Homework 5

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Question 7:

A. Exercise 8.2.2

(b) Let $f(n) = n^3 + 3n^2 + 4$ and $g(n) = n^3$. Choose $n_0 = 1$ and assume $n > 1$, then

$$\frac{f(n)}{g(n)} = \frac{n^3+3n^2+4}{n^3} < \frac{n^3+3n^3+4n^3}{n^3} = 8$$

Therefore choose $c = 8$. Also, $3n^2 < 3n^3$ and $4 < 4n^3$

Therefore, when $c = 8$, $n_0 = 1$ and $n > 1$, $f(n) = n^3 + 3n^2 + 4 <= 8n^3$

Therefore, $f = \theta(n^3)$

B.
We have to show that $c1g(n) <= \sqrt{7n^2 + 2n - 8} <= c2g(n)$
Now, $c1n <= \sqrt{7n^2 + 2n - 8}$, take c1 = 1

for n = 1, 1 $<= \sqrt{7 + 2 - 8}$

for n = 2, 2 $<= \sqrt{28 + 4 - 8}$

thus $c1n <= \sqrt{7n^2 + 2n - 8}$ for c1 = 1 Now, $\sqrt{7n^2 + 2n - 8} <= c2n$ take c2 = 4

for n = 1, $\sqrt{7 + 2 - 8} <= 4$

for n = 2, $\sqrt{28 + 4 - 8} <= 8$

thus $\sqrt{7n^2 + 2n - 8} <= c2n$ for c2 = 4

Therefore, $c1n <= \sqrt{7n^2 + 2n - 8} <= c2n$ for c1 = 1, c2 = 4, n ¿=1

Therefore, from def, $\sqrt{7n^2 + 2n - 8} = \theta(n)$

C. Exercise 8.3.5

(a) The algorithm sorts all the numbers in the sequence less than p on one side and the numbers greater than p on other side.

(b) Yes, the answer depends on the actual values of the numbers. Counters increase on the basis of the value and p.

Maximum number of times i gets incremented is when all the integers are negative in the sequence and p = 0. It will iterate till the end.

Minimum = 0, when all the numbers are positive and p = 0.

The effect will be vice versa for j.

(c) Yes, the answer depends on the numbers, because if the list is of all negative integers and p = 0, then i will increase till the value of n and when j = n, swap will not be executed.

If the list is all positive integers, then j will decrease to i. No swap will be executed.

Swap will be executed only when there is a balance between the numbers in sequence.

count = count of numbers less than p towards the end of sequence.

If only 2 negative integers are there at the end of sequence, then count is 2 which will be minimum.

Swap will be maximum for $\frac{n}{2}$ numbers.

(d) The Time Complexity for the Inner while loop will always be n. The counters will be executed n times (i + j = n + 1). The loop will be executed till i < j. Hence, Time Complexity is n times.

Lower bound = O(n)

(e) The matching upper bound for the time complexity of the MysteryAlgorithm is given as O(n)

Question 8:

A. Exercise 5.1.1

(b) Let D be the set of digits, L the set of letters, and S the set of special characters. The three sets are mutually disjoint, so the total number of characters is

$$|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$$

Strings can be of length 7 or 8 or 9.

When length = 7, each of the 7 characters in the string can be any of the 40 characters. So a total of 40^7 strings of length 7.

Similarly, a total of 40^8 strings of length 8.

Similarly, a total of 40^9 strings of length 9.

$$\text{Therefore, Total strings of lengths 7 or 8 or 9} = 40^7 + 40^8 + 40^9$$

(c) Let D be the set of digits, L the set of letters, and S the set of special characters. The three sets are mutually disjoint, so the total number of characters is

$$|D \cup L \cup S| = |D| + |L| + |S| = 10 + 26 + 4 = 40$$

Strings can be of length 7 or 8 or 9.

But the strings cannot have their first character as a letter. Therefore 1st character can be any of digits or special characters.

$$|D| + |S| = 10 + 4 = 14$$

When length = 7, 1st character has 14 options and each of the remaining 6 characters in the string can be any of the 40 characters. So total 14×40^6 strings of length 7.

Similarly, a total of 14×40^7 strings of length 8.

Similarly, a total of 14×40^8 strings of length 9.

Therefore, Total strings of lengths 7 or 8 or 9 = $(14 \times 40^6) + (14 \times 40^7) + (14 \times 40^8)$

B. Exercise 5.3.2

(a) We have 3 letters a,b,c. First letter can be filled in 3 ways.

The remaining 9 letters can be filled only in 2 ways since we need to exclude the letter at current position each time to avoid repetition.

$$\text{Therefore, Total strings} = 3 \times 2^9$$

C. Exercise 5.3.3

(b) There are 10 ways to fill in first digit location and 9 ways and 8 ways for second and third digit location because no digit appears more than once. There are 26 ways to fill in each of the letter locations. Therefore, using product rule, total ways to fill in all 7 locations will be,

$$10 \times 9 \times 8 \times 26^4$$

(c) There are 10 ways to fill in first digit location and 9 ways and 8 ways for second and third digit location because no digit appears more than once. There are 26 ways to fill in first letter location and 25 ways and 24 ways and 23 ways for second and third and fourth letter locations because no letter appears more than once. Therefore, using product rule, total ways to fill in all 7 locations will be,

$$10 \times 9 \times 8 \times 26 \times 25 \times 24 \times 23$$

D. Exercise 5.2.3

(a) The bijection f(x) is,

$$f(x) = \begin{cases} x + 0 & \text{if } x \in E(9) \\ x + 1 & \text{if } x \notin E(9) \end{cases}$$

Thus the function concatenates 0 to a 9 bit string if it has even number of 1s and a 1 if it does not.

Bijection Proof:
For a function to be bijective, it should be both one-to-one and onto.

Let $f(a) = f(b) = c$

There are two cases:

(i) The last bit of c is 1:

$$\text{a} + 1 = \text{b} + 1 = \text{c}$$
$$\text{a} = \text{b}$$

(ii) The last bit of c is 0:

$$\text{a} + 0 = \text{b} + 0 = \text{d}$$
$$\text{a} = \text{b}$$

Since f(a) = f(b) gives a = b, f is one-one.

Let $d \in E(10)$ then there are two cases:

(i) The last bit is 1. Let e be the string with the last bit removed.

$$\text{d} = \text{e} + 1$$

But since d has even 1s, e must have odd 1s

$$\text{d} = \text{f(e)}$$

(ii) The last bit is 0. Let e be the string with the last bit removed.

$$\text{d} = \text{e} + 0$$

But since d has even 1s, e must have even 1s

$$\text{d} = \text{f(e)}$$

Thus every image in E_{10} has a pre-image in B^9, hence f is onto.

Therefore f is a bijection.

(b) Every string has 10 bits in E_{10}. Each bit can have 2 characters (1 or 0). Therefore, total number of strings possible = 2^{10}. The number of strings with even 1s = the number of strings with odd 1s.

Therefore $|E^{10}| = \frac{2^{10}}{2} = 2^9$

Question 9:

A. Exercise 5.4.2

(a) Total no of digits = 7. The first 3 digits are always 824 or 825, so they can be chosen in 2 ways. Last 4 digits can be chosen in $10 \times 10 \times 10 \times 10$ ways. Therefore, total different phone numbers possible are,

$$2 \times 10^4$$

(b) Total no of digits = 7. The first 3 digits are always 824 or 825, so they can be chosen in 2 ways. Last 4 digits can be chosen in $10 \times 9 \times 8 \times 7$ ways since the digits cannot be repeated. Therefore, total different phone numbers possible are,

$$2 \times 10 \times 9 \times 8 \times 7$$

B. Exercise 5.5.3

(a) No restrictions = 2^{10}

(b) The string starts with 001 = 2^7

(c) The string starts with 001 or 10 = $2^7 + 2^8$

(d) The first two bits are the same as the last two bits = 2^8

(e) The string has exactly six 0's = $\binom{10}{6}$

(f) The string has exactly six 0's and the first bit is 1 = $\binom{9}{6}$

(g) There is exactly one 1 in 1st half and exactly three 1's in 2nd half. = $\binom{5}{1} \cdot \binom{5}{3}$

C. Exercise 5.5.5

(a) There are $\binom{30}{10}$ ways to select a subset of 10 boys from a set of 30 boys. There are $\binom{35}{10}$ ways to select a subset of 10 girls from a set of 35 girls. Since the choir director must select the girls and the boys for the chorus, the product rule applies, and there are a total of $\binom{30}{10} \cdot \binom{30}{10}$ ways to make the selection.

D. Exercise 5.5.8

(c) There are 26 hearts and 26 diamonds in the deck. So the total number of ways to select five card hands that are made entirely of hearts and diamonds will be

$$= \binom{26}{5}$$

(d) There are 13 ranks in total and each rank have 4 cards. So there are 13 ways to select four cards of same rank. Also, Total number of cards are 52. So the remaining 5th card can be anything from the 52 other than the chosen 4. So the fifth card can be any of the 48 (52 - 4) cards. Therefore total number of ways in which five-card hands can have four cards of the same rank will be,

$$= 13 \times 48$$

(e) Number of ways to select two cards of same rank is 13.

Number of ways to select three cards of same rank and the chosen rank is not the same as previous chosen rank is 12.

Two cards of same rank can be chosen $\binom{4}{2}$ ways.

Three cards of same rank can be chosen $\binom{4}{3}$ ways.

Therefore total number of ways five-card hands can contain a full house will be,

$$= 13 \times 12 \times \binom{4}{2} \times \binom{4}{3}$$

(f) The number of ways to choose 5 different ranks is $\binom{13}{5}$. For each rank, there are 4 different cards available to choose from the deck. Therefore the total number of five-card hands that do not have any two cards of the same rank will be,

$$\binom{13}{5} \times 4^5$$

E. Exercise 5.6.6

(a) Senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Rupudiators. To select a committee of 10 senate members with the same number of Demonstrators and Repudiators, we select 5 from 44 Demonstrators ($\binom{44}{5}$ ways to do that) and 5 from 56 Rupudiators ($\binom{56}{5}$ ways to do that). So the total ways to form the committee will be

$$\binom{44}{5} \cdot \binom{56}{5}$$

(b) Each party must select a speaker and a vice speaker. From Rupudiators, We have 56 options to select speaker and 55 options to select vice speaker (1 already selected for speaker. Also, from Demonstrators, we have 44 options to select speaker and 43 options to select vice speaker (1 already selected for speaker. So the total number of ways for the two speakers and two vice speakers to be selected will be

$$\binom{56}{1} \cdot \binom{55}{1} \cdot \binom{44}{1} \cdot \binom{43}{1}$$

Question 10:

A. Exercise 5.7.2

(a) There are 52 total cards in the deck. There are $\binom{52}{5}$ 5-card hands possible from the total cards. Counting by complement, there are $\binom{39}{5}$ possible cards without club. Therefore, total number of 5-card hands with atleast one club will be,

$$\binom{52}{5} - \binom{39}{5}$$

(b) Counting by complement, we'll find 5-card hands which have no 2 cards of same rank.

The number of ways to choose 5 different ranks is $\binom{13}{5}$. For each rank, there are 4 different cards available to choose from the deck. Therefore the total number of five-card hands that do not have any two cards of the same rank are, $\binom{13}{5} \times 4^5$

Therefore total number of 5-card hands that have at least two cards with the same rank will be

$$\binom{52}{5} - \binom{13}{5} \times 4^5$$

B. Exercise 5.8.4

(a) 20 different comic books are distributed to kids and each kid can get any number of books.

So 1st book can be given to any number of kids (1 to 5 kids) which can be done in 5 ways. Similary 2nd book in 5 ways, 3rd book in 5 ways upto 20th book. Therefore, total ways to distribute the comic books if there are no restrictions on how many go to each kid will be,

$$5^{20}$$

(b) 20 books are divided evenly among kids, so each kids get 4 books. The number of ways to do that will be,

$$\frac{20!}{4!4!4!4!4!}$$

Question 11:

(a) 0

Since the domain has 5 elements which is one more than the target which has only 4 elements, it is not possible to have even a single one-to-one function because there will be one element left unmapped in the domain if all other elements in the domain gets mapped to an element in the target.

(b) Domain and target have equal number of elements which is equal to 5. Each element from the domain gets mapped to one element in the target for it to be one-to-one. So,

1st element = 5 ways to map

2nd element = 4 ways to map

3rd element = 3 ways to map

4th element = 2 ways to map

5th element = 1 ways to map

So, in total, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ one-to-one functions are possible.

(c) Domain has 5 elements and target has 6 elements. Each element from the domain gets mapped to one element in the target for it to be one-to-one. So,

1st element = 6 ways to map

2nd element = 5 ways to map

3rd element = 4 ways to map

4th element = 3 ways to map

5th element = 2 ways to map

So, in total, there are $6 \times 5 \times 4 \times 3 \times 2 = 720$ one-to-one functions are possible.

(d) Domain has 5 elements and target has 7 elements. Each element from the domain gets mapped to one element in the target for it to be one-to-one. So,

1st element = 7 ways to map

2nd element = 6 ways to map

3rd element = 5 ways to map

4th element = 4 ways to map

5th element = 3 ways to map

So, in total, there are $7 \times 6 \times 5 \times 4 \times 3 = 2520$ one-to-one functions are possible.
