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Homework 2

1. **Exercise 8.4.1**: Components of an inductive proof. (a-f)
   Define P(n) to be the assertion that:
   $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+\ )}{6}$$

   (a) Verify that P(3) is true.
   - $1^2 + 2^2 + 3^2 \quad ? \quad \frac{3(3+1)(2(3)+1)}{6}$

     $1 + 4 + 9 \quad ? \quad \frac{3(4)(7)}{6}$

     $14 \quad = \quad 14$

   (b) Express P(k).
   - $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$

   (c) Express P(k+1).
   - $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

     $= \frac{(k+1)(k+2)(2k+3)}{6}$

   (d) In an inductive proof that for every positive integer n,
   $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
   What must be proven in the base case?
   - For the base case, it has to be proven that formula is true for n = 1.
   - When n = 1, the left side of the equation is $\sum_{j=1}^{1} j^2 = 1$.

     When n = 1, the right side of the equation is $\frac{1(1+1)(2+1)}{6} = 1$

     Therefore, $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2+1)}{6}$.

   (e) In an inductive proof that for every positive integer n,
   $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$
   What must be proven in the inductive step?
   - Suppose that for positive integer k, $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$, then we will show that $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$
   - Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 \quad \text{by separating out the last term}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the inductive hypothesis}$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{by algebra}$$

Therefore, $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

- The inductive hypothesis would be this → $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$

2. **Exercise 8.4.3**: Proving inequalities by induction. (a)
Prove each of the following statements using mathematical induction.

(a) Prove that for $n \geq 2$, $3^n > 2^n + n^2$.

- Base case: $n = 2$
$3^2 = 9 > 8 = 2^2 + 2^2$
Therefore, for $n = 2$, $3^n > 2^n + n^2$
- Inductive step: Suppose that for positive integer $k \geq 2$, if $3^k > 2^k + k^2$, then $3^{k+1} > 2^{k+1} + (k+1)^2$
Starting with the left side of the inequality to be prove:
$3^{k+1} = 3 \cdot 3^k$ by algebra
$> 3 \cdot (2^k + k^2)$ by the inductive hypothesis
$= 3 \cdot 2^k + 3 \cdot k^2$
$> 2^k + 2^k + 2^k + k^2 + k^2 + k^2$
$> 2^k + 2^k + 2^0 + k^2 + k^2 + k^2$ because $k \geq 2 \geq 0$
$= 2^k + 2^k + k^2 + 2k^2 + 1$
$> 2^k + 2^k + (k+1)^2$
$= 2 \cdot 2^k + (k+1)^2$
$= 2^{k+1} + (k+1)^2$ by algebra
Therefore, $3^{k+1} > 2^{k+1} + (k+1)^2$

3. **Exercise 8.5.1**: Proving divisibility results by induction. (b)
Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

- Theorem: For every positive integer n, 4 evenly divides $3^{2n} - 1$.
- Proof: By induction on n.
- Base case: n = 1
  $3^{2(1)} - 1 = 3^2 – 1 = 8$. Since 4 evenly divides 8, the theorem holds for the case n = 1.
- Inductive step: Suppose that for positive integer k, 4 evenly divides $3^{2k} – 1$. Then we will show that 4 evenly divides $3^{2(k+1)} – 1$.
  By the inductive hypothesis, 4 evenly divides $3^{2k} – 1$, which means that $3^{2k} – 1 = 4m$ for some integer m. By adding 1 to both sides of the equation $4m = 3^{2k} – 1$, we get $4m + 1 = 3^{2k}$ which is an equivalent statement of the inductive hypothesis.
  We must show that $3^{2(k+1)} – 1$ can be expressed as 4 times an integer.
  $$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 9 \cdot 3^{2k} - 1 && \text{by algebra} \\ &= 9(4m + 1) - 1 && \text{by the inductive hypothesis} \\ &= 4 \cdot 9m + 9 - 1 \\ &= 4(9m + 2) && \text{by algebra} \end{aligned}$$
  Since m is an integer, (9m + 2) is also an integer. Therefore $3^{2(k+1)} – 1$ is equal to 4 times an integer which means that $3^{2(k+1)} – 1$ is divisible by 4.

(b) Prove that for any positive integer n, 6 evenly divides $7^n - 1$.

- Theorem: For every positive integer n, 6 evenly divides $7^n - 1$.
- Proof: By induction on n.
- Base case: n = 1
  $7^1 - 1 = 6$. Since 6 evenly divides 6, the theorem holds for the case n = 1.
- Inductive step: Suppose that for positive integer k, 6 evenly divides $7^k – 1$. Then we will show that 6 evenly divides $7^{(k+1)} – 1$.
  By the inductive hypothesis, 6 evenly divides $7^k – 1$, which means that $7^k – 1 = 6m$ for some integer m. By adding 1 to both sides of the equation $6m = 7^k – 1$, we get $6m + 1 = 7^k$ which is an equivalent statement of the inductive hypothesis.
  We must show that $7^{(k+1)} – 1$ can be expressed as 6 times an integer.
  $$\begin{aligned} 7^{(k+1)} - 1 &= 7 \cdot 7^k - 1 && \text{by algebra} \\ &= 7(6m + 1) - 1 && \text{by the inductive hypothesis} \\ &= 6 \cdot 7m + 7 - 1 \end{aligned}$$

$= 6(7m + 1)$ by algebra

- Since m is an integer, $(7m + 1)$ is also an integer. Therefore $7^{(k+1)} - 1$ is equal to 6 times an integer which means that $7^{(k+1)} - 1$ is divisible by 6.

4. **Exercise 8.5.2**: Proving explicit formulas for recurrence relations by induction. (b)
   Prove each of the following statements using mathematical induction.

   (b) Define the sequence $\{b_n\}$ as follows:
   $b_0 = 1$
   $b_k = 2b_{k-1} + 1$ for $k \geq 1$
   Prove that for $n \geq 0$, $b_n = 2^{n+1} - 1$.
   - Proof: By induction on n
   - Base case: $n = 0$
     We must show that $b_0 = 2^{0+1} - 1$. Since $2^{0+1} - 1 = 1$ and the initial condition states that $b_0 = 1$, the theorem holds for $n = 0$.
   - Inductive step:
     Suppose that for any integer $k \geq 0$, $b_k = 2^{k+1} - 1$. Then we will show that $b_{k+1} = 2^{k+2} - 1$.
     For any integer $k \geq 0$:
     $$\begin{aligned} b_{k+1} &= 2b_k + 1 \\ &= 2\cdot(2^{k+1} - 1) + 1 \\ &= 2^{k+1+1} - 2 + 1 \\ &= 2^{k+2} - 1 \end{aligned}$$
     Therefore, $b_{k+1} = 2^{k+2} - 1$

5. **Exercise 8.6.1**: Proofs by strong induction – combining stamps. (a)
   Prove each of the following statements using strong induction.

   (a) Prove that any amount of postage worth 8 cents or more can be made from 3-cent or 5-cent stamps.
   - Base case: P(8), P(9), P(10)
     $n = 8$: one 3-cent stamp and one 5-cent stamp
     $n = 9$: three 3-cent stamps
     $n = 10$: two 5-cent stamps
   - Inductive step: For $k \geq 10$, assume P(j) is true for any j in the range 8 through k.
     Prove P(k + 1)

$k \geq 10 \Rightarrow k - 2 \geq 8$
$k - 2$ is in the range 8 through k
$(k - 2) + 3 = k + 1$ stamps $P(k + 1)$ is true.

Since $k \geq 10$, then $k - 2 \geq 6$. Therefore, $k - 2$ is in the range 8 through k and by the inductive hypothesis, $k - 2$ stamps can be purchased.

6. **Exercise 8.6.2**: Proofs by strong induction – explicit formulas for recurrence relations. (c)
Prove each of the following statements using strong induction.

(c) Define the sequence $\{g_n\}$ as follows:

$g_0 = 51$
$g_1 = 348$
$g_n = 5g_{n-1} - 6g_{n-2} + 20 \cdot 7^n$ for $n \geq 2$

Prove that for $n \geq 0$, $g_n = 2^n + 3^n + 7^{n+2}$

- Base case: $n = 0$, $n = 1$
  - ➔ We must show that $g_0 = 2^0 + 3^0 + 7^{0+2}$. Since $2^0 + 3^0 + 7^{0+2} = 51$ and the initial condition states that $g_0 = 51$, the theorem holds for $n = 0$.
  - ➔ We must show that $g_1 = 2^1 + 3^1 + 7^{1+2}$. Since $2^1 + 3^1 + 7^{1+2} = 348$ and the initial condition states that $g_1 = 348$, the theorem holds for $n = 1$.
- Inductive step:
Suppose that for any integer $k \geq 0$, $g_k = 2^k + 3^k + 7^{k+2}$. Then we will show that $g_{k+1} = 2^{k+1} + 3^{k+1} + 7^{k+3}$.
For any integer $k \geq 0$:

$$
\begin{aligned}
g_{k+1} &= 5g_k - 6g_{k-1} + 20 \cdot 7^{k+1} \\
&= 5(2^k + 3^k + 7^{k+2}) - 6(2^{k-1} + 3^{k-1} + 7^{k+1}) + 20 \cdot 7^{k+1} \\
&= 5 \cdot 2^k + 5 \cdot 3^k + 5 \cdot 7^{k+2} - 6 \cdot 2^{k-1} - 6 \cdot 3^{k-1} - 6 \cdot 7^{k+1} + 20 \cdot 7^{k+1} \\
&= (5 \cdot 2^k - 6 \cdot 2^{-1} \cdot 2^k) + (5 \cdot 3^k - 6 \cdot 3^{-1} \cdot 3^k) + (5 \cdot 7 \cdot 7 \cdot 7^k - 6 \cdot 7 \cdot 7^k + \\
&\quad 20 \cdot 7 \cdot 7^k) \\
&= 2^k(5 - 6 \cdot 2^{-1}) + 3^k(5 - 6 \cdot 3^{-1}) + 7^k(5 \cdot 7 \cdot 7 - 6 \cdot 7 + 20 \cdot 7) \\
&= 2^k(5 - 3) + 3^k(5 - 2) + 7^k(343) \\
&= 2 \cdot 2^k + 3 \cdot 3^k + 7 \cdot 7 \cdot 7 \cdot 7^k \\
&= 2^{k+1} + 3^{k+1} + 7^{k+3}
\end{aligned}
$$

Therefore, $g_{k+1} = 5g_k - 6g_{k-1} + 20 \cdot 7^{k+1}$