Parametric Surface fit to Mesh

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1 Triangulated Mesh Data

Suppose we have a triangulated mesh in three-dimensional space:

$$f_i = (j_i,\ k_i,\ \ell_i) \tag{1.1}$$

$$v_p = (x_p,\ y_p,\ z_p) \tag{1.2}$$

with $i = 1 \cdots N$, N the number of triangle faces in the mesh, and $p = 1 \cdots M$, M the number of vertices in the data[1]. The i-th face consists of vertices $(v_j, v_k, v_\ell)_i$.

2 The Question: Fit a parameterized differentiable surface to the data?

Recall that differential geometry for surfaces in 3D is phrased as a surface taking a differentiable function $\mathbf{x}$ mapping an open set to an open set: $\mathbf{x} : U \subset \mathbb{R}^2 \to \Omega \subset \mathbb{R}^3$.

Then one looks at $\mathbf{x}(u^1, u^2) = \big(x(u^1, u^2), y(u^1, u^2), z(u^1, u^2)\big)$ and derivatives $\partial x / \partial u^i$, and so on.

Notice that getting to this *nice* description from Equation 1.1-Equation 1.2 is not straightforward. How do you make such a parameterization, that is differentiable, given only a bunch of vertices and faces defining connectivity?

Task: determine how to form a differentiable parametric surface description $\mathbf{x}(u^1, u^2)$ given data of the form in Equation 1.1-Equation 1.2. It might be defined to exactly fit the vertices, or to be 'close' if smoothing is desired.

3 Purpose

To step back for a moment, let us think about why having a parameterized surface would be nice. If such a thing existed, all of the surface derivatives and geometrical quantities could be written down in closed form, rather than as a numerical fit.

[1] all M vertices are not necessarily used by the faces, and there could be duplicates

Furthermore, a problem that keeps coming up in the current mesh contour segmentation work is local minima of geometrical quantities. For example, there can arise a patch of high curvature due to a jagged piece of mesh, thus throwing off a contour evolution. It would be great if somehow a 'regularization' factor could be included in the construction of a parametric surface, thus keeping only large-scale geometric features.

Lastly is data compression- if we only need a relatively small number of basis functions with which to do large-scale geometric computations, great.

4 Observations and Thoughts

This should be rather easy if the surface is assumed to be either the graph of a function $\mathbf{x}(u^1, u^2) = (u^1, u^2, f(u^1, u^2))$.

It is just slightly harder if it is a hypersurface of the form $\mathbf{x}(u^1, u^2) = g^{-1}(0)$ where there is some function $g : \mathbb{R}^3 \to \mathbb{R}$ and the surface is g's zero level set. Example: $g(x, y, z) = x^2 + y^2 + z^2 - 1$ in which case $\mathbf{x}$ is the unit sphere.

Unfortuneately, given observations of how segmentation methods in medical imaging create data, a practical / useable approach has to allow for surfaces that fit neither of these cases. For example, something topologically similar to a torus cut in half.

Some methods to consider:

- least-squares fitting of basis functions to the given vertices. A basis that makes sense has to be chosen, if it is to work for general meshes
- wavelet-shrinkage : this relates to the first item, it seems to be similar and there's some experience with this in the lab
- calculus of variations: we are trying to determine an unknown differentiable function with some constraints (fitting the data), a variational method might make sense

*****Warning***** I have not yet done a sufficient amount of background reading on this question; it may be already solved. In fact it is definately solved for the *graph of a function* case (interpolation!), and probably somewhere for the *hypersurface* case. The more general case would be harder to find, but there has to be a period of 4-8 hours of digging through papers before one concludes that it is or is not solved already.