shortest paths problem

$G=(V,E) \quad w: E \to \mathbb{Z}, \quad s \in V, \quad t \in V$

Solve shortest paths problem with s as source.

$$u \xrightarrow{(u,v).\text{weight}} v$$

$\delta(s,u) = u.\text{distance} \qquad \delta(s,v) = v.\text{distance}$

Define an edge $e = (u,v)$ to be tight if

$$v.\text{distance} = u.\text{distance} + (u,v).\text{weight}.$$

$\Rightarrow$ there is a shortest path from s to v in which $v.\pi = u$

Consider subgraph H of G that contains all tight edges of G.

Fact: Any path from s to u in H is a shortest path from s to u in G.

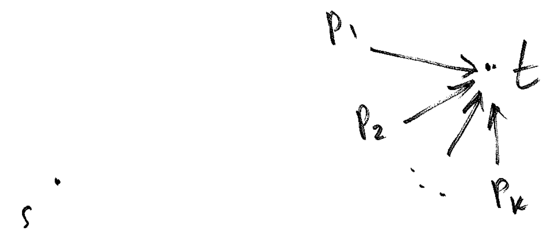
If G has no cycles of negative or zero weight, then H is acyclic (a DAG).

Counting shortest paths from s to t in G $\equiv$ Counting # of paths from s to t in H

Count # of paths in a DAG (H)

Define $N(t)$ = # of paths from s to t in H.

$N(s) = 1$



$$N(t) = \sum_{i=1}^{k} N(P_i)$$ where $P_1 \cdots P_k$ are nodes that have edges into t

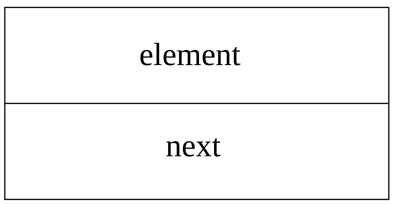
In any topological ordering of H, $P_1 \cdots P_k$ all must precede t.

Go through nodes in topological order: calculate N.

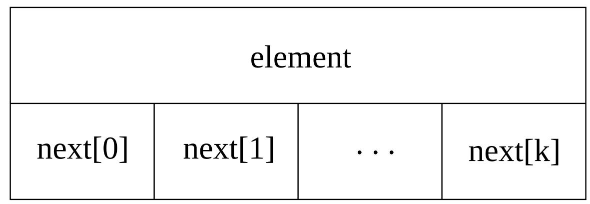
Enumeration — similar to enumeration of topological orders.

Skip Lists

Generalization of sorted linked lists for implementing Dictionary ADT (insert, delete, find, min, succ) in $O(\log n)$ expected time per operation, where the n is the size of the dictionary. Skip lists compete with balanced search trees like AVL, Red-Black, and B-Trees.

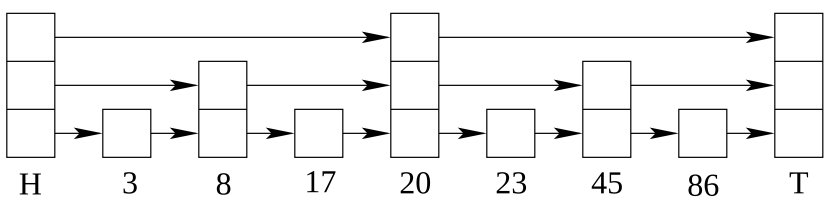


List Entry



Skip List Entry

The elements are stored in sorted order, in a linked list of nodes. Each skip list entry has an array of next pointers, where $next[i]$ points to an element that is roughly 2^i nodes away from it. The $next$ array at each entry has random size between 1 and $maxLevel$, the maximum number of levels in the current skip list. Ideally, $maxLevel \approx \log n$. Each skip list has dummy head and tail nodes, both of $maxLevel$ height, storing sentinels $-\infty$ and $+\infty$, respectively. Iterating through the list using $next[0]$ will go through the nodes in sorted order. A reference to the previous element can also be stored by adding a $prev$ field to Skip List Entry.



Search starts at $maxLevel$, goes as far as possible at each level, without going past target, descending one level at a time, until reaching the target node. Addition/Removal of nodes makes it difficult to maintain an ideal skip list, in which $next[i]$ of a node points to a node that is exactly 2^i away from it. Skip lists solve this problem by selecting the number of levels (size of $next[\,]$) of a new node probabilistically. When the size exceeds a threshold, elements are reorganized into an ideal skip list, with a new choice of $maxLevel$.

```
find(x): // Helper function
// return prev[0..maxLevel] of nodes at which search went down one level, looking for x
p ← head
for i ← maxLevel downto 0 do
    while p.next[i].element < x do
        p ← p.next[i]
    prev[i] ← p
return prev
```

```
chooseLevel(lev): // Choose number of levels for a new node randomly
// Prob(choosing level i) = 1/2 Prob(choosing level i − 1)
i ← 0
while i < lev do
    b ← random.nextBoolean()
    if b then i++ else break
return i
```

```
add(x):
prev ← find(x)
if prev[0].next[0].element = x then
    prev[0].next[0].element ← x
else
    lev ← chooseLevel(maxLevel)
    n ← new SkipListEntry(x, lev)
    for i ← 0 to lev do
        n.next[i] ← prev[i].next[i]
        prev[i].next[i] ← n
    size + +
```

```
contains(x):
prev ← find(x)
return prev[0].next[0].element = x ?
```

```
remove(x):
prev ← find(x)
n ← prev[0].next[0]
if n.element ≠ x then
    return null
else
    for i ← 0 to maxLevel do
        if prev[i].next[i] = n then
            prev[i].next[i] ← n.next[i]
        else break
    size − −
    return n.element
```