Homework 8

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Question 5:

a. Use mathematical induction to prove that for any positive integer n, 3 divide $n^3 + 2n$ (leaving no remainder).

Proof. Proof by induction on n.

Base case: $n = 1$

$$P(1) = 1^3 + 2 = 3$$

3 evenly divides 3

Therefore $P(1)$ is true

Inductive step:

For any $k >= 1$, suppose that, 3 evenly divides $k^3 + 2k$, we need to prove that, 3 evenly divides $(k+1)^3 + 2(k+1)$

Since 3 evenly divides $k^3 + 2k$,

$$k^3 + 2k = 3 \cdot m \qquad \text{for some integer m}$$

$$\begin{aligned} P(k+1) &= (k+1)^3 + 2(k+1) \\ &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3k^2 + 3k + 3 \\ &= 3m + 3k^2 + 3k + 3 \qquad \text{By Inductive Hypothesis.} \\ &= 3(m + k^2 + k + 1) \end{aligned}$$

Since m and k are integers, $m + k^2 + k + 1$ is also an integer

Therefore 3 evenly divides $(k+1)^3 + 2(k+1)$

Therefore $P(k+1)$ is true

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b. Use strong induction to prove that any positive integer $n(n >= 2)$ can be written as a product of primes.

Proof. Proof by strong induction on n.

Base case: $n = 2$. Since 2 is a prime number, it already is a product of one prime number: 2.

Inductive step:

Assume that for $k \geq 2$, any integer j in the range from 2 through k can be expressed as a product of prime numbers. We will show that $k+1$ can be expressed as a product of prime numbers.

If $k+1$ is prime, then it is a product of one prime number, $k+1$. If $k+1$ is not prime, $k+1$ is composite and can be expressed as the product of two integers, a and b, that are each at least 2. We need to show that both a and b are at most k in order to apply in the inductive hypothesis.

Since $k+1 = a \cdot b$, then $a = \frac{(k+1)}{b}$. Furthermore, since $b \geq 2$, then $a = \frac{(k+1)}{b} < k+1$. If a is an integer which is strictly less than $k+1$, then $a \leq k$. The symmetric argument can be used to show that $b = \frac{k+1}{a} \leq k$. Thus a and b both fall in the range from 2 through k which means that the inductive hypothesis can then be applied and they can each be expressed as a product of primes:

$$a = p_1 \cdot p_2 \ldots p_l$$

$$b = q_1 \cdot q_2 \ldots q_m$$

Now k + 1 can be expressed as a product of primes:

$$k + 1 = a \cdot b = (p_1 \cdot p_2 \ldots p_l) \cdot (q_1 \cdot q_2 \ldots q_m)$$

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Question 6:

A. Exercise 7.4.1

(a) Verify that $P(3)$ is true.

Solution:

$$P(n) = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

when n = 3,

$$\begin{aligned}P(3) = \sum_{j=1}^{3} j^2 &= 1^2 + 2^2 + 3^2 \\ &= 1 + 4 + 9 \\ &= 14\end{aligned}$$

$$\begin{aligned}\frac{n(n+1)(2n+1)}{6} &= \frac{3(3+1)((2 \times 3)+1)}{6} \\ &= \frac{3 \times 4 \times 7}{6} \\ &= \frac{3 \times 4 \times 7}{6} \\ &= 14\end{aligned}$$

Therefore $P(3)$ is true

(b) Express $P(k)$

Solution:

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(c) Express $P(k+1)$.

Solution:

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

Solution:

In the base case we need to prove that when $n = 1, P(1)$ is true

$$P(1) = \sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{1(1+1)((2+1)}{6} \\ &= \frac{2 \times 3}{6} \\ &= 1 \end{aligned}$$

Therefore $P(1)$ is true

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

Solution:

For any $k >= 1$, suppose that,

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

we need to prove that,

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6} \text{ is true.}$$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

Solution:

Inductive Hypothesis is the supposition that for any $k >= 1$,

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. Proof by induction on n.

Base case: $n = 1$

$$P(1) = \sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= \frac{1(1+1)((2+1)}{6} \\ &= \frac{2 \times 3}{6} \\ &= 1 \end{aligned}$$

Therefore $P(1)$ is true

Inductive step:

For any $k >= 1$, suppose that,

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

we need to prove that,

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6} \text{ is true.}$$

Starting with the left side of the inequality to be proven:

$$
\begin{aligned}
P(k+1) = \sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 && \text{By separating the last term.} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{By Inductive Hypothesis.} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\,[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)\,[2k^2 + 7k + 6)]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{By finding the factors.}
\end{aligned}
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Therefore $P(k+1)$ is true ■

B. Exercise 7.4.3

(c) Prove that for $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof. Proof by induction on n.

Base case: $n = 1$

$$P(1) = \sum_{j=1}^{1} \frac{1}{j^2} = \frac{1}{1^2} = 1$$

$$2 - \frac{1}{n} = 2 - \frac{1}{1} = 1$$

$$\sum_{j=1}^{1} \frac{1}{j^2} \leq 2 - \frac{1}{1}$$

Therefore $P(1)$ is true

Inductive step:

For any $k >= 1$, suppose that,

$$P(k) = \sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$$

we need to prove that,

$$P(k+1) = \sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1} \text{ is true.}$$

Starting with the left side of the inequality to be proven,

$$\begin{aligned}
P(k+1) = \sum_{j=1}^{k+1} \frac{1}{j^2} &= \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} && \text{By separating the last term.} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{By Inductive Hypothesis.} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} && \text{By Inductive Hypothesis.} \\
&\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} && (\frac{1}{k+1} \leq \frac{1}{k} \text{ since } k \geq 1) \\
&= 2 - \frac{1-k-1}{k(k+1)} \\
&= 2 - \frac{k}{k(k+1)} \\
&= 2 - \frac{1}{k+1}
\end{aligned}$$

Therefore $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$ is true

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C. Exercise 7.5.1

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$

Proof. Proof by induction on n.

Base case: $n = 1$

$$P(1) = 3^{2 \cdot 1} - 1 = 9 - 1 = 8$$

$$\text{4 evenly divides 8}$$

$$\text{Therefore } P(1) \text{ is true}$$

Inductive step:

For any $k >= 1$, suppose that, 4 evenly divides $3^{2k} - 1$, we need to prove that, 4 evenly divides $3^{2(k+1)} - 1$ is true.

Since 4 evenly divides $3^{2k} - 1$,

$$3^{2k} - 1 = 4 \cdot m \qquad \text{for some integer } m$$

$$3^{2k} = 4m + 1$$

Starting with the left side of the inequality to be proven:

$$\begin{aligned} P(k+1) &= 3^{2k+2} - 1 \\ &= 9 \cdot 3^{2k} - 1 \\ &= 9 \cdot (4m + 1) - 1 && \text{By Inductive Hypothesis.} \\ &= 9 \cdot 4m + 9 - 1 \\ &= (9 \cdot 4m) + 8 \\ &= 4 \cdot (9m + 2) \end{aligned}$$

Since m is an integer, $9m + 2$ is also an integer

Therefore 4 evenly divides $3^{2(k+1)} - 1$

Therefore $P(k + 1)$ is true

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