because it may violate human rights and training someone to distinguish between the rumor and truth requires support of social resources such as accessing to security documents or reaching of a group of experts. Similarly, injecting truth to OSNs needs to provide fact-based proofs to media or influential users and strive for their cooperation. For example, incentive is generally needed when persuading someone to be an "anti-rumor" spreader.

Unfortunately, the *blocking rumors* and *spreading truth* strategies are individually considered for most of the exiting works on restraining the spread of rumors in OSNs. The popular key approach of the current works is to identify a minimum subset of individuals as rumor blockers or truth spreaders to combat rumors, such as blocking the links in [3] and removing influential nodes in [4] [5] [6]. On the other hand, the implementation costs of those two strategies are also overlooked by most of the existing works. The work in [9] studies which method has better performance for *blocking rumors* and *spreading truth*. The work in [5] studies the NP-hard problem of minimizing the cost of immunizing nodes for preventing the spread of a rumor. In this paper, we demonstrate that the two restraining methods, *blocking rumors* and *spreading truth*, have different efficiencies in different environments, *i.e.*, the percentage of the infected individuals in an OSN. The goal of our optimization method is to minimize the implementation cost by combining the two restraining methods such that a rumor can be extinct within an expected time period.

In order to study rumor spreading in a more practical sense, we first design a heterogeneous network based epidemic model which mathematically models the propagation dynamics of rumors in heterogeneous OSNs. Although some efforts

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have been spent on adopting the epidemic model to simulate information diffusion in OSNs [10] [11], some important social properties are overlooked by the existing works. In OSNs, individuals have different impacts on rumor spreading. For example, public figures can spread news to a large number of people while unsocial persons only have few social contacts [12]. The individuals' differences on *Degree*, *Betweenness* and *Core* are called network heterogeneity. Our extended epidemic model incorporates network heterogeneity and based on which the optimized countermeasures are designed. The main contributions of this work are summarized as follows:

- To the best of our knowledge, this is the first work that proposes an epidemic model incorporating both the network heterogeneity and various countermeasures. Based on this model, we derive the critical conditions that determines whether a rumor continuously propagates or becomes extinct as time passes.
- We jointly consider the *blocking rumors* and *spreading truth* methods in order to restrain the spread of rumors. By employing the Pontryagin's maximum principle, we obtain the optimized countermeasures with the minimum cost to effectively restrain further propagation of rumors.
- We conduct extensive experiments to evaluate the effectiveness of the proposed model and the efficiency of the optimized countermeasures based on the real Digg2009 dataset.

The remainder of this paper is organized as follows. Section II describes the dynamic model and problem definition. Section III presents the heterogeneous network based epidemic model and derives the critical conditions that determine whether a rumor continuously propagates or becomes extinct. The optimized countermeasures are illustrated in Section IV. The validation of the proposed model and optimized countermeasures are presented in Section V. The related works are addressed in Section VI. Finally, Section VII concludes this paper.

II. Preliminaries

In this section, we first give the network model and then define the investigated problems.

A. The dynamic model

As rumor spreads, network users can be divided into three classes: *Susceptible* (*S*) represents the individuals who have not known a rumor yet but are susceptible to it; *Infected* (*I*) denotes the individuals who believe in a rumor and act as rumor spreaders; *Recovered* (*R*) denotes the individuals who are blocked or immunized by truth. Since different users (such as public figures or unsocial persons) have difference in accepting information (rumor or truth), we characterize individuals based on their social connectivity, *i.e.*, degree. Although degree based heterogeneity is not optimal, it can effectively characterize information diffusion in scale free networks [4] [6] [13], (*i.e.*, power-law degree distribution).

Based on degree, the users in a network can be classified into n groups so that the individuals in one group have the same social connectivity, *i.e.* node degree. For group i $(i = 1, 2, \cdots, n)$, k_i denotes the social connectivity of the individuals in this group. Let $S_{k_i}(t)$, $I_{k_i}(t)$ and $R_{k_i}(t)$ denote the density of the susceptible, infected and recovered individuals in group i at time t respectively, so that $S_{k_i}(t)+I_{k_i}(t)+R_{k_i}(t) = 1$. As a rumor spreads, the countermeasures are carried out to combat it. Thus, an individual's state is transformed among S, I and R as rumor and countermeasurs combat with each other in the network. For an arbitrary group i with node degree k_i, the state transformation of nodes (Fig.1) are explained as follows:

1) Susceptible individuals are transformed to infected ones if they believe the rumor (*i.e.*, $S_{k_i}(t) \rightarrow I_{k_i}(t)$ in Fig.1). The rumor acceptance rate for the individuals in group i is $\lambda(k_i)$ $(0 < \lambda(k_i) < 1)$, that is, a susceptible individual with connectivity k_i is infected with probability $\lambda(k_i)$ if it is connected to an infected individual. A susceptible individual is connected to one or more infected individuals with rate $\Theta(t)$ at time t. Thus, the infected rate for any susceptible individual with connectivity k_i at any time t is $\lambda(k_i)\Theta(t)$.
2) At any time t, susceptible individuals are immunized by truth to transform to recovered individuals with rate $\varepsilon_1(t)$ (*i.e.*, $S_{k_i}(t) \rightarrow R_{k_i}(t)$ in Fig.1). Meanwhile, infected individuals are blocked to transform to recovered individuals with rate $\varepsilon_2(t)$ at time t (*i.e.*, $I_{k_i}(t) \rightarrow R_{k_i}(t)$ in Fig.1).
3) As a rumor spreads, more and more new individuals begin to concern about the rumor with adding rate α. Supposing these individuals neither have been infected by rumors nor immunized by truth, they should be susceptible ones.

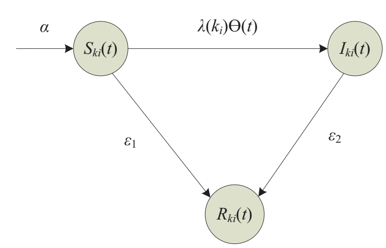

Fig. 1. State transition in the group i with degree k_i. $\lambda(k_i)\Theta(t)$: infected rate for any susceptible individual with connectivity k_i at any time t; $\varepsilon_1(t)$: immunization rate on susceptible individuals; $\varepsilon_2(t)$: blocking rate on infected individuals.

Note that $\Theta(t)$ couples the dynamics of n groups of SIR subsystems (Susceptible-Infected-Recovered), which will be discussed in Section III in details. At any time t, the values of the state variables are bounded in the following solution space: Ω=$\{(S_{k_1}(t), I_{k_1}(t), R_{k_1}(t)), \cdots, (S_{k_n}(t), I_{k_n}(t), R_{k_n}(t))$: $S_{k_i} \geq 0,\ I_{k_i} \geq 0,\ R_{k_i} \geq 0,\ S_{k_i} + I_{k_i} + R_{k_i} = 1,\ 1 \leq i \leq n\}$. At the initial time, there are a small number of rumor spreaders and most individuals are susceptible. As time passes, more and more susceptible individuals are infected by the rumor and then transform to infected individuals. Meanwhile, as a rumor attacks the public opinion progressively, corresponding countermeasures are carried out to restrain its further propagation. Thus, the initial condition of the model is $I_{k_i}(t_0) > 0$, $S_{k_i}(t_0) = 1 - I_{k_i}(t_0)$ and $R_{k_i}(t_0) = 0$, where $t_0 = 0$.

B. Problem definition

Based on the SIR dynamic model, two questions need to be addressed: 1) Under what conditions do rumors continuously propagate or become extinct as time passes? and 2) What are the optimized countermeasures, *i.e.*, optimized proportional combination of *blocking rumors* and *spreading truth*, to restrain rumor spreading with the minimum cost in an expected time period? Note that, an expected time period implies a deadline to restrain the spread of rumor. Specifically, the problem studied in this paper is defined as follows.

Input:

1) $I_{k_i}(t_0)$, $S_{k_i}(t_0)$ and $R_{k_i}(t_0)$, where $t_0 = 0$ and $i = 1, 2, \cdots, n$.

2) The dynamic model with immunized rate ε_1 and blocked rate ε_2.

3) The expected time period to restrain the spread of rumors: $(0, t_f]$.

Output:

1) Critical conditions.

2) Optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$, $t \in (0, t_f]$.

III. Heterogeneous Network based SIR Model

In this section, we first describe the dynamic model as meanfield rate equations. Then, we address the critical conditions that determine whether a rumor continuously propagates or becomes extinct through analyzing the existence and stability of the equilibrium solutions.

Ever since Daley and Kendall introduced a standard model of rumor spreading in 1965 which is called DK model [14], many variants have been proposed, such as the famous Maki-Thompson (MK) model [15]. We also adopt epidemic model as basis to describe the rumor spreading. Based on the spreading rules described in Section II, the heterogeneous network based SIR model can be described as the following Ordinary Differential Equation (ODE) system:

$$\begin{aligned}
&\frac{dS_{k_i}(t)}{dt} = \alpha - \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_1 S_{k_i}(t)\\
&\frac{dI_{k_i}(t)}{dt} = \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t)\\
&\frac{dR_{k_i}(t)}{dt} = \varepsilon_1 S_{k_i}(t) + \varepsilon_2 I_{k_i}(t)\\
&i = 1, 2, \cdots, n,\ t > 0
\end{aligned} \tag{1}$$

For convenience, Table I summarizes the major parameters in System (1). The three ordinary differential equations describe the time evolution of the densities of susceptible, infected and recovered individuals respectively. Take the first equation for example, The derivative of $S_{k_i}(t)$ with respect to time is the changing rate of $S_{k_i}(t)$. The adding rate of susceptible individuals at time t is α. Meanwhile, the susceptible ones are infected by a rumor with a part of $\lambda(k_i)S_{k_i}(t)\Theta(t)$ and immunized by truth to transform to recovered ones with a part of $\varepsilon_1 S_{k_i}(t)$. The remaining two equations obey the similar rules.

TABLE I. Major parameters in the dynamic model

Parameter	Definition
k_i	Social connectivity of the individuals in group i (*i.e.*, degree)
α	Rate of new individuals entering an OSN
$\lambda(k_i)$	Rumor acceptance rate of the susceptible individuals in group i
$\varepsilon_1(t)$	Proportion of the susceptible individuals being immunized at time t
$\varepsilon_2(t)$	Proportion of the infected individuals being blocked at time t
$P(k_i)$	Probability of a node with degree k_i
$\langle k \rangle$	Average degree of an OSN
$\omega(k_i)$	Infectivity of an infected individual with degree k_i

When susceptible individuals in group i contact with infected ones, rumors can be propagated with rate $\lambda(k_i)\Theta(t)$. $\Theta(t)$ is the average infectivity of rumors at time t, which is the proportion of the social connection of infected individuals over the entire social connection. Note that $\Theta(t)$ couples the whole impact of n groups of SIR subsystems. we have

$$\Theta(t) = \frac{1}{\langle k \rangle}\sum_{i=1}^{n}\omega(k_i)P(k_i)I_{k_i}(t)$$

where $P(k_i)$ is the probability that a node has degree k_i so that $\sum_{i=1}^{n} P(k_i) = 1$, and $\langle k \rangle$ is the average node degree of the network, thus $\langle k \rangle = \sum_{i=1}^{n} k_i P(k_i)$. $\omega(k_i)$ measures the infectivity of a node with degree k_i. Several types of $\omega(k_i)$ have been considered such as $\omega(k_i) = C$ where C is a constant [16], $\omega(k_i) = k_i$ [17] or nonlinear function $k_i^{\beta}/(1 + k_i^{\gamma})$ [18]. Intuitively, the more social connections an infected individual has, the higher infectivity this individual has. However, the infectivity will saturate when the social connectivity reaches some extent. Thus, regarding rumor spreading, a non-linear function is more reasonable.

The ODE system mathematically describes the temporal dynamics of rumor spreading. In order to figure out the rumor propagation condition at arbitrary time t, the solution of System (1), *i.e.*, $S_{k_i}(t), I_{k_i}(t), R_{k_i}(t)$, needs to be derived. A significant problem emerges naturally, that is, under what levels of countermeasures a rumor continuously propagates or becomes extinct as time passes. To answer this question, we first provide the definition of equilibrium solution as preparation.

A. The definition of equilibrium solution

In System (1), dS/dt (or $dI/dt, dR/dt$) is the changing rate of the density of susceptible nodes (or infected nodes, recovered nodes). As a rumor spreads progressively, corresponding countermeasures are conducted to restrain its further propagation. If the countermeasures are effective, the infection is no longer epidemic and the rumor will be extinct (in the case that the number of the infected person keep being less than a threshold). Otherwise, the rumor will continuously propagate and its quantity will converge to a stable level. Stability Theory for Ordinary Differential Equations (ODEs) [19] indicates that system only converges to stable equilibrium solutions. Thus, the density of the three types of nodes will always converge to a stable level at time t, which is the equilibrium solution of System (1). These two cases correspond to the zero-equilibrium solution of System (1) and the positive-equilibrium solution of System (1), respectively. For clarity, the definitions of the equilibrium solutions are presented as follows. More details can be found in [20].

Definition 1: **Equilibrium solution.** Mathematically, the solution $x^* \in \mathbb{R}$ is an equilibrium solution for the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if $f(t, x^*) = 0$ for all t.

Definition 2: **Zero-equilibrium solution.** E_0 is called a zero-equilibrium solution of System (1) if the solution of System (1) $\{(S_{k_1}(t), I_{k_1}(t), R_{k_1}(t)), \cdots, (S_{k_n}(t), I_{k_n}(t), R_{k_n}(t))\}$ converges to $E_0 = \left\{(S^0_{k_1}, I^0_{k_1}, R^0_{k_1}), \cdots, (S^0_{k_n}, I^0_{k_n}, R^0_{k_n})\right\}$, where $S^0_{k_i} > 0$, $R^0_{k_i} > 0$ and $I^0_{k_i} = 0$ $(i = 1, 2, \cdots, n)$.

Definition 3: **Positive-equilibrium solution.** E_+ is called a positive-equilibrium solution of System (1) if the solution of System (1) $\{(S_{k_1}(t), I_{k_1}(t), R_{k_1}(t)), \cdots, (S_{k_n}(t), I_{k_n}(t), R_{k_n}(t))\}$ converges to $E_+ = \left\{(S^+_{k_1}, I^+_{k_1}, R^+_{k_1}), \cdots, (S^+_{k_n}, I^+_{k_n}, R^+_{k_n})\right\}$, where $S^+_{k_i} > 0$, $R^+_{k_i} > 0$ and $I^+_{k_i} > 0$ $(i = 1, 2, \cdots, n)$.

Thus, to investigate the problem of whether a rumor continuously propagates or not, we need to investigate the critical conditions that determine the existence and stability of equilibrium solution of System (1).

B. The existence of equilibrium solution

For simplicity, we simplify the $\Theta(t)$ by letting $\varphi(k_i) = \omega(k_i)P(k_i)$. For the existence of the equilibrium solution of System (1), we have the following theorem.

Theorem 1: For parameter

$$r_0 = \frac{\alpha}{\langle k \rangle} \sum_{i=1}^{n} \frac{\lambda(k_i)\varphi(k_i)}{\varepsilon_1 \varepsilon_2},$$

Case 1: If $r_0 \leq 1$, System (1) only has a zero-equilibrium solution denoted by $E_0 = \left\{(S^0_{k_1}, I^0_{k_1}, R^0_{k_1}), \cdots, (S^0_{k_n}, I^0_{k_n}, R^0_{k_n})\right\}$, where $S^0_{k_i} = \alpha/\varepsilon_1$, $I^0_{k_i} = 0$ and $R^0_{k_i} = 1 - \alpha/\varepsilon_1$ $(i = 1, 2, \cdots, n)$.

Case 2: If $r_0 > 1$, System (1) has either a zero-equilibrium solution or a positive-equilibrium solution denoted by $E_+ = \left\{(S^+_{k_1}, I^+_{k_1}, R^+_{k_1}), \cdots, (S^+_{k_n}, I^+_{k_n}, R^+_{k_n})\right\}$, where

$$\begin{aligned} S^+_{k_i} &= \frac{\varepsilon_2 I^+_{k_i}}{\lambda(k_i)\Theta^+}, \\ I^+_{k_i} &= \frac{\alpha\lambda(k_i)\Theta^+}{\varepsilon_2(\lambda(k_i)\Theta^+ + \varepsilon_1)}, \\ R^+_{k_i} &= 1 - S^+_{k_i} - I^+_{k_i}, \end{aligned}$$

$\Theta^+ = \langle k \rangle^{-1} \sum_{i=1}^{n} \varphi(k_i) I^+_{k_i}$ $(i = 1, 2, \cdots, n)$.

Proof. From System (1), we observe that the first and second equations are independent of the third one. Thus, we first analyze the first and second equations, and then derive $R_{k_i}(t)$ based on the third one naturally $(i = 1, 2, \cdots, n)$. System (2) is the result of extracting the first and second equations from System (1)

$$\begin{aligned} \frac{dS_{k_i}(t)}{dt} &= \alpha - \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_1 S_{k_i}(t) \\ \frac{dI_{k_i}(t)}{dt} &= \lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t) \end{aligned} \tag{2}$$

An ODE system converging to equilibrium solutions E_*, $E_* = \left\{(S^*_{k_1}, I^*_{k_1}, R^*_{k_1}), \cdots, (S^*_{k_n}, I^*_{k_n}, R^*_{k_n})\right\}$, means that the system variables $S_{k_i}(t)$, $I_{k_i}(t)$ and $R_{k_i}(t)$ are fixed at $S^*_{k_i}$, $I^*_{k_i}$ and $R^*_{k_i}$ as time passes, *i.e.*, $dS_{k_i}(t)/dt = 0$, $dI_{k_i}(t)/dt = 0$ and $dR_{k_i}(t)/dt = 0$. Thus, when System (1) converges to E_*, System (2) should satisfy

$$\begin{aligned} &\alpha - \lambda(k_i)S^*_{k_i}\Theta^* - \varepsilon_1 S^*_{k_i} = 0 \\ &\lambda(k_i)S^*_{k_i}\Theta^* - \varepsilon_2 I^*_{k_i} = 0 \end{aligned} \tag{3}$$

where $\Theta^* = \langle k \rangle^{-1} \sum_{i=1}^{n} \varphi(k_i) I^*_{k_i}$. From System (3), we have

$$I^*_{k_i} = \frac{\alpha\lambda(k_i)\Theta^*}{\varepsilon_2(\lambda(k_i)\Theta^* + \varepsilon_1)}. \tag{4}$$

It is easy to find that $I^*_{k_i} = 0$ $(i = 1, 2, \cdots, n)$ always satisfy Equation (4). Substituting $I^*_{k_i} = 0$ into System (3), we can derive the corresponding $S^*_{k_i}$ and $R^*_{k_i}$ as presented in Case 1 of Theorem 1. Substituting Equation (4) into Θ^* and moving the right item to left, we have

$$\Theta^* \left(1 - \frac{1}{\langle k \rangle} \sum_{i=1}^{n} \frac{\alpha\lambda(k_i)\varphi(k_i)}{\varepsilon_2(\lambda(k_i)\Theta^* + \varepsilon_1)}\right) = 0. \tag{5}$$

For Equation (5), let

$$F(\Theta^*) = 1 - \frac{1}{\langle k \rangle} \sum_{i=1}^{n} \frac{\alpha\lambda(k_i)\varphi(k_i)}{\varepsilon_2(\lambda(k_i)\Theta^* + \varepsilon_1)}.$$

Since $F'(\Theta^*) > 0$ for all Θ^* and $\lim_{\Theta^* \to \infty} F(\Theta^*) = 1$, there exists a non-trivial solution for equation $F(\Theta^*) = 0$ if and only if $\lim_{\Theta^* \to 0^+} F(\Theta^*) < 0$. $\lim_{\Theta^* \to 0^+} F(\Theta^*) < 0$ is satisfied only on the condition that $r_0 > 1$. Namely, System (1) has a positive-equilibrium solution when $r_0 > 1$. Substituting Equation (4) into System (3), we can obtain the positive-equilibrium solution of System (3) as presented in Case 2 of Theorem 1. □

Theorem 1 quantifies a restrictive correlation between the existence of equilibrium solution and countermeasures $(\varepsilon_1, \varepsilon_2)$. We find that zero-equilibrium solution always exists and positive-equilibrium solution exists iff $r_0 > 1$. However, as pointed out in [19] (Stability Theory for ODE), system state only converges to stable equilibrium solution, so Theorem 1 is not enough to determine the propagation state of rumors. Thus, we need to analyze the stability of two equilibrium suctions respectively.

C. The stability of equilibrium solution

There are two types of stability, *i.e.*, locally asymptotically stability and globally asymptotically stability. If the state variables coverage to the equilibrium solution just on the condition that its initial values are bounded in a certain region, this equilibrium solution is local asymptotically stable. Correspondingly, the equilibrium solution is global asymptotically stable if state variables coverage to it under any values of initial variable. Lyapunov's second stability method is generally used to determine the global asymptotic stability of the equilibrium solution of ODE system. For clarity, it is introduced as follows and more details can be found in [19].

Definition 4: **Lyapunov's second method for stability.** Mathematically, the equilibrium solution $x^* = 0$ is global asymptotically stable for the differential equation

$$\frac{dx}{dt} = f(t, x)$$

if there exists a scalar function $V(x) : \mathbb{R}^\ltimes \to \mathbb{R}$ which has continuous first partial derivative and satisfies

1) $V(x) \geq 0$ with equality if and only if $x = 0$.
2) $dV(x)/dt \leq 0$ with equality not constrained to only $x = 0$.

We next investigate the locally asymptotically stability and globally asymptotically stability of E_0 and E_* respectively.

1) Stability for E_0: For the stability of System (1) at E_0, we have the following Theorem 2 and Theorem 3.

Theorem 2: If $r_0 < 1$, E_0 is local asymptotically stable. If $r_0 > 1$, E_0 is unstable.

Proof. According to the stability theory of ODEs [19], if and only if all the eigenvalues of the characteristic equation of $J(E_*)$ are less than zero, the system is local asymptotically stable at E_*, where $J(E_*)$ is the Jacobian matrix of dynamic system at E_*. Thus, to analyze the stability of System (1) at E_0, we need to linearize the proposed ODE system firstly to obtain the eigenvalues of the characteristic equation of $J(E_0)$. As the first and second equations are independent of the third one in system (1), we also turn to analyze System (2) first. The result of linearizing System (2) at E_0 is as follows

$$\frac{dS_{k_i}(t)}{dt} = J_{k_i}^{1,1}(S - S_{k_i}^0) + J_{k_i}^{1,2}(I - I_{k_i}^0)$$
$$\frac{dI_{k_i}(t)}{dt} = J_{k_i}^{2,1}(S - S_{k_i}^0) + J_{k_i}^{2,2}(I - I_{k_i}^0)$$

where $J_{k_i}^{p,q}$ (p and $q \in \{1, 2\}, i = 1, 2, \cdots, n$) are the elements of the Jacobian matrix of System (2) at E_0 for group i. Then, the Jacobian matrix of System (2) at E_0 can be written as

$$J(E_0) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}.$$

For each $A_{i,j}$ where $i = j$, *i.e.*, the diagonal elements of $J(E_0)$, we have

$$A_{i,j} = \begin{pmatrix} -\varepsilon_1 & -\frac{\lambda(k_i)\alpha}{\varepsilon_1}\frac{1}{\langle k \rangle}\varphi(k_i) \\ 0 & \frac{\lambda(k_i)\alpha}{\varepsilon_1}\frac{1}{\langle k \rangle}\varphi(k_i) - \varepsilon_2 \end{pmatrix}.$$

For $A_{i,j}$ where $i \neq j$, we have

$$A_{i,j} = \begin{pmatrix} 0 & -\frac{\lambda(k_i)\alpha}{\varepsilon_1}\frac{1}{\langle k \rangle}\varphi(k_i) \\ 0 & \frac{\lambda(k_i)\alpha}{\varepsilon_1}\frac{1}{\langle k \rangle}\varphi(k_i) \end{pmatrix}.$$

The characteristic equation of $J(E_0)$ can be expressed as

$$(\chi + \varepsilon_1)^3(\chi + \varepsilon_2)^2(\chi - (\Gamma - \varepsilon_2)) = 0$$

where

$$\Gamma = \frac{\alpha}{\langle k \rangle}\sum_{i=1}^{n}\frac{\lambda(k_i)\varphi(k_i)}{\varepsilon_1}.$$

Thus, the eigenvalues of the characteristic equation of $J(E_0)$ are $\chi_1 = -\varepsilon_1$, $\chi_2 = -\varepsilon_2$ and $\chi_3 = \Gamma - \varepsilon_2$. Since $-\varepsilon_1 < 0$ and $-\varepsilon_2 < 0$, the local stability of E_0 is completely determined by the sign of $\Gamma - \varepsilon_2$. If $r_0 < 1$, we have $\Gamma - \varepsilon_2 < 0$ so that System (1) is local asymptotically stable at E_0. If $r_0 > 1$, we have $\Gamma - \varepsilon_2 > 0$, thus System (1) is unstable at E_0. □

To verify the global asymptotic stability of E_0, we first present Lemma 1 as preparation.

Lemma 1: As System (1) asymptotically converges to E_+ (*i.e.*, $r_0 > 1$), ε_2 satisfies

$$\lim_{E^* \to E^+} \varepsilon_2 = \frac{1}{\langle k \rangle}\sum_{i=1}^{n}\lambda(k_i)\varphi(k_i)S_{k_i}^+. \tag{6}$$

Proof. Based on the definition of $\Theta(t)$ and combining the second equation of System (1), we have

$$\begin{aligned} \Theta'(t) &= \frac{1}{\langle k \rangle}\sum_{i=1}^{n}\varphi(k_i)I'_{k_i}(t) \\ &= \frac{1}{\langle k \rangle}\sum_{i=1}^{n}\varphi(k_i)(\lambda(k_i)S_{k_i}(t)\Theta(t) - \varepsilon_2 I_{k_i}(t)) \\ &= \Theta(t)\left(\frac{1}{\langle k \rangle}\sum_{i=1}^{n}(\lambda(k_i)\varphi(k_i)S_{k_i}(t)) - \varepsilon_2\right) \end{aligned} \tag{7}$$

When System (1) converges to E_+, $\Theta'(t) = 0$. Since $\Theta(t) > 0$, from Equation (7) we can derive Equation (6). □

Theorem 3: If $r_0 < 1$, E_0 is global asymptotically stable.

Proof. To investigate the global asymptotic stability of the ODE system at E_* where E_* is the equilibrium point, we need to construct a Lyapunov function $V(t)$. According to Definition 4, we construct the Lyapunov function for E_0 as

$$V(t) = \frac{1}{\varepsilon_2}\Theta(t).$$

Then, based on Lemma 1 and combining the equilibrium solution $S_{k_i}^0 = \alpha/\varepsilon_1$, the time derivative of $V(t)$ computed in the solution space of System (1) for $t > 0$ is

$$\begin{aligned} \frac{dV(t)}{dt} &= \frac{1}{\varepsilon_2}\Theta'(t) \\ &= \frac{1}{\varepsilon_2}\Theta(t)\left(\frac{1}{\langle k \rangle}\sum_{i=1}^{n}(\lambda(k_i)\varphi(k_i)S_{k_i}(t)) - \varepsilon_2\right) \\ &\leq \frac{1}{\varepsilon_2}\Theta(t)\left(\frac{1}{\langle k \rangle}\sum_{i=1}^{n}\left(\lambda(k_i)\varphi(k_i)S_{k_i}^0\right) - \varepsilon_2\right) \\ &= \frac{1}{\varepsilon_2}\Theta(t)\left(\frac{1}{\langle k \rangle}\sum_{i=1}^{n}\frac{\alpha\lambda(k_i)\varphi(k_i)}{\varepsilon_1} - \varepsilon_2\right) \\ &= \Theta(t)\left(\frac{\alpha}{\langle k \rangle}\sum_{i=1}^{n}\frac{\lambda(k_i)\varphi(k_i)}{\varepsilon_1\varepsilon_2} - 1\right) \\ &= \Theta(t)(r_0 - 1) \end{aligned}$$

When $r_0 < 1$, we have $dV(t)/dt < 0$. Thus, as time approaches to infinity, E_0 is global asymptotically stable. □

2) Stability for E_+: For the stability of System (1) at E_+, we have the following Theorem 4.

Theorem 4: If $r_0 > 1$, E_+ is global asymptotically stable.

Proof. We construct the Lyapunov function $V(t)$ as

$$V(t) = \frac{1}{2}\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{S_{k_i}^{+}}\left(\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})^2\right) + \left(\Theta(t)-\Theta^{+}-\Theta^{+}\ln(\frac{\Theta(t)}{\Theta^{+}})\right).$$

Then the time derivative of $V(t)$ computed in the solution space of System (1) is

$$V'(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})S_{k_i}'\right) + \frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\Theta'(t). \tag{8}$$

For clarity, we split Equation (8) into two parts. When System (2) converges to E_+, from the first equation of System (3), we have $\alpha = \lambda(k_i)S_{k_i}^{+}\Theta^{+} + \varepsilon_1 S_{k_i}^{+}$. Thus, for the first part of Equation (8),

$$\begin{aligned}
&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})S_{k_i}'\right)\\
=&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})(\alpha-\lambda(k_i)S_{k_i}(t)\Theta(t)\right.\\
&\left.-\varepsilon_1 S_{k_i}(t))\right)\\
=&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(\frac{1}{S_{k_i}^{+}}\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})(\lambda(k_i)S_{k_i}^{+}\Theta^{+}\right.\\
&\left.+\varepsilon_1 S_{k_i}^{+}-\lambda(k_i)S_{k_i}(t)\Theta(t)-\varepsilon_1 S_{k_i}(t))\right)\\
=&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon_1)(S_{k_i}(t)-S_{k_i}^{+})^2\right.\\
&\left.-\varphi(k_i)\lambda(k_i)(\Theta(t)-\Theta^{+})(S_{k_i}(t)-S_{k_i}^{+})\right)
\end{aligned} \tag{9}$$

and for the second part of Equation (8),

$$\begin{aligned}
&\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\Theta'(t)\\
=&\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\left[\Theta(t)\left(\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}(t)-\varepsilon_2\right)\right]\\
=&(\Theta(t)-\Theta^{+})\left[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}(t)\right.\\
&\left.-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)S_{k_i}^{+}\right]\\
=&\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)(\Theta(t)-\Theta^{+})(S_{k_i}(t)-S_{k_i}^{+})
\end{aligned} \tag{10}$$

Combining Equation (9) and Equation (10), we can derive the result of Equation (8) as follows

$$\begin{aligned}
V'(t) =& \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left\{\frac{1}{S_{k_i}^{+}}\varphi(k_i)(S_{k_i}(t)-S_{k_i}^{+})S_{k_i}'\right\}\\
&+\frac{\Theta(t)-\Theta^{+}}{\Theta(t)}\Theta(t)'\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon_1)(S_{k_i}(t)-S_{k_i}^{+})^2\right.\\
&\left.-\varphi(k_i)\lambda(k_i)(\Theta(t)-\Theta^{+})(S_{k_i}(t)-S_{k_i}^{+})\right)\\
&+\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\varphi(k_i)\lambda(k_i)(\Theta(t)-\Theta^{+})(S_{k_i}(t)-S_{k_i}^{+})\\
&=\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\left(-\frac{1}{S_{k_i}^{+}}\varphi(k_i)(\lambda(k_i)\Theta(t)+\varepsilon)(S_{k_i}(t)-S_{k_i}^{+})^2\right)\\
&\le 0
\end{aligned} \tag{11}$$

Thus, E_+ is global asymptotically stable. □

D. The critical conditions

Based on the above discussions about the existence and stability of equilibrium solutions, it is easy to obtain the following theorem about the critical conditions:

Theorem 5: If the countermeasures are effective resulting to $r_0 \le 1$, the infection is no longer epidemic and the rumor will be extinct. Otherwise, if the countermeasures result in $r_0 > 1$, the rumor will continuously propagate and the infected individuals will converge to a stable level.

Theorem 5 indicates that the propagation dynamics of rumors are determined by threshold r_0, which quantifies a restrictive correlation between rumor spreading behaviour and social network parameters such as $\alpha, \lambda(k_i), \langle k\rangle, \varphi(k_i)$ and countermeasure parameters such as ε_1 and ε_2.. To restrain the spread of rumor, countermeasures should be taken to let $r_0 < 1$.

IV. OPTIMIZED COUNTERMEASURES

In this section, we specify the optimized countermeasures, based on which the spread of rumors will be restrained at the end of an expected time period with lowest cost. As mentioned earlier, the implementation cost of *blocking rumors* and *spreading truth* are expensive and different. To specify the cost, we employ c_1 and c_2 to represent the unit cost of *spreading truth* to immunize susceptible individuals and that of *blocking rumors* at infected individuals, respectively. Assuming the expected time period is (0, t_f], the optimization objective motivates the following objective function:

$$\begin{aligned}
&J(\varepsilon_1(t),\varepsilon_2(t))\\
&=\min\left\{\sum_{i=1}^{n}I_{k_i}(t_f)\right.\\
&\left.+\int_0^{t_f}\sum_{i=1}^{n}(c_1F(\varepsilon_1(t),S_{k_i}(t))+c_2G(\varepsilon_2(t),I_{k_i}(t)))dt\right\}
\end{aligned} \tag{12}$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ act as control variables. The feasible region of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ is $U = \{(\varepsilon_1(t), \varepsilon_2(t)) | 0 \leq \varepsilon_1(t) \leq \varepsilon_1^{max}, 0 \leq \varepsilon_2(t) \leq \varepsilon_2^{max}, t \in (0, t_f]\}$ where ε_1^{max} and ε_2^{max} are the upper bound of ε_1 and ε_2 respectively.

The objective function (12) incorporates two objectives. First, the part of $\sum_{i=1}^{n} I_{k_i}(t_f)$ sums the number of infected individuals across all groups at t_f; thus, minimizing it ensures the spread of rumors can be restrained at t_f. Second, the integral process of $\int_0^{t_f} \sum_{i=1}^{n} (c_1 F(\varepsilon_1(t), S_{k_i}(t)) + c_2 G(\varepsilon_2(t), I_{k_i}(t)))dt$ describes the cost of countermeasures in the whole rumor combating process from the initial time to t_f; taking it as optimization objective ensures the cost of countermeasures is lowest during $(0, t_f]$. Functions F and G are used to describe the real-time cost of *spreading truth* and *blocking rumors* respectively. Here we introduce a quadratic function to illustrate the non-linear property of F and G. Thus, the objective function can be rewritten as follows

$$
\begin{aligned}
&J(\varepsilon_1(t), \varepsilon_2(t)) \\
&= \min \left\{ \sum_{i=1}^{n} I_{k_i}(t_f) \right. \\
&\left. + \int_0^{t_f} \sum_{i=1}^{n} \left(c_1 \varepsilon_1^2(t) S_{k_i}^2(t) + c_2 \varepsilon_2^2(t) I_{k_i}^2(t) \right) dt \right\}
\end{aligned} \tag{13}
$$

The main challenge in computing the optimized values of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ is that the ODE System (1) can be solved provided $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are known. An exhaustive search in U is impossible since an infinite number of such $\varepsilon_1(t)$'s and $\varepsilon_2(t)$'s are in U. To conquer this problem, the Pontryagin's Maximum Principle is utilized based on which the optimized countermeasures can be derived. The Pontryagin's maximum principle is widely used in the optimal control region to find the optimized control strategies that let an objective get optimized values based on a dynamical system when transforming from one state to another. The optimized control problem involves several entities: a dynamic control system, the initial and transversality conditions, admissible controls and objective function. More details can be found in [21]. Based on this theory, our optimized problem can be described as follows.

Input:

(1) Dynamic control system: Heterogeneous network based SIR Model, *i.e.*, System (1).

(2) Initial conditions: $I_{k_i}(t_0) > 0$, $S_{k_i}(t_0) = 1 - I_{k_i}(t_0)$ and $R_{k_i}(t_0) = 0$, where $t_0 = 0$ and $i = 1, 2, \cdots, n$.

(3) Transversality conditions: $\psi_i(t_f) = 0$ and $\phi_i(t_f) = 1$ ($i = 1, 2, \cdots, n$), where ψ_i and ϕ_i are co-state functions of group i.

(4) Admissible controls: $\varepsilon_1(t)$ and $\varepsilon_2(t)$, where $t \in (0, t_f]$, $0 \leq \varepsilon_1(t) \leq \varepsilon_1^{max}, 0 \leq \varepsilon_2(t) \leq \varepsilon_2^{max}$ and $i = 1, 2, \cdots, n$.

Output: The optimized controls ($\varepsilon_1^*(t)$, $\varepsilon_2^*(t)$), which satisfy the objective function (13).

According to the Pontryagin's Maximum Principle, we set $((S(t), I(t)), \varepsilon_1(t), \varepsilon_2(t))$ as the optimized solution. Then, the Hamiltonian function H can be defined as follows

$$
\begin{aligned}
H = &\sum_{i=1}^{n} \left(c_1 \varepsilon_1^2(t) S_{k_i}^2(t) + c_2 \varepsilon_2^2(t) I_{k_i}^2(t) \right) \\
&+ \left(\sum_{i=1}^{n} \psi_i(t) (\alpha - \lambda(k_i) S_{k_i}(t) \Theta(t) - \varepsilon_1(t) S_{k_i}(t)) \right) \\
&+ \left(\sum_{i=1}^{n} \phi_i(t) (\lambda(k_i) S_{k_i}(t) \Theta(t) - \varepsilon_2(t) I_{k_i}(t)) \right)
\end{aligned} \tag{14}
$$

where $\psi_1(t)$ and $\psi_2(t)$ are the co-state functions defined as follows

$$
\begin{aligned}
\frac{d\psi_i(t)}{dt} &= -\frac{\partial H}{\partial S_{k_i}(t)} \\
&= -2c_1 \varepsilon_1^2(t) S_{k_i}(t) - \psi_i(t) \left(-\lambda(k_i) \Theta(t) - \varepsilon_1(t) \right) \\
&- \phi_i(t) \lambda(k_i) \Theta(t)
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
\frac{d\phi_i(t)}{dt} &= -\frac{\partial H}{\partial I_{k_i}(t)} \\
&= -2c_2 \varepsilon_2^2(t) I_{k_i}(t) + \psi_i(t) \langle k \rangle^{-1} \varphi(k_i) \lambda(k_i) S_{k_i}(t) \\
&- \phi_i(t) \left(\langle k \rangle^{-1} \varphi(k_i) \lambda(k_i) S_{k_i}(t) - \varepsilon_2(t) \right)
\end{aligned} \tag{16}
$$

Objective function (13) shows that the density of the infected individuals should be minimized at t_f so that the transversality conditions are

$$
\psi_i(t_f) = 0, \phi_i(t_f) = 1.
$$

According to the Pontryagin 's Maximum Principle, if there exist continuous and piecewise differentiable co-state functions $\psi_i(t)$ and $\phi_i(t)$ at each $t \in (0, t_f]$ where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are continuous, satisfying (15) and (16), the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ should satisfy

$$
\begin{aligned}
&(\varepsilon_1(t), \varepsilon_2(t)) \in \\
&\arg\max H \left\{ (\psi_i(t), \phi_i(t)), (S_{k_i}(t), I_{k_i}(t)), U(\varepsilon_1(t), \varepsilon_2(t)) \right\}
\end{aligned}
$$

where $U(\varepsilon_1(t), \varepsilon_2(t))$ is the universal set of admissible control of $\varepsilon_1(t)$ and $\varepsilon_2(t)$. Since H is a quadratic convex function of $\varepsilon_1(t)$ and $\varepsilon_2(t)$, the maximum value of H is obtained at the stationary point (*i.e.*, $\frac{\partial H}{\partial \varepsilon_1(t)} = 0$ and $\frac{\partial H}{\partial \varepsilon_2(t)} = 0$) or at the start point and end point. We first compute the stationary point of H as

$$
\begin{aligned}
\frac{\partial H}{\partial \varepsilon_1(t)} &= 2c_1 \varepsilon_1(t) \sum_{i=1}^{n} S_{k_i}^2(t) - \sum_{i=1}^{n} \psi_i(t) S_{k_i}(t) \\
\frac{\partial H}{\partial \varepsilon_2(t)} &= 2c_2 \varepsilon_2(t) \sum_{i=1}^{n} I_{k_i}^2(t) - \sum_{i=1}^{n} \phi_i(t) I_{k_i}(t)
\end{aligned} \tag{17}
$$

Let $\partial H / \partial \varepsilon_1(t) = 0$ and $\partial H / \partial \varepsilon_2(t) = 0$. From (17), we have

$$
\varepsilon_1(t) = \frac{\sum_{i=1}^{n} \psi_i(t) S_{k_i}(t)}{2c_1 \sum_{i=1}^{n} S_{k_i}^2(t)}, \varepsilon_2(t) = \frac{\sum_{i=1}^{n} \phi_i(t) I_{k_i}(t)}{2c_2 \sum_{i=1}^{n} I_{k_i}^2(t)}. \tag{18}
$$

Finally, the optimized countermeasures under the objective function (13) in $(0, t_f]$ are

$$\begin{aligned} \varepsilon_1(t) &= \min\{\max(0, \varepsilon_1(t)), \varepsilon_1^{max}\} \\ \varepsilon_2(t) &= \min\{\max(0, \varepsilon_2(t)), \varepsilon_2^{max}\} \end{aligned} \tag{19}$$

Thus, Equations (19) provides the optimized $\varepsilon_1(t)$ and $\varepsilon_2(t)$ to restrain the spread of rumors at the end of expected time period $(0, t_f]$ with lowest cost.

V. Model Validation

In this section, we evaluate the performance of the dynamic model and optimized countermeasures. The employed data is from "Digg2009 datasite"collected in [22]. As one of the most popular social news aggregation sites, Digg is also a staging ground of rumors and truth. Digg users can submit links of news coming from any other news website and comment or vote on the submitted news. A friendship link is formed if a user follows another user. The Digg2009 data set provides 1731658 friendship links of 71367 users who have voted on the submitted news. Based on these friendship links, a directed social network graph is formed. According to different social connectivity degrees, we can classify these 71367 users into 848 groups. The maximum degree of this data set is 995 and the minimum degree is 1. The average degree of this data set is close to 24, *i.e.*, $\langle k \rangle = 24$.

We first evaluate the effectiveness of the critical conditions that determine whether a rumor continuously propagates or becomes extinct. Next, the impact of the optimized countermeasures on restraining the spread of rumors is validated.

A. *Effectiveness of Critical Conditions*

According to Theorem 5, the critical conditions are determined by threshold r_0, which is the function of social network parameters such as $\alpha, \lambda(k_i), \langle k \rangle, \varphi(k_i)$ and countermeasure parameters such as ε_1 and ε_2.

Assuming the rumor acceptance rate grows linearly with its social connectivity, we let $\lambda(k_i) = k_i$. Meanwhile, a non-linear infectivity for an infected individual is adopted as illustrated in Section III: $\omega(k_i) = k_i^\beta/(1 + k_i^\gamma)$ with $\beta = 0.5$ and $\gamma = 0.5$. Other parameters in System (1) are set as $\alpha = 0.01$, $\varepsilon_1 = 0.2$, $\varepsilon_2 = 0.05$. We can compute that $r_0 = 0.7220 < 1$. In this case, E_0 is global asymptotically stable (as indicated by Theorem 3). According to Theorem 5, the infection is no longer epidemic and the rumor will be extinct.

Assuming $E(t)$ is an arbitrary solution of System (1), we employ $Dist_0(t)$ to denote the Euclidean Distance between $E(t)$ and E_0:

$$Dist_0(t) = \|E(t) - E_0\|_\infty$$

Under 10 different initial values (*i.e.*, different $S_{k_i}(0)$ and $I_{k_i}(0)$, $R_{k_i}(0) = 0$), the time evolution of $Dist_0(t)$ is shown in Fig.2(a). As shown in Fig.2(a), the 10 curves represents time evolution of the Euclidean Distance between $E(t)$ and E_0 under 10 initial conditions. We observe that $Dist_0(t)$ converge to zero under different initial conditions, which means that E_0 is global asymptotically stable. Next, Fig.2(b), Fig.2(c) and Fig.2(d) shows the time evolutions of

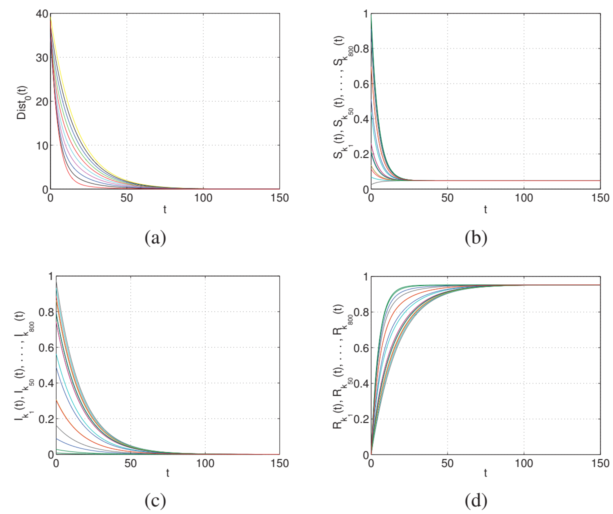

Fig. 2. (a): The time evolution of $Dist_0(t)$ under 10 different initial conditions. The time evolution of (b): $S_{k_i}(t)$; (c): $I_{k_i}(t)$; (d): $R_{k_i}(t)$, where $i = 1, 50, \ldots, 800$, $r_0 = 0.7220 < 1$.

$S_{k_i}, I_{k_i}, R_{k_i}, i = 1, 50, 100, \ldots, 800$ under one arbitrary initial condition respectively. We can observe that the infection is no longer epidemic and the rumor will be extinct under this level of countermeasures.

Keeping the other parameters unchanged, we set $\alpha = 0.002, \varepsilon_1 = 0.002, \varepsilon_2 = 0.0001$. We can compute that $r_0 = 2.1661 > 1$. In this case, E_+ is global asymptotically stable (as indicated by Theorem 4). According to Theorem 5, the rumor will continuously propagate and system variables converge to E_+. Similarly, to verify it, the Euclidean Distance between $E(t)$ and E_+ is measured by

$$Dist_+(t) = \|E(t) - E_+\|_\infty$$

The time evolutions of $Dist_+(t)$ are shown in Fig.3(a). As shown in Fig.3(a), the 10 curves represents time evolution of $Dist_+(t)$ under 10 initial conditions. We observe that $Dist_+(t)$ converges to zero under different initial conditions, which means that E_+ is global asymptotically stable. Then, setting an arbitrary initial condition, the time evolution of $S_{k_i}, I_{k_i}, R_{k_i}, i = 1, 2, \ldots, 20$ are shown in Fig.3(b), 3(c) and 3(d) respectively. We can observe that under this level of countermeasures, a rumor will continuously propagate and the system variables converge to E_+.

B. *Efficiency of Optimized Countermeasures*

In this subsection, we will investigate the framework of the optimized countermeasures, *i.e.*, under what level of countermeasures, the densities of the infected individuals will be minimized by t_f with the lowest cost. We assume the expected time period is $(0, 100]$ and the cost of *blocking rumors* is larger than that of *spreading truth*, specifically, $c_1 = 5$ and $c_2 = 10$. Keeping the other parameters unchanged, the optimized values of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ ($t \in (0, 100]$) to satisfy the objective function (13) are shown in Fig.4(a). As shown in Fig.4(a), the method of *spreading truth* should play a dominant role (*i.e.*, $\varepsilon_1 > \varepsilon_2$) in the initial rumor restraining phase. Then, when

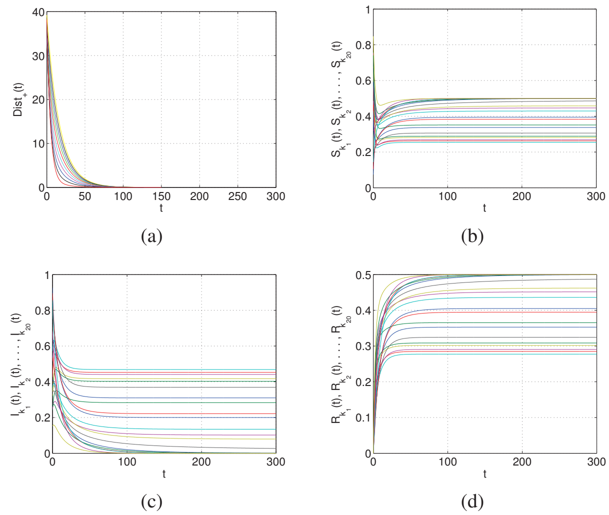

Fig. 3. (a): The time evolution of $Dist_{+}(t)$ under 10 different initial conditions. The time evolution of (b): $S_{k_i}(t)$; (c): $I_{k_i}(t)$; (d): $R_{k_i}(t)$, where $i = 1, 2, \ldots, 20$, $r_0 = 2.1661 > 1$.

approaching to the end of expected time period, the method of *blocking rumors* should be taken intensively, (*i.e.*, $\varepsilon_1 < \varepsilon_2$). Specifically, the real-time implementation proportion of two methods provides an effective decision reference to restrain rumors.

To further learn the rumor propagation dynamics on the impact of optimized countermeasures, we can observe the time evolution of threshold r_0 as shown in Fig.4(b). We observe that, r_0 decreases with the time as countermeasures are carried out, which indicates that the rumor tend to be extinct. Moreover, r_0 is larger than 1 at the early stage and less than 1 lately, *i.e.*, the optimized countermeasures allow the rumor first propagate mildly then limit its propagation when approaching to the end of the expected time period.

To verify the efficiency of the proposed optimized countermeasures, we compare the cost of the heuristic and optimized countermeasures when controlling the number of infected individuals to a same level within a same expected time period. The heuristic countermeasures restrain the spread of rumors just based on the current infection state, *i.e.*, there do not have a global control. We set a set of t_f such as $t_f = 10, 20, ..., 100$ and let the densities of infected individuals at t_f be less than 0.0001. Thus, under these 10 different time periods, the cost comparison is shown in Fig.4(c). Compared with the heuristic countermeasures, optimized countermeasures have lower costs when achieving the same effects.

VI. Related Works

With the popularity of OSNs, extensive studies have been conducted on the issue of information diffusion in OSNs. The problem of influence maximization is well known and many significant results have been obtained [23] [24]. The key approach to influence maximization is to identify an optimized set of seed nodes as information spreaders. Meanwhile, the reverse problem of restraining the spread of rumors in OSNs has also attracted increasing attentions with the flood of rumors in OSNs. Similarly, the key approach of the current works is to identify a minimum subset of individuals as rumor blockers or truth spreaders to combat rumors [3] [4] [5] [6]. However, most of existing works do not consider the implementation cost of restraining the spread of rumors. [5] studied how to minimize the cost of immunizing the individuals to restrain the rumors and which is shown to be an NP-Hard problem. Moreover, based on the two types of strategies of *blocking rumors* and *spreading truth*, the work in [12] studies which method has better performance. The work in [25] develops an analytical model to present temporal dynamics of propagation of positive and negative information considering social individuals' preferences. The work in [26] proposes the immunization strategies for rumor spreading based on the weighted trust networks.

Thus, to minimize the implementation cost by combining the two restraining methods such that a rumor can be extinct within an expected time period, the first concern is how to model the propagation dynamics of rumors. The investigation to model the propagation dynamic is initially influenced by the study of epidemics. Many epidemic models are proposed to investigate temporal dynamic based on ODEs [27], [11], temporal-spatial dynamics based on partial differential equations (PDEs) [28] [29] or Markov chain [30]. The work in [27] proposes an SIR model to describe rumor spreading in OSNs and discusses the effectiveness of random and targeted inoculations. Similarly, an SIR model based on user browsing behavior was proposed in [11] to investigate the efficiency of different rumor countermeasures. However, most of these works assume that the social individuals mix homogeneously without considering the heterogeneity of OSNs. For all the concerns, we proposes an SIR epidemic model incorporating both the network heterogeneity and various countermeasures to investigate the propagation dynamics and optimized countermeasures.

VII. Conclusions

This paper proposes a heterogeneous network based SIR Model to describe the propagation dynamics of rumors in OSNs and analyzes the optimized countermeasures that can restrain the spread of rumors with lowest implementation cost. Based on our mathematical model, we derive the critical conditions that determine whether a rumor continuously propagates or becomes extinct, which formulate the relationship between the rumor spreading and the level of countermeasures. Moreover, we investigate the framework of the optimized countermeasures, which provide a real-time decision reference to restrain rumor spreading. Finally, the proposed model and optimized countermeasures are validated through the real Digg2009 data set. The experimental results verify the effectiveness of the dynamic model and the efficiency of the optimized countermeasures.

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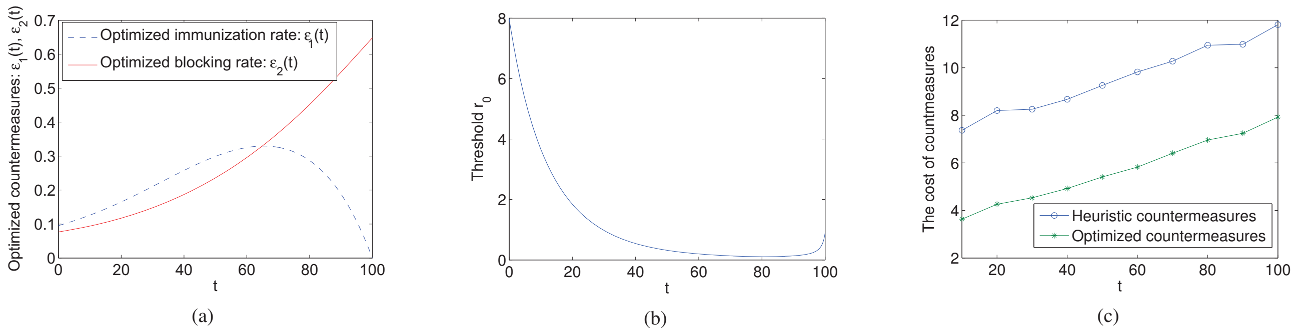

Fig. 4. The time evolution of (a): optimized countermeasures: ε_1 and ε_2; (b): threshold r_0 under optimized countermeasures; (c) cost comparison of heuristic and optimized countermeasures where the horizontal axis is t_f.

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