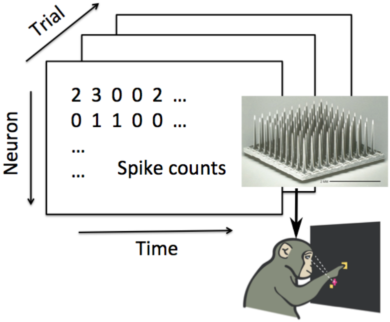
Statistical Machine Learning Methods for High-dimensional Neural Population Data Analysis

Yuanjun Gao

Department of Statistics
Columbia University

Overview



- Neuroscience + Big data = Opportunities!

Overview

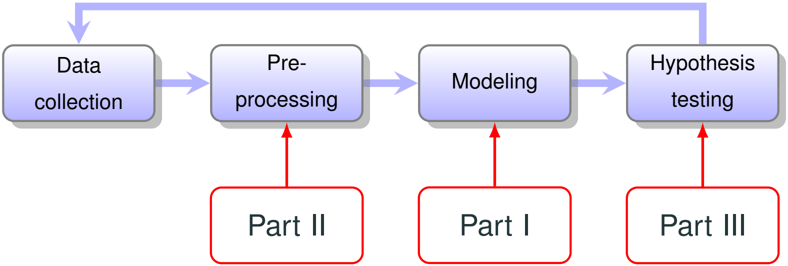
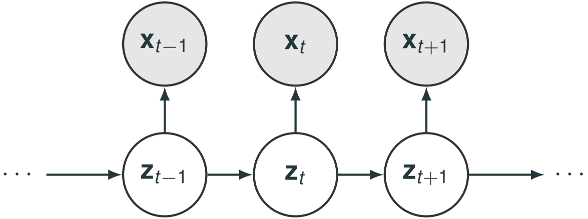

Data collection
Pre-processing
Modeling
Hypothesis testing
Part II
Part I
Part III

Table of Contents

Table of Contents

State space models

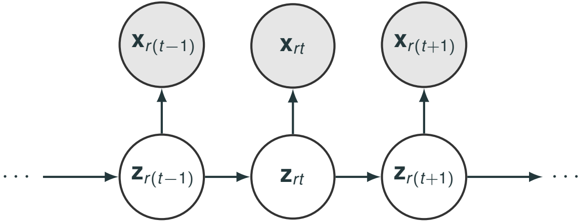


- $\mathbf{x}_t \in \mathbb{N}^n$: spike counts; $\mathbf{z}_t \in \mathbb{R}^m$: latent variables
- Joint distribution

$$p(\mathbf{x}, \mathbf{z}) = \underbrace{p(\mathbf{z}_1)}_{\text{Initial distribution}} \underbrace{\prod_{t=1}^{T-1} p(\mathbf{z}_{t+1}|\mathbf{z}_t)}_{\text{Transition model}} \underbrace{\prod_{t=1}^{T} p(\mathbf{x}_t|\mathbf{z}_t)}_{\text{Observation model}}$$

- Common input; Dynamical view of motor data (TODO: elaborate this line)

State space models: multiple trials



- $r = 1, ..., R$: trial number
- $\mathbf{x}_{rt} \in \mathbb{N}^n$: spike counts; $\mathbf{z}_{rt} \in \mathbb{R}^m$: latent variables
- Joint distribution

$$p(\mathbf{x}, \mathbf{z}) = \prod_{r=1}^{R} \left[\underbrace{p(\mathbf{z}_{r1})}_{\text{Initial distribution}} \underbrace{\prod_{t=1}^{T-1} p(\mathbf{z}_{r(t+1)} | \mathbf{z}_{rt})}_{\text{Transition model}} \underbrace{\prod_{t=1}^{T} p(\mathbf{x}_{rt} | \mathbf{z}_{rt})}_{\text{Observation model}} \right]$$

Common parameterization and our extensions

- Common assumptions for latent dynamics: linear Gaussian dynamical system (LDS)

$$\mathbf{z}_1 \sim \mathcal{N}(\mu_1, Q_1)$$
$$\mathbf{z}_{t+1} | \mathbf{z}_t \sim \mathcal{N}(A\mathbf{z}_t, Q)$$

- Common observation models:

$$\mathbf{x}_t | \mathbf{z}_t \sim \underbrace{\underbrace{\mathcal{N}(C\mathbf{z}_t + d, \Sigma)}_{\text{model mismatch}} \text{ or } \underbrace{\text{Poisson}\,(\exp(C\mathbf{z}_t + d))}_{\text{equal dispersion}}}_{\text{stringent assumptions}}$$

- Our extensions for observation model:
  - Generalized count distribution (GCLDS) (Gao et al. 2015)
  - Flexible nonlinear observation (fLDS) (Gao et al. 2016)

Table of Contents

Motivation

- Doubly stochastic Poisson model implies overdispersion

$$\left.\begin{aligned} \mathbf{z} &\sim p(\mathbf{z}) \\ \mathbf{x} &\sim \text{Poisson}(f(\mathbf{z})) \end{aligned}\right\} \Rightarrow \text{var}(\mathbf{x}) \geq E(\mathbf{x})$$

- Need a more flexible distribution to separate firing rate variability with noise variability.

$$\text{var}(\mathbf{x}) = \underbrace{\text{var}\left(E(\mathbf{x}|\mathbf{z})\right)}_{\text{firing rate variability}} + \underbrace{E\left(\text{var}(\mathbf{x}|\mathbf{z})\right)}_{\text{noise variability}}$$

Generalized count distribution family

- Generalized count (GC) distribution family

$$p_{\text{Poisson}}(x;\lambda) \propto \frac{\exp\{\log\lambda \cdot x\}}{x!}, \quad x \in \mathbb{N}$$

$$\Downarrow$$

$$p_{\mathcal{GC}}(x;\theta, g(\cdot)) \propto \frac{\exp(\theta \cdot x + g(x))}{x!}, \quad x \in \mathbb{N}$$

  where $\theta \in \mathbb{R}$, $g(\cdot) : \mathbb{N} \to \mathbb{R}$.
- Parameterizes all the count distributions redundantly.
- Given $g(\cdot)$, θ controls the expectation.
- $g(\cdot)$ controls the “shape” of the distribution.
  Convex/concave/linear $g(\cdot)$ implies overdispersed/underdispered/Poisson distribution.

Model formulation

- Linear dynamical systems with generalized count observation

$$\begin{aligned} \mathbf{z}_{r1} &\sim \mathcal{N}(\mu_1, Q_1) \\ \mathbf{z}_{r(t+1)}|\mathbf{z}_{rt} &\sim \mathcal{N}(A\mathbf{z}_{rt}, Q) \\ x_{rti} &\sim \mathcal{GC}(c_i^T\mathbf{z}_{rt}, g_i(\cdot)), i = 1, ..., n \end{aligned}$$

- Practical considerations
  - Set $g_i(k) = -\infty$ for $k > K$ to facilitate computation;
  - Ridge penalty on the 2^{nd} difference of $g_i(\cdot)$ to avoid overfitting;
  - Set $g_i(0) = 0$ without loss of generality.

Variational Bayes Expectation Maximization (VBEM)

- $\mathbf{x}$: data, $\mathbf{z}$: latent variables, θ: model parameters,
- Often hard to compute $p_\theta(\mathbf{x}) = \int p_\theta(\mathbf{x}, \mathbf{z}) d\mathbf{z}$ and $p_\theta(\mathbf{z}|\mathbf{x})$.
- Approximate the posterior by a tractable distribution family.

$$p_\theta(\mathbf{z}|\mathbf{x}) \approx q(\mathbf{z}) \in \mathcal{Q}$$

- Optimize a lower bound of log likelihood, or ELBO

$$\begin{aligned} \mathrm{ELBO}(\theta, q) &= \int \left[\log p_\theta(\mathbf{x}, \mathbf{z}) - \log q(\mathbf{z})\right] q(\mathbf{z}) d\mathbf{z} \\ &= \log p_\theta(\mathbf{x}) - \mathrm{KL}(q(\mathbf{z}) || p_\theta(\mathbf{z}|\mathbf{x})) \leq \log p_\theta(\mathbf{x}) \end{aligned}$$

Variational Bayes Expectation Maximization (VBEM)

- VBEM: Optimize ELBO$(\theta, q) \leq \log p_\theta(\mathbf{x})$ iteratively
  - E-step: For a fixed θ, optimize q
  - M-step: For a fixed q, optimize θ
- VBEM for GCLDS
  - We set q to be multivariate Gaussian
  - We derive a looser but tractable ELBO
  - E-step: fast Laplace approximation initialization + dual optimization
  - M-step: convex optimization + analytical solution

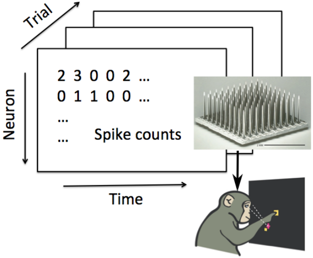
Experiments

- For both simulated and real dataset, we compare GCLDS with PLDS (Poisson observation model)

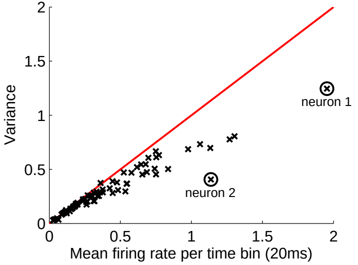
	Mean	Variance	Likelihood
PLDS	✓	✗	✗
GCLDS	✓	✓	✓

Real data analysis: data

Data



Variance and mean of spike counts

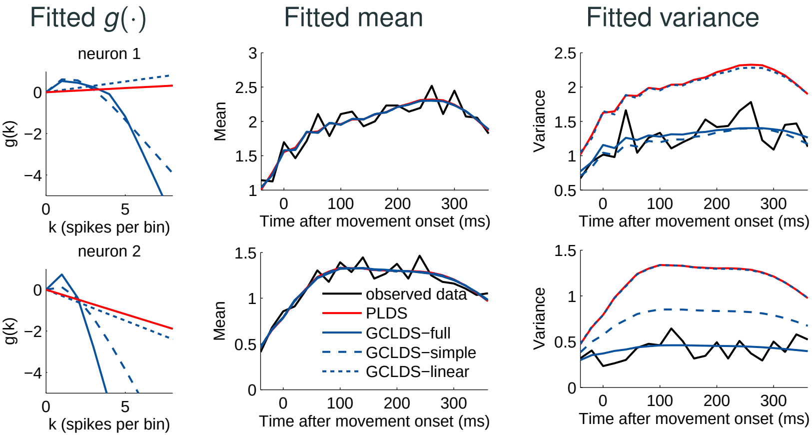


- Center-out reaching experiments
- Multi-electrode array recording
- Strong under-dispersion

Real data analysis: algorithms

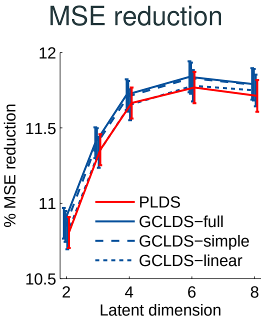
- Main algorithms to be compared
  - PLDS: Poisson observation
  - GCLDS-full: Generalized count observation, individual $g(\cdot)$ across neurons
- Two control cases for GCLDS
  - GCLDS-linear: truncated linear $g(\cdot)$ (truncated Poisson)
  - GCLDS-simple: $g(\cdot)$ shared across neurons (up to a linear function)

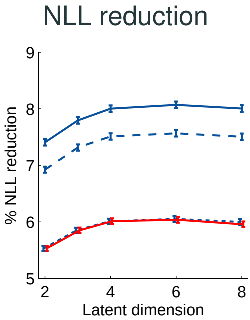
Real data analysis: single neuron fit



Real data analysis: population fit

- Leave-one-neuron-out prediction





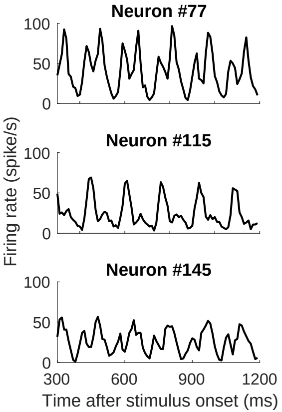
Conclusion and discussion

- Summary
  - Incorporated generalized count family into state space models.
  - Developed VBEM algorithm.
  - Observed superior fitted result on real neural data.
- Extensions
  - $g(\cdot)$ vary across time?
  - Share information of $g(\cdot)$ across neurons? (hierarchical model?)
  - Generative models for under-dispersion?
- Gao Y, Buesing L, Shenoy KV, Cunningham JP (2015) High-dimensional neural spike train analysis with generalized count linear dynamical systems. NIPS 2015.

Table of Contents

Motivation

- Neural activities lie in a low-dimensional nonlinear manifold rather than a linear subspace
- Flexible observation model makes the state space model more expressive



Model formulation: fLDS

- Linear dynamical systems with nonlinear link and count observation

$$
\begin{aligned}
\mathbf{z}_{r1} &\sim \mathcal{N}(\mu_1, Q_1) \\
\mathbf{z}_{r(t+1)} | \mathbf{z}_{rt} &\sim \mathcal{N}(A\mathbf{z}_{rt}, Q) \\
x_{rti} &\sim \text{Poisson}(f_i(\mathbf{z}_{rt})) \text{ (PfLDS)} \\
&\quad \text{or } \mathcal{GC}(f_i(\mathbf{z}_{rt}), g_i(\cdot)) \text{ (GCfLDS)}
\end{aligned}
$$

  where f_i is a nonlinear function parameterized by a neural network

- Linear dynamics: simple, tractable, interpretable
- Nonlinear observation: flexibility

Inference algorithm: AEVB (high level idea)

- Auto-encoding Variational Bayes (AEVB)
- Learn a mapping (recognition model) from data to the approximate posterior distribution of latent variable.
- Jointly optimize the generative model parameters and recognition model parameters.
- Naturally incorporate stochastic optimization to handle large datasets.

Inference algorithm: AEVB (algorithm)

- Decompose ELBO by trials

$$\mathrm{ELBO}(\theta, q) = \sum_{r=1}^{R} \int \left[\log p_\theta(\mathbf{x}_r, \mathbf{z}_r) - \log q(\mathbf{z}_r)\right] q(\mathbf{z}_r) d\mathbf{z}_r$$

- Map data $\mathbf{x}_r$ to $q(\mathbf{z}_r)$ by a parameterized function

$$q(\mathbf{z}_r) = q_\phi(\mathbf{z}_r; \mathbf{x}_r) = \mathcal{N}\left(\mu_\phi(\mathbf{x}_r), \Sigma_\phi(\mathbf{x}_r)\right)$$

- Learn both θ and ϕ by optimizing ELBO

$$\mathrm{ELBO}(\theta, \phi) = \sum_{r=1}^{R} \int \left[\log p_\theta(\mathbf{x}_r, \mathbf{z}_r) - \log q_\phi(\mathbf{z}_r; \mathbf{x}_r)\right] q_\phi(\mathbf{z}_r; \mathbf{x}_r) d\mathbf{z}_r$$

- Do stochastic optimization with gradient of a single trial

Inference algorithm: AEVB (important details)

- Specific parameterization of the recognition model

$$q(\mathbf{z}_r) = q_\phi(\mathbf{z}_r; \mathbf{x}_r) = \mathcal{N}\left(\mu_\phi(\mathbf{x}_r), \Sigma_\phi(\mathbf{x}_r)\right)$$

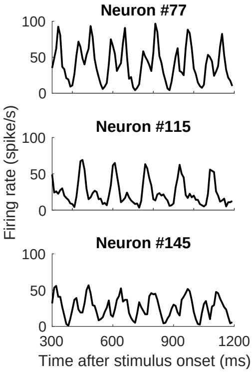
  - Block tri-diagonal precision matrix that agrees with Markovian structure
  - Potentially useful to perform filtering in an online fashion
- Reparameterization trick for stochastic optimization
  - Easy implementation
  - Low variance

Experiments

	Mean	Variance	Likelihood	Concise representation
PLDS	✓	✗	✗	✗
GCLDS	✓	✓	✓	✗
PfLDS	✓	✗	✗	✓
GCfLDS	✓	✓	✓	✓

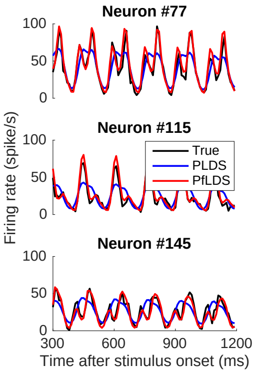
Real data analysis: primate visual cortex

Firing rate



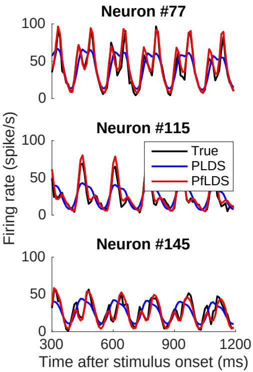
Real data analysis: primate visual cortex

Firing rate



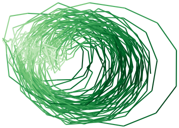
Real data analysis: primate visual cortex

Firing rate

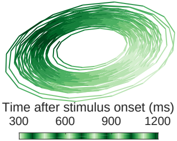


Latent projection

PLDS

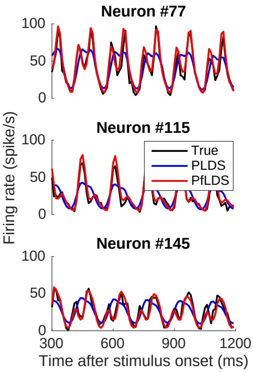


PfLDS

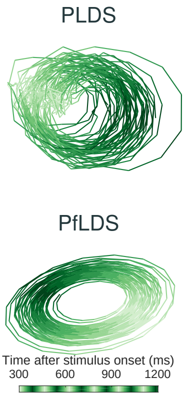


Real data analysis: primate visual cortex

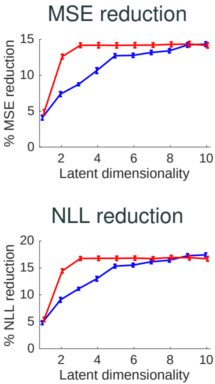
Firing rate



Latent projection

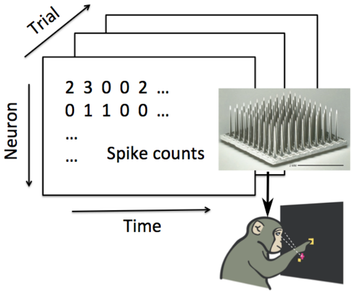


1-step-ahead prediction

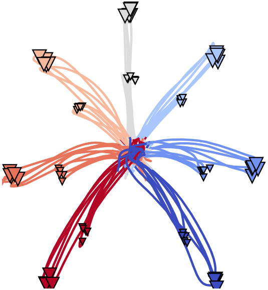


Real data analysis: Primate motor cortex

Data

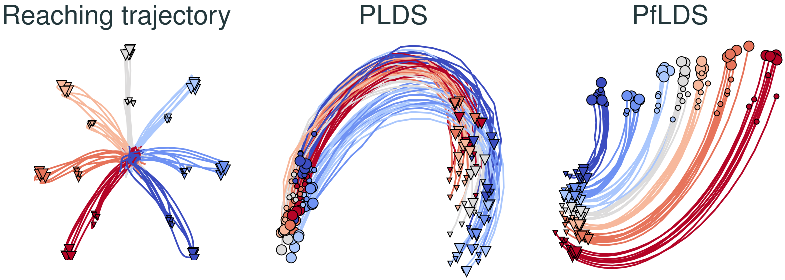


Reaching trajectory



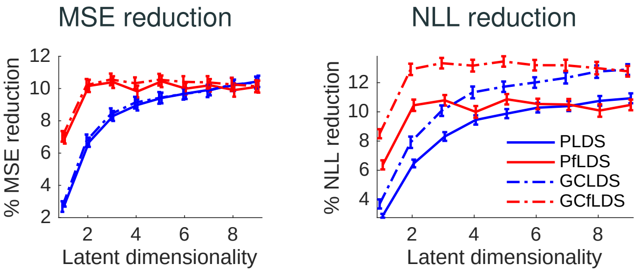
Real data analysis: Primate motor cortex

- Latent projection with 2 latent dimensions



Real data analysis: Primate motor cortex

- One-step-ahead predictive performance



Conclusion and discussion

- Summary
  - Incorporated nonlinear observation into state space models.
  - Developed AEVB algorithm (flexible and scalable).
  - Obtain concise latent representation.
- Future work
  - Better stochastic optimization scheme
  - Interpretable nonlinearity
  - Application on more complex datasets
- Gao Y*, Archer E*, Paninski L, Cunningham JP (2016) Linear dynamical neural population models through nonlinear embeddings. NIPS 2016. (* = equal contribution)

Table of Contents

Introduction: calcium imaging data

- Basic principle: the spiking activity of a neuron induce a transient increase in calcium concentration, which can be indirectly observed by recording the fluorescent properties of certain calcium indicators.
- Allows simultaneous recording from hundreds of thousands of neurons.

Model formulation: idea

- $X \in \mathbb{R}^{N \times T}$ represents the calcium imaging data, where each column is a (vectorized) frame that contains N pixels
- Decompose X into a product of K spatial component and temporal component

$$X = D \cdot A^T + \text{noise}$$

  - $D = [D_1, ..., D_K] \in \mathbb{R}^{N \times K}$ represents the neuron shapes
  - $A = [A_1, ..., A_K] \in \mathbb{R}^{T \times K}$ is the neural activities
- Further exploit structure of the components (localized neuron shapes)

Model formulation: objective

- Structured matrix factorization

$$\begin{aligned} \underset{D,A}{\text{minimize}} \quad & \|X - DA^T\|_2^2 + f_D(D), \\ \text{subject to} \quad & D_k \in \mathcal{D}_w^+; k = 1, \ldots, K, \\ & \|A_k\|_2 \leq c_k, \end{aligned}$$

- $\mathcal{D}_w^+$: non-negative vectors whose nonzero values is within a $w \times w$ window
- $f_D(D)$ regularizes the neuron shape (discussed later)
- $\|A_k\|_2 \leq c_k$ avoids degenerate solution

Greedy algorithm

- Scan the each frame of the video with a small Gaussian kernel
- At iteration k, given the current residue (unexplained by existing ROI)
    - Greedy identification: Identify the location p_k where the Gaussian kernel explains most of the data (across time)
    - Shape fine tuning: Locally optimize the spatial and temporal component
    - Residue update: Subtract the newly identified ROI

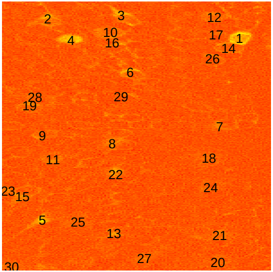
Shape fine tuning

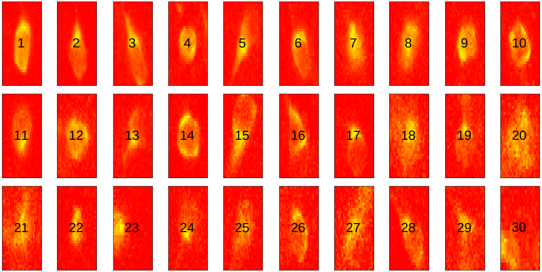
- Given current residue R, an identified center pixel p_k, denote S_k as a $w \times w$ window centered at p_k

$$\begin{aligned} \underset{D_k, A_k}{\text{minimize}} \quad & \|R - D_k A_k^T\|^2 + f(D_k), \\ \text{subject to} \quad & D_{kp} \geq 0, p \in S_k, \\ & D_{kp} = 0, p \notin S_k, \\ & \|A_k\|_2 \leq c_k, \end{aligned}$$

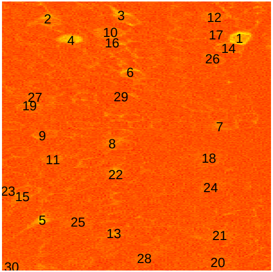
- $f(D_k) = \sum_{i=1}^{3} \lambda_i f_i(D_k)$
  - $f_1(D_k) = \sum_p \tau_{(p,p_k)} |D_{kp}|$ encourages sparsity
  - $f_2(D_k) = \sum_p (D_{kp} - G_{p_k})^2$ encourage Gaussian shape
  - $f_3(D_k) = \sum_{p_1 \text{ and } p_2 \text{ are neighbors}} (D_{kp_1} - D_{kp_2})^2$ encourages smoothness
- Optimize D_k and A_k by block coordinate descent

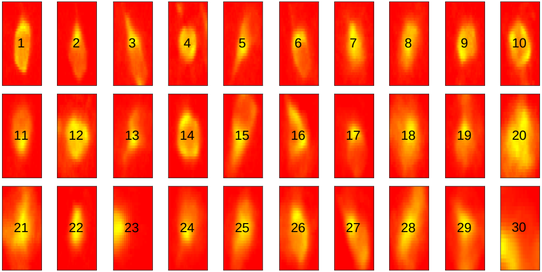
Real data analysis: sample patch, no shape regularization





Real data analysis: sample patch, shape regularization





Conclusion and discussion

- Summary
  - Formulating calcium imaging ROI detection as a structure matrix factorization problem
  - Greedy algorithm with shape regularization
  - Fast ROI detection algorithm
- Future work
  - More spatial and temporal structure
  - Overlapping neuron
  - Online ROI detection
  - Motion correction, background elimination

- Pnevmatikakis EA, Soudry D, Gao Y, Machado TA, Merel J, Pfau D, Reardon T, Mu Y, Lacefield C, Yang W, Ahrens M, Bruno R, Jessell TM, Peterka DS, Yuste R, Paninski L (2016) Simultaneous denoising, deconvolution, and demixing of calcium imaging data. Neuron, 89(2), 285-299.

Table of Contents

Maximum entropy principle

- Entropy: for a continuous distribution with density $p(\mathbf{z})$ where $\mathbf{z} \in \mathbb{R}^d$, the entropy is defined as

$$H(p) = -\int p(\mathbf{z}) \log p(\mathbf{z}) d\mathbf{z} = E_{Z \sim p} \left[-\log p(Z) \right].$$

  A popular measure of diversity or information content.
- Maximum entropy (ME) principle: Subject to some given prior knowledge (moment or support constraints), the distribution that makes minimal additional assumptions is that which has the largest entropy of any distribution obeying those constraints

Maximum entropy problem

- Maximum entropy (ME) problem

$$
\begin{aligned}
p^* \quad = \quad & \text{maximize} \quad H(p) \\
& \text{subject to} \quad E_{\mathbf{Z}\sim p}[T(\mathbf{Z})] = 0 \\
& \qquad\qquad\qquad \mathrm{supp}(p) = \mathcal{Z},
\end{aligned}
$$

where $T(\mathbf{z}) = (T_1(\mathbf{z}), ..., T_m(\mathbf{z})) : \mathcal{Z} \to \mathbb{R}^m$ is the vector of known statistics, and $\mathcal{Z}$ is the given support.

Applications of maximum entropy

- Texture modeling: generate an image with a certain texture by specifying expected value of features relevant to texture.
- Neuroscience: generate a distribution of neural activity by specifying a set of features (pairwise correlation etc.) for hypothesis testing.
- Ecology: Fit species distribution with certain feature constraints (altitude, temperature etc.).
- Finance: Fit the risk-neutral distribution of an asset given a list of option prices.
- ...

Gibbs distribution

- Under standard regularity conditions, the maximum entropy problem can be solved by Lagrange multipliers, yielding an exponential family p^* of the form (Gibbs distribution):

$$p^*(\mathbf{z}) \propto e^{<\eta, T(\mathbf{z})>} \mathbb{1}(\mathbf{z} \in \mathcal{Z})$$

- Identifying $\eta \in \mathbb{R}^m$ can be hard in high-dimensional setting.
- Sampling from the distribution can be hard. MCMC methods can take long to mix.
- Question: is there a better way to do this?

Idea: normalizing flow

- Considering a family of smooth and invertible transformation (normalizing flow)

$$\mathcal{F} = \{f_\phi : \mathbb{R}^d \to \mathbb{R}^d, \phi \in \mathbb{R}^q\}$$

- Identify a transformation $f_{\phi^*} \in \mathcal{F}$ that transforms a simple distribution p_0 to approximate the maximum entropy distribution.

$$\begin{aligned} \phi^* \quad = \quad & \text{maximize} \quad H(p_\phi) \\ & \text{subject to} \quad E_{\mathbf{Z}_0 \sim p_0}[T(f_\phi(\mathbf{Z}_0))] = 0 \\ & \qquad\qquad\quad\; \text{supp}(p_\phi) = \mathcal{Z}. \end{aligned}$$

where $p_\phi(\mathbf{z})$ is the distribution of $f_\phi(Z_0)$ for $Z_0 \sim p_0$.

$$p_\phi(\mathbf{z}) = p_0(f_\phi^{-1}(\mathbf{z}))|\det(J_\phi(\mathbf{z}))|^{-1}$$

Augmented Lagrangian method

- Denote $R(\phi) = E\left(T(f_\phi(\mathbf{Z}_0))\right) \in \mathbb{R}^m$, augmented Lagrangian method minimizes the objective
$$L(\phi; \lambda, c) = -H(p_\phi) + \lambda^\top R(\phi) + \frac{c}{2}||R(\phi)||^2$$
for a sequence of $\lambda \in \mathbb{R}^m$ and $c \geq 0$.
- Update rule: at iteration k, given λ_k and c_k, suppose ϕ_k optimizes $L(\phi; \lambda_k, c_k)$, update λ and c by
$$\begin{aligned} \lambda_{k+1} =& \lambda_k + c_k R(\phi_k) \\ c_{k+1} =& \begin{cases} \beta c_k & ||R(\phi_k)|| > \gamma ||R(\phi_{k-1})|| \\ c_k & \text{otherwise} \end{cases} \end{aligned}$$
for some $\gamma \in (0, 1)$, $\beta > 1$

Augmented Lagrangian method in stochastic setting

- Denote $R(\phi) = E\left(T(f_\phi(\mathbf{Z}_0))\right) \in \mathbb{R}^m$, augmented Lagrangian method minimizes the objective

$$L(\phi; \lambda, c) = -H(p_\phi) + \lambda^\top R(\phi) + \frac{c}{2}||R(\phi)||^2$$

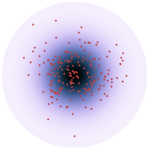
  for a sequence of $\lambda \in \mathbb{R}^m$ and $c \geq 0$.
- Here $R(\phi) = E\left(T(f_\phi(\mathbf{Z}_0))\right) \in \mathbb{R}^m$ is intractable, but we can approximate with a sampled version.

$$R(\phi) \approx \frac{1}{n}\sum_{i=1}^{n} T(f_\phi(\mathbf{z}^{(i)})), \mathbf{z}^{(i)} \sim p_0$$

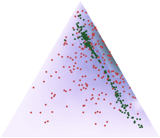
  We can then optimize the objective by stochastic gradient descent.

Experiment: Dirichlet

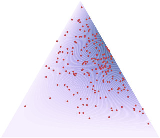
- Dirichlet distribution is the ME distribution on a simplex $\mathcal{S} = \{\mathbf{z} = (z_1, \ldots, z_{d-1}) : z_i \geq 0$ and $\sum_{k=1}^{d-1} z_k \leq 1\}$ with expetation on the log of each coordinate $E[\log Z_k] = \kappa_k (k = 1, ..., d)$, where $Z_d = 1 - \sum_{k=1}^{d-1} Z_k$.



Initial distribution p_0



MEFN result p_{ϕ^*} (Control case: moment matching)



Ground truth p^*

Experiment: Texture modeling

- Sample from the space of images $\mathbf{z} \in [0,1]^{224\times224\times3}$ where $T(\mathbf{z}) : [0,1]^{224\times224\times3} \to \mathbb{R}$ is a complicated "texture loss" defined in Ulyanov et al. 2016.
- Ulyanov et al. 2016 proposes texture net, which solves the problem by

$$\min E_{\mathbf{Z}\sim p_0}\left[T(f_\phi(\mathbf{Z}))\right]$$

  without considering entropy term.
- We compare our formulation with texture net. We use real-nvp (Dinh, Sohl-Dickstein, and Bengio 2016) as the normalizing flow structure

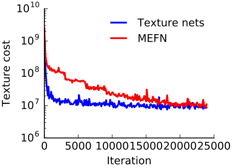
Experiment: Texture modeling

Input

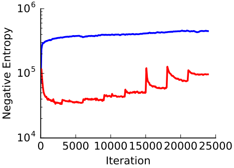
Texture net
(Ulyanov et al. 2016)

MEFN (ours)

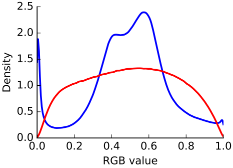
Texture cost



Negative entropy



RGB histogram



Experiment: Texture modeling

- We provide two numerical measure for assessing sample diversity given a set of images $\{\mathbf{z}^{(1)}, ..., \mathbf{z}^{(n)}\}$.

Method	d_{L^2}	SST	SSW	SSB
Texture net	11534	128680	109577	19103
MEFN	17014	175604	161639	13964

- Mean Euclidean distance: $d_{L^2} = \text{mean}_{i \neq j} \|\mathbf{z}^{(i)} - \mathbf{z}^{(j)}\|_2^2$
- ANOVA: $\text{SST} = \text{SSW} + \text{SSB}$
  - Total: $\text{SST} = \sum_{i,k} (z_k^{(i)} - \bar{z})^2$
  - Within group: $\text{SSW} = \sum_{i,k} (z_k^{(i)} - \bar{z}_k)^2$ (larger $\Rightarrow$ better)
  - Between group: $\text{SSB} = \sum_k n(\bar{z}_k - \bar{z})^2$ (smaller $\Rightarrow$ better)

Conclusion and discussion

- Summary
    - Bridging information theory and deep learning
    - Solve maximum entropy problem by optimizing a normalizing flow
    - Combining augmented Lagrangian optimization with stochastic optimization
    - Promising result on simulation and real data
- Future work
    - Better normalizing flow structure
    - Better constrained stochastic optimization algorithm
- Loaiza G*, Gao Y*, Cunningham JP (2017) Maximum entropy flow networks. ICLR 2017. (*=equal contribution)

Table of Contents

fLDS: AEVB form (backup)

MEFN: Normalizing flow structures (backup)