Homework 4

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Question 9:

A. Exercise 4.1.3

(b) $f(x) = \frac{1}{(x^2-4)}$

f is not a function

$$\text{When } x = 2 \text{ or} - 2, f(x) = \frac{1}{0} \text{ which is not defined.}$$

(c) $f(x) = \sqrt{x^2}$

f is a function.

$$\text{Range} = [0, \infty)$$

B. Exercise 4.1.5

(b) Let $A = \{2, 3, 4, 5\}$, $f : A \to \mathbb{Z}$, such that $f(x) = x^2$

$$\{4, 9, 16, 25\}$$

(d) $f : \{0, 1\}^5 \to \mathbb{Z}$. For $x \in \{0, 1\}^5, f(x)$ is the number of 1's that occur in x.

$$\{0, 1, 2, 3, 4, 5\}$$

(h) Let $A = \{1, 2, 3\}$, $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$

$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Range = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$$

(i) Let $A = \{1, 2, 3\}$, $f : A \times A \to \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

$$Range = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4)\}$$

(l) Let $A = \{1, 2, 3\}$, $f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - \{1\}$

$$P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

$$Range = \{\varnothing, \{2\}, \{3\}, \{2,3\}\}$$

Question 10:

I (a) Exercise 4.2.2

(c) $h : \mathbb{Z} \to \mathbb{Z}$. $h(x) = x^3$

Not onto.

Let y be an integer such that it is not a perfect cube. Then there is no x in the integers such that $x^3 = y$. For example, there is no $x \in \mathbb{Z}$ such that $x^3 = 2$.

One-to-one.

(g) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. $f(x, y) = (x + 1, 2y)$

Not onto.

Since $2y$ is always even for any integer y from $\mathbb{Z}$, all the ordered pairs in the target with y as odd integers will not have any element mapped to them from the domain. For example, there is no $y \in \mathbb{Z}$ such that $2y = 1$.

One-to-one.

(k) $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$. $f(x, y) = 2^x + y$

Not onto.

Let $f(x, y) = z$. When z is 1 or 2, there is no (x, y) in $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $2^x + y = z$

One-to-one.

I (b) Exercise 4.2.4

(b) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.

Not onto.

Let y be an element from the target $\{0, 1\}^3$ and $y = 001$, then there is no x in the domain $\{0, 1\}^3$, which can make $f(x) = y = 001$

Not one-to-one. $f(101) = f(001) = 101$

(c) $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

Onto and **One-to-one.**

(d) $f : \{0,1\}^3 \to \{0,1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.

Not onto.

Let y be an element from the target $\{0,1\}^4$ and $y = 0001$, then there is no x in the domain $\{0,1\}^3$, which can make $f(x) = y = 0001$

One-to-one.

(g) Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}.f : P(A) \to P(A)$. For $X \subseteq A, f(X) = X - B$.

Not onto.

When the set Y has the element 1, there is no set $X \subseteq A$ that can make $X - B = Y$. For example, when $Y = \{1\}$, no set $X \subseteq A$ can make $X - B = Y = \{1\}$

Not one-to-one. $f(\{1,2,3\}) = f(\{2,3\}) = \{2,3\}$

II $f : \mathbb{Z} \to \mathbb{Z}^+$

(a) one-to-one but not onto

$$f(x) = \begin{cases} 2 \cdot |x| & \text{if } x \text{ is negative} \\ 2x + 3 & \text{if } x \text{ is non-negative} \end{cases}$$

(b) onto but not one-to-one

$$f(x) = |x| + 1$$

(c) one-to-one and onto

$$f(x) = \begin{cases} 2 \cdot |x| & \text{if } x \text{ is negative} \\ 2x + 1 & \text{if } x \text{ is non-negative} \end{cases}$$

(d) neither one-to-one nor onto

$$f(x) = x^2 + 1$$

Question 11:

A. Exercise 4.3.2

(c) $f : \mathbb{R} \to \mathbb{R}. f(x) = 2x + 3$

The function has well defined inverse.

$$f^{-1}(x) = \frac{x-3}{2}$$

(d) Let A be defined to be set $\{1, 2, 3, 4, 5, 6, 7, 8\}. f : P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. For $X \subseteq A, f(X) = |X|$

The function is not one-to-one. So f^{-1} is not well defined.

(g) $f : \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

The function has well defined inverse.

$$f^{-1} = f = \{0, 1\}^3$$

(i) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$

The function has well defined inverse.

$$f^{-1}(x, y) = (x - 5, y + 2)$$

B. Exercise 4.4.8

(c) $f \circ h$

$$f(h(x)) = 2x^2 + 5$$

(d) $h \circ f$

$$\begin{aligned} h(f(x)) &= (2x + 3)^2 + 1 \\ &= 4x^2 + 9 + 12x \\ &= 4x^2 + 12x + 10 \end{aligned}$$

C. Exercise 4.4.2

(b) Evaluate $f \circ h(52)$

$$\begin{aligned} h(52) &= \left\lceil \frac{52}{5} \right\rceil \\ &= 11 \end{aligned}$$

$$\begin{aligned} f(11) &= 11^2 \\ f \circ h(52) &= 121 \end{aligned}$$

(c) Evaluate $g \circ h \circ f(4)$

$$\begin{aligned} f(4) &= 4^2 \\ &= 16 \end{aligned}$$

$$\begin{aligned} h(16) &= \left\lceil \frac{16}{5} \right\rceil \\ &= 4 \end{aligned}$$

$$\begin{aligned} g(4) &= 2^4 \\ g \circ h \circ f(4) &= 16 \end{aligned}$$

(d) Give a mathematical expression for $h \circ f$

$$f(x) = x^2$$

$$h \circ f(x) = \left\lceil \frac{x^2}{5} \right\rceil$$

D. Exercise 4.4.6

(c) What is $h \circ f(010)$

$$f(010) = 110$$

$$h \circ f(010) = 111$$

(d) What is the range of $h \circ f$

$$\text{Range} = \{101, 111\}$$

(e) What is the range of $g \circ f$

$$\text{Range} = \{001, 011, 101, 111\}$$

E. Exercise 4.4.4

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one?

No. We will show that if f is not one-to-one, then $g \circ f$ must not be one to one.
If f is not one-to-one, then $\exists\, x_1, x_2 \in X : x_1 \neq x_2 \wedge f(x_1) = f(x_2)$.
Also, $g(f(x_1)) = g(f(x_2))$ and $x_1 \neq x_2$. Therefore $g \circ f$ is not one-to-one.

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one?

Yes. The diagram below illustrates an example.