Homework 6

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Question 7:

A. Exercise 6.1.5

(b)

$$p(E) = \frac{|E|}{|S|}$$

$$|E| = \binom{13}{1} \cdot \binom{4}{3} \cdot \binom{49}{2}$$

$$|S| = \binom{52}{5}$$

$$p(E) = \frac{(13 \cdot 4 \cdot 49 \cdot 24)}{\binom{52}{5}} = \frac{61152}{2598960} = 0.0235$$

(c)

$$p(E) = \frac{|E|}{|S|}$$

$$|E| = \binom{13}{5} \cdot 4$$

$$|S| = \binom{52}{5}$$

$$p(E) = \frac{(1287 \cdot 4)}{\binom{52}{5}} = \frac{5148}{2598960} = 0.00198$$

(d)

$$p(E) = \frac{|E|}{|S|}$$

$$|E| = \binom{13}{1} \cdot \binom{4}{2} \cdot \binom{50}{3}$$

$$|S| = \binom{52}{5}$$

$$p(E) = \frac{(13 \cdot 6 \cdot 19600)}{\binom{52}{5}} = \frac{1528800}{2598960} = 0.5882$$

B. Exercise 6.2.4

(a)

$$E = \text{The hand has at least one club.}$$

$$\overline{E} = \text{The hand has no club.}$$

$$|\overline{E}| = \binom{39}{5}$$

$$|S| = \binom{52}{5}$$

$$p(\overline{E}) = \frac{\binom{39}{5}}{\binom{52}{5}}$$

$$p(E) = 1 - P(\overline{E})$$

$$p(E) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

$$= 1 - \frac{575757}{2598960} = 1 - 0.2215 = 0.7785$$

(b)

$$E = \text{The hand has at least two cards with the same rank.}$$

$$|S| = \binom{52}{5}$$

$$p(E) = 1 - P(0 \text{ cards with same rank}) - P(1 \text{ card with same rank})$$

$$p(E) = 1 - \frac{\binom{13}{0} \cdot \binom{39}{5}}{\binom{52}{5}} - \frac{\binom{13}{1} \cdot \binom{39}{4}}{\binom{52}{5}}$$

$$= 1 - 0.2215 - 0.4114 = 0.3671$$

(c)

$$C = \text{Exactly 1 club}$$

$$S = \text{Exactly 1 spade}$$

$$S' = \text{Sample space}$$

$$C \cup S = \text{The hand has exactly one club or exactly one spade.}$$

$$|C| = \binom{13}{1} \cdot \binom{39}{4}$$

$$|S| = \binom{13}{1} \cdot \binom{39}{4}$$

$$|S'| = \binom{52}{5}$$

$$p(C \cup S) = p(C) + p(S)$$

$$p(C \cup S) = \frac{2 \cdot 13 \cdot \binom{39}{4}}{\binom{52}{5}}$$

$$= 26 \times \frac{82251}{2598960} = 0.8228$$

(d)

$$
\begin{aligned}
C &= \text{Atleast 1 club} \\
S &= \text{Atleast 1 spade} \\
S' &= \text{Sample space} \\
(\overline{C} \cap \overline{S}) &= \text{No clubs and No spades} \\
C \cup S &= \text{Atleast 1 club or Atleast 1 spade} \\
|S'| &= \binom{52}{5} \\
|(\overline{C} \cap \overline{S})| &= \binom{26}{5} \\
p(C \cup S) &= 1 - \frac{\binom{26}{5}}{\binom{52}{5}} \\
&= 1 - \frac{65780}{2598960} = 1 - 0.0253 = 0.9747
\end{aligned}
$$

Question 8:

A. Exercise 6.3.2

(a)

$$p(A) = \frac{6!}{7!} = \frac{1}{7}$$

$$p(B) = \frac{1}{2}$$

$$p(C) = \frac{5!}{7!} = \frac{1}{42}$$

(b)

$$p(A|C) = \frac{p(A \cap C)}{p(C)}$$

$$p(A \cap C) = \frac{12}{7!}$$

$$p(C) = \frac{5!}{7!}$$

$$p(A|C) = \frac{12}{7!} \times \frac{7!}{5!}$$

$$= \frac{12}{5!} = \frac{1}{10}$$

(c)

$$p(B|C) = \frac{p(B \cap C)}{p(C)}$$

$$p(B \cap C) = \frac{5!}{2} \times \frac{1}{7!}$$

$$p(C) = \frac{5!}{7!}$$

$$p(B|C) = \frac{5!}{2} \times \frac{1}{7!} \times \frac{7!}{5!} = \frac{1}{2}$$

(d)

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

$$p(A \cap B) = \frac{6!}{2} \times \frac{1}{7!}$$

$$p(B) = \frac{1}{2}$$

$$p(A|B) = \frac{\frac{6!}{2} \times \frac{1}{7!}}{\frac{1}{2}}$$

$$= \frac{6!}{7!}$$

(e)

$$p(A) = \frac{6!}{7!} = \frac{1}{7}$$

$$p(B) = \frac{1}{2}$$

$$p(C) = \frac{5!}{7!} = \frac{1}{42}$$

$$p(A \cap B) = \frac{6!}{2} \times \frac{1}{7!}$$

$$p(B \cap C) = \frac{5!}{2} \times \frac{1}{7!}$$

$$p(A \cap B) = p(A) \cdot p(B) \therefore \text{A and B are independent.}$$

$$p(B \cap C) = p(B) \cdot p(C) \therefore \text{B and C are independent.}$$

B. Exercise 6.3.6

(b)

$$(\frac{1}{3})^5 \times (\frac{2}{3})^5$$

(c)

$$\frac{1}{3} \times (\frac{2}{3})^9$$

C. Exercise 6.4.2

(a)

$$\text{For fair die, } p(X_1) = p(X_2) = p(X_3) = p(X_4) = p(X_5) = p(X_6) = \frac{1}{6}$$

$$\text{For biased die, } p(X_1) = p(X_2) = p(X_3) = p(X_4) = p(X_5) = 0.15, p(X_6) = 0.25$$

$$p(F) = p(\overline{F}) = \frac{1}{2}$$

$$p(X|F) = \frac{1}{6^6}$$

$$\begin{aligned} p(X|\overline{F}) &= p(X_4) \cdot p(X_3) \cdot p(X_6) \cdot p(X_6) \cdot p(X_5) \cdot p(X_5) \\ &= (0.15) \cdot (0.15) \cdot (0.25) \cdot (0.25) \cdot (0.15) \cdot (0.15) \\ &= (0.15)^4 \cdot (0.25)^2 = 0.00003164 \end{aligned}$$

$$p(F|X) = \frac{p(X|F) \cdot p(F)}{p(X|F) \cdot p(F) + p(X|\overline{F}) \cdot p(\overline{F})}$$

$$\begin{aligned} p(F|X) &= \frac{\frac{1}{6^6} \cdot \frac{1}{2}}{(\frac{1}{6^6} \cdot \frac{1}{2}) + (0.00003164 \cdot \frac{1}{2})} \\ &= 0.4038 \end{aligned}$$

Question 9:

A. Exercise 6.5.2

(a)

$$\{0, 1, 2, 3, 4\}$$

(b)

$$\{(0, \frac{\binom{48}{5}}{\binom{52}{5}}), (1, 4 \cdot \frac{\binom{48}{4}}{\binom{52}{5}}), (2, 6 \cdot \frac{\binom{48}{3}}{\binom{52}{5}}), (3, 4 \cdot \frac{\binom{48}{2}}{\binom{52}{5}}), (4, \frac{48}{\binom{52}{5}}\}$$

B. Exercise 6.6.1

(a)

$$G = 0, 1, 2$$

$$p(G = 0) = \frac{3}{45}$$

$$p(G = 1) = \frac{21}{45}$$

$$p(G = 2) = \frac{21}{45}$$

$$E[G] = (0 \times \frac{3}{45}) + (1 \times \frac{21}{45}) + (2 \times \frac{21}{45})$$

$$E[G] = 1.4$$

C. Exercise 6.6.4

(a)

Probability of any number on a die $= \frac{1}{6}$

$$X = 1, 4, 9, 16, 25, 36$$

$$E[X] = \frac{1}{6} \times (1 + 4 + 9 + 16 + 25 + 36)$$

$$E[X] = 15.17$$

(b)

$$Y = 0, 1, 4, 9$$

$$p(Y=0) = \frac{1}{8},\ p(Y=1) = \frac{3}{8},\ p(Y=4) = \frac{3}{8},\ p(Y=9) = \frac{1}{8}$$

$$E[Y] = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (4 \times \frac{3}{8}) + (9 \times \frac{1}{8})$$

$$E[Y] = 3$$

D. Exercise 6.7.4

(a)

$$C_i = i^{\text{th}} \text{ child who get his or her own coat}$$

$$\text{Total children } (n) = 10$$

$$E[\sum_{i=1}^{10} C_i] = \sum_{i=1}^{10} E[C_i]$$

$$E[C] = E[C_1] + E[C_2] + E[C_3] + \ldots + E[C_{10}]$$

$$E[C] = 10 \times E[C_1]$$

$$E[C_1] = (1 \times \frac{1}{10}) + (0 \times \frac{9}{10}) = \frac{1}{10}$$

$$E[C] = 10 \times \frac{1}{10} = 1$$

Question 10:

A. Exercise 6.8.1

(a)

$$n = 100$$

$$k = 98$$

$$p = 0.99$$

$$q = 0.01$$

$$b(k; n, p) = \binom{n}{k} \cdot p^k \cdot q^{n-k}$$

$$\text{Probability that exactly 2 have defects} = \binom{100}{98} \cdot 0.99^{98} \cdot 0.01^2 = 0.18486$$

(b)

Probability that at least 2 have defects = 1 − Probability of 0 defects − Probability of 1 defect

$$\begin{aligned} b(k; n, p) &= \binom{n}{k} \cdot p^k \cdot q^{n-k} \\ &= 1 - ((\binom{100}{100} \times (0.99)^{100}) + (\binom{100}{99} \times (0.99)^{99} \times (0.01)^1)) \\ &= 1 - 0.736 \end{aligned}$$

Probability that at least 2 have defects = 0.264

(c)

$$\begin{aligned} C &= \text{circuit boards with defects out of the 100 made} \\ E[C] &= 100 \times 0.01 \\ &= 1 \end{aligned}$$

(d)

$$\text{Probability that a batch has a defect} = 0.01$$

$$\text{Probability of at least 2 batches with defects} = 1 - \text{Probability of at most 1 batch with defect}$$

$$\text{Probability of at most 1 batch with defect} = (\binom{50}{0} \cdot (0.01)^0 \cdot (0.99)^{50}) + (\binom{50}{1} \cdot (0.01)^1 \cdot (0.99)^{49})$$

$$= 0.605 + 0.305 = 0.91$$

$$\text{Probability of at least 2 batches with defects} = 1 - 0.91 = 0.089$$

$$E[X] = 50 \times 0.01 = 0.5$$

Expected number of circuit boards with defects when made in batches is half the number of circuit boards with defects than made individually. Therefore the probability of atleast two defects is reduced while done in batches.

B. Exercise 6.8.3

(b) The incorrect conclusion is reached if there are atleast four heads.

$$P(X >= 4) = 1 - P(X < 4)$$

$$= 1 - (0.7^{10} + (10 \times 0.3 \times 0.7^9) + (\binom{10}{2} \times 0.3^2 \times 0.7^8) + (\binom{10}{3} \times 0.3^3 \times 0.7^7))$$

$$= 1 - (0.0824 \times 7.888)$$

$$\approx 0.35$$
