JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
Volume 11, Number 1, January 2015

doi:10.3934/jimo.2015.11.105
pp. 105–133

STATISTICAL PROCESS CONTROL OPTIMIZATION WITH VARIABLE SAMPLING INTERVAL AND NONLINEAR EXPECTED LOSS

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(Communicated by Kok Lay Teo)

Abstract. The optimization of a statistical process control with a variable sampling interval is studied, aiming in minimization of the expected loss. This loss is caused by delay in detecting process change and depends nonlinearly on the sampling interval. An approximate solution of this optimization problem is obtained by its decomposition into two simpler subproblems: linear and quadratic. Two approaches to the solution of the quadratic subproblem are proposed. The first approach is based on the Pontryagin's Maximum Principle, leading to an exact analytical solution. The second approach is based on a discretization of the problem and using proper mathematical programming tools, providing an approximate numerical solution. Composite solution of the original problem is constructed. Illustrative examples are presented.

1. **Introduction.** The statistical process control (SPC) (see e.g. [18]) is widely used in monitoring processes in industry, medicine, environment etc. Its objective is to minimize the losses, caused by delay in the detection of undesirable accidents, keeping acceptable inspection expenses. The idea of using variable/adaptive sampling SPC intervals to achieve process stability is known in literature. For the first time it appeared in the work of [22]. Then it was developed in a number of works (see e.g. [1, 2, 7, 8, 9, 10, 11, 20, 21]). A detailed review on the topic, including the above-mentioned works, is presented in [4].

Following [22], usually the reaction time of SPC to process change is considered as the main criterion of optimality. The alternative criterion, proposed in [25], and further developed in [4], is the expected loss, caused by delay in detecting process change. This criterion seems to be more general and at the same time more usable, from the engineering viewpoint. The relation between such a loss and reaction time (detection delay) is not necessarily a linear one. There are various processes, clearly demonstrating the non-linear character of damage, caused by a detection

2010 *Mathematics Subject Classification.* Primary: 49N90, 49M25; Secondary: 90C20.
Key words and phrases. Statistical process control, optimization, variable sampling interval, nonlinear loss, optimal control, approximation, quadratic programming.

delay. For instance, we can mention fires propagation [3], oil spills spreading [24], cholesterol plaque growth [5], epidemics propagation [6], fatigue crack growth in ship hull structures [16] and others.

Genichi Taguchi proposed the quadratic dependence of the loss on the critical performance parameter whose desired value is intended to be as low as possible within the existing constraints (see e.g. [25], "smaller the best" loss function). In the modern industry, the process control becomes an indispensable part of the process itself. Therefore, the detection delay, being a critical performance parameter of the process control, becomes a critical performance parameter of the process. Thus, the Taguchi model yields a quadratic dependence of the loss on the detection delay. It should be noted that the delay time in SPC is a random variable, which distribution depends on the process change extent. Therefore, expected (in the statistical sense) loss becomes the actual optimization criterion.

In this paper, the SPC with a variable sampling time is formulated as a calculus of variations problem, in which the expected loss should be minimized by a proper choice of a sampling interval. In this problem, two types of constraints are imposed. The first type of constraints is an isoperimetric (weak) constraint, meaning that the average variable sampling time is equal to a properly prechosen constant nominal sampling time. The second type of constraints is two geometric (hard) constraints, which determine the lower and upper bounds of the sampling interval. The latter is not addressed by the classical calculus of variations theory. This makes the extremal problem to be non-standard.

Moreover, solving this problem is a rather complicated task. Therefore, in this paper, its decomposition into two simpler subproblems is proposed. Both subproblems are still non-standard variational problems: one is linear, and the other is quadratic. The linear subproblem was solved in [4]. In this paper, we concentrate on solving the second (quadratic) subproblem, and on constructing a composite solution of the original problem.

Two approaches to the solution of this subproblem are proposed. In the first approach, it is transformed to an auxiliary optimal control problem [19]. The latter is treated by applying the Pontryagin Maximum Principle (PMP), leading to an exact analytical solution. The second approach is based on a discretization of the problem and using proper mathematical programming tools, providing an approximate numerical solution.

The paper is organized as follows. The next section is devoted to the detailed problem statement. In Section 3, the original problem is decomposed into two simpler problems, linear and quadratic ones. The solution of the linear problem is briefly outlined in Section 4. The detailed analytical solution of the quadratic problem is presented in Section 5. In Section 6, the approximate numerical solution of the quadratic problem is developed. Numerical evaluation of the theoretical results is presented in Section 7. Section 8 of the paper is devoted to conclusions. Proofs of main statements are placed into Section 9.

2. **Problem statement.**

2.1. **Process monitoring.**

Similarly to [4], we consider the case, where the process monitoring of a performance parameter x is carried out according to a sample mean $\bar{x} \sim N(\mu, \sigma/\sqrt{n})$. This assumption is reasonable, because, due to the Central Limiting Theorem, the normal distribution provides a good approximation of $\bar{x}$

distribution, even if x does not strictly fit a normal distribution. The sample size n is fixed. Thus, the standard score in an in-control state is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1). \tag{1}$$

Note that the upper and lower control limits of a standard Shewhart control chart for z are $z_{\min} = -3$ and $z_{\max} = 3$, respectively [18]. Therefore, the false alarm probability α (type I error), i.e. the probability of the event $z \notin [-3,3]$, is

$$\alpha = 1 - \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} \exp(-z^2/2)dz = 1 - [\Phi(3) - \Phi(-3)] = 0.0027, \tag{2}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-\zeta^2/2)d\zeta. \tag{3}$$

If the performance parameter mean value shifts by Δ, i.e. a new mean value is $\mu' = \mu + \Delta$, and the value of σ remains unchanged, then the distribution of z becomes

$$z \sim N(\delta, 1), \qquad \delta = \frac{\Delta}{\sigma/\sqrt{n}} \tag{4}$$

where δ is the so-called *signal-to-noise ratio*. The probability of discovering the shift (receiving the signal) by a single sample is the probability of the event $z \notin [-3,3]$ subject to (4):

$$1 - \beta = 1 - \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} \exp(-(z-\delta)^2/2)dz = 1 - [\Phi(3-\delta) - \Phi(-3-\delta)], \tag{5}$$

where β is the probability of a type II error (not discovering the shift), depending on a normalized shift δ.

2.2. **Constraints on variable sampling interval.** Consider the adaptive statistical process control with a variable sampling time $u(z)$, where z is a current value of a standard score, given by (1). Note that the function $u(z)$ is even ($u(-z) = u(z)$), because the value of the sampling time should depend only on the absolute value of the standard score. Therefore, in the sequel, we consider the function $u(z)$ to be defined on the interval $[0,3]$. It is assumed that the expected sampling interval in the case of unshifted z is equal to a prescribed nominal value T:

$$E(u)|_{\delta=0} = T, \tag{6}$$

yielding the isoperimetric (integral) constraint [4]

$$\int_{0}^{3} \exp(-z^2/2)[u(z) - T]dz = 0. \tag{7}$$

It is also assumed that the function $u(z)$ is bounded by

$$0 < u_{\min} \le u \le u_{\max}, \tag{8}$$

where $u_{\min} < T$ and $u_{\max} > T$.

In what follows, for the sake of convenience, we set

$$T = 1, \tag{9}$$

yielding

$$u_{\min} < 1, \quad u_{\max} > 1. \tag{10}$$

2.3. **Quadratic loss model.** If the process shift remains constant, the time T_d, needed for discovering the shift (so-called *time to signal*), is the sum of a random amount N_d of random independent and identically distributed sampling intervals u_i, conditionally independent of N_d:

$$T_d = \sum_{i=1}^{N_d} u_i. \tag{11}$$

The number of samples N_d, needed for the shift detection, is a random value, distributed geometrically [4] with the success probability $1-\beta$, given by (5). Its expectation and variance are [23]:

$$\mathrm{E}(N_d) = \frac{1}{1-\beta}, \quad \mathrm{Var}(N_d) = \frac{\beta}{(1-\beta)^2}. \tag{12}$$

The cost functional, to be minimized by a properly chosen sampling interval u, is the mathematical expectation $\mathrm{E}(L)$ of the loss L, caused by the shift detection delay (expected loss). Due to the assumption that the loss L is proportional to T_d^2, ($L = kT_d^2$, $k > 0$), we obtain

$$\mathrm{E}(L) = k\mathrm{E}(T_d^2). \tag{13}$$

Thus, we can formulate the optimization problem:

$$\mathrm{E}(T_d^2) \to \min_u \quad \text{s.t. (7) − (9).} \tag{14}$$

Due to [23],

$$\mathrm{E}(T_d) = \mathrm{E}(N_d)\,\mathrm{E}(u), \tag{15}$$

$$\mathrm{Var}(T_d) = \mathrm{E}(N_d)\,\mathrm{Var}(u) + \mathrm{Var}(N_d)\mathrm{E}^2(u). \tag{16}$$

Thus,

$$\begin{gathered}\mathrm{E}(T_d^2) = \mathrm{E}^2(T_d) + \mathrm{Var}(T_d) = \mathrm{E}^2(N_d)\mathrm{E}^2(u) + \mathrm{E}(N_d)\,\mathrm{Var}(u) + \mathrm{Var}(N_d)\mathrm{E}^2(u) = \\ \mathrm{E}^2(N_d)\mathrm{E}^2(u) + \mathrm{E}(N_d)(\mathrm{E}(u^2) - \mathrm{E}^2(u)) + \mathrm{Var}(N_d)\mathrm{E}^2(u) = \\ \mathrm{E}(N_d)\,\mathrm{E}(u^2) + \left[\mathrm{E}^2(N_d) - \mathrm{E}(N_d) + \mathrm{Var}(N_d)\right]\mathrm{E}^2(u). \end{gathered} \tag{17}$$

By (12), the equation (17) leads to

$$\mathrm{E}(T_d^2) = \frac{1}{1-\beta}\,\mathrm{E}(u^2) + \frac{2\beta}{(1-\beta)^2}\mathrm{E}^2(u). \tag{18}$$

Due to [4],

$$\mathrm{E}(u) = \frac{\exp(-\delta^2/2)}{\sqrt{2\pi}\beta} \int_0^3 \psi(z,\delta)u(z)dz, \tag{19}$$

$$\mathrm{E}(u^2) = \frac{\exp(-\delta^2/2)}{\sqrt{2\pi}\beta} \int_0^3 \psi(z,\delta)u^2(z)dz, \tag{20}$$

where

$$\psi(z,\delta) = 2\exp(-z^2/2)\cosh(\delta z). \tag{21}$$

Thus, by virtue of (18) – (20),

$$\mathrm{E}(T_d^2) = A\left[\int_0^3 \psi(z,\delta)u^2(z) + B\left(\int_0^3 \psi(z,\delta)u(z)dz\right)^2\right], \tag{22}$$

where

$$A \triangleq \frac{\exp(-\delta^2/2)}{(1-\beta)\beta\sqrt{2\pi}}, \qquad B \triangleq \frac{2\exp(-\delta^2/2)}{(1-\beta)\sqrt{2\pi}}. \tag{23}$$

Due to (22), the optimization problem (14) is equivalent to the problem

$$J(u) \triangleq \int_0^3 \psi(z,\delta)u^2(z) + B\left(\int_0^3 \psi(z,\delta)u(z)dz\right)^2 \to \min_u,$$

$$\text{s.t. } (7) - (9). \tag{24}$$

The objective of the paper is to solve the optimization problem (24).

3. **Problem decomposition and composite solution.**

The cost functional $J(u)$ in (24) consists of two addends, which contribution into the entire functional $J(u)$ depends on the value of the coefficient $B = B(\delta)$, given in (23). Namely, for large values of B, the second addend dominates in $J(u)$, while for small values of B, the first addend is dominant.

In Fig. 1, the coefficient B as a function of δ is depicted. It is seen that this function decreases monotonically from considerably large to negligibly small values. Thus, it is reasonable to assume that there exists $\delta = \delta^*$ such that for $\delta \in [0, \delta^*)$, the second addend of $J(u)$ dominates, while for $\delta \in [\delta^*, 3]$, the first addend does. Note that $\delta^* \geq \bar{\delta} = 1.89$, where $B(\bar{\delta}) = 1$.

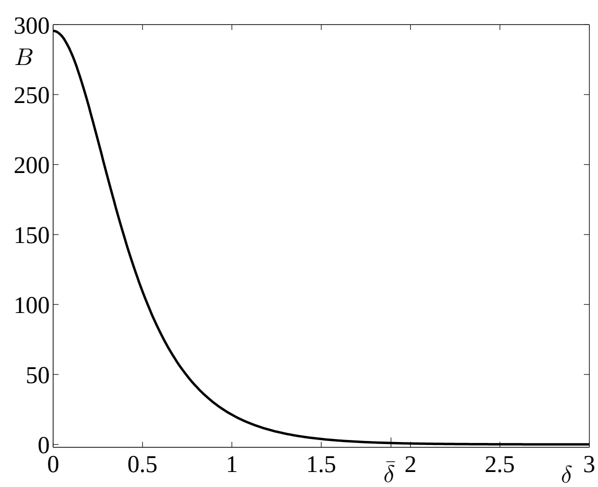


FIGURE 1. Function $B = B(\delta)$

Thus, the above mentioned inspires us to decompose the original problem (24) into two simpler problems. Namely, the first problem is

$$J_1(u) \triangleq \left(\int_0^3 \psi(z,\delta)u(z)dz\right)^2 \to \min_u \quad \text{s.t. } (7)-(9), \tag{25}$$

to be solved for small δ.

The second problem is

$$J_2(u) \triangleq \int_0^3 \psi(z,\delta)u^2(z)dz \to \min_u \quad \text{s.t. } (7)-(9), \tag{26}$$

to be solved for large δ.

The approximate composite solution of the original problem (24) is

$$u_c(z) = \begin{cases} u_l^*(z), & \delta \in [0,\delta^*), \\ u_{nl}^*(z), & \delta \in [\delta^*, 3], \end{cases} \qquad z \in [0,3], \tag{27}$$

where $u_l^*(z)$ and $u_{nl}^*(z)$ are the solutions of the problems (25) and (26), respectively. The choice of δ^* will be presented in Section 7.1.

4. **Solution of (25).** The problem (25) is equivalent to

$$\int_0^3 \psi(z,\delta)u(z)dz \to \min_u \quad \text{s.t. } (7)-(9). \tag{28}$$

Due to (15) and (19), the problem (28) is in fact the problem of minimizing the loss, linearly depending on the time to signal T_d.

The problem (28) was considered in details in [4]. It was solved there by its transforming to a linear optimal control problem and applying to the latter the Pontryagin's Maximum Principle [19].

This solution is obtained as a bang-bang (step) function

$$u_l^*(z) = \begin{cases} u_{\max}, & z \in [0, z_{sw}^*), \\ u_{\min}, & z \in [z_{sw}^*, 3], \end{cases} \tag{29}$$

where the switch value z_{sw}^* is

$$z_{sw}^* = \Phi^{-1}(b), \tag{30}$$

$$b = \frac{(1-u_{\min})\Phi(3) + (u_{\max}-1)\Phi(0)}{u_{\max}-u_{\min}}, \tag{31}$$

$\Phi^{-1}(\cdot)$ is the inverse function to Φ.

5. **Analytical solution of (26).** In this section, the problem of minimizing the cost functional $J_2(u)$ in (26) with respect to the function $u(z)$, satisfying the integral (soft) constraint (7) and the geometric (hard) constraint (8) for $T = 1$, is solved. This problem is a nonstandard variational calculus problem with two types of constraints, an isoperimetric constraint (see (7)) and a geometric constraint (see (8)), imposed on the minimizing function. Note that the classical calculus of variations theory does not study the extremal problems with the geometric constraints (see, e.g., [14]). If we omit the geometric constraints, then the Euler–Lagrange equation becomes algebraic, because the integrands in the functional $J_2(u)$ and in

the isoperimetric constraint of this problem are independent of the derivative of the minimizing function. This equation admits a solution, not necessarily satisfying the geometric constraints. Here, we propose another approach to solving this problem. This approach consists in its transformation into an optimal control problem. The latter is analyzed by application of Pontryagin's Maximum Principle (PMP) [19].

5.1. **Transforming (26) into an optimal control problem.** Let us introduce the following auxiliary functions of $z \in [0,3]$:

$$w_1(z) = \int_0^z \psi(\zeta,\delta)u^2(\zeta)d\zeta, \tag{32}$$

$$w_2(z) = \int_0^z \exp(-\zeta^2/2)u(\zeta)d\zeta. \tag{33}$$

These functions satisfy the differential equations

$$\frac{dw_1}{dz} = \psi(z,\delta)u^2(z), \tag{34}$$

$$\frac{dw_2}{dz} = \exp(-z^2/2)u(z), \tag{35}$$

and the initial conditions

$$w_1(0) = 0, \qquad w_2(0) = 0. \tag{36}$$

Moreover, due to the integral constraint (7), $w_2(z)$ satisfies the terminal condition

$$w_2(3) = a, \tag{37}$$

where

$$a = \int_0^3 \exp(-z^2/2)dz. \tag{38}$$

By using (32), the cost functional $J_2(u)$ becomes

$$J_2(u) = w_1(3). \tag{39}$$

Thus, we have transformed the problem (26) into the following equivalent optimal control problem: to find the control function $u(z)$, transferring the system (34) – (35) from the initial position (36) to the terminal position (37) and minimizing the cost functional (39), subject to the geometric constraint (8). This control problem is non-linear with respect to $u(z)$, and in what follows, it is called the Non-linear Optimal Control Problem (NOCP).

5.2. **NOCP solution by using PMP.** The Variational Hamiltonian of the NOCP has the form

$$H = H(w_1, w_2, u, \lambda_1, \lambda_2, z) = \lambda_1\psi(\delta,z)u^2 + \lambda_2\exp(-z^2/2)u, \tag{40}$$

where $\lambda_1 = \lambda_1(z)$ and $\lambda_2 = \lambda_2(z)$ are the costate variables.

The costate variables λ_1 and λ_2 satisfy the differential equations

$$\frac{d\lambda_1}{dz} = -\frac{\partial H}{\partial w_1} = 0, \quad z \in [0,3], \tag{41}$$

$$\frac{d\lambda_2}{dz} = -\frac{\partial H}{\partial w_2} = 0, \quad z \in [0,3], \tag{42}$$

and the transversality condition for λ_1,

$$\lambda_1(3) = -\frac{\partial J_2}{\partial w_1(3)} = -1. \tag{43}$$

Due to PMP, an optimal control $u^*_{nl}(z)$ of the NOCP necessarily satisfies the following condition

$$u^*_{nl}(z) = \arg \max_{u_{\min} \le u(z) \le u_{\max}} H(w_1(z), w_2(z), u(z), \lambda_1(z), \lambda_2(z), z). \tag{44}$$

Thus, any control $u(z)$, satisfying the equations (40), (44), (34) –(37) and (41) – (43) is an optimal control candidate in the NOCP. Let us start with the obtaining such a control by solving the equations (41) – (43). These equations yield the solution

$$\lambda_1(z) = -1, \quad \lambda_2(z) = C = \text{const}, \quad z \in [0, 3]. \tag{45}$$

By substituting (45) into (40), the Variational Hamiltonian becomes

$$H = \exp(-z^2/2) G(u, z, \delta, C), \tag{46}$$

where the function $G(u, z, \delta, C)$ has the form

$$G(u, z, \delta, C) = Cu - 2\cosh(\delta z) u^2. \tag{47}$$

Remark 1. Due to [15, Theorem 3], the NOCP has a solution (optimal control).

Due to (44), (46) – (47) and Remark 1, the optimal control of the NOCP is

$$u^*_{nl}(z, C) = \begin{cases} u_{\min}, & \bar{u}_{nl}(z, \delta, C) \le u_{\min}, \\ \bar{u}_{nl}(z, \delta, C), & u_{\min} < \bar{u}_{nl}(z, \delta, C) \le u_{\max}, \\ u_{\max}, & \bar{u}_{nl}(z, \delta, C) > u_{\max}, \end{cases} \tag{48}$$

where

$$\bar{u}_{nl}(z, \delta, C) = \frac{C}{4\cosh(\delta z)} \tag{49}$$

is the unique solution of the following equation with respect to u:

$$\frac{\partial G(u, z, \delta, C)}{\partial u} = 0. \tag{50}$$

In order to use the equation (48), one has to know the constant C. This constant should be chosen in such a way that the resulting control (48) will transfer the system (34) – (35) from the initial position (36) to the terminal position (37). Due to (34) – (37), this means that the value C should satisfy the algebraic equation

$$\Lambda(C) \triangleq \int_0^3 \exp(-z^2/2) u^*_{nl}(z, C) dz = a, \tag{51}$$

where a is given by (38).

Lemma 5.1. *There exists the unique solution C^* of the equation (51).*

Proof. Note that for $C \le 4u_{\min}$: $\bar{u}_{nl}(z, \delta, \lambda) \le u_{\min}$ for all $z \in [0, 3]$, i.e., by (48), $u^*_{nl}(z, C) \equiv u_{\min}$, leading to

$$\Lambda(C) = u_{\min} \int_0^3 \exp(-z^2/2) dz = u_{\min} a. \tag{52}$$

For $C > 4u_{\max}\cosh(3\delta)$: $\bar{u}_{nl}(z_i,\delta,\lambda) > u_{\max}$ for all $z \in [0,3]$, i.e., by (48), $u^*_{nl}(z,C) \equiv u_{\max}$ leading to

$$\Lambda(C) = u_{\max}\int_0^3 \exp(-z^2/2)dz = u_{\max}a. \tag{53}$$

From (48), it immediately follows that for any $z \in [0,3]$, the control $u^*_{nl}(z,C)$ is continuous and monotonically increasing function of C. Therefore, the function $\Lambda(C)$, defined in (51), also is continuous and monotonically increasing.

Due to (10) and (52) – (53), $\Lambda(4u_{\min}) < a$ and $\Lambda(4u_{\max}\cosh(3\delta)) > a$. Thus, the equation (51) has the unique solution $C^* \in (4u_{\min}, 4u_{\max}\cosh(3\delta))$. □

By virtue of Remark 1 and Lemma 5.1, the optimal control of the NOCP is $u^*_{nl}(z,C^*)$.

In what follows, we assume that the parameters $u_{\min}$, $u_{\max}$ and δ satisfy the inequality

$$u_{\max} < u_{\min}\cosh(3\delta). \tag{54}$$

Indeed, due to results of Section 3, the NOCP is considered for large values of δ (not smaller than $\bar{\delta}$). The inequality (54) is fulfilled for $\delta \in [\bar{\delta},3]$ if

$$\frac{u_{\max}}{u_{\min}} < \cosh(3\bar{\delta}) \approx 145.02. \tag{55}$$

Due to [22], the condition (55) is more than reasonable.

The following theorem presents the NOCP solution $u^*_{nl}(z,C^*)$ in details.

Let us denote

$$\Lambda^* \triangleq u_{\max}\int_0^{z^*} \frac{\exp(-z^2/2)}{\cosh(\delta z)}dz + u_{\min}\int_{z^*}^3 \exp(-z^2/2)dz, \tag{56}$$

and

$$\Lambda^{**} \triangleq u_{\max}\int_0^{z^{**}} \exp(-z^2/2)dz + \cosh(3\delta)u_{\min}\int_{z^{**}}^3 \frac{\exp(-z^2/2)}{\cosh(\delta z)}dz, \tag{57}$$

where

$$z^* = Z_1(0), \qquad z^{**} = Z_2(3), \tag{58}$$

$$Z_1(z) \triangleq \frac{1}{\delta}\operatorname{arccosh}\left(\frac{u_{\max}}{u_{\min}}\cosh(\delta z)\right), \quad Z_2(z) \triangleq \frac{1}{\delta}\operatorname{arccosh}\left(\frac{u_{\min}}{u_{\max}}\cosh(\delta z)\right).. \tag{59}$$

Note that due to (54), $z^* \in [0,3)$, while due to $u_{\min} < u_{\max}$, also $z^{**} \in [0,3)$.

Theorem 5.2. (I) *If*

$$\Lambda^* \geq a, \tag{60}$$

then

$$u^*_{nl}(z,C^*_I) = \begin{cases} \bar{u}_{nl}(z,\delta,C^*_I), & 0 \leq z \leq z_I, \\ u_{\min}, & z_I < z \leq 3, \end{cases} \tag{61}$$

where

$$C^*_I \triangleq 4u_{\min}\cosh(\delta z_I), \tag{62}$$

$z_I \in (0,3)$ *is the unique solution of the algebraic equation with respect to* z:

$$G_I(z) \triangleq \cosh(\delta z) \int\limits_0^z \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta + \int\limits_z^3 \exp(-\zeta^2/2) d\zeta - \frac{a}{u_{\min}} = 0. \tag{63}$$

(II) *If*

$$\Lambda^* < a \le \Lambda^{**}, \tag{64}$$

then

$$u_{nl}^*(z, C_{II}^*) = \begin{cases} u_{\max}, & 0 \le z \le z_{II}, \\ \bar{u}_{nl}(z, \delta, C_{II}^*), & z_{II} < z \le Z_1(z_{II}), \\ u_{\min}, & Z_1(z_{II}) < z \le 3, \end{cases} \tag{65}$$

where

$$C_{II}^* \triangleq 4u_{\max}\cosh(\delta z_{II}), \tag{66}$$

$z_{II} \in (0,3)$ *is the unique solution of the algebraic equation with respect to* z:

$$G_{II}(z) \triangleq \int\limits_0^z \exp(-\zeta^2/2) d\zeta + \cosh(\delta z) \int\limits_z^{Z_1(z)} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta +$$

$$+ \frac{u_{\min}}{u_{\max}} \int\limits_{Z_1(z)}^3 \exp(-\zeta^2/2) d\zeta - \frac{a}{u_{\max}} = 0. \tag{67}$$

(III) *If*

$$\Lambda^{**} < a, \tag{68}$$

then

$$u_{nl}^*(z, C_{III}^*) = \begin{cases} u_{\max}, & 0 \le z \le z_{III}, \\ \bar{u}_{nl}(z, \delta, C_{III}^*), & z_{III} < z \le 3, \end{cases} \tag{69}$$

where

$$C_{III}^* \triangleq 4u_{\max}\cosh(\delta z_{III}), \tag{70}$$

$z_{III} \in (0,3)$ *is the unique solution of the algebraic equation with respect to* z:

$$G_{III}(z) \triangleq \int_0^z \exp(-\zeta^2/2) d\zeta + \cosh(\delta z) \int\limits_z^3 \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta - \frac{a}{u_{\max}} = 0. \tag{71}$$

The proof of the theorem is presented in Section 9.1.

6. Numerical solution of (26).

6.1. Problem discretization.

Let divide the interval $[0,3]$ into N equal subintervals by the collocation points

$$z_i = i\Delta z, \quad i = 0, 1, \ldots, N, \tag{72}$$

where

$$\Delta z = 3/N. \tag{73}$$

Then, the integrals in the cost functional $J_2(u)$ and in the isoperimetric constraint (7) are approximated by using the left rectangles formula [12]. The cost functional is approximated as

$$J_2 \approx \tilde{J}_2^N(U) \triangleq \Delta z \sum_{i=0}^{N-1} \psi(z_i, \delta) U_i^2, \tag{74}$$

where the vector $U \in \mathbb{R}^N$ is

$$U = (U_0, U_1, \dots, U_{N-1})^T = (u(z_0), u(z_1), \dots, u(z_{N-1}))^T. \tag{75}$$

The constraint (7) is approximated as

$$\Delta z \sum_{i=0}^{N-1} \exp(-z_i^2/2) U_i = \Delta z \sum_{i=0}^{N-1} \exp(-z_i^2/2), \tag{76}$$

where the right-hand side expression approximates the value a, given by (38).

Thus, dividing (74) and (76) by Δz and taking into account the geometric constraint (8) yield the following finite-dimensional Quadratic Programming Problem (QPP):

$$J_2^N(U) \triangleq \sum_{i=0}^{N-1} \psi(z_i, \delta) U_i^2 \to \min_U \tag{77}$$

subject to

$$\sum_{i=0}^{N-1} \exp(-z_i^2/2) U_i = \sum_{i=0}^{N-1} \exp(-z_i^2/2) \triangleq a_N, \tag{78}$$

$$u_{\min} \le U_i \le u_{\max}, \quad i = 0, 1, \dots, N-1. \tag{79}$$

The problem (77) – (79) can be easily solved by using standard optimization tools, for example, the MATLAB function "quadprog". However, in order to justify this approximate solution, we show that this solution approaches the exact solution of Section 5 for $\Delta z \to 0$.

6.2. **Solution of QPP (77) – (79).** Due to [13], the problem (77) – (79) is replaced with the problem for the Lagrangian

$$L(U_0, \dots, U_{N-1}, \lambda) \triangleq \sum_{i=0}^{N-1} \psi(z_i, \delta) U_i^2 + \lambda \left(a_N - \sum_{i=0}^{N-1} \exp(-z_i^2/2) U_i \right) \to \min \quad \text{s.t. (79)}, \tag{80}$$

where λ is the Lagrange multiplier. It can be shown that for a fixed λ, the solution of the problem (80) is

$$U_i^* = U_i^*(z_i, \lambda) = \begin{cases} u_{\min}, & \bar{u}_{nl}(z_i, \delta, \lambda) \le u_{\min}, \\ \bar{u}_{nl}(z_i, \delta, \lambda), & u_{\min} < \bar{u}_{nl}(z_i, \delta, \lambda) \le u_{\max}, \\ u_{\max}, & \bar{u}_{nl}(z_i, \delta, \lambda) > u_{\max}, \end{cases} \tag{81}$$

where $i = 0, 1, \dots, N-1$; the function $\bar{u}_{nl}(z, \delta, C)$ is given by (49).

Substituting $U_i^*(z_i, \lambda)$ into the condition (78) yields the algebraic equation w.r.t to λ:

$$\Lambda_N(\lambda) \triangleq \sum_{i=0}^{N-1} \exp(-z_i^2/2) U_i^*(z_i, \lambda) = a_N. \tag{82}$$

Lemma 6.1. *There exists the unique solution λ^* of the equation (82).*

Proof. The lemma is proved similarly to Lemma 5.1. □

Due to [13] and Lemma 6.1, the solution of the problem (77) – (79) is $U_i^*(z_i, \lambda^*)$, $i = 0, 1, \dots, N-1$.

As in Section 5.2, the condition (54) is valid.

Let us define two values:

$$\Lambda_N^* \triangleq u_{\max} \sum_{i=0}^{i^*-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + u_{\min} \sum_{i=i^*}^{N-1} \exp(-z_i^2/2), \tag{83}$$

and

$$\Lambda_N^{**} \triangleq u_{\max} \sum_{i=0}^{i^{**}-1} \exp(-z_i^2/2) + \cosh(3\delta) u_{\min} \sum_{i=i^{**}}^{N-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}, \tag{84}$$

where

$$i^* = \mathcal{Z}_1(0), \quad i^{**} = \mathcal{Z}_2(N), \tag{85}$$

$\mathcal{Z}_1(i)$ is the minimal natural number j, satisfying

$$z_j \geq Z_1(z_i), \tag{86}$$

$\mathcal{Z}_2(i)$ is the minimal natural number j, satisfying

$$z_j \geq Z_2(z_i), \tag{87}$$

where the functions $Z_1(z)$ and $Z_2(z)$ are defined by (59).

Theorem 6.2. ($\mathrm{I}_{\mathbf{N}}$) *If*

$$\Lambda_N^* \geq a_N, \tag{88}$$

then

$$U_i^* = U_i^I(z_i, \lambda_I^*) \triangleq \begin{cases} \bar{u}_{nl}(z_i, \delta, \lambda_I^*), & i = 0, 1, \dots, K_I - 1, \\ u_{\min}, & i = K_I, \dots, N-1, \end{cases} \tag{89}$$

where

$$\lambda_I^* \triangleq 4 \frac{a_N - u_{\min} \sum\limits_{i=K_I}^{N-1} \exp(-z_i^2/2)}{\sum\limits_{i=0}^{K_I-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}}, \tag{90}$$

K_I is such a natural number that

$$\cosh(\delta z_K) \sum_{i=0}^{K-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=K}^{N-1} \exp(-z_i^2/2) - \frac{a_N}{u_{\min}} < 0, \quad K < K_I, \tag{91}$$

$$\cosh(\delta z_K) \sum_{i=0}^{K-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=K}^{N-1} \exp(-z_i^2/2) - \frac{a_N}{u_{\min}} \geq 0, \quad K \geq K_I. \tag{92}$$

($\mathrm{II}_{\mathbf{N}}$) *If*

$$\Lambda_N^* < a_N \leq \Lambda_N^{**}, \tag{93}$$

then

$$U_i^* = U_i^{II}(z_i, \lambda_{II}^*) \triangleq \begin{cases} u_{\max}, & i = 0, 1, \dots, M_{II} - 1, \\ \bar{u}_{nl}(z_i, \delta, \lambda_{II}^*), & i = M_{II}, \dots, K_{II} - 1, \\ u_{\min}, & i = K_{II}, \dots, N-1, \end{cases} \tag{94}$$

where

$$\lambda^*_{II} \triangleq 4\frac{a_N - u_{\max}\sum\limits_{i=0}^{M_{II}-1}\exp(-z_i^2/2) - u_{\min}\sum\limits_{i=K_{II}}^{N-1}\exp(-z_i^2/2)}{\sum\limits_{i=M_{II}}^{K_{II}-1}\dfrac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}}, \tag{95}$$

$K_{II} = \mathcal{Z}_1(M_{II})$, M_{II} *is such a natural number that*

$$\cosh(\delta z_M)\sum_{i=M}^{\mathcal{Z}_1(M)-1}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}+$$

$$\sum_{i=0}^{M-1}\exp(-z_i^2/2) + \frac{u_{\min}}{u_{\max}}\sum_{i=\mathcal{Z}_1(M)}^{N-1}\exp(-z_i^2/2) - \frac{a_N}{u_{\max}} < 0, \quad M < M_{II}, \tag{96}$$

$$\cosh(\delta z_M)\sum_{i=M}^{\mathcal{Z}_1(M)-1}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}+$$

$$\sum_{i=0}^{M-1}\exp(-z_i^2/2) + \frac{u_{\min}}{u_{\max}}\sum_{i=\mathcal{Z}_1(M)}^{N-1}\exp(-z_i^2/2) - \frac{a_N}{u_{\max}} \geq 0, \quad M \geq M_{II}. \tag{97}$$

($\mathrm{III}_{\mathbf{N}}$) *If*

$$\Lambda^{**}_N < a_N, \tag{98}$$

then

$$U_i^* = U_i^{III}(z_i, \lambda^*_{III}) \triangleq \begin{cases} u_{\max}, & i = 0, 1, \dots, M_{III}-1, \\ \bar{u}_{nl}(z_i, \delta, \lambda^*_{III}), & i = M_{III}, \dots, N-1, \end{cases} \tag{99}$$

where

$$\lambda^* = \lambda^*_{III} \triangleq 4\frac{a_N - u_{\max}\sum\limits_{i=0}^{M_{III}-1}\exp(-z_i^2/2)}{\sum\limits_{i=M_{III}}^{N-1}\dfrac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}}, \tag{100}$$

M_{III} *is such a natural number that*

$$\cosh(\delta z_M)\sum_{i=M}^{N-1}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=0}^{M-1}\exp(-z_i^2/2) - \frac{a_N}{u_{\max}} < 0, \quad M < M_{III}, \tag{101}$$

$$\cosh(\delta z_M)\sum_{i=M}^{N-1}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=0}^{M-1}\exp(-z_i^2/2) - \frac{a_N}{u_{\max}} \geq 0, \quad M \geq M_{III}. \tag{102}$$

The proof of the theorem is presented in Section 9.2.

6.3. **Error estimate.** In this subsection, we derive an estimate of error caused by replacing the original problem (26) with its discretized version (77) – (79). We consider two types of the error: (1) error in the solution $u(z)$; (2) error in the optimal value of the cost functional $J_2(u)$, defined in (26). Here, we restrict ourselves by the case, where the inequalities (60) and (64) are fulfilled strictly. Thus, we do not consider the cases $\Lambda^* = a$ and $\Lambda^{**} = a$, because these cases are rather rare and of less practical significance.

Theorem 6.3. *Let $u_{\max}$, $u_{\min}$ and δ be such that*

$$\Lambda^* \neq a, \quad \Lambda^{**} \neq a. \tag{103}$$

Then there exist a natural number $\bar{N}$ and positive constants C_1, C_2, independent of N, such that for all $N \geq \bar{N}$,

$$\epsilon_u(N) \triangleq \max_{i=0,\ldots,N-1} |u^*_{nl}(z_i) - U^*_i| \leq \frac{C_1}{N}, \tag{104}$$

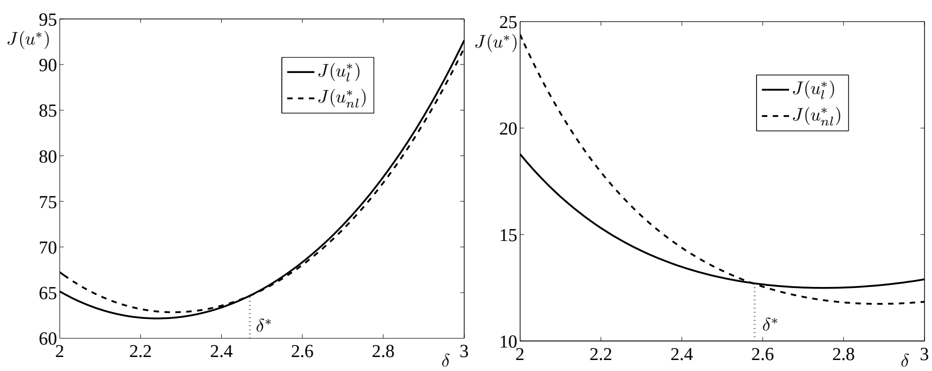
$$\epsilon_J(N) \triangleq |J_2(u^*_{nl}) - \Delta z J^N_2(U^*)| \leq \frac{C_2}{N}. \tag{105}$$

The proof of the theorem is presented in Section 9.3.

7. **Numerical evaluation.** In this section, we assume that the condition (55), which is justified in Section 5.2 for the problem (26), is satisfied for the general problem (24). Thus, the numerical evaluation is carried out for the values of $u_{\min}$ and $u_{\max}$, satisfying (55).

7.1. **Searching δ^* in composite solution (27).** The objective of this subsection is to determine the value $\delta^* \in (\bar{\delta}, 3]$ in the composite solution (27). For this value, the switch occurs from the solution of (25) $u^*_l(z)$, given by (29) – (31), to the solution of (26) $u^*_{nl}(z)$, given by Theorem 5.2. For determining δ^*, the following practical condition is proposed: for $\delta \in [0, \delta^*)$, $J(u^*_l) < J(u^*_{nl})$, while for $\delta \in [\delta^*, 3]$, $J(u^*_{nl}) \leq J(u^*_l)$.

In Figs. 2a – 2b, the graphs of $J(u^*_l)$ and $J(u^*_{nl})$ are depicted as functions of δ for two sets of parameters: $u_{\min} = 0.5$, $u_{\max} = 3.5$ and $u_{\min} = 0.1$, $u_{\max} = 2.5$. Note that in both cases, the condition (55) is satisfied. For the first set, $\delta^* = 2.47$, for the second set, $\delta^* = 2.58$.



a) $u_{\min} = 0.5$, $u_{\max} = 3.5$ b) $u_{\min} = 0.1$, $u_{\max} = 2.5$

FIGURE 2. Searching δ^* in composite solution

7.2. **Evaluation of estimates (104) – (105).** The evaluation of the estimates (104) – (105) is carried out for two data sets, where $u_{\min}$ and $u_{\max}$ are as in Section 7.1. In the first data set, $\delta = 2.5$, while in the second data set, $\delta = 2.9$. In both cases, $\delta > \delta^*$.

The value of a, given by (38) is $a = 1.25$. For the first data set, $\Lambda^* = 1.998$, $\Lambda^{**} = 4.33$, i.e. the inequality (60) is satisfied and the case (I) of Theorem 5.2 is valid. For the second data set, $\Lambda^* = 1.216$, $\Lambda^{**} = 3.033$, i.e. the inequality (64) is satisfied and the case (II) of Theorem 5.2 is valid. Note that the inequalities (60) and (64) are satisfied strictly.

In Fig. 3, the value $\epsilon_u(N)$ for the first data set is depicted for $N = 100 \div 1000$. It is seen that the estimate (104) of Theorem 6.3 is valid for $\bar{N} = 100$, $C_1 = 3.4$.

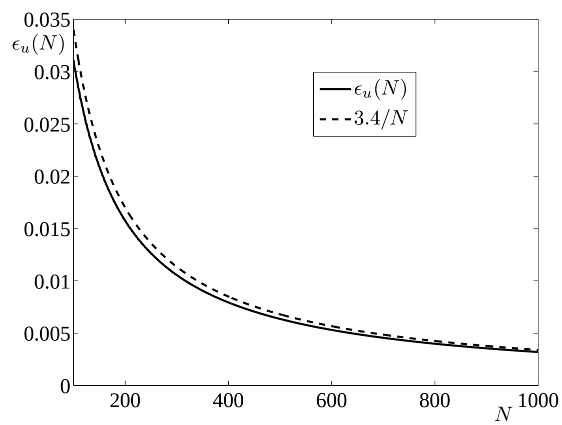
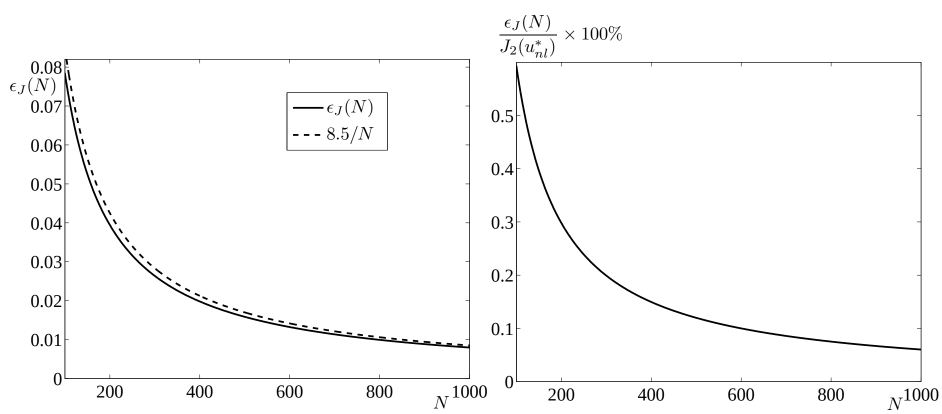


FIGURE 3. Behavior of $\epsilon_u(N)$ in Case (I)

In Fig. 4a, the value $\epsilon_J(N)$ (the absolute error of $J_2^N(U^*)$) for the first data set is depicted for $N = 100 \div 1000$. It is seen that the estimate (105) of Theorem 5.2 is valid for $\bar{N} = 100$, $C_2 = 8.5$. In Fig. 4b, the relative error of $J_2^N(U^*)$ (the percentage of $\epsilon_J(N)$ in $J_2(u^*_{nl})$) is depicted. It is seen that when N increases from 100 to 1000, the relative error decrease from 0.59% to 0.06%.



a) b)

FIGURE 4. Behavior of $\epsilon_J(N)$ in Case (I)

In Figs. 5 – 6, the results of the numerical evaluation for the second data set are presented. It is seen that the estimates (104) – (105) of Theorem 6.3 are valid for $\bar{N} = 100$, and $C_1 = 5.9$ and $C_2 = 6.4$. The relative error of $J_2^N(U^*)$ decreases from 0.8% to 0.08%.

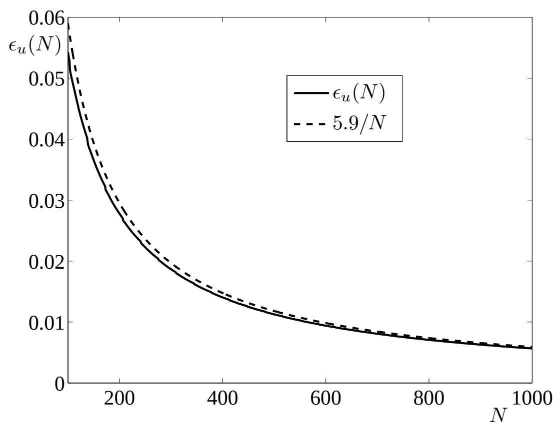


FIGURE 5. Behavior of $\epsilon_u(N)$ in Case (II)

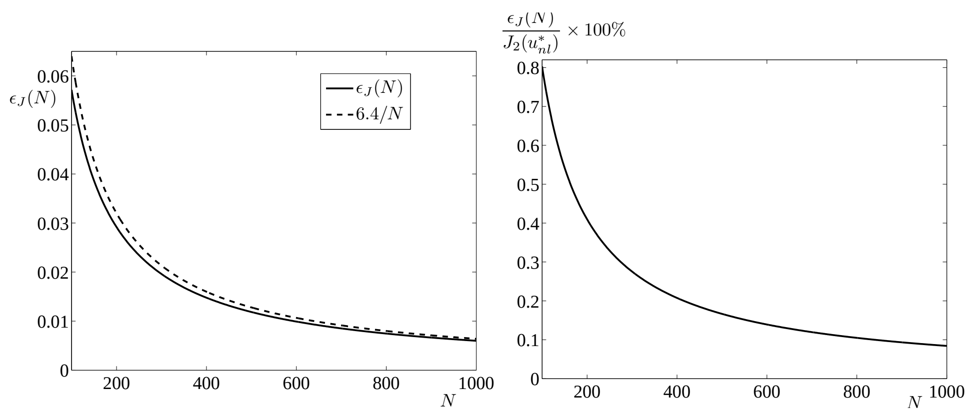


FIGURE 6. Behavior of $\epsilon_J(N)$ in Case (II)

In Fig. 7, the exact and the approximate solutions are compared for two data sets (Figs. 7a and 7b, respectively). The approximate solution is obtained for $N = 100$. It is seen that the approximate solution provides a sufficient accuracy.

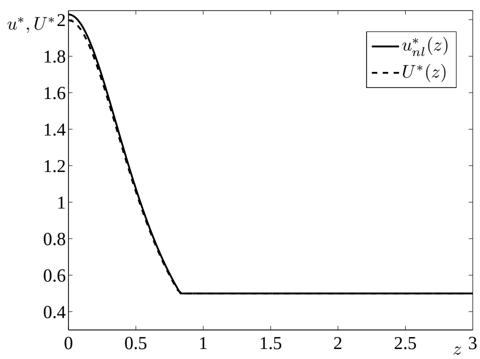
7.3. **Evaluation of composite solution.** In this subsection, the composite solution $u = u_c$ (27) of the problem (24), representing the suboptimal variable sampling SPC interval, is compared with the constant sampling interval $u = \bar{u} \equiv 1$ and with the random sampling interval $u = u_r$, satisfying $\mathrm{E}(u_r) = 1$.

Since for the constant sampling interval, $\mathrm{E}(\bar{u}) = \mathrm{E}(\bar{u}^2) = 1$, then, due to (18),

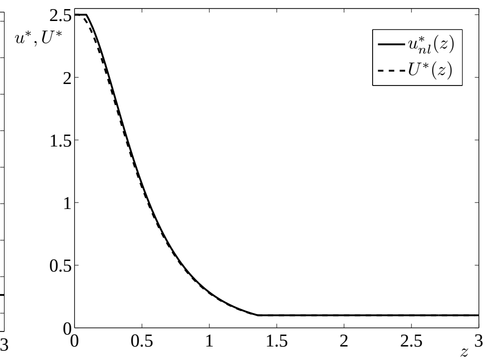
$$\mathrm{E}(T_d^2 \mid u = \bar{u}) = \frac{1}{1-\beta} + \frac{2\beta}{(1-\beta)^2} = \frac{1+\beta}{(1-\beta)^2}. \tag{106}$$

For the random sampling interval, $\mathrm{E}(u_r) = 1$, $\mathrm{E}(u_r^2) = \mathrm{E}^2(u_r) + \mathrm{Var}(u_r) = 1 + \mathrm{Var}(u_r)$. Therefore, by virtue of (18) and (106),

$$\mathrm{E}(T_d^2 \mid u = u_r) = \frac{1 + \mathrm{Var}(u_r)}{1-\beta} + \frac{2\beta}{(1-\beta)^2} = \mathrm{E}(T_d^2 \mid u = \bar{u}) + \frac{\mathrm{Var}(u_r)}{1-\beta}. \tag{107}$$



a)



b)

FIGURE 7. Exact vs. approximate solutions

It is seen from (107) that the expected loss for the random sampling interval is larger than for the constant one. Therefore, we present the numerical comparison of $J(u_c)$ only with $J(\bar{u})$.

In Fig. 8, the value

$$\eta \triangleq \frac{J(u_c)}{J(\bar{u})} \tag{108}$$

is depicted as a function of δ for two sets of parameters $u_{\min}$ and $u_{\max}$. It is seen that in both cases, using the suboptimal sampling interval provides smaller value of the expected loss. Moreover, for larger δ, the advantage of the suboptimal sampling interval is larger.

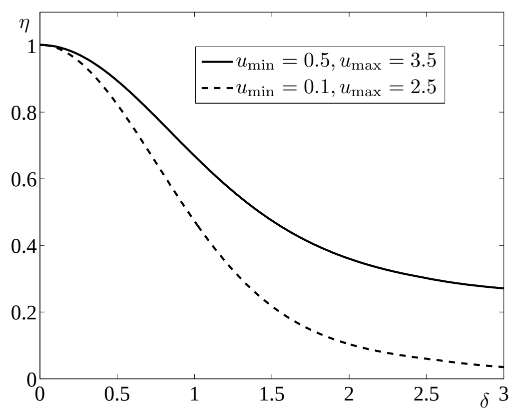


FIGURE 8. Suboptimal variable sampling interval vs. constant

8. **Conclusions.** In this paper, the problem of SPC optimization with respect to the sampling interval was considered. As an optimization criterion, an expected loss, quadratically dependent on the time to signal, was chosen. An approximate analytical solution was obtained, based on the problem decomposition into two simpler subproblems, linear and quadratic. Two approaches to the solution of the

quadratic subproblem were proposed. The first approach is based on the problem transformation into the optimal control problem and using the Pontryagin's Maximum Principle, which leads to an exact analytical solution. The second approach is based on a discretization of the problem and using proper mathematical programming tools, providing an approximate numerical solution. The discretization error estimates were obtained. It was shown that both errors, in the solution and in the optimal value of the cost functional, are of the order of the discretization step. Based on the solutions of these subproblems, the composite suboptimal solution was designed. The numerical evaluation showed a high accuracy of the second subproblem discretization. The numerical evaluation also indicated a considerable superiority of the suboptimal variable sampling interval, represented by the composite solution of the original problem, over the constant sampling interval. The approach to the SPC optimization, proposed in this paper, can be extended to the case of non-gaussian distribution of the performance parameter sample mean, which will be a subject of a future research.

9. **Appendix: Proof of theorems.**

9.1. **Proof of Theorem 5.2.** In order to obtain an explicit form of C^*, let us express the function $\Lambda(C)$ in the interval $(4u_{\min}, 4u_{\max}\cosh(3\delta))$ explicitly.

Due to (48), (49) and (54), three different intervals of C, yielding three different types of the control (48), and the respective expressions for the function $\Lambda(C)$, can be distinguished.

I. $4u_{\min} < C \leq 4u_{\max}$. Since $\bar{u}_{nl}(0,\delta,C) = \dfrac{C}{4}$, in this case,

$$u_{\min} < \bar{u}_{nl}(0,\delta,C) \leq u_{\max}. \tag{109}$$

Moreover, due to (54),

$$\bar{u}_{nl}(3,\delta,C) = \frac{C}{4\cosh(3\delta)} \leq u_{\min}. \tag{110}$$

Note that the function $\bar{u}_{nl}(z,\delta,C)$ decreases monotonically w.r.t. z. Then due to the left-hand inequality in (109) and (110), there exists the unique $z_m = z_m(C) \in (0,3]$, satisfying the equation

$$\bar{u}_{nl}(z_m,\delta,C) = \frac{C}{4\cosh(\delta z_m)} = u_{\min}. \tag{111}$$

Therefore, due to (48) and the right-hand inequality in (109),

$$u^*_{nl}(z,C) = \begin{cases} \bar{u}_{nl}(z,\delta,C), & 0 \leq z_m(C), \\ u_{\min}, & z_m(C) < z \leq 3, \end{cases} \tag{112}$$

leading to

$$\Lambda(C) = \frac{C}{4}\int_0^{z_m(C)} \frac{\exp(-z^2/2)}{\cosh(\delta z)}dz +$$

$$u_{\min}\int_{z_m(C)}^{3} \exp(-z^2/2)dz, \quad 4u_{\min} < C \leq 4u_{\max}. \tag{113}$$

Note that, by (111),

$$z_m(C) = \frac{1}{\delta}\text{arccosh}\left(\frac{C}{4u_{\min}}\right), \tag{114}$$

and it is non-decreasing function of C.

II. $4u_{\max} < C \le 4u_{\min}\cosh(3\delta)$. In this case,

$$\bar{u}_{nl}(0,\delta,C) > u_{\max} > u_{\min}, \tag{115}$$

and

$$\bar{u}_{nl}(3,\delta,C) \le u_{\min} < u_{\max}. \tag{116}$$

Then the equation

$$\bar{u}_{nl}(z_M,\delta,C) = \frac{C}{4\cosh(\delta z_M)} = u_{\max} \tag{117}$$

has the unique solution

$$z_M(C) = \frac{1}{\delta}\text{arccosh}\left(\frac{C}{4u_{\max}}\right). \tag{118}$$

Note that in this case, the value $z_m(C)$, given by (114), still is the solution of the equation (111) and $z_m(C) \in (0,3]$. Moreover, due to (114) and (118),

$$z_m(C) > z_M(C). \tag{119}$$

Thus, due to (48) and (115) – (116),

$$u^*_{nl}(z,C) = \begin{cases} u_{\max}, & 0 \le z \le z_M(C), \\ \bar{u}_{nl}(z,\delta,C), & z_M(C) < z \le z_m(C), \\ u_{\min}, & z_m(C) < z \le 3, \end{cases} \tag{120}$$

leading to

$$\Lambda(C) = u_{\max}\int_0^{z_M(C)} \exp(-z^2/2)dz + \frac{C}{4}\int_{z_M(C)}^{z_m(C)} \frac{\exp(-z^2/2)}{\cosh(\delta z)}dz +$$

$$u_{\min}\int_{z_m(C)}^{3} \exp(-z^2/2)dz, \quad 4u_{\max} < C \le 4u_{\min}\cosh(3\delta). \tag{121}$$

Note that $z_M(C)$ is also non-decreasing function of C.

III. $4u_{\min}\cosh(3\delta) < C < 4u_{\max}\cosh(3\delta)$. In this case, $\bar{u}_{nl}(z,\delta,C) > u_{\min}$ for all $z \in [0,3]$. Note that in this case, the value $z_M(C)$, given by (118), still is the solution of the equation (117) and $z_M(C) \in [0,3)$. Thus, due to (48),

$$u^*_{nl}(z,C) = \begin{cases} u_{\max}, & 0 < z \le z_M(C), \\ \bar{u}_{nl}(z,\delta,C), & z_M(C) < z \le 3, \end{cases} \tag{122}$$

leading to

$$\Lambda(C) = u_{\max}\int_0^{z_M(C)} \exp(-z^2/2)dz +$$

$$\frac{C}{4}\int\limits_{z_M(C)}^{3}\frac{\exp(-z^2/2)}{\cosh(\delta z)}dz, \quad 4u_{\min}\cosh(3\delta) < C < 4u_{\max}\cosh(3\delta). \tag{123}$$

Now, let us consider the cases (I), (II) and (III) of the theorem separately.

(I) Note that by virtue of (114),

$$z_m(4u_{\max}) = \frac{1}{\delta}\operatorname{arccosh}\left(\frac{u_{\max}}{u_{\min}}\right) = z^*, \tag{124}$$

where z^* is given by (58) – (59). Therefore, due to (113),

$$\Lambda(4u_{\max}) = \Lambda^*, \tag{125}$$

where the value Λ^* is given by (56).

Remember that the function $\Lambda(C)$ is continuous and monotonically increasing for $C \in (4u_{\min}, 4u_{\max}\cosh(3\delta))$. Moreover, due to (60) and (125),

$$\Lambda(4u_{\max}) \geq a. \tag{126}$$

Then the solution of the equation (51) satisfies the inclusion $C^* \in (4u_{\min}, 4u_{\max}]$. Therefore, due to (113),

$$\frac{C^*}{4}\int\limits_0^{z_m(C^*)}\frac{\exp(-z^2/2)}{\cosh(\delta z)}dz + u_{\min}\int\limits_{z_m(C^*)}^{3}\exp(-z^2/2)dz = a. \tag{127}$$

The equations (111) and (127) imply that $z_m(C^*)$ satisfies the equation

$$G_I(z) = 0, \tag{128}$$

where the function $G_I(z)$ is defined in (63), and

$$C^* = 4u_{\min}\cosh(\delta z_m(C^*)). \tag{129}$$

In order to complete the proof in the case (I), it is sufficient to show that the solution of the equation (128) on the interval $(0, 3]$ is unique. The latter is a direct consequence of the following inequality:

$$\frac{dG_I(z)}{dz} = \frac{d(\cosh(\delta z))}{dz}\int\limits_0^{z}\frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta > 0, \qquad z \in (0, 3]. \tag{130}$$

Thus, $C^* = C_I^*$ and $z_m(C^*) = z_I$.

(II) Note that by virtue of (114) and (118),

$$z_m(4u_{\min}\cosh(3\delta)) = 3, \tag{131}$$

and

$$z_M(4u_{\min}\cosh(3\delta)) = \frac{1}{\delta}\operatorname{arccosh}\left(\cosh(3\delta)\frac{u_{\min}}{u_{\max}}\right) = z^{**}, \tag{132}$$

where z^{**} is given by (58) – (59). Therefore, due to (121),

$$\Lambda(4u_{\min}\cosh(3\delta)) = \Lambda^{**}, \tag{133}$$

where the value Λ^{**} is given by (57). Thus, due to (64), (125) and (133),

$$\Lambda(4u_{\max}) < a \leq \Lambda(4u_{\min}\cosh(3\delta)), \tag{134}$$

meaning that the solution of the equation (51) satisfies the inclusion $C^* \in (4u_{\max}, 4u_{\min}\cosh(3\delta)]$. Therefore, due to (121),

$$\frac{C^*}{4}\int\limits_{z_M(C^*)}^{z_m(C^*)}\frac{\exp(-z^2/2)}{\cosh(\delta z)}dz + u_{\max}\int\limits_{0}^{z_M(C^*)}\exp(-z^2/2)dz +$$

$$u_{\min}\int\limits_{z_m(C^*)}^{3}\exp(-z^2/2)dz = a. \tag{135}$$

The equations (59), (111) and (117) yield that

$$z_m(C^*) = Z_1(z_M(C^*)). \tag{136}$$

This equation, along with (117) and (135), imply that $z_M(C^*)$ satisfies the equation

$$G_{II}(z) = 0, \tag{137}$$

where the function $G_{II}(z)$ is defined in (67), and

$$C^* = 4u_{\max}\cosh(\delta z_M(C^*)). \tag{138}$$

The uniqueness of the solution of (137) follows from the inequality

$$\frac{dG_{II}(z)}{dz} = \frac{d\cosh(\delta z)}{dz}\int\limits_{z}^{Z_1(z)}\frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta > 0, \qquad z \in (0,3]. \tag{139}$$

Thus, $C^* = C^*_{II}$ and $z_M(C^*) = z_{II}$.

(III) In this case, the proof is similar to the cases (I) and (II).

This completes the proof of the theorem. □

9.2. **Proof of Theorem 6.2.** In order to obtain an explicit form of λ^*, let us express the function $\Lambda_N(\lambda)$ in the interval $(4u_{\min}, 4u_{\max}\cosh(3\delta))$ explicitly.

Due to (49), (81) and (54), three different domains of λ, yielding three different types of the solution (81), and the respective expressions for the function $\Lambda_N(\lambda)$, can be distinguished.

I. $4u_{\min} < \lambda \le 4u_{\max}$. In this case, $\bar{u}_{nl}(z_0,\delta,\lambda) = \bar{u}_{nl}(0,\delta,\lambda) > u_{\min}$, and, due to (54), $\bar{u}_{nl}(z_N,\delta,\lambda) = \bar{u}_{nl}(3,\delta,\lambda) \le u_{\min}$. Moreover, the function $\bar{u}_{nl}(z,\delta,\lambda)$ decreases monotonically w.r.t. z. Let $K = K(\lambda) \in \{1,\dots,N-1\}$ be the minimal natural number i, satisfying the inequality

$$\bar{u}_{nl}(z_i,\delta,\lambda) = \frac{\lambda}{4\cosh(\delta z_i)} \le u_{\min}. \tag{140}$$

Then, due to (81),

$$U_i^*(z_i,\lambda) = \begin{cases} \bar{u}_{nl}(z_i,\delta,\lambda), & i = 0,1,\dots,K(\lambda)-1, \\ u_{\min}, & i = K(\lambda),\dots,N-1, \end{cases} \tag{141}$$

leading to

$$\Lambda_N(\lambda) = \lambda\sum_{i=0}^{K(\lambda)-1}\frac{\exp(-z_i^2/2)}{4\cosh(\delta z_i)} +$$

$$u_{\min}\sum_{i=K(\lambda)}^{N-1}\exp(-z_i^2/2),\quad 4u_{\min}<\lambda\le 4u_{\max}. \tag{142}$$

Note that $K(\lambda)$ is piecewise constant non-decreasing function of λ, because the function $\bar{u}_{nl}(z,\delta,\lambda)$ increases monotonically w.r.t. λ.

II. $4u_{\max}<\lambda\le 4u_{\min}\cosh(3\delta)$. In this case, $\bar{u}_{nl}(z_0,\delta,\lambda)=\bar{u}_{nl}(0,\delta,\lambda)>u_{\max}>u_{\min}$, and $\bar{u}_{nl}(z_N,\delta,\lambda)=\bar{u}_{nl}(3,\delta,\lambda)\le u_{\min}<u_{\max}$. In addition to $K(\lambda)$, let us define $M=M(\lambda)\in\{1,\dots,N-1\}$, $M\le K$, as the minimal natural number i, satisfying the inequality

$$\bar{u}_{nl}(z_i,\delta,\lambda)=\frac{\lambda}{4\cosh(\delta z_i)}\le u_{\max}. \tag{143}$$

Then

$$U_i^*(z_i,\lambda)=\begin{cases} u_{\max}, & i=0,1,\dots,M(\lambda)-1,\\ \bar{u}_{nl}(z_i,\delta,\lambda), & i=M(\lambda),\dots,K(\lambda)-1,\\ u_{\min}, & i=K(\lambda),\dots,N-1,\end{cases} \tag{144}$$

leading to

$$\Lambda_N(\lambda)=u_{\max}\sum_{i=0}^{M(\lambda)-1}\exp(-z_i^2/2)+\lambda\sum_{i=M(\lambda)}^{K(\lambda)-1}\frac{\exp(-z_i^2/2)}{4\cosh(\delta z_i)}+$$

$$u_{\min}\sum_{i=K(\lambda)}^{N-1}\exp(-z_i^2/2),\quad 4u_{\max}<\lambda\le 4u_{\min}\cosh(3\delta). \tag{145}$$

Note that $M(\lambda)$ is also piecewise constant non-decreasing function of λ.

III. $4u_{\min}\cosh(3\delta)<\lambda\le 4u_{\max}\cosh(3\delta)$. In this case, $\bar{u}_{nl}(z_i,\delta,\lambda)>u_{\min}$ for all $i=0,1,\dots,N-1$, and

$$U_i^*(z_i,\lambda)=\begin{cases} u_{\max}, & i=0,1,\dots,M(\lambda)-1,\\ \bar{u}_{nl}(z_i,\delta,\lambda), & i=M(\lambda),\dots,N-1,\end{cases} \tag{146}$$

leading to

$$\Lambda_N(\lambda)=u_{\max}\sum_{i=0}^{M(\lambda)-1}\exp(-z_i^2/2)+$$

$$\lambda\sum_{i=M(\lambda)}^{N-1}\frac{\exp(-z_i^2/2)}{4\cosh(\delta z_i)},\quad 4u_{\min}\cosh(3\delta)<\lambda\le 4u_{\max}\cosh(3\delta). \tag{147}$$

The rest of the proof consists of a separate analysis of the cases $(\mathrm{I_N})$, $(\mathrm{II_N})$ and $(\mathrm{III_N})$ of Theorem 6.2. Since this analysis is similar to the analysis of the respective cases (I), (II) and (III) in the proof of Theorem 5.2, we restrict ourselves by the detailed analysis of the case $(\mathrm{I_N})$.

By virtue of (140), $K(4u_{\max})$ is the minimal natural number i, satisfying the inequality

$$\frac{u_{\max}}{\cosh(\delta z_i)}\le u_{\min}, \tag{148}$$

meaning by (85) – (86) that $K(4u_{\max}) = \mathcal{Z}_1(0) = i^*$. Therefore, due to (83) and (142),

$$\Lambda_N(4u_{\max}) = \Lambda_N^*. \tag{149}$$

Remember that the function $\Lambda_N(\lambda)$ is continuous and monotonically increasing for $\lambda \in (4u_{\min}, 4u_{\max}\cosh(3\delta))$. Moreover, due to (88) and (149),

$$\Lambda_N(4u_{\max}) \geq a_N. \tag{150}$$

Then the solution of the equation (82) satisfies the inclusion $\lambda^* \in (4u_{\min}, 4u_{\max}]$. Therefore, due to (142),

$$\lambda^* = 4\frac{a_N - u_{\min}\sum\limits_{i=K(\lambda^*)}^{N-1}\exp(-z_i^2/2)}{\sum\limits_{i=0}^{K(\lambda^*)-1}\dfrac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}} \in (4u_{\min}, 4u_{\max}]. \tag{151}$$

Substituting (151) instead of λ into the inequality (140) yields that $K(\lambda^*)$ is the minimal K, for which the inequality (92) is satisfied. In order to complete the proof in the case ($\mathrm{I_N}$), it is sufficient to show that the inequality (92) is valid for $K > K(\lambda^*)$. For this purpose, it is sufficient to show that

$$\mathcal{G}_I(K+1) > \mathcal{G}_I(K), \quad K = 0, \ldots, N-1, \tag{152}$$

where

$$\mathcal{G}_I(K) \triangleq \cosh(\delta z_K)\sum_{i=0}^{K-1}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=K}^{N-1}\exp(-z_i^2/2). \tag{153}$$

Indeed, by a simple algebra, the inequality (152) is equivalent to the inequality

$$\Big(\cosh(\delta z_{K+1}) - \cosh(\delta z_K)\Big)\sum_{i=0}^{K}\frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} > 0, \tag{154}$$

which is obviously fulfilled due to the monotonic increasing of the function $\cosh(\delta z)$ w.r.t. z.

Thus, $\lambda^* = \lambda_I^*$ and $K(\lambda^*) = K_I$, which completes the proof in the case ($\mathrm{I_N}$). □

9.3. **Proof of Theorem 6.3.** In this section, the detailed proof of Theorem 6.3 is presented in the case (I) of Theorem 5.2. The error estimates in the cases (II) and (III) are derived similarly.

The proof is based on the following auxiliary lemmas.

9.3.1. *Auxiliary lemmas.*

Lemma 9.1. *The following is valid:*

$$\lim_{N\to\infty} z_{i^*} = z^*, \tag{155}$$

where the natural number i^* is defined in (85) – (86), z^* is defined in (58) – (59).

Proof. Due to (58) – (59), (72) – (73) and (85) – (86),

$$z_{i^*} = \frac{3i^*}{N}, \tag{156}$$

where

$$i^* = \min\left\{j \in \{0, 1, \ldots, N-1\} : \frac{3j}{N} \geq z^*\right\}. \tag{157}$$

Therefore,

$$z^* \in [z_{i^*-1}, z_{i^*}]. \tag{158}$$

The inclusion (158) yields

$$|z_{i^*} - z^*| \le \frac{3}{N}, \tag{159}$$

which proves (155). □

Lemma 9.2. *The following is valid:*

$$\lim_{N\to\infty} \Delta z \Lambda_N^* = \Lambda^*, \tag{160}$$

$$\lim_{N\to\infty} \Delta z a_N = a. \tag{161}$$

where a and a_N are given by (38) and (78), respectively, Λ^* and Λ_N^* are given by (56) and (83), respectively.

Proof. Denote

$$\gamma_1 \triangleq \sum_{i=0}^{i^*-1} \Delta z \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} - \int_0^{z_{i^*}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta, \tag{162}$$

$$\gamma_2 \triangleq \sum_{i=i^*}^{N-1} \Delta z \exp(-z_i^2/2) - \int_{z_{i^*}}^{3} \exp(-\zeta^2/2) d\zeta, \tag{163}$$

$$\gamma_3 \triangleq \Delta z a_N - a. \tag{164}$$

Then, due to [12], there exist a natural number $\hat{N}_1$ and a positive constant $\hat{C}_1$, independent of N, such that for all $N \ge \hat{N}_1$:

$$0 < \gamma_i < \frac{\hat{C}_1}{N}, \quad i = 1, 2, 3. \tag{165}$$

Due to (83) and (162) – (163),

$$\Delta z \Lambda_N^* = u_{\max} \int_0^{z_{i^*}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta + u_{\min} \int_{z_{i^*}}^{3} \exp(-\zeta^2/2) d\zeta + u_{\max}\gamma_1 + u_{\min}\gamma_2. \tag{166}$$

Moreover, due to Lemma 9.1,

$$\lim_{N\to\infty} \int_0^{z_{i^*}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta = \int_0^{z^*} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta, \tag{167}$$

$$\lim_{N\to\infty} \int_{z_{i^*}}^{3} \exp(-\zeta^2/2) d\zeta = \int_{z^*}^{3} \exp(-\zeta^2/2) d\zeta. \tag{168}$$

Thus, (160) directly follows from (165) for $i = 1, 2$ and (166) – (168), while (161) directly follows from (165) for $i = 3$. □

Lemma 9.3. *Let the condition*

$$\Lambda^* > a, \tag{169}$$

hold. Then there exist a natural number $\tilde{N}_0$ such that for all $N \ge \tilde{N}_0$, the condition (88) holds.

Proof. The statement of the lemma directly follows from Lemma 9.2. □

Lemma 9.4. *There exist a natural number $\tilde{N}_1$ and a positive constant $\tilde{C}_1$, independent of N, such that for all $N \geq \tilde{N}_1$,*

$$|z_{K_I} - z_I| \leq \frac{\tilde{C}_1}{N}, \tag{170}$$

where $K_I \in \{1, \dots, N-1\}$ is defined by (91) – (92), $z_I \in (0,3)$ is the unique solution of the equation (63).

Proof. From the inequalities (91) – (92), it directly follows that

$$\cosh(\delta z_{K_I}) \sum_{i=0}^{K_I-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=K_I}^{N-1} \exp(-z_i^2/2) - \frac{a_N}{u_{\min}} - \varepsilon_1 = 0, \tag{171}$$

where ε_1 satisfies the inequality

$$0 \leq \varepsilon_1 \leq \mathcal{G}_I(K_I) - \mathcal{G}_I(K_I - 1), \tag{172}$$

the function $\mathcal{G}_I(K)$ is given by (153). By using (72) – (73) and (153), the inequality (172) can be rewritten as

$$0 \leq \varepsilon_1 \leq 2 \sinh\left(\frac{\delta(z_{K_I} + z_{K_I-1})}{2}\right) \sinh\left(\frac{3\delta}{2N}\right) \sum_{i=0}^{K_I-1} \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)}. \tag{173}$$

Let us multiply the equation (171) by Δz:

$$\cosh(\delta z_{K_I}) \sum_{i=0}^{K_I-1} \Delta z \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} + \sum_{i=K_I}^{N-1} \Delta z \exp(-z_i^2/2) - \frac{a_N \Delta z}{u_{\min}} - \varepsilon_1 \Delta z = 0. \tag{174}$$

Denote

$$\gamma_4 \triangleq \sum_{i=0}^{K_I-1} \Delta z \frac{\exp(-z_i^2/2)}{\cosh(\delta z_i)} - \int_0^{z_{K_I}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)} d\zeta, \tag{175}$$

$$\gamma_5 \triangleq \sum_{i=K_I}^{N-1} \Delta z \exp(-z_i^2/2) - \int_{z_{K_I}}^{3} \exp(-\zeta^2/2) d\zeta. \tag{176}$$

Then there exist a natural number $\hat{N}_2$ and a positive constant $\hat{C}_2$, independent of N, such that for all $N \geq \hat{N}_2$:

$$0 < \gamma_i < \frac{\hat{C}_2}{N}, \quad i = 4, 5. \tag{177}$$

Moreover, the inequality (173), the equality (175) and the inequality (165) imply the existence of a natural number $\hat{N}_3$ and a positive constant $\hat{C}_3$, independent of N, such that for all $N \geq \hat{N}_3$:

$$0 \leq \varepsilon_1 \Delta z < \frac{\hat{C}_3}{N}. \tag{178}$$

Due to (63), (164) and (175) – (176), the equation (174) can be rewritten as

$$G_I(z_{K_I}) = \gamma_6 \tag{179}$$

where

$$\gamma_6 \triangleq -\gamma_4 \cosh(\delta z_{K_I}) - \gamma_5 + \frac{\gamma_3}{u_{\min}} + \varepsilon_1 \Delta z. \tag{180}$$

By virtue of the estimates (175) – (178), there exist a natural number $\hat{N}_4 \geq \max\{\hat{N}_1, \hat{N}_2, \hat{N}_3\}$ and a positive constant $\hat{C}_4$, independent of N, such that for all $N \geq \hat{N}_4$,

$$0 < |\gamma_6| < \frac{\hat{C}_4}{N}. \tag{181}$$

Note that, due to Theorem 5.2,

$$G_I(z_I) = 0. \tag{182}$$

By using the equations (179), (182), the Lagrange Mean Value Theorem [17] and the inequality (181),

$$|G_I(z_{K_I}) - G_I(z_I)| = |G_I'(\hat{z})||z_{K_I} - z_I| \leq \frac{\hat{C}_4}{N}, \tag{183}$$

where $\hat{z}$ lies between z_{K_I} and z_I.

The equation (179) and the inequality (181), as well as the uniqueness of the solution of the equation (63), imply the existence of $\bar{z} \in (0, 3)$ such that

$$z_{K_I} \geq \bar{z}, \quad N \geq \hat{N}_4. \tag{184}$$

Therefore, due to (130),

$$|G_I'(\hat{z})| \geq \hat{C} > 0, \quad N \geq \hat{N}_4, \tag{185}$$

where the number $\hat{C}$ is independent of N. The inequalities (183) and (185) directly yield the statement of the lemma for $\tilde{N}_1 = \hat{N}_4$, $\tilde{C}_1 = \hat{C}_4/\hat{C}$. □

Lemma 9.5. *There exist a natural number $\tilde{N}_2$ and a positive constant $\tilde{C}_2$, independent of N, such that for all $N \geq \tilde{N}_2$,*

$$|\lambda_I^* - C_I^*| \leq \frac{\tilde{C}_2}{N}, \tag{186}$$

where λ_I^* and C_I^* are given by (90) and (62), respectively.

Proof. By multiplying the numerator and the denominator of (90) with Δz and by using the equations (164) and (175) – (176),

$$\lambda_I^* = 4\frac{a - u_{\min}\displaystyle\int_{z_{K_I}}^{3} \exp(-\zeta^2/2)d\zeta + \gamma_3 - u_{\min}\gamma_5}{\displaystyle\int_0^{z_{K_I}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta + \gamma_4}. \tag{187}$$

Denote

$$\gamma_7 \triangleq \int_{z_{K_I}}^{3} \exp(-\zeta^2/2)d\zeta - \int_{z_I}^{3} \exp(-\zeta^2/2)d\zeta, \tag{188}$$

$$\gamma_8 \triangleq \int_0^{z_{K_I}} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta - \int_0^{z_I} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta. \tag{189}$$

Due to Lemma 9.4, for all $N \geq \tilde{N}_1$:

$$0 < |\gamma_i| < \frac{\tilde{C}_1}{N}, \quad i = 7, 8. \tag{190}$$

By virtue of (188) – (189), the equation (187) can be rewritten as

$$\lambda_I^* = 4\frac{a - u_{\min}\int\limits_{z_I}^{3} \exp(-\zeta^2/2)d\zeta + \gamma_3 - u_{\min}\gamma_5 - u_{\min}\gamma_7}{\int\limits_{0}^{z_I} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta + \gamma_4 + \gamma_8}. \tag{191}$$

Finally, taking into account that z_I satisfies the equation (63), the equation (191) yields

$$\lambda_I^* = 4u_{\min}\frac{\cosh(\delta z_I)\int\limits_{0}^{z_I} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta + \gamma_3 - u_{\min}\gamma_5 - u_{\min}\gamma_7}{\int\limits_{0}^{z_I} \frac{\exp(-\zeta^2/2)}{\cosh(\delta\zeta)}d\zeta + \gamma_4 + \gamma_8}. \tag{192}$$

Now, the equations (62) and (192), along with the inequalities (165), (177) and (190), directly lead to the the statement of the lemma. □

9.3.2. *Main part of the proof.* Remember that here we present the detailed proof for the case (169). In this case, the NOCP solution is given by (61) – (63). Moreover, due to Lemma 9.3, for $N \geq \tilde{N}_0$, the QPP solution is given by (89) – (92).

Let start with the proof of (104). Let $N \geq \max\{\tilde{N}_0, \tilde{N}_1, \tilde{N}_2\}$. For such N, two cases can be distinguished: (i) $z_{K_I} > z_I$; (ii) $z_{K_I} \leq z_I$.

Case (i). Denote

$$R \triangleq \min\Big\{j \in \{0, 1, \ldots, N-1\} : z_j \geq z_I\Big\}, \tag{193}$$

i.e. z_R is the collocation point, nearest to z_I from the right. Then, due to (49), (61) and (89)

$$u_{nl}^*(z_i) - U_i^* = \begin{cases} \dfrac{C_I^* - \lambda_I^*}{4\cosh(\delta z_i)}, & i = 0, \ldots, R-1, \\[2ex] u_{\min} - \dfrac{\lambda_I^*}{4\cosh(\delta z_i)}, & i = R, \ldots, K_I - 1, \\[2ex] 0, & i = K_I, \ldots, N-1. \end{cases} \tag{194}$$

Note that, due to (62) and Lemmas 9.4 – 9.5, there exists a positive constant $\hat{C}_5$, independent of N, such that

$$\left| u_{\min} - \frac{\lambda_I^*}{4\cosh(\delta z_i)} \right| \leq \frac{\hat{C}_5}{N}, \quad i = R, \ldots, K_I - 1. \tag{195}$$

Thus, (104) is a direct consequence of (194), Lemma 9.5 and the inequality (195).

Case (ii). Denote

$$L \triangleq \max\Big\{j \in \{0, 1, \ldots, N-1\} : z_j \leq z_I\Big\}, \tag{196}$$

i.e. z_L is the collocation point, nearest to z_I from the left. Then, due to (49), (61) and (89)

$$u_{nl}^*(z_i) - U_i^* = \begin{cases} \dfrac{C_I^* - \lambda_I^*}{4\cosh(\delta z_i)}, & i = 0, \dots, K_I - 1, \\ \dfrac{C_I^*}{4\cosh(\delta z_i)} - u_{\min}, & i = K_I, \dots, L, \\ 0, & i = L+1, \dots, N-1. \end{cases} \tag{197}$$

Note that, due to (62) and Lemma 9.4, there exists a positive constant $\hat{C}_6$, independent of N, such that

$$\left| \frac{C_I^*}{4\cosh(\delta z_i)} - u_{\min} \right| \le \frac{\hat{C}_6}{N}, \quad i = K_I, \dots, L. \tag{198}$$

Thus, (104) is a direct consequence of (197), Lemma 9.5 and the inequality (198). This completes the proof of (104).

Proceed to the proof of (105). Denote

$$\gamma_9 \triangleq \Delta z \sum_{i=0}^{N-1} \psi(z_i, \delta)(u_{nl}^*)^2(z_i) - \int_0^3 \psi(z, \delta)(u_{nl}^*)(z)^2 dz. \tag{199}$$

Then, due to [12], there exist a natural number $\breve{N}$ and a positive constant $\breve{C}$, independent of N, such that for all $N \ge \breve{N}$:

$$|\gamma_9| < \frac{\breve{C}}{N}. \tag{200}$$

Due to (26), (77) and (199),

$$\left| J_2(u_{nl}^*) - \Delta z J_2^N(U^*) \right| \le \Delta z \sum_{i=0}^{N-1} \psi(z_i, \delta) \left| (u_{nl}^*(z_i))^2 - (U_i^*)^2 \right| + |\gamma_9|. \tag{201}$$

Now, (104) and (200) – (201) directly yield (105), which completes the proof of the theorem. □

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Received April 2013; 1st revision November 2013; 2nd revision December 2013.

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