Homework 2

Gowtham Rajeshshekaran (gr2180@nyu.edu)

Question 5:

A.1: Exercise 1.12.2

(b)
$$\begin{array}{l} p \to (q \land r) \\ \neg q \\ \hline \therefore \neg p \end{array}$$

1.	$p \to (q \land r)$	Hypothesis
2.	$p \to q$	Simplification, 1
3.	$\neg q$	Hypothesis
4.	$\neg p$	Modus tollens, 2, 3

(e)
$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore r \end{array}$$

1.	$p \lor q$	Hypothesis
2.	$\neg p \lor r$	Hypothesis
3.	$q \lor r$	Resolution, 1, 2
4.	$\neg q$	Hypothesis
5.	r	Disjunctive syllogism, 3, 4

A.2: Exercise 1.12.3

(c)
$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

1.	$p \lor q$	Hypothesis
2.	$\neg\neg p \lor q$	Double negation, 1
3.	$\neg p \to q$	Conditional identity, 2
4.	$\neg p$	Hypothesis
5.	q	Modus ponens, 3, 4

A.3: Exercise 1.12.5

(c)
I will buy a new car and a new house only if I get a job.
I am not going to get a job.
$\therefore$ I will not buy a new car.

- j: I will get a job
- c: I will buy a new car
- h: I will buy a new house

The form of the argument is

$$\begin{array}{l} (c \land h) \to j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

The argument is **not valid**. When c = T, and h = j = F, the hypotheses are both true and the conclusion $\neg c$ is false.

(d)
I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house.
$\therefore$ I will not buy a new car.

- j: I will get a job
- c: I will buy a new car
- h: I will buy a new house

The form of the argument is

$$\begin{array}{l} (c \land h) \to j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

The argument is **valid.**

1.	$(c \land h) \to j$	Hypothesis
2.	$\neg(c \land h) \lor j$	Conditional identity, 1
3.	$j \lor \neg(c \land h)$	Commutative law, 2
4.	$\neg j$	Hypothesis
5.	$\neg(c \land h)$	Disjunctive syllogism, 3, 4
6.	$\neg c \lor \neg h$	De morgans law, 5
7.	$\neg h \lor \neg c$	Commutative law, 6
8.	h	Hypothesis
9.	$\neg\neg h$	Double negation law, 8
10.	$\neg c$	Disjunctive syllogism, 7, 9

B.1: Exercise 1.13.3

(b)
$$\begin{array}{l} \exists x(P(x) \lor Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	P	**Q**
a	F	F
b	F	T

$\exists x(P(x) \lor Q(x))$ is true because when x = b, Q(b) is true.

$\exists x \neg Q(x)$ is true because when x = a, $\neg Q(a)$ is true.

However, since P(a) = P(b) = F, $\exists x P(X)$ is false. Therefore both hypotheses are true and the conclusion is false.

B.2: Exercise 1.13.5

(d)
Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

∴ Penelope did not get a detention.

- D(x): x got a detention.
- M(x): x missed class.

The form of the argument is
$$\begin{array}{l} \forall x(M(x) \rightarrow D(x)) \\ \textit{Penelope}\text{, a student in the class} \\ \neg M(\textit{Penelope}) \\ \hline \therefore \neg D(\textit{Penelope}) \end{array}$$

The argument is **not valid.** *Penelope* is a student in the class (Hypothesis).

When $M(\textit{Penelope})$ = F and $D(\textit{Penelope})$ = T, then the hypotheses are all true and the conclusion is false. In other words, *Penelope* did not miss the class and could have got a detention.

(e)

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

$\therefore$ Penelope did not get a detention..

- A(x): x received an A.
- D(x): x got a detention.
- M(x): x missed class.

The form of the argument is

$$\begin{array}{l} \forall x((M(x) \vee D(x)) \rightarrow \neg A(x)) \\ Penelope\text{, a student in the class} \\ A(Penelope) \\ \hline \therefore \neg D(Penelope) \end{array}$$

The argument is **valid.**

1.	$\forall x((M(x) \vee D(x)) \rightarrow \neg A(x))$	Hypothesis
2.	$Penelope$, a student in the class	Hypothesis
3.	(M(Penelope) $\vee D(Penelope)) \rightarrow \neg A(Penelope)$	Universal instantiation, 1, 2
4.	$A(Penelope)$	Hypothesis
5.	$\neg\neg A(Penelope)$	Double negation law, 4
6.	$\neg((M(Penelope) \vee D(Penelope))$	Modus tollens, 3, 5
7.	$\neg M(Penelope) \wedge \neg D(Penelope)$	De morgans law, 6
8.	$\neg D(Penelope) \wedge \neg M(Penelope)$	Commutative law, 7
9.	$\neg D(Penelope)$	Simplification, 8

Question 6:

Exercise 2.2.1

(c) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

Proof. Direct proof. Let's assume x is a real number and $x \leq 3$ and show that $12 - 7x + x^2 \geq 0$.

Subtract x on both sides from $x \leq 3$,

$$\begin{aligned} x - x &\leq 3 - x \\ 0 &\leq 3 - x \end{aligned}$$

This can also be written as $(3 - x) \geq 0$. Add 1 on both sides,

$$\begin{aligned} 3 - x + 1 &\geq 1 \\ 4 - x &\geq 1 \end{aligned}$$

Since $(4 - x) \geq 1$, we can say that $(4 - x) > 0$.
Since $(4 - x) > 0$, and $(3 - x) \geq 0$, on multiplying both terms,

$$\begin{aligned} (4 - x) \cdot (3 - x) &\geq 0 \\ 12 - 4x - 3x + x^2 &\geq 0 \\ 12 - 7x + x^2 &\geq 0 \end{aligned}$$

■

(d) The product of two odd integers is an odd integer.

Proof. Direct proof. Assume p and q are two odd integers. We will show that $p \cdot q$ is also an odd integer.

If p and q are odd integers,

$$\begin{aligned} p &= 2m + 1 \text{ for some integer } m \\ q &= 2n + 1 \text{ for some integer } n \end{aligned}$$

On multiplying both the integers we get,

$$\begin{aligned} p \cdot q &= (2m + 1) \cdot (2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \; [k = 2mn + m + n] \end{aligned}$$

Since m and n are both integers, $k = 2mn + m + n$ is also an integer.
Since $p \cdot q = 2k + 1$ for some integer k $[k = 2mn + m + n]$, $p \cdot q$ is an odd integer. ■

Question 7:

Exercise 2.3.1

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof. Proof by contrapositive. We assume that n is an even integer and show that $n^2 - 2n + 7$ is odd.

Since n is even,

$$n = 2k \text{ for some integer } k$$

Substitute $n = 2k$ in $n^2 - 2n + 7$,

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 4k^2 - 4k + 6 + 1 \\ &= 2(2k^2 - 2k + 3) + 1 \\ &= 2m + 1 \; [m = 2k^2 - 2k + 3] \end{aligned}$$

Since k is an integer, $m = 2k^2 - 2k + 3$ is also an integer.
Since $n^2 - 2n + 7$ can be expressed as 2 times some integer + 1, $n^2 - 2n + 7$ is odd. ■

(f) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Proof. Proof by contrapositive. Let x be a non-zero real number. We assume that $\frac{1}{x}$ is not irrational and show that x must be rational.

Every real number is either rational or irrational. Since $\frac{1}{x}$ is a real number and is not irrational and $x \neq 0$, $\frac{1}{x}$ is rational.

$$\therefore \; \frac{1}{x} = \frac{a}{b} \text{ where } a \text{ and } b \text{ are integers and } b \neq 0$$

Taking reciprocal on both sides,

$$x = \frac{b}{a}$$

Since b can be written as $b = x \cdot a$ and $b \neq 0$, therefore $a \neq 0$.
Since x is the ratio of two integers b and a, and $a \neq 0$, therefore x is rational. ■

(g) For every pair of real numbers x and y, if $x^3 + xy^2 \le x^2y + y^3$, then $x \le y$.

Proof. Proof by contrapositive. We assume for every pair of real numbers x and y, $x > y$ and show that $x^3 + xy^2 > x^2y + y^3$

Since x and y are real numbers and $x > y$, either x or y is not 0, therefore $x^2 + y^2$ is also a real number and $x^2 + y^2 \neq 0$ and also $x^2 + y^2 > 0$ (since square of any non-zero real number is greater than 0).

Multiplying on both sides of the equation $x > y$ by $x^2 + y^2$,

$$x \cdot (x^2 + y^2) > y \cdot (x^2 + y^2)$$
$$x^3 + xy^2 > x^2y + y^3$$

■

(l) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Proof. Proof by contrapositive. We assume for every pair of real numbers x and y, $x \le 10$ and $y \le 10$ and show that $x + y \le 20$.

Adding both terms $x \le 10$ and $y \le 10$,

$$x + y \le 10 + 10$$
$$x + y \le 20$$

■

Question 8:

Exercise 2.4.1

(c) The average of three real numbers is greater than or equal to at least one of the numbers.

Proof. Proof by contradiction. Let x, y, z be three real numbers. Assume that the average of these three numbers is less than all the three numbers.

$$\frac{x+y+z}{3} < x, \frac{x+y+z}{3} < y, \frac{x+y+z}{3} < z$$

Adding these three equations,

$$\begin{aligned}
\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} &< x+y+z \\
\frac{3x+3y+3z}{3} &< x+y+z \\
\cancel{3} \cdot (\frac{x+y+z}{\cancel{3}}) &< x+y+z \\
x+y+z &< x+y+z
\end{aligned}$$

Since the sum of three real numbers cannot be less than their sum, this contradicts the assumption that the average of three numbers is less than all the three numbers. Therefore, The average of three real numbers is greater than or equal to at least one of the numbers. ■

(e) There is no smallest integer.

Proof. Proof by contradiction. Assume there is a smallest integer x.

Subtracting x by 1 gives $x - 1$ which is an integer, and also $(x - 1) < x$.
This contradicts the assumption that x is the smallest integer. Therefore, there is no smallest integer. ■

Question 9:

Exercise 2.5.1

(c) If integers x and y have the same parity, then $x + y$ is even.

Proof. Proof by cases.

Case 1: x and y are even integers.
If x and y are even integers,

$$x = 2m \quad \text{for some integer m}$$
$$y = 2n \quad \text{for some integer n}$$

Adding x and y,

$$x + y = 2m + 2n$$
$$x + y = 2(m + n)$$
$$x + y = 2k \quad [k = m + n]$$

Since m and n are both integers $k = m + n$ is also an integer.
Since $x + y$ can be expressed as 2 times some integer, $x + y$ is even.

Case 2: x and y are odd integers.
If x and y are odd integers,

$$x = 2p + 1 \quad \text{for some integer p}$$
$$y = 2q + 1 \quad \text{for some integer q}$$

Adding x and y,

$$x + y = 2p + 1 + 2q + 1$$
$$x + y = 2p + 2q + 2$$
$$x + y = 2(p + q + 1)$$
$$x + y = 2l \quad [l = m + n]$$

Since p and q are both integers $l = p + q + 1$ is also an integer.
Since $x + y$ can be expressed as 2 times some integer, $x + y$ is even. ∎
