Indirect Inference for Stable Distributions and Income Distribution

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Introduction to Stable Distributions

Generalized Central Limit Theorem
[Gnedenko and Kolmogorov, 1954].

Definition

A random variable X is said to have a stable distribution if there is a sequence of i.i.d. random variables Y_1, Y_2, $\cdots$, Y_n and sequences of positive numbers of d_n and real number a_n, such that $\frac{Y_1+Y_2+\ldots+Y_n}{d_n} + a_n \xrightarrow{D} X$, where $\xrightarrow{D}$ denotes convergence in distribution.

Introduction to Stable Distribution

Definition

A random variable X follows stable distribution $S(\alpha, \beta, \sigma, \mu)$ if its characteristic function $\varphi(t) = Ee^{itX}$ has the following form, where $0 < \alpha \leq 2$ measures the tail thickness, $-1 \leq \beta \leq 1$ determines skewness, and $\mu \in \mathbb{R}$, $\sigma > 0$ are location and scale parameters in the sense that $\frac{X-\mu}{\sigma} \sim S(\alpha, \beta, 1, 0)$. $\varphi(t) =$

$$\begin{cases} exp(-\sigma^{\alpha}|t|^{\alpha}(1 - i\beta\frac{t}{|t|}tan(\frac{\pi\alpha}{2})) + i\mu t), \alpha \neq 1 \\ exp(-\sigma|t|^{\alpha}(1 + i\beta\frac{2}{|\pi|}\ln(t)) + i\mu t), \alpha = 1 \end{cases} \tag{1}$$

Existing Estimation Methods

- MLE [Nolan, 2002].
- Regression method based on characteristic function [Koustrouvellis, 1980].
- The quantile method of [McCulloch, 1986].
- Bayesian perspective: Gibbs sampling [Buckle, 1995].

Spacings Estimation Idea

- $x_1, \ldots, x_n$ from a univariate distribution with distribution function $F(x; \boldsymbol{\theta})$.
- $D_i(\boldsymbol{\theta}) = F(x_{(i)}; \boldsymbol{\theta}) - F(x_{(i-1)}; \boldsymbol{\theta}), i = 1, \ldots, n+1.$
- For any convex function $h : (0, \infty) \to \Re$, find $\boldsymbol{\theta}$ which minimizes the quantity $T_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} h(nD_i(\boldsymbol{\theta}))$.
- This method gives estimates as good as MLE for large samples (computational savings).

Monte Carlo Evaluation: $(\alpha, \beta) = (1.5, 0.2)$

N=500 B=1000				
	MLE method	Spacing method	Regression method	Quantile method
$(\hat{\alpha}, \hat{\beta})$	(1.5016,0.2159)	(1.4951,0.1944)	(1.5031,0.2058)	(1.5039,0.2242)
MSE	(0.0138,0.0567)	(0.0173,0.0741)	(0.0179,0.0833)	(0.0250,0.0822)

Table: Monte Carlo mean and mean square error of $(\hat{\alpha}, \hat{\beta})$ with different methods

Quantile method of McCulloch

X_p is the *pth* quantile if $F(x_p) = p$, where $F(x)$ is the distribution function. Define two functions of theoretical quantiles:

$$\begin{cases} v_\alpha = \dfrac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}} = \phi_1(\alpha, \beta) = \phi_1(\alpha, -\beta) \\ v_\beta = \dfrac{(x_{0.95} - x_{0.5}) - (x_{0.5} - x_{0.05})}{x_{0.95} - x_{0.05}} = \phi_2(\alpha, \beta) = -\phi_2(\alpha, -\beta) \end{cases} \tag{2}$$

Introduction to Indirect Inference

- First introduced by [C.Gourierou and Renault, 1993].
- A simulation-based inference method requires only that data could be simulated from the model.
- Useful when likelihood and/or moments are intractable.
- Idea is close to the Generalized Method of Moments [Hansen, 1982].

An Estimation Framework

- Suppose data $\boldsymbol{X}$ comes from distribution $F_{\boldsymbol{\theta}}$.
- $\boldsymbol{\pi}(\boldsymbol{\theta})$: auxiliary parameter, not required to have a explicit expression.
- $\hat{\boldsymbol{\pi}}$: auxiliary statistics, the empirical counterpart of $\boldsymbol{\pi}(\boldsymbol{\theta})$.
- In general, an estimator could be defined:

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\operatorname{argmin}} \ (\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}(\boldsymbol{\theta}))^T \boldsymbol{\Omega} (\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}(\boldsymbol{\theta})) \tag{3}$$

- If $E(\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}(\boldsymbol{\theta})) = \mathbf{0}$: this is the Generalized method of moments.

Indirect Inference

$\boldsymbol{\pi}(\boldsymbol{\theta})$ can be approximated by $\boldsymbol{\pi}^*(\boldsymbol{\theta})$ via parametric bootstrap [Efron, 1979].

Step 1 H samples of sample size N is simulated from $F_{\boldsymbol{\theta}}$.

Step 2 For each sample $h, h = 1, 2, \ldots, H$, its $\boldsymbol{\pi}^{*h}(\boldsymbol{\theta})$ is calculated based on its empirical distribution function.

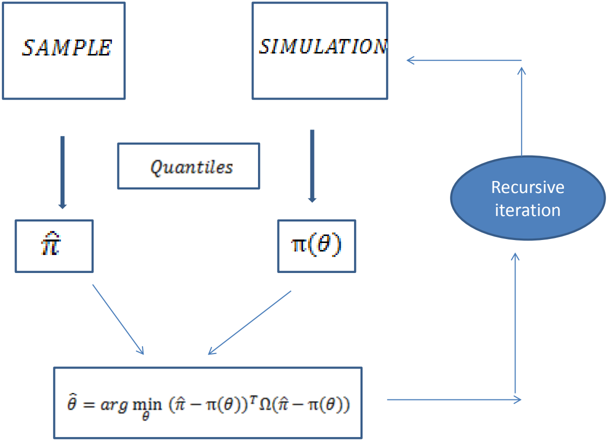
Step 3 $\boldsymbol{\pi}(\boldsymbol{\theta})$ could be approximated by $\boldsymbol{\pi}^*(\boldsymbol{\theta}) = \frac{1}{H} \sum_{h=1}^{H} \boldsymbol{\pi}^{*h}(\boldsymbol{\theta})$

Quantile based Indirect Inference

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\operatorname{argmin}} \ (\hat{\boldsymbol{q}} - \boldsymbol{q}^*(\boldsymbol{\theta}))^T \boldsymbol{\Omega} (\hat{\boldsymbol{q}} - \boldsymbol{q}^*(\boldsymbol{\theta})) \tag{4}$$

- $\boldsymbol{q}(\boldsymbol{\theta}) = (X_{p_1}, X_{p_2}, \cdots, X_{p_m})^T$, where $F_\theta(X_{p_i}) = p_i, 0 < p_i < 1, i = 1, 2, \cdots, m$.
- These quantiles are usually equally spaced: $p_{i+1} - p_i = p_i - p_{i-1}$, for $1 < i < m-1, p_0 = 0$.

Estimation Algorithm



Choice of Weight Matrix

- Optimal choice: $\Omega = (Var(\boldsymbol{q}(\hat{\boldsymbol{\theta}})))^{-1}$.
- Two-step procedure[Hansen, 1982]:

Step 1 : Use $\boldsymbol{\Omega} = \boldsymbol{I}$ initially, with $\boldsymbol{I}$ denoting the identity matrix, to solve the optimization problem in (2) and obtain initial estimate $\boldsymbol{\theta}_1$.

Step 2 : Use the weight matrix with $\hat{\boldsymbol{\Omega}} = (Var(\hat{\boldsymbol{q}}(\boldsymbol{\theta}_1)))^{-1}$ as the next approximation.

- $Var(\hat{\boldsymbol{q}}(\boldsymbol{\theta}_1)))^{-1}$ may need to be approximated by parametric bootstrap.

Asymptotic Normality of Auxiliary Parameter

Lemma

([Cramer, 1946], page 369) Let $0 < p_1 < \cdots < p_m < 1$. *Suppose that cumulative distribution function* F *has a density* f *in neighborhoods of quantiles* $\boldsymbol{q} = (X_{p_1}, \cdots, X_{p_m})^T$ *and that* f *is positive and continuous at* $\boldsymbol{q}$. *Then the empirical quantiles* $\hat{\boldsymbol{q}} = (\hat{X}_{p_1}, \cdots, \hat{X}_{p_m})^T$ *have asymptotically normal distribution:* $\sqrt{n}\,(\hat{\boldsymbol{q}} - \boldsymbol{q}) \xrightarrow{\mathcal{D}} \mathcal{N}\,(\mathbf{0}, \mathbf{V})$. *where the* (i, j)*-th element of covariance matrix* $\mathbf{V}$ *is*

$$\mathbf{V}_{ij} = \frac{p_i(1-p_j)}{f(X_{p_i})f(X_{p_j})} = \frac{p_i(1-p_j)}{f(F^{-1}(p_i))f(F^{-1}(p_j))}, \text{ for } 1 \le i \le j \le m.$$

Conditions needed for Asymptotic Normality of $\hat{\theta}$

(A1) Asymptotic normality of auxiliary parameter:
$\boldsymbol{\xi}_n = \sqrt{n}(\hat{\boldsymbol{q}} - \boldsymbol{q}(\boldsymbol{\theta}_0)) \xrightarrow{D} N(\mathbf{0}, \boldsymbol{V})$ where $\boldsymbol{V} = \lim_{n\to\infty} Var(\boldsymbol{\xi}_n)$

(A2) There is a unique $\boldsymbol{\theta}_0$ such that sample quantiles equal the theoretical ones: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ if and only if $\hat{\boldsymbol{q}} = \boldsymbol{q}(\boldsymbol{\theta_0})$.

(A3) If $\boldsymbol{\Omega}$ is estimated by $\hat{\boldsymbol{\Omega}}$, then $\hat{\boldsymbol{\Omega}} \xrightarrow{P} \boldsymbol{\Omega}$, where $\boldsymbol{\Omega} > 0$

(A4) $\boldsymbol{q}(\boldsymbol{\theta})$ is a differentiable function with $\mathbf{D}(\boldsymbol{\theta}) = \partial \boldsymbol{q}(\boldsymbol{\theta})/\partial \boldsymbol{\theta}^T$.

(A5) The matrix $\mathbf{D}^T(\boldsymbol{\theta})\boldsymbol{\Omega}\mathbf{D}(\boldsymbol{\theta})$ is full rank.

(A6) $\boldsymbol{\Theta}$ is compact.

(A7) The choice of the initial value of $\boldsymbol{\theta}$ is independent of the estimation algorithm.

Asymptotic Normality of $\hat{\theta}$

Theorem

[C.Gourierou and Renault, 1993] [Dominicy and Veredas, 2012] Under the regularity conditions (A1)-(A7), the indirect estimator is asymptotically normal, when n goes to infinity (with H fixed):

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xmapsto{\mathcal{D}} \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Lambda}\right)$$

with $\boldsymbol{\Lambda} = (1 + \frac{1}{H})\boldsymbol{\Gamma}\mathbf{V}\boldsymbol{\Gamma}^T$ *where*
$\boldsymbol{\Gamma} = \left(\mathbf{D}^T\left(\theta_0\right)\boldsymbol{\Omega}\mathbf{D}\left(\theta_0\right)\right)^{-1}\mathbf{D}^T\left(\theta_0\right)\boldsymbol{\Omega}$.

Robustness Property

- If the influence function of an estimator is bounded, then the estimator is said to be "robust".
- [Genton and Ronchetti, 2003] showed that if the auxiliary parameter $\boldsymbol{\pi}(\boldsymbol{\theta})$ in indirect inference approach has a bounded influence function, then so does the indirect estimator $\hat{\boldsymbol{\theta}}$
- We showed that $\boldsymbol{q}(\boldsymbol{\theta})$ does have a bounded influence function in our case.

Choice of Quantiles

- McCulloch (1986) suggested using:

$$\pi(\theta) = (\frac{X_{0.95} - X_{0.05}}{X_{0.75} - X_{0.25}}, \frac{(X_{0.95} - X_{0.50}) - (X_{0.5} - X_{0.25})}{X_{0.95} - X_{0.05}}, X_{0.75} - X_{0.25}, X_{0.5})^T \qquad (5)$$

- More generally, we use m equally spaced quantiles:

$$\pi(\theta) = (X_{p_1}, X_{p_2}, \cdots, X_{p_m})^T \qquad (6)$$

Simulation Studies

- MSE of our estimator $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma}, \hat{\mu})^T$ is approximated by Monte Carlo with B replications.
- $MSE(\hat{\boldsymbol{\theta}}) = E(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2 \approx \frac{1}{B}\sum_{i=1}^{B}(\hat{\boldsymbol{\theta}}_{\boldsymbol{i}} - \boldsymbol{\theta})^2$ where each $\hat{\boldsymbol{\theta}}_{\boldsymbol{i}}$ is estimated by its individual simulated sample with sample size N.

Simulation Studies

- For simulations we set $\boldsymbol{\theta} = (\alpha, \beta, \sigma, \mu)^T = (1.5, -0.2, 1, 0)^T$.
- Add mean $\bar{X}$ to the auxiliary statistics:

$$\hat{\boldsymbol{\pi}} = (\hat{X}_{p_1}, \hat{X}_{p_2}, \cdots, \hat{X}_{p_m}, \bar{X})^T \tag{7}$$

- Evaluate the best m by Monte Carlo. Sample size N=1000, Monte Carlo replication B=1000.
- m is chosen to be odd to include the median among the auxiliary statistics.

How many quantiles should be selected

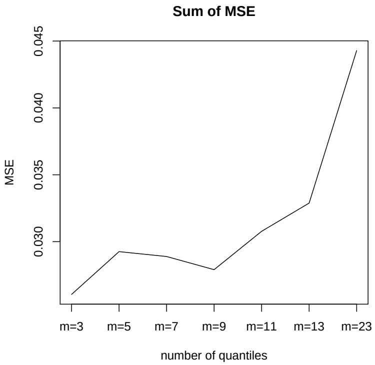


Figure: Sum of MSE

$$\text{Sum of MSE} = MSE(\hat{\alpha}) + MSE(\hat{\beta}) + MSE(\hat{\sigma}) + MSE(\hat{\mu})$$

Location parameter

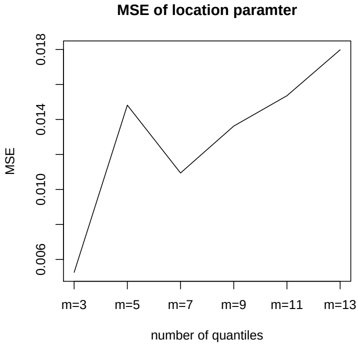


Figure: MSE of location $\hat{u}$ by different q

$MSE(\hat{\mu}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\mu}_i - \mu)^2$, where true parameter $\mu = 0$. B=1000.

MSE comparison: Identity matrix vs 2-step weight matrix

$N = 1000, B = 1000$ $\beta = -0.2, \sigma = 1, \mu = 0$		
	Identity matrix	2-step weight matrix
$\alpha = 1.2$	0.0092	0.0005
$\alpha = 1.3$	0.0067	0.0005
$\alpha = 1.4$	0.0052	0.0009
$\alpha = 1.5$	0.0047	0.0013
$\alpha = 1.6$	0.0040	0.0023
$\alpha = 1.7$	0.0033	0.0037
$\alpha = 1.8$	0.0023	0.0030
$\alpha = 1.9$	0.0014	0.0025

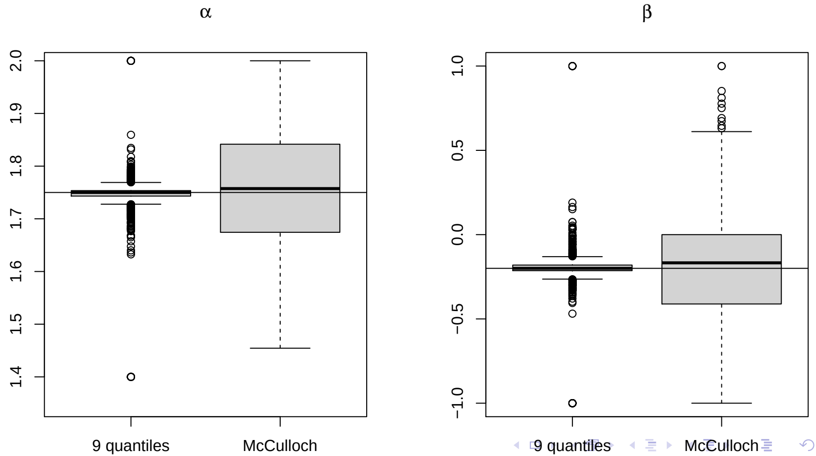
MSE comparison: Identity matrix vs 2-step weight matrix

$N = 1000, B = 1000$ $\alpha = 1.5, \sigma = 1, \mu = 0$		
	Identity matrix	2-step weight matrix
$\beta = 0$	0.0045	0.0057
$\beta = -0.1$	0.0050	0.0051
$\beta = -0.2$	0.0082	0.0069
$\beta = -0.3$	0.0121	0.0103
$\beta = -0.4$	0.0235	0.0177
$\beta = -0.5$	0.0247	0.0169

Final Choice

- $\hat{\pi} = (\hat{X}_{0.1}, \hat{X}_{0.2}, \cdots, \hat{X}_{0.9}, \bar{X})^T$
- Two-step weight matrix is chosen for $\alpha \leq 1.6$, $\beta \leq -0.2$, otherwise, the identity matrix performs better.

Monte Carlo Evaluation: N=1000, B=1000



MSE Comparison Between Methods

$\beta = -0.2$
$\sigma = 1$
$\mu = 0$

	Indirect inference	MLE	McCulloch	Regression method
$\alpha = 1.2$	0.0005	0.0018	0.0027	0.0029
$\alpha = 1.3$	0.0005	0.0020	0.0033	0.0031
$\alpha = 1.4$	0.0009	0.0025	0.0036	0.0034
$\alpha = 1.5$	0.0013	0.0026	0.0046	0.0035
$\alpha = 1.6$	0.0023	0.0026	0.0057	0.0036
$\alpha = 1.7$	0.0033	0.0024	0.0075	0.0030
$\alpha = 1.8$	0.0023	0.0019	0.0091	0.0027
$\alpha = 1.9$	0.0014	0.0013	0.0080	0.0018

MSE Comparison Between Methods

$\alpha = 1.5$, $\sigma = 1$, $\mu = 0$	Indirect inference	MLE	McCulloch	Regression method
$\beta = 0$	0.0045	0.0096	0.0126	0.0161
$\beta = -0.1$	0.0050	0.0099	0.0137	0.0147
$\beta = -0.2$	0.0069	0.0095	0.0117	0.0158
$\beta = -0.3$	0.0103	0.0097	0.0127	0.0144
$\beta = -0.4$	0.0177	0.0084	0.0136	0.0153
$\beta = -0.5$	0.0169	0.0078	0.0156	0.0144

Data

USA 's Income share by deciles(%)										
Year	lowest	2nd	3rd	4th	5th	6th	7th	8th	9th	highest
2010	1.70	3.40	4.56	5.73	7.00	8.44	10.19	12.52	16.25	30.19

Poverty index				
Year	mean($/month)	pov.line	headcount(%)	Gini index(%)
2010	1917.38	1.90	1.00	41.06

Table: Original Data

Lorenz Curve

- Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be ordered data. The empirical Lorenz Curve is defined as
$$L(i/n) = s_i/s_n \tag{8}$$
where $s_i = x_1 + x_2 + \cdots + x_i, L(0) = 0, i = 0, \cdots, n.$
- If x_i is drawn from the distribution $F(x)$ with mean μ, p is a particular quantile where $0 \leq p \leq 1$, the theoretical Lorenz Curve is defined by
$$L(p) = \mu^{-1} \int_0^p F^{-1}(t)dt \tag{9}$$

Gini Index

- $Gini := \frac{E|X - Y|}{2 * E(X)}$ where X, Y are two random points drawn from the distribution F.
- The sample version:

$$Gini(S) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j|}{2(n-1) \sum_{i=1}^{n} x_i} \tag{10}$$

- It could also be calculated via Lorenz Curve: [Gastwirth, 1972]

$$G(t) = 2 * \int_0^1 (t - L(t))\, dt \tag{11}$$

Transform Data to Empirical LC

USA Empirical LC									
By deciles(%)									
p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\hat{L}(p)$	1.70	5.10	9.66	15.39	22.39	30.83	41.02	53.54	69.77

Table: Transform Data to empirical LC

Indirect Inference Framework

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}\in\boldsymbol{\Theta}}{\operatorname{argmin}} \; (\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}^*(\boldsymbol{\theta}))^T \, \boldsymbol{\Omega} \, (\hat{\boldsymbol{\pi}} - \boldsymbol{\pi}^*(\boldsymbol{\theta})) \tag{12}$$

- $\hat{\boldsymbol{\pi}} = (\bar{X}, \hat{L}(0.1), \cdots, \hat{L}(0.9))$

Parametric Distribution

- The income distribution is heavily positively skewed and has a long right tail
- Generalized Beta-2 distribution, Gamma distribution and lognormal distribution are usually chosen.
- [Cowell, 1995] suggests not to use more complicated four parameters densities because of over-fitting problem.
- Lognormal is chosen in our case.

Properties of This Estimator

- $L(\boldsymbol{p})$ and $F^{-1}(\boldsymbol{p})$ share the similar properties:
- [Goldie, 1977] proved that the Lorenz process $l_n(\boldsymbol{p}) = \sqrt{n}[L_n(\boldsymbol{p}) - L(\boldsymbol{p})], 0 \leq \boldsymbol{p} \leq 1$, converges to a Gaussian process if $L(\boldsymbol{p})$ is continuous at the empirical points.
- Thus the asymptotical property of our auxiliary parameters $L(\boldsymbol{p}, \boldsymbol{\theta})$ is established.
- The asymptotic normality of our estimator $\hat{\boldsymbol{\theta}}$ is guaranteed by Theorem 17 if stated regularity conditions (A2)-(A7) hold.

A test for goodness of fit

- Quadratic form: If $\boldsymbol{y} \sim N(\boldsymbol{0}, \boldsymbol{I}_p)$ and $\boldsymbol{A}$ is an idempotent matrix with rank q, then $\boldsymbol{y}^T \boldsymbol{A}\boldsymbol{y} \sim \chi^2_q$.
- J-test:

$$J_N = N\left(\hat{L}(\boldsymbol{p}) - L(\boldsymbol{p}, \hat{\boldsymbol{\theta}})\right)^T \hat{\boldsymbol{\Omega}} \left(\hat{L}(\boldsymbol{p}) - L(\boldsymbol{p}, \hat{\boldsymbol{\theta}})\right) \xrightarrow{D} \chi^2_{P-K} \tag{13}$$

P is the dimension of auxiliary parameters, K is the number of estimated parameters in the distribution. The sample size N is number of the surveyed citizens. [Hajargasht et al., 2012] assume $N = 10000$.

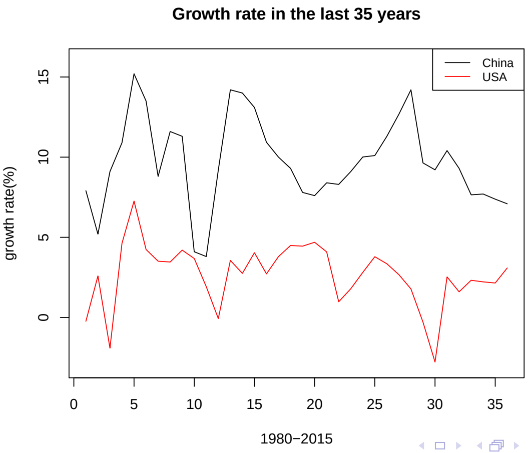
Comparison between $\hat{L}(\boldsymbol{p})$ and $L(\boldsymbol{p}, \hat{\theta})$

USA 2010									
p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\hat{L}(\boldsymbol{p})$	1.70	5.10	9.66	15.39	22.39	30.83	41.02	53.54	69.77
$L(\boldsymbol{p}, \hat{\theta})$	2.15	5.63	10.14	15.63	22.31	30.59	40.66	52.98	69.17

Table: Goodness of Fit Assessment

Case study

GDP 2015. USA: $18,287 billion China: $11,285 billion



Case study

USA 's Income share by deciles(%)										
Year	lowest	2nd	3rd	4th	5th	6th	7th	8th	9th	highest
2010	1.70	3.40	4.56	5.73	7.00	8.44	10.19	12.52	16.25	30.19
1981	1.81	3.59	5.00	6.23	7.51	8.95	10.69	12.96	16.40	26.86

Poverty Index				
Year	mean($/month)	pov.line($/day)	headcount(%)	Gini index(%)
2010	1917.38	1.9	1	41.06
1981	1581.81	1	0.67	37.73

Table: Income inequality of USA: 1981 v.s 2010

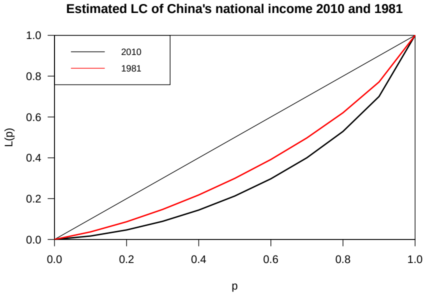
Case study

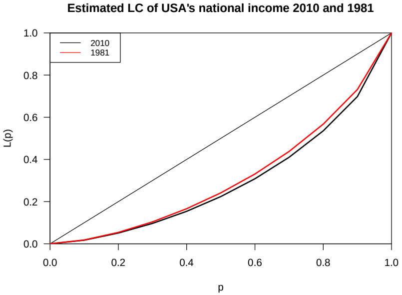
China 's Income share by deciles(%)										
Year	lowest	2nd	3rd	4th	5th	6th	7th	8th	9th	highest
2010	1.69	2.98	4.23	5.51	6.88	8.43	10.31	12.88	17.11	29.98
1981	3.72	4.96	6.05	7.08	8.12	9.25	10.58	12.31	15.08	22.86

Poverty Index				
Year	mean($/month)	pov.line($/day)	headcount(%)	Gini index(%)
2010	218.54	1.9	11.18	42.06
1981	34.64	1	88.32	18.46

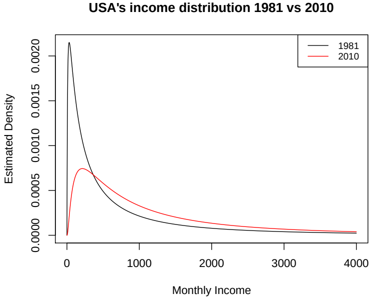
Table: Income inequality of China: 1981 v.s 2010

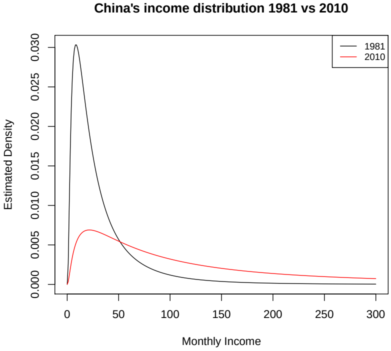
Case study


Estimated LC of China's national income 2010 and 1981
2010
1981
L(p)
p
0.0
0.2
0.4
0.6
0.8
1.0


Estimated LC of USA's national income 2010 and 1981
2010
1981
L(p)
p
0.0
0.2
0.4
0.6
0.8
1.0

Case study


USA's income distribution 1981 vs 2010
Estimated Density
0.0000
0.0005
0.0010
0.0015
0.0020
0
1000
2000
3000
4000
Monthly Income
1981
2010


China's income distribution 1981 vs 2010
Estimated Density
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0
50
100
150
200
250
300
Monthly Income
1981
2010

Main Contributions

- Spacings-based estimation performs as good as the MLE for large samples and is flexible.
- Quantile based indirect inference extends the idea and the estimation framework of [McCulloch, 1986].
- A practical estimation framework of analyzing income distribution is developed.

Further Work

- Explore indirect inference for wrapped stable distribution used in circular data [Gatto and Jammalamadaka, 2003].
- Explore "Tempered stable distribution" which patches thinner tails to a stable distribution[Rosiński, 2002].
- Indirect inference in time series, and correlated data.

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